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# THE TOPOLOGY OF DISCRETE GROUPS

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In this paper we explore and exploit a relation between group theory and topology whose existence is suggested by a recent result of D. Kan and W. Thurston [14]: they show that for any connected CW-complex $X$, there exist a group $TX$ and a map $t_x : K(TX, 1) \to X$ which induces a homomorphism of $TX = \pi_1 K(TX, 1)$ onto $\pi_1 X$ and gives rise to isomorphisms of homology with respect to all local coefficient systems on $X$.

We amplify this result by showing that the category of pointed, connected CW-complexes and homotopy classes of maps is equivalent to a category of fractions of the category of pairs $(G, P)$, where $G$ is a group and $P$ is a perfect normal subgroup. In other words, homotopy theory can be reconstructed within group theory.

In view of this close connection, it seems appropriate to use methods of a geometric character in studying groups. We do this, in particular, for groups corresponding to finite CW-complexes, which we shall call geometrically finite groups; these are the groups for which $K(G, 1)$ has the homotopy type of a finite complex. This implies that $G$ has a finite presentation but is in fact a good deal stronger. For example, such a group must be torsion free.

Central for these constructions, as well as those of Kan and Thurston, is the fact that any group can be imbedded in a larger group which has the integral homology of a point; this can be thought of as being analogous to the cone over a topological space. Thus for example a "suspension" of a group can be constructed by amalgamating two cones along the original group. These "acyclic" groups evidently form an important set of building blocks and we investigate both their construction and properties. We show, for example, that geometrically finite groups can be imbedded in geometrically finite acyclic groups, finitely generated groups in finitely generated acyclic groups, and locally finite groups in locally finite acyclic groups. Among the applications of our methods is the result that algebraically closed groups are acyclic.

We construct, finally, a functor $L$ of simplicial complexes sharing the properties adduced above for the functor of Kan and Thurston, but with the added property of being effective; i.e., of providing presentations for the groups $LX$ and of taking finite complexes into geometrically finite groups. This answers a question asked in [14]. The functor $L$ conserves dimension also, the geometric dimension of a group $G$ being the smallest of those of complexes of the homotopy type of $K(G, 1)$.

It is our hope that this paper will be of interest to people working in several areas: group theory, homological algebra, and topology. While we have tried to make it self-contained, we have occasionally had to import some results from the literature, especially from topology. The latter part, moreover, is couched in categorical language. This seems to be essential to the exposition, but it is used only as an organizing principle.

## 1. Geometrically finite groups

A pathwise connected space is said to be *aspherical* if its homotopy groups in all dimensions greater than 1 are trivial. It is said to be *acyclic* provided its singular homology groups with integer coefficients are the same as those of a point.

For any discrete group $G$ there is an aspherical space $K(G, 1)$ with basepoint $*$ which is a simplicial complex and whose fundamental group $\pi_1(K(G, 1), *)$ is isomorphic to $G$. For the group $G$ any two such spaces have the same pointed homotopy type. The homology groups of $K(G, 1)$ with local coefficients in a $Z[G]$-module $A$ can be identified with the homology groups $H_*(G; A)$ of the group $G$ as defined in homological algebra. A similar remark holds for cohomology.

There are a number of general techniques for forming spaces $K(G, 1)$, some of them functorial in $G$, but none pretends to any sort of economy of construction.

The group $G$ is said to be *geometrically finite* if there is a $K(G, 1)$ which is a finite simplicial complex. The *geometrical dimension* of the group $G$, written $\mathrm{g\,dim}\, G$, is the minimal dimension of the various spaces $K(G, 1)$. Finally, the *cohomological dimension* of $G$, written $\mathrm{ch\,dim}\, G$, if finite is the supremum of the integers $n$ for which there is a $Z[G]$-module $A$ with $H^n(G; A) \neq 0$.

A theorem of S. Eilenberg and T. Ganea [7] states that for $\mathrm{ch\,dim}\, G \geqslant 3$, $\mathrm{g\,dim}\, G = \mathrm{ch\,dim}\, G$, and a theorem of J. Stallings and R. G. Swan ([28] and [30]) states that $\mathrm{ch\,dim}\, G = 1$ if and only if $\mathrm{g\,dim}\, G = 1$. It is unknown whether there is a group $G$ with $\mathrm{ch\,dim}\, G = 2$ and $\mathrm{g\,dim}\, G = 3$.

For a geometrically finite group $G$ we define the *geometrically finite dimension* of $G$, written $\mathrm{gf\,dim}\, G$, to be the minimal dimension of the various spaces $K(G, 1)$ which are finite simplicial complexes. Clearly, $\mathrm{g\,dim}\, G \leqslant \mathrm{gf\,dim}\, G$. We return to this inequality later in this section.

Also in this section we collect a number of elementary facts about geometrically finite groups, which are either known or easily deducible from published results. The principal references are papers of J.P. Serre [27] and C.T.C. Wall [32].

J.H.C. Whitehead showed [34] that if the connected simplicial complex $X$ is the union of subcomplexes $A$ and $B$ such that each of $A$, $B$, and $A \cap B$ is aspherical and the homomorphisms $\pi_1(A \cap B) \to \pi_1 A$ and $\pi_1(A \cap B) \to \pi_1 B$ induced by inclusions are both injections, then $X$ is also aspherical. This result and the Seifert–van Kampen Theorem imply

**Proposition 1.1.** *If $G = A *_C B$ is a free product with amalgamation of geometrically finite groups $A$, $B$ and $C$, then $G$ is also geometrically finite. Furthermore,*

$$\text{gf dim } G \leq \max(\text{gf dim } A, \text{gf dim } B, 1 + \text{gf dim } C).$$

**Proof.** Realize the injections $a : C \to A$ and $b : C \to B$ by simplicial maps

$$K(C, 1) \to K(A, 1) \quad \text{and} \quad K(C, 1) \to K(B, 1)$$

which induce $a$ and $b$ on fundamental groups, form the mapping cylinders of these maps, and unite them along $K(C, 1)$. □

There is a geometrically finite version of the Eilenberg–Ganea Theorem referred to above.

**Theorem 1.2.** *If $G$ is geometrically finite and* ch dim $G \neq 2$, *then*

$$\text{ch dim } G = \text{gf dim } G,$$

*If* ch dim $G = 2$, *then* $2 \leq \text{gf dim } G \leq 3$.

The proof of this result requires some preparation. Note that if in a connected, finite simplicial complex, we select a maximal tree and collapse it to a point, we obtain a finite CW-complex of the same pointed homotopy type (using any vertex of the original complex) with just one point in its 0-skeleton. If we start with a pointed, connected finite CW-complex, then going inductively up the dimensions of the skeleta, using simplicial approximation and triangulation, we can obtain a pointed, connected, finite simplicial complex of the same pointed homotopy type and of the same dimension. Thus we can restate questions about geometrically finite groups in terms of connected CW-complexes having only one vertex.

**Proposition 1.3.** *Let $X$ be an aspherical* CW*-complex having only one vertex, $*$, and only finitely many cells. For $G = \pi_1(X, *)$ there is a free $Z[G]$ resolution of $Z$*

$$0 \to F_n \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} Z[G] \xrightarrow{\varepsilon} Z \to 0 \tag{1.1}$$

*with $F_i$ having a basis of cardinality the number of $i$-dimensional cells of $X$ and $d_2$ being the Fox derivatives of the presentation of $G$ given by the 2-skeleton of $X$.*

*Conversely, let*

$$\langle x_1, \ldots, x_s; r_1, \ldots, r_t \rangle$$

*be a finite presentation of a group $G$ and let* (1.1) *be a finitely generated free resolution* (1.1) *of $Z$ with $d_2$ being the Fox derivative of the given presentation. Then there exists a*

*CW-complex $X$ which is aspherical, which has only one vertex and only finitely many cells, which has fundamental group isomorphic to $G$, and whose universal covering space has chain-complex $\{C_*(\tilde{X}; Z), d_*\}$ isomorphic to* (1.1).

**Proof.** Let $\tilde{X}$ be the universal covering space of $X$ with respect to the point $*$. Then $G$ operates freely on $\tilde{X}$, which has a CW-structure induced by that of $X$. The group $G$ permutes the various cells of $\tilde{X}$ which lie over a given cell of $X$ freely among themselves. Thus, $C_i(\tilde{X}; Z)$ is a free $Z[G]$-module with one generator for each $i$-cell of $X$. The space $\tilde{X}$ is pathwise connected and all of its homotopy groups are trivial. Thus by the classical theorem of W. Hurewicz, all of its homology groups are trivial; and so, the chain-complex of $\tilde{X}$

$$0 \to C_n(\tilde{X}; Z) \to \cdots \to C_2(\tilde{X}; Z) \xrightarrow{d_2} C_1(\tilde{X}; Z) \xrightarrow{d_1} Z[G] \xrightarrow{\varepsilon} Z \to 0$$

is a finitely generated free resolution of $Z$ over $Z[G]$.

Since $X$ has only finitely many cells in dimensions 1 and 2, $G$ has a finite presentation with one generator corresponding to each 1-cell of $X$ and one relator given by the attachment of each 2-cell of $X$ to the 1-skeleton. It has been shown in this case that $d_2$ expressed in terms of the specified generators and relators is the Fox derivative (see R.C. Lyndon [18]).

For the converse, assume given the presentation

$$P = \langle x_1, \ldots, x_s; r_1, \ldots, r_t \rangle$$

of $G$ and resolution (1.1) of $Z$ over $Z[G]$. Let $X^{(2)} = C(P)$ be the cell-complex corresponding to $P$. Then for $i = 1, 2$,

$$C_i(\tilde{X}^{(2)}; Z) \approx F_i$$

as $Z[G]$-modules and $d_2$ and $d_1$ are as in (1.1). Since $\pi_1(\tilde{X}^{(2)}) = 0$, there are $Z[G]$-isomorphisms

$$\pi_2(X^{(2)}) \approx \pi_2(\tilde{X}^{(2)}) \approx H_2(\tilde{X}^{(2)}; Z) \leftarrow \ker d_2.$$

But $\ker d_2 = \operatorname{im} d_3$; and so, corresponding to each generator $\alpha$ of $F_3$ there is a map

$$\hat{\alpha} : S^2 \to X^{(2)}$$

so that the images in $\pi^2(X^{(2)})$ generate $\ker d_2$. Attach 3-cells by the maps $\hat{\alpha}$ to obtain $X^{(3)}$.

Then for $i = 1, 2, 3$

$$C_i(\tilde{X}^{(3)}; Z) \approx F_i$$

as $Z[G]$-modules and $d_3$, $d_2$, $d_1$ are as in (1.1). This process clearly continues until we obtain $X = X^{(n)}$ with similar properties. Thus,

$$H_i(\tilde{X}; Z) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

by (1.1). Since $\tilde{X}$ is connected and simply connected, again by the Hurewicz Theorem, all of the homotopy groups of $\tilde{X}$ are trivial. Thus $X$ is an aspherical CW-complex satisfying the required conditions. □

We shall use the following well-known result (see Kaplansky [15]).

**Lemma 1.4.** *Let $G$ be a group, $A$ be a $Z[G]$-module and*

$$0 \to K \to P_i \to P_{i-1} \to \cdots \to P_0 \to A \to 0$$

*and*

$$0 \to K' - P_i' \to P_{i-1}' \to \cdots \to P_0' \to A \to 0$$

*be exact sequences with the $P_i$ and $P_i'$ all being projective $Z[G]$-modules. Then*

$$K \oplus P_i' \oplus P_{i-1} \oplus P_{i-2}' \oplus \cdots \approx K' \oplus P_i \oplus P_{i-1}' \oplus P_{i-2} \oplus \cdots,$$

*the direct sums ending in $P_0$ and $P_0'$ correctly placed according to the parity of $i$.*

**Lemma 1.5.** *If $G$ has a finite presentation and there exists a finitely generated free resolution of $Z$ over $Z[G]$ terminating at dimension $n \geq 3$, then there is a resolution of the form* (1.1).

**Proof.** Let

$$0 \to C_n \xrightarrow{d_n'} \cdots \to C_2 \xrightarrow{d_2'} C_1 \xrightarrow{d_1'} C_0 \xrightarrow{\varepsilon} Z \to 0$$

be a finitely generated free resolution of $Z$ with $n \geq 3$. Using any finite presentation of $G$, start a resolution

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} Z[G] \xrightarrow{\varepsilon} Z \to 0$$

with boundaries $d_2$ and $d_1$ corresponding to the presentation.

Then by Lemma 1.4,

$$\ker d_2 \oplus C_2 \oplus F_1 \oplus C_0 \approx \operatorname{im} d_3' \oplus F_2 \oplus C_1 \oplus F_0.$$

Thus there exist a finitely generated free $Z[G]$-module $F_3$ and a homomorphism

$$d_3 : F_3 \to F_2$$

with $\operatorname{im} d_3 = \ker d_2$. Continue in this way up to

$$\ker d_{n-1} \oplus C_{n-1} \oplus F_{n-2} \oplus \cdots \approx C_n \oplus F_{n-1} \oplus C_{n-2} \oplus \cdots$$

Then

$$0 \to (\ker d_{n-1}) \oplus (C_{n-1} \oplus F_{n-2} \oplus \cdots) \to (F_{n-1}) \oplus (C_{n-1} \oplus F_{n-2} \oplus \cdots)$$
$$\to F_{n-2} \to \cdots \to F_1 \to Z[G] \to Z \to 0$$

is a resolution of the form of (1.1). □

**Question 1.** Is it true that if there is a finitely generated free resolution of $Z$ over $Z[G]$ terminating at some dimension $n$, then $G$ is finitely presented? [13].

**Proposition 1.6.** *If $G$ is finitely presented,* $\operatorname{ch}\dim G = k$, *and there is a finitely generated free resolution of $Z$ over $Z[G]$ terminating at some finite dimension, then*

$$\operatorname{gf}\dim G \leqslant \max(k, 3).$$

**Proof.** Let

$$0 \to C_n \xrightarrow{d_n'} \cdots \to C_0 \to Z \to 0$$

be a finitely generated free resolution of $Z$ over $Z[G]$. (We can assume $n \geqslant 3$.) By the previous lemma there is a resolution

$$0 \to F_n \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} Z[G] \xrightarrow{\varepsilon} Z \to 0$$

of the form (1.1). Since $\operatorname{ch}\dim G = k$, $\ker d_{k-1}$ is projective. Then (1.1) splits into

$$0 \to P_k \to F_{k-1} \to \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} Z[G] \xrightarrow{\varepsilon} Z \to 0 \tag{1.2}$$

and

$$0 \to F_n \to \cdots \to F_k \to P_k \to 0 \tag{1.3}$$

with $P_k = \ker d_{k-1}$ being projective. From (1.3) we can conclude that there exists a finitely generated free $Z[G]$-module $F$ such that $P_k \oplus F$ is also a finitely generated free $Z[G]$-module. Thus, by (1.2),

$$0 \to P_k \oplus F \to F_{k-1} \oplus F \to F_{k-2} \to \cdots \to Z \to 0$$

is also a resolution of the form (1.1) provided $k \geqslant 3$. If $k < 3$, we apply the above argument at 3. (The argument at lower values of $k$ would alter $d_2$ corresponding to the Fox derivative of a presentation. Even at 3 we must adjoin a number of 2-spheres equal to the number of generators of $F$ and corresponding 3-cells with those spheres as boundaries.)

Theorem 1.2 now follows from the Eilenberg–Ganea Theorem, the Stallings–Swan Theorem, Proposition 1.3, and Proposition 1.6.

Since finite-sheeted covering spaces correspond to subgroups of the fundamental group of finite index, and a finite sheeted covering space of a finite complex is a finite complex, we have

**Proposition 1.7.** *If $H$ is a subgroup of finite index of a geometrically finite group $G$, then $H$ is geometrically finite and* $\operatorname{gf}\dim H = \operatorname{gf}\dim G$, *except perhaps* $\operatorname{gf}\dim H = 2$ *and* $\operatorname{gf}\dim G = 3$.

**Proof.** It is clear that $H$ is geometrically finite and $\operatorname{gf}\dim H \leqslant \operatorname{gf}\dim G$. By a theorem of Serre [27], $\operatorname{ch}\dim H = \operatorname{ch}\dim G$. The remainder of the conclusion then follows from Theorem 1.2.

**Question 2.** Suppose $H$ is geometrically finite and is a subgroup of finite index in a torsion-free group $G$. Is $G$ geometrically finite? [27].

One can show rather easily that if $H$ is a subgroup of finite index in the torsion-free group $G$ and $H \times Z$ is geometrically finite, then $G \times Z$ is also geometrically finite. This appears to be far short of answering the above question.

**Question 3.** Suppose $G$ is geometrically finite and $H$ is a retract of $G$. Is $H$ geometrically finite?

**Proposition 1.8.** *If* $1 \to K \to G \to H \to 1$ *is exact and* $K$ *and* $H$ *are geometrically finite, then so is* $G$.

**Proof.** Applying a construction of C.T.C. Wall [33] and Proposition 1.3, this is immediate.

**Question 4.** Does it then follow that $\operatorname{gf}\dim G = \operatorname{gf}\dim K + \operatorname{gf}\dim H$?

We mention here one further result about geometrically finite groups, but its proof is deferred until Section 11.

**Theorem 1.9.** *If* $G$ *is geometrically finite and* $\operatorname{gf}\dim G = n$, *then* $G$ *is a subgroup of an acyclic, geometrically finite group* $cG$ *with* $\operatorname{gf}\dim cG = 2n+1$.

The change of dimension is a consequence of our method of proof. We do not know in general whether it is best possible. It bears an intriguing resemblance to the classical imbedding theorem for simplicial complexes of dimension $n$.

## 2. Some basic group theoretic constructions

In this section we recall some standard group theoretic constructions which will be used in the sequel. This material is included as an aid to the reader who does not routinely work with possibly non-abelian groups. Proofs are omitted or are very brief.

We write $H \leqslant A$ for the statement that $H$ is a subgroup of $A$, $H \lhd A$ if $H$ is a normal subgroup of $A$, and $\to$, $\twoheadrightarrow$ *and* $\rightarrowtail$ for homomorphism, epimorphism and monomorphism, respectively.

Suppose $H \leqslant A$, $K \leqslant B$ and that $\varphi : H \twoheadrightarrow K$. Then we denote the *free product P of A and B with H and K amalgamated according to* $\varphi$ by

$$P = \{A * B; H \overset{\varphi}{=} K\}.$$

(This is also called the generalized free product of $A$ and $B$ identifying $H$ with $K$ by $\varphi$.) If $\varphi$ is the identity isomorphism, we write simply $P = \{A * B; H\}$. Of course, the group $P$ is a pushout in the category of groups. We then call a product

$$p_1 p_2 \cdots p_n$$

*a strictly alternating* $(A \cup B)$*-product* provided

(i) $n > 1$,

(ii) $p_i \in (A \cup B) - H$, $i = 1, 2, \ldots, n$,

(iii) if $p_i \in A$ and $i < n$, then $p_{i+1} \in B$, and

(iv) if $p_i \in B$ and $i < n$, then $p_{i+1} \in A$.

The following lemma will be useful (see B.H. Neumann [23]).

**Lemma 2.1.** *Suppose that the group P is generated by its subgroups A and B and that* $A \cap B = H$. *Then*

$$P = \{A * B; H\}$$

*if and only if every strictly alternating* $(A \cup B)$*-product*

$$p_1 p_2 \cdots p_n \neq 1.$$

This lemma has an immediate consequence (B.H. Neumann [23]).

**Corollary 2.2.** *Suppose* $P = \{A * B; H\}$, *that* $A_1 \leqslant A$, $B_1 \leqslant B$ *and*

$$A_1 \cap H = H_1 = B_1 \cap H.$$

*Then*

$$\mathrm{gp}(A_1, B_1) = \{A_1 * B_1; H_1\}.$$

(In general, if $L$ is a set of elements of a group $G$, gp $L$ denotes the subgroup of $G$ generated by the elements of $L$.)

Another consequence of Lemma 2.1 that will be useful is

**Lemma 2.3.** *Suppose that* $P = \{A * B; H\}$, *that* $a \in A$ *and* $a \notin H$, *and that*

$$a^{-1} h a = h \quad \text{for all } h \in H.$$

*Then*

$$\mathrm{gp}(B, a^{-1} B a) = \{B * a^{-1} B a; H\}.$$

**Proof.** It suffices to observe that every strictly alternating $(B \cup a^{-1}Ba)$-product can be viewed as a strictly alternating $(A \cup B)$-product in $P$. Lemma 2.3 then follows immediately by applying Lemma 2.1. $\square$

We can form $K(P, 1)$ from $K(A, 1)$, $K(B, 1)$ and $K(H, 1)$ as shown in the proof of Proposition 1.1.

Let $G$ be a group, $A$ and $B$ be subgroups of $G$, and $\varphi : A \twoheadrightarrow B$. Suppose $G$ is presented by

$$\langle X; R \rangle.$$

Then the group $E$ presented by

$$\langle X \cup \{t\}; R \cup \{t^{-1} at \varphi(a)^{-1} | a \in A\} \rangle$$

is called an HNN-*extension of* $G$, $t$ the *stable letter* of $E$, $A$ and $B$ the *associated subgroups*, $\varphi$ the *associating isomorphism*, and $E$ itself is denoted by

$$\langle G, t; t^{-1} at = \varphi a, a \in A \rangle.$$

The group $E$ can be identified with a subgroup of a suitably chosen free product with an amalgamated subgroup. We recall the details. Let $C$ and $D$ be the ordinary free products

$$C = G * \langle \tilde{t}_1 \rangle \quad \text{and} \quad D = G * \langle \tilde{t}_2 \rangle,$$

and put

$$H = \mathrm{gp}(G, \tilde{t}_1^{-1} A \tilde{t}_1) \quad \text{and} \quad K = \mathrm{gp}(G, \tilde{t}_2 B \tilde{t}_2^{-1}).$$

Then

$$H = G * \tilde{t}_1^{-1} A \tilde{t}_1 \quad \text{and} \quad K = G * \tilde{t}_2 B \tilde{t}_2^{-1}.$$

Let $\theta : H \twoheadrightarrow K$ be defined by

$$g \to g \quad \text{for } g \in G$$

and

$$\tilde{t}_1^{-1} a \tilde{t}_1 \to \tilde{t}_2 \varphi a \tilde{t}_2^{-1} \quad \text{for } a \in A.$$

Then form

$$P = \{C * D; H \overset{\theta}{=} K\}.$$

The following lemma is crucial (G. Higman, B.H. Neumann and Hanna Neumann [12]).

**Lemma 2.4.** *The group $E$ and the subgroup*

$$\tilde{E} = \mathrm{gp}(G, \tilde{t}_1 \tilde{t}_2)$$

*of P are isomorphic; indeed the function from E to $\tilde{E}$ defined by $g \to g$, for $g \in G$, and $t \to \tilde{t}_1 \tilde{t}_2$ is an isomorphism.*

This lemma is very useful; it allows us to deduce properties of HNN-extensions immediately from corresponding properties of free products with amalgamation. In particular, it follows that $G$ is naturally imbedded in $E$. Also, Lemma 2.4 together with Corollary 2.2 immediately imply

**Lemma 2.5.** *Let*

$$E = \langle G, t; t^{-1} at = \varphi a, a \in A \rangle.$$

*Let $G_1 \leq G$ and for $A_1 = G_1 \cap A$ and $B_1 = G_1 \cap B$, suppose that $\varphi_1 = \varphi | A_1$ is an isomorphism of $A_1$ with $B_1$. Then*

$$E_1 = \mathrm{gp}(G_1, t) = \langle G_1, t; t^{-1} at = \varphi_1 a, a \in A_1 \rangle$$

*is again an* HNN-*extension with stable letter t, associated subgroups $A_1$ and $B_1$ and associated isomorphism $\varphi_1$.*

Finally, we observe the simple

**Lemma 2.6.** *The subgroup $\tilde{E}$ is a free factor of P. Indeed*

$$P = \tilde{E} * \langle \tilde{t}_1 \rangle.$$

**Proof.** It suffices to observe that $\mathrm{gp}(\tilde{E}, \tilde{t}_1) = P$ and that every strictly alternating $(\tilde{E} \cup \langle \tilde{t}_1 \rangle)$-product is non-trivial. (We are using here Lemma 2.1 in the special case where the amalgamated subgroup is the identity subgroup.) $\square$

One can form spaces $K(E, 1)$ as follows: let $\alpha : A \to G$ be the inclusion of $A$ in $G$ and $\beta : A \to G$ be the composite of the isomorphism $\varphi$ and the inclusion of $B$ in $G$. Realize $\alpha$ and $\beta$ by simplicial mappings

$$\bar{\alpha} : K(A, 1) \to K(G, 1) \quad \text{and} \quad \bar{\beta} : K(A, 1) \to K(G, 1)$$

which induce $\alpha$ and $\beta$ on fundamental groups. Form the (non-reduced) mapping cylinders

$$M(\bar{\alpha}) = K(A, 1) \times I + K(G, 1)/a, 1 \sim \bar{\alpha} a$$

and $M(\bar{\beta})$; in $M(\bar{\alpha}) + M(\bar{\beta})$ identify the two copies of $K(A, 1) \times 0$ and identify the two copies of $K(G, 1)$. The resulting quotient space is $K(E, 1)$.

## 3. Some examples of acyclic groups

We begin our study of *acyclic groups* (groups with integral homology that of a point) by giving several examples of such groups.

Our first example is an acyclic group $A$ with gf dim $A = 2$ which has a three generator, three relator presentation. Let

$$F = \langle a, b \rangle$$

be the free group of rank two. Put

$$C = \mathrm{gp}(u = a,\ v = b^{-1}\underline{a}^{-1}b\underline{a}b,\ w = b^{-2}\underline{a}b^{-1}a^{-2}b\underline{a}b^2,\ x = b^{-3}\underline{a}b^{-1}a^{-2}b\underline{a}b^3).$$

Observe that in any reduced product of the given generators $u$, $v$, $w$, and $x$ of $C$ the underlined letters do not cancel. So $C$ is free of rank four on $u$, $v$, $w$, and $x$. Notice also that both $w$ and $x$ lie in the derived group of $F$.

Let $F_i$, $i = 1, 2$, be isomorphic copies of $F$ and define $\varphi : C_1 \twoheadrightarrow C_2$ by

$$u_1 \to x_2, \qquad v_1 \to w_2, \qquad w_1 \to v_2, \qquad x_1 \to u_2.$$

Put

$$A = \{F_1 * F_2;\ C_1 \overset{\varphi}{=} C_2\}.$$

It follows by Proposition 1.1 that $A$ is geometrically finite and that gf dim $A \leqslant 2$. The group $A$ is non-trivial, $F_1$ being a subgroup, and by construction its abelianization, $H_1(A; Z)$, is trivial. Thus $A$ is not free and gf dim $A = 2$.

The construction of $K(A, 1)$ shows its homology occurs in a Mayer–Vietoris Sequence

$$\cdots \to H_{n+1}A \to H_nC \to H_nF_1 \oplus H_nF_2 \to H_nA \to \cdots.$$

(This sequence is valid with coefficients in any $Z[A]$-module. Here we shall use only coefficients $Z$.) Since free groups have trivial homology except in dimensions 0 and 1, it follows that $H_nA = 0$ for $n > 2$. Thus the sequence

$$0 \to H_2A \to H_1C \to H_1F_1 \oplus H_1F_2 \to 0$$

is exact. But both $H_1C$ and $H_1F_1 \oplus H_1F_2$ are free abelian of rank four. Hence $H_2A = 0$ and $A$ is acyclic. It is clear also that the generalized free product $A$ can be generated by three elements and is defined in terms of them by three defining relations.

**Question 5.** Does there exist a two generator, two relator presentation of a non-trivial, geometrically finite acyclic group? Of any non-trivial acyclic group?

Our second example is the well-known group

$$H = \langle a, b, c, d;\ b^{-1}ab = a^2,\ c^{-1}bc = b^2,\ d^{-1}cd = c^2,\ a^{-1}da = d^2 \rangle$$

of G. Higman [11]. Recall that if

$$A = \langle a, b, c;\ b^{-1}ab = a^2,\ c^{-1}bc = b^2 \rangle$$

and

$$B = \langle c', d, a';\ d^{-1}c'd = c'^2,\ a'^{-1}da' = d^2 \rangle,$$

then

$$C = \mathrm{gp}(a, c) \quad \text{and} \quad D = \mathrm{gp}(a', c')$$

are both free of rank two. So the map

$$a \to a', \qquad c \to c'$$

defines an isomorphism $\varphi : C \twoheadrightarrow D$. Hence

$$H = \{A * B; C \overset{\varphi}{=} D\}.$$

It follows that the homology of $H$ can be computed by a Mayers–Vietoris Sequence. This computation leads to the desired conclusion that $H$ is acyclic. This was observed by E. Dyer and A.T. Vasquez in [6], who show there also that the cell complex of the given presentation of $H$ is aspherical; thus,

$$\text{gf dim } H = 2.$$

As a third example, we exhibit an acyclic group $G$ having a non-trivial finite quotient group. This group is not geometrically finite, but ch dim $G = 2$.

A group is said to be locally free if each of its finitely generated subgroups is free. Since homology commutes with directed colimits, it follows immediately (see e.g. K.W. Gruenberg [9]) that

**Lemma 3.1.** *A locally free group $G$ is acyclic if and only if it is perfect; i.e., $G = [G, G]$.*

Since perfect locally free groups abound, (A.G. Kurosh [17]), this lemma provides us with a host of locally free acyclic groups (see also G. Baumslag and K.W. Gruenberg [3]). We shall construct such a group having a non-trivial finite quotient group. Since non-trivial finite groups are never acyclic (R.G. Swan [31]), we obtain an acyclic group having a non-acyclic quotient group.

Recall first some notation. If $x$ and $y$ are elements of a group, put

$$[x, y] = x^{-1}y^{-1}xy \quad \text{and} \quad x^y = y^{-1}xy.$$

Consider the one-relator group

$$H = \langle s, t; s = ([s, s^{t^3}]^t [s, s^{t^3}]^{t^4} [s, s^{t^3}]^{t^4})^2 \rangle.$$

Since this relator is not a proper power, it follows that $H$ is geometrically finite and gf dim $H = 2$ (see also W.H. Cockroft [5]; E. Dyer and A.T. Vasquez [6]). Let $G$ be the normal subgroup of $H$ generated by $s$:

$$G = \mathrm{gp}_H(s).$$

It follows from the usual method of W. Magnus that $G$ is locally free and perfect (see e.g. W. Magnus, A. Karrass and D. Solitar [20]). So $G$ is acyclic and ch dim $G = 2$. To see that $G$ has a non-trivial finite quotient, consider the alternating group $A_5$ of degree 5. The map

$$s \to \sigma = (1\,2\,3), \qquad t \to \tau = (1\,2\,3\,4\,5)$$

defines a homomorphism $\varphi$ of $H$ onto $A_5$ because

$$\sigma = ([\sigma, \sigma^{\tau^3}]^{\tau}[\sigma, \sigma^{\tau^3}]^{\tau^4}[\sigma, \sigma^{\tau^3}]^{\tau^4})^2.$$

Since $\sigma \in \varphi G$, $A_5$ is a finite quotient of $G$.

As our final example in this section we construct an acyclic group $G$ which contains the fundamental groups of all closed orientable surfaces of genus $\geqslant 2$. A surprising feature of this example is that it is geometrically finite with gf dim $= 2$.

Let

$$F = \langle a, b, c\,;\, a^2b^2 = c^3 \rangle.$$

Since $F$ is a one-relator group and the relator $a^2b^2c^{-3}$ is not a proper power, $F$ is geometrically finite and gf dim $F = 2$. It is easy to verify that

$$H_1(F; Z) \approx Z \oplus Z \approx \mathrm{gp}(a[F, F]) \times \mathrm{gp}(bc^{-1}[F, F]).$$

Also, $F$ is a free product with amalgamation

$$F = \langle A * B\,;\, H\},$$

where

$$A = \langle c \rangle, \qquad B = \langle a, b \rangle \quad \text{and} \quad H = \mathrm{gp}(c^3).$$

It follows from Lemma 2.3 that

$$S = \mathrm{gp}(B, c^{-1}Bc) = \{B * c^{-1}Bc\,;\, H\}.$$

It follows that

$$S \approx \langle u, v, w, x\,;\, u^2v^2w^2x^2 = 1 \rangle.$$

Hence $S$ contains the fundamental group of every orientable surface of genus $\geqslant 2$.

Now let

$$K = \langle \alpha, \beta, \gamma, \delta\,;\, \alpha^{\beta} = \alpha^2, \beta^{\gamma} = \beta^2, \gamma^{\delta} = \gamma^2, \delta^{\alpha} = \delta^2 \rangle$$

be a Higman group. Put $L = \mathrm{gp}(a, bc^{-1})$ and $M = \mathrm{gp}(\alpha, \gamma)$. Both $L$ and $M$ are free of rank two and the map

$$a \to \alpha, \qquad bc^{-1} \to \gamma$$

defines an isomorphism $\varphi$ from $L$ to $M$. Let

$$G = \{F * K\,;\, L \stackrel{\varphi}{=} M\}.$$

The homology of $G$ can be readily computed by a Mayer–Vietoris Sequence remembering that $K$ is acyclic. The acyclicity of $G$ follows readily. Proposition 1.1 shows that $G$ is geometrically finite and that gf dim $G = 2$.

## 4. Mitotic groups

The major objective of this and the next section is to imbed groups in acyclic groups, in a functorial manner, preserving finite generation. There are two central ideas involved in this process, embodied in the following two definitions.

**Definition 4.1.** A supergroup $M$ of a group $B$ is called a *mitosis* of $B$ if there exist elements $s$ and $d$ of $M$ such that

(i) $M = \mathrm{gp}(B, s, d)$,
(ii) $b^d = bb^s$ for all $b \in B$, and
(iii) $[b', b^s] = 1$ for all $b, b' \in B$.

A mitosis of $B$ can be regarded as a process for dividing $B$ into two copies of itself. This leads us to

**Definition 4.2.** A group $M$ is *mitotic* if it contains a mitosis of every one of its finitely generated subgroups.

Roughly speaking, some of the homology of $B$ is annulled in any mitosis. This results in the acyclicity of every mitotic group. Since every group can be imbedded in a mitotic group by repeated HNN-extensions, this implies that every group can be imbedded in an acyclic group. The details of this process, as well as some refinements, constitute the contents of this and the next section.

Although it is difficult to obtain precise information about the homology of a mitosis, it suffices for our purposes to prove.

**Proposition 4.1.** *Let $\varphi : A \to B$ be a homomorphism, $M$ be a mitosis of $B$, and $\mu : B \rightarrowtail M$ be the injection of $B$ into $M$. Furthermore, let $\mathbf{k}$ be a field and suppose that*

$$H_i(\varphi; \mathbf{k}) = 0 \quad \text{for } i = 1, 2, \ldots, n-1.$$

*Then*

$$H_i(\mu\varphi; \mathbf{k}) = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

Before beginning the proof of Proposition 4.1, we establish some notation. First observe that since $M$ is a mitosis of $B$, there are elements $s$ and $d$ of $M$ satisfying conditions (i) and (ii) above. We can therefore define a homomorphism

$$\kappa : B \times B \to M$$

by $\kappa(b', b) = b'b^s$. Furthermore, define

$$\lambda : B \to B \times B$$

by $\lambda b = (b, 1)$. Then clearly,

$$\mu = \kappa\lambda.$$

Next define

$$\lambda' : A \to A \times A \quad \text{and} \quad \rho' : A \to A \times A$$

by $\lambda' a = (a, 1)$ and $\rho' a = (1, a)$. We shall denote by $\hat{m}$ the inner automorphism $x \to x^m$, $x \in M$, induced by $m \in M$. Finally, if $\sigma$ is a homomorphism of a group $X$ into a group $Y$, we denote $H_n(\sigma; \mathbf{k})$ simply by $\sigma_*$, where $n$ and $\mathbf{k}$ are specified in Proposition 4.1.

**Proof of Proposition 4.1.** Since it is clear that $H_i(\mu\varphi; \mathbf{k}) = 0$ for $i = 1, 2, \ldots, n-1$, we need verify only that $H_n(\mu\varphi; \mathbf{k}) = 0$.

By the Künneth formula for homology of a direct product

$$H_n(B \times B; \mathbf{k}) \approx \sum_{i+j=n} H_i(B; \mathbf{k}) \otimes H_j(B; \mathbf{k}).$$

Since $\mu\varphi = \kappa(\varphi \times \varphi)\lambda'$, it follows for $\alpha \in H_n(A; \mathbf{k})$ that

$$(\mu\varphi)_*\alpha = \kappa_*(\varphi_*\alpha \otimes 1). \tag{4.1}$$

On the other hand, observe that

$$\hat{d}\mu\varphi = \kappa(\varphi \times \varphi)\Delta_A,$$

where $\Delta_A : A \to A \times A$ is the diagonal map $a \to (a, a)$. Hence,

$$(\hat{d}\mu\varphi)_*\alpha = \kappa_*(\varphi_*\alpha \otimes 1 + 1 \otimes \varphi_*\alpha). \tag{4.2}$$

Since conjugation induces the identity on homology, (4.2) yields

$$(\mu\varphi)_*\alpha = \kappa_*(\varphi_*\alpha \otimes 1) + \kappa_*(1 \otimes \varphi_*\alpha). \tag{4.3}$$

Similarly,

$$\hat{s}\mu\varphi = \kappa(\varphi \times \varphi)\rho';$$

and so,

$$(\mu\varphi)_*\alpha = (\hat{s}\mu\varphi)_*\alpha = \kappa_*(1 \otimes \varphi_*\alpha).$$

Combining (4.1), (4.3) and (4.4), we find

$$(\mu\varphi)_*\alpha = 0,$$

as required. □

Proposition 4.1 enables us to prove

**Theorem 4.2.** *Mitotic groups are acyclic.*

**Proof.** Let $G$ be mitotic. If $K \leqslant G$ is a finitely generated subgroup, then $K = K_0 \leqslant K_1 \leqslant K_2 \leqslant \cdots \leqslant G$ where each injection $K_n \to K_{n+1}$ is a mitosis. It follows at once from Proposition 4.1 that for any field $\mathbf{k}$,

$$H_i(K;\mathbf{k}) \to H_i(K_n;\mathbf{k})$$

is zero for $i=1,\ldots,n-1$, and thus

$$H_i(K;\mathbf{k}) \to H_i(G;\mathbf{k})$$

is zero for all $i>0$.

But $G$ is the directed colimit of its finitely generated subgroups, and the colimit of their inclusion maps is $1_G$. Since homology commutes with directed colimits, we have $H_i(1_G;\mathbf{k})=0$ for $i>0$; in other words

$$H_i(G;\mathbf{k})=0 \quad \text{for } i>0.$$

It follows from the Universal Coefficient Theorem that

$$H_i(G;Z)=0 \quad \text{for } i>0;$$

i.e., $G$ is acyclic. $\square$

It is perhaps worth pointing out that the class of mitotic groups is closed under quotients and direct products, both restricted and unrestricted. As noted in the previous section, the class of all acyclic groups is not closed under quotients. It is, by the Künneth formula, closed under restricted direct products. It seems unlikely that all direct products of acyclic groups will turn out to be acyclic.

We recall that a group $G$ is *algebraically closed* if every finite set of equations

$$f_i(g_1,\ldots,g_n,x_1,\ldots,x_m)=1, \quad i=1,\ldots,k,$$

in the variables $x_1,\ldots,x_m$ and constants $g_1,\ldots,g_n \in G$, which has a solution in some supergroup of $G$ already has a solution in $G$.

**Theorem 4.3.** *Algebraically closed groups are mitotic.*

**Proof.** Let $G$ be algebraically closed and let

$$A=\mathrm{gp}(g_1,\ldots,g_n)$$

be a finitely generated subgroup of $G$. Let

$$D=G\times G$$

and let

$$\bar{G}=\{(g,1);\, g\in G\}, \qquad H=\{(g,g);\, g\in G\} \quad \text{and} \quad K=\{(1,g);\, g\in G\}.$$

Form the HNN-extensions

$$E=\langle D,t;\, t^{-1}(g,1)t=(g,g),\, g\in G\rangle \tag{4.5}$$

and

$$m(G)=\langle E,u;\, u^{-1}(g,1)u=(1,g),\, g\in G\rangle. \tag{4.6}$$

We view $G$ as a subgroup of $m(G)$ by identifying $g \in G$ with $(g, 1) \in m(G)$. Thus $G$ is identified with $\bar{G}$. Observe that the finitely many equations

$$g_i^{x_1}(g_i g_i^{x_2})^{-1} = 1, \qquad [g_i, g_j^{x_2}] = 1, \quad i, j \in \{1, 2, \ldots, n\}$$

have a solution $x_1 = t$, $x_2 = u$ in $m(G)$. Thus they have a solution $x_1 = d$, $x_2 = s$ in $G$ itself. But then

$$\mathrm{gp}(A, d, s)$$

is a mitosis of $A$ in $G$. Hence $G$ is mitotic. □

**Corollary 4.4.** *Algebraically closed groups are acyclic.*

Every infinite group can be imbedded in an algebraically closed group of the same cardinality. Thus Corollary 4.4 yields a somewhat different proof of the following result first proved by D. Kan and W. Thurston [14].

**Corollary 4.5.** *Every infinite group can be imbedded in an acyclic group of the same cardinality.*

A group $G$ is called *locally finite* if every finitely generated subgroup of $G$ is finite. As we noted earlier, a non-trivial finite group is not acyclic (R.G. Swan [31]). However, we have

**Theorem 4.6.** *Every locally finite group can be imbedded in a mitotic locally finite group.*

Instead of appealing to HNN-extensions, as we did in the proof of Theorem 4.3, we use a generalization of a theorem of P. Hall as detailed by O. Kegel and B. Wehrfritz (Theorems 6.5 and 6.1 of [16]). This readily provides an imbedding of any locally finite group in a locally finite mitotic group.

**Corollary 4.7.** *Every locally finite group can be imbedded in a locally finite acyclic group.*

It follows in much the same way (see Lemma 6.3 of O. Kegel and B. Wehrfritz [16]) that there exists a colimit of type $\omega$ of finite alternating groups which is acyclic. This contrasts with the fact that $A_\infty$, the finitary alternating group on $\{1, 2, \ldots\}$, has a lot of non-trivial integral homology (M. Nakaoka [22]).

## 5. Imbedding countable groups in finitely generated acyclic groups

Let $G$ be a group. We begin by imbedding $G$ in a mitosis $m(G)$ of $G$ as in the proof of Theorem 4.3 (see equations (4.5) and (4.6)). To this end put

$$D = G \times G,$$

$$E = \langle D, t; t^{-1}(g, 1)t = (g, g), g \in G\rangle, \tag{5.1}$$

and

$$m(G) = \langle E, u; u^{-1}(g, 1)u = (1, g), g \in G\rangle. \tag{5.2}$$

We shall view $G$ as a subgroup of $m(G)$ by identifying $g$ in $G$ with $(g, 1)$ in $m(G)$. We have then

**Lemma 5.1.** *Suppose $G$ is presented in the form*

$$G = \langle X; R\rangle.$$

*Then $m(G)$ has the presentation*

$$m(G) = \langle X \cup \{t, u\}; R \cup \{x' = xx'', [x', x''] \| x', x \in X\}\rangle.$$

The proof of Lemma 5.1 follows readily from the discussion in Section 2. Notice that

$$m(G) = \mathrm{gp}(G, t, u).$$

The following lemma will be useful in the sequel.

**Lemma 5.2.** *Let $\theta : G_1 \to G$ be a homomorphism of $G_1$ into $G$. Furthermore let*

$$m(G_1) = \mathrm{gp}(G_1, t_1, u_1) \quad \textit{and} \quad m(G) = \mathrm{gp}(G, t, u).$$

*Then the map defined by*

$$g_1 \to \theta g_1, \qquad t_1 \to t \quad \textit{and} \quad u_1 \to u \tag{5.3}$$

*extends to a homomorphism*

$$m(\theta) : m(G_1) \to m(G).$$

*Moreover, if $\theta$ is a monomorphism, so is $m(\theta)$.*

**Proof.** It follows immediately from Lemma 5.1 that the mapping defined by (5.3) extends uniquely to a homomorphism $m(\theta)$.

To prove that $m(\theta)$ is monic if $\theta$ is, we invoke Lemma 2.5. We observe that the conditions of Lemma 2.5 are satisfied first in $E$ of equation (5.1) and also in $m(G)$ of equation (5.2). More precisely, if we put

$$D_1 = \theta(G_1) \times \theta(G_1),$$

then we note first, by Lemma 2.5, that

$$E_1 = \mathrm{gp}(D_1, t) = \langle D_1, t; t^{-1}(g, 1)t = (g, g), g \in \theta G_1\rangle.$$

Next we find, again by Lemma 2.5, that

$$\mathrm{gp}(E_1, u) = \langle E_1, u;\, u^{-1}(g, 1)u = (1, g), g \in \theta G_1\rangle.$$

So the natural map from $m(G_1)$ to $\mathrm{gp}(E_1, u)$ is an isomorphism; hence $m(\theta)$ is a monomorphism. □

Let us now define inductively

$$m^{k+1}(G) = m(m^k(G)), \quad k = 1, 2, \ldots,$$

and for a homomorphism $\theta : G_1 \to G$,

$$m^{k+1}(\theta) = m(m^k(\theta)), \quad k = 1, 2, \ldots.$$

As before, there is a natural injection

$$i : G \to m(G).$$

This gives rise to a sequence of injections

$$G \xrightarrow{i} m(G) \xrightarrow{m(i)} m^2(G) \xrightarrow{m^2(i)} \cdots.$$

Define $m^\infty(G) = \mathrm{colim}\, m^i(G)$. For $\theta : G_1 \to G$, define

$$m^\infty(\theta) = \mathrm{colim}\, m^i(\theta).$$

An immediate consequence of Lemma 5.2 is

**Lemma 5.3.** *For every $i = 1, 2, \ldots, \infty$, $m^i$ is a functor from groups to groups which preserves injections.*

Letting $l = m^\infty$, we have

**Lemma 5.4.** *There is a functor from groups $G$ to injections of groups*

$$G \to l(G),$$

*with $l(G)$ a mitotic group.*

Let now $G$ be any group. Consider the injection

$$i : G \to m(G).$$

By Lemma 5.3 we have then an injection

$$l(i) : l(G) \to l(m(G)) = l(G).$$

Observe that (adopting the obvious notation) if

$$l(G) = \mathrm{gp}(G, t_1, u_1, t_2, u_2, \ldots) = H,$$

then the injection $l(i)$ is just the injection of

$$\mathrm{gp}(G, t_2, u_2, \ldots)$$

into $l(G)$. Hence the map $\varphi : l(G) \to H$ defined by

$$g \to g, \qquad t_i \to t_{i+1}, \qquad u_i \to u_{i+1}, \quad i = 1, 2, \ldots,$$

is an injection. Put

$$k(G) = \langle l(G), t; t^{-1}at = \varphi a, a \in l(G) \rangle. \tag{5.4}$$

By Lemma 2.6, $k(G)$ is a free factor of the free product with amalgamation

$$P = \{U * V; W\},$$

where

$$U \approx l(G) * \langle x \rangle, \qquad V \approx l(G) * \langle y \rangle \quad \text{and} \quad W \approx l(G) * l(G).$$

Since $l(G)$ is mitotic, and hence acyclic, $W$ is acyclic. Also, $U$ and $V$ have trivial integral homology in dimension greater than 1. It follows by the Mayer–Vietoris Sequence that

$$H_n(P; Z) = 0 \quad \text{for } n > 1;$$

and thus

$$H_n(k(G); Z) = 0 \quad \text{for } n > 1.$$

It is clear by equation (5.4) that the abelianization of $k(G)$ is the infinite cyclic group generated by the image of $t$. There is a non-trivial torsion-free acyclic group $A$ generated by three elements $a_1$, $b_1$, $b_2$ (see Section 3). Form the free product with amalgamation

$$\mathscr{A}(G) = \{k(G) * A; t = a_1\}.$$

Again using the Mayer–Vietoris Sequence, we see that $\mathscr{A}(G)$ is acyclic. This proves

**Theorem 5.5.** *There is a functor $\mathscr{A}$ associating with each group $G$ an injection*

$$G \to \mathscr{A}(G)$$

*of groups with $\mathscr{A}(G)$ acyclic. If $G$ is infinite, $\mathscr{A}(G)$ has the same cardinality as $G$. If $G$ is an $n$-generator group, then $\mathscr{A}(G)$ is an $(n+5)$-generator group.*

Since every countable group can be imbedded in a 2-generator group (G. Higman, B.H. Neumann and Hanna Neumann [12]), we have

**Corollary 5.6.** *Every countable group can be imbedded in a 7-generator acyclic group.*

This leaves unresolved

**Question 6.** Can every finitely presented group be imbedded in a finitely presented acyclic group?

We cannot even determine whether there is a finitely presented acyclic group containing a non-trivial element of finite order. In particular, the functor $\mathscr{A}$ never suffices.

**Theorem 5.6.** *Let $G$ be a group. Then $\mathscr{A}(G)$ is not finitely presented.*

On reviewing the construction of $\mathscr{A}(G)$, we find, supposing that

$$G = \mathrm{gp}(X),$$

the following set of generators for $\mathscr{A}(G)$:

$$\mathscr{A}(G) = \mathrm{gp}(X, t_1, u_2, t_2, u_2, \ldots, t, a_1, b_1, b_2).$$

It is clear that many of these generators are redundant. However, we have

**Lemma 5.7.** *If $G$ is not finitely generated, then $\mathscr{A}(G)$ is not either.*

**Proof.** Suppose $\mathscr{A}(G)$ is finitely generated. Then the above set of generators contains a finite subset which suffices to generate $\mathscr{A}(G)$. Therefore, we can find a finite set $\{x_1, \ldots, x_m\}$ of elements of $X$ such that

$$\mathscr{A}(G) = \mathrm{gp}(x_1, \ldots, x_m, t_1, u_1, t_2, u_2, \ldots, t, a_1, b_1, b_2).$$

Since $G$ is not finitely generated, there exists $g \in G$ such that

$$g \notin \mathrm{gp}(x_1, \ldots, x_m).$$

We now trace through the construction of $\mathscr{A}(G)$, step by step, repeatedly employing Lemma 2.4. It follows that

$$\mathrm{gp}(x_1, \ldots, x_m, t_1, u_1, \ldots, t_n, u_n) \cap G = \mathrm{gp}(x_1, \ldots, x_m).$$

Hence $\mathscr{A}(G) \cap G = \mathrm{gp}(x_1, \ldots, x_m)$ and $g \notin \mathscr{A}(G)$. $\square$

We can now prove Theorem 5.6.

**Proof.** Suppose that $\mathscr{A}(G)$ is finitely presented. Then by Lemma 5.7, $G$ must be finitely generated, say

$$G = \mathrm{gp}(x_1, \ldots, x_m), \quad m < \infty.$$

Then

$$\mathscr{A}(G) = \mathrm{gp}(x_1, \ldots, x_m, t_1, u_1, t_2, u_2, \ldots, t, a_1, b_1, b_2)$$

Notice that

$$t_i = t_1^{t^i} \quad \text{and} \quad u_i = u_1^{t^i} \quad \text{for } i = 0, 1, 2, \ldots.$$

Remembering that $t = a_1$, we find

$$\mathscr{A}(G) = \mathrm{gp}(x_1, \ldots, x_m, t_1, u_1, t, b_1, b_2).$$

For ease of notation we put

$$t_1 = d, \qquad u_1 = s, \qquad b_1 = f, \quad \text{and} \quad b_2 = g.$$

Then

$$\mathscr{A}(G) = \mathrm{gp}(x_1, \ldots, x_m, d, s, t, f, g). \tag{5.5}$$

In terms of these generators, $\mathscr{A}(G)$ has an obvious presentation. In order to describe it, put

$$Z_0 = \{x_1, \ldots, x_m\}$$

and for $n \geq 1$,

$$Z_n = \{x_1, \ldots, x_m, d, s, d^t, s^t, \ldots, d^{t^{n-1}}, s^{t^{n-1}}\}.$$

The defining relations for $\mathscr{A}(G)$ can be grouped together as follows. First there are the three relations between $t, f$ and $g$, which define the group $A$ of Section 3, which we lump together simply as $S$, and the defining relations $R$ of $G$. These give us

$$R \cup S. \tag{5.6}$$

Then for each $n = 0, 1, \ldots$ we have the further relations

$$x^{d^{t^n}} = x x^{s^{t^n}} \quad \text{and} \quad [x, y^{s^{t^n}}] = 1, \quad x, y \in Z_n. \tag{5.7}$$

The relations (5.6) and (5.7) constitute a set of defining relations for $\mathscr{A}(G)$ in terms of the generators (5.5).

According to a theorem of B.H. Neumann ([25]), finitely many of the given defining relations of $\mathscr{A}(G)$ suffice to define $\mathscr{A}(G)$, since we have assumed that $\mathscr{A}(G)$ is finitely presented. It follows therefore that the relations (5.6) together with finitely many of the relations (5.7) will also define $\mathscr{A}(G)$. Thus we can find a positive integer $q$ such that $R \cup S$ together with all relations of the form (5.7) for $n \leq q$ define $\mathscr{A}(G)$. Put

$$\tilde{\mathscr{A}}(G) = \langle x_1, \ldots, xm, d, s, t, f, g; R \cup S \cup T \rangle$$

with

$$T = \{x^{d^{t^n}} = x x^{s^{t^n}}, [x, y^{s^{t^n}}] = 1 \mid x, y \in Z_n, n = 0, 1, \ldots, q\}. \tag{5.8}$$

We claim that

$$\mathscr{A}(G) \neq \tilde{\mathscr{A}}(G);$$

indeed we shall show that in $\tilde{\mathscr{A}}(G)$

$$[d, d^{s^{t^{q+1}}}] \neq 1. \tag{5.9}$$

It is a little simpler to accomplish this by adding the relations

$$x_1 = x_2 = \cdots = x_m = 1$$

to $\tilde{\mathscr{A}}(G)$ and then proving that in the quotient group, $B$, (5.9) is valid. Notice that

$$B = \langle d, s, t, f, g; S \cup T\rangle \tag{5.10}$$

with $T$ given by (5.8).

Now put

$$L = \mathrm{gp}(d, s, t) \quad \text{and} \quad M = \mathrm{gp}(t, f, g).$$

It is clear from the presentation (5.10) (see also (5.8)) that $B$ is the free product of $L$ and $M$ amalgamating $\mathrm{gp}(t)$:

$$B = \{L*M; \mathrm{gp}(t)\}.$$

Now

$$L = \langle d, s, t; T\rangle,$$

which means that it is enough to prove (5.9) in $L$ itself. We shall construct $L$ as an HNN-extension of a colimit of certain free products with amalgamations. To this end let $\langle d_0, s_0\rangle$ be the free group on $d_0$ and $s_0$. Form the mitosis

$$m(\langle d_0, s_0\rangle) = \mathrm{gp}(d_0, s_0, d_1, s_1)$$

as in the first paragraph of this section. Our notation puts $d_1$ in role of $t$ in (5.1) and $s_1$ in the role of $u$ in (5.2). We then form $m^2(\langle d_0, s_0\rangle)$, etc., ending with

$$m^q(\langle d_0, s_0\rangle) = \mathrm{gp}(d_0, s_0, d_1, s_1, \ldots, d_q, s_q).$$

It follows from properties of the functor $m^q$ (see Lemma 5.3) and from its definition (see Lemma 2.1) that

$$\mathrm{gp}(d_1, s_1, \ldots, d_q, s_q = m^{q-1}(\mathrm{gp}(d_1, s_1))$$

and that

$$\mathrm{gp}(d_1, s_1) \text{ is free on } d_1 \text{ and } s_1.$$

Put

$$N_0 = m^q(\langle d_0, s_0\rangle) \quad \text{and} \quad N_1 = m^q(\langle d_1, s_1\rangle)$$

where we can assume, using the obvious notation, that

$$N_1 = \mathrm{gp}(d_1, s_1, \ldots, d_{q+1}, s_{q+1}).$$

We can then form the free product with amalgamation

$$N_{0,1} = \{N_0*N_1; m^{q-1}(\langle d_1, s_1\rangle)\}.$$

Notice that it follows by Lemma 2.1 that

$$[d_0, d_{s+1}] \neq 1.$$

We continue in this manner forming

$$N_2 = m^q(\langle d_2, s_2\rangle), \qquad N_{0,2} = \{N_{0,1} * N_2;\ m^{q-1}(\langle d_2, s_2\rangle)\},$$

etc., ending with

$$N = N_{0,\infty} = \operatorname{colim} N_{0,i}.$$

The mapping $d_i \to d_{i+1}$, $s_i \to s_{i+1}$, $i = 0, 1, \ldots,$ defines a monomorphism $\varphi$ of $N$ into a subgroup $\mathbf{0}$ of $N$. We now form

$$\tilde{L} = \langle N, t;\ a^t = \varphi a \quad \text{for} \quad a \in N\rangle.$$

It is clear from the construction of $\tilde{L}$ that there is a homomorphism (in fact an isomorphism) of $L$ into $\tilde{L}$ given by

$$t \to t, \qquad d \to d_0 \quad \text{and} \quad s \to s_0.$$

Then

$$[d, d^{s^{q+1}}] \to [d_0, d_{q+1}] \neq 1;$$

which establishes (5.9) and completes the proof of Theorem 5.6. $\square$

There is a consequence of Theorem 5.6 that is worth drawing attention to. Suppose that $\sigma = \{\sigma_i\}$ is a properly ascending sequence of positive integers. Let $Z(j)$ denote the cyclic group of order $p_j$, the $j$th prime, and put

$$A_\sigma = \bigoplus_i Z(\sigma_i).$$

Now $A_\sigma$ is countable. So it can, by the method of G. Higman, B.H. Neumann and Hanna Neumann [12], be imbedded in a 2-generator group $C_\sigma$, where the orders of the elements of finite order in $C_\sigma$ are precisely those of the elements of $A_\sigma$. Consider now the groups $\mathscr{A}(C_\sigma)$. Again the orders of the elements of finite order in $\mathscr{A}(C_\sigma)$ are those of the elements of $A_\sigma$. If $\sigma \neq \tau$, then $A_\sigma$ and $A_\tau$ do not have the same orders for their elements. Hence,

$$\mathscr{A}(C_\sigma) \not\cong \mathscr{A}(C_\tau) \quad \text{for } \sigma \neq \tau.$$

However, the number of such properly increasing sequences is the cardinality of the real line. Thus we have

**Corollary 5.8.** *There exist continuously many 7-generator acyclic groups.*

This contrasts rather favourably with G. Baumslag [1] (see also G. Baumslag [2] and G. Baumslag and R. Strebel [4]) where the first example of a finitely generated group $G$ which is not finitely presented with $H_2(G; Z) = 0$ was constructed. Indeed Corollary 5.8 ends all possibility that information about the defining relations of a group can be crudely gleaned from its integral homology.

## 6. Suspensions

Let $G$ be a group and suppose

$$G \rightarrowtail cG$$

is an injection of $G$ into an acyclic group $cG$. In analogy with the topological construction we form a "suspension" of $G$ by taking the pushout

$$\begin{tikzcd} G \arrow[r] & cG \\ cG \arrow[u] \arrow[r] & \Sigma G \arrow[u] \end{tikzcd}$$

Since $\Sigma G = \{cG * cG; G\}$, it is clear from the Mayer–Vietoris Sequence that

$$H_1(\Sigma G; Z) = 0$$

and

$$H_{i+1}(\Sigma G; Z) \approx H_i(G; Z) \quad \text{for } i \geqslant 1.$$

It is this property that allows one to think of $\Sigma G$ as a sort of group-theoretic suspension of $G$. In order to be able to repeat this process we need an injection of $\Sigma G$ into an acyclic group. This can be done economically, and it is important that it can be done economically (cf. the proof of Lemma 10.1), as follows.

**Theorem 6.1.** *Suppose $F \rightarrowtail G$, where $G$ is acyclic. Then $\{G * G; F\}$ injects into the acyclic group*

$$P = \{(A \times F) * G; F\},$$

*where $A$ is any non-trivial acyclic group.*

**Proof.** It is clear from the Mayer–Vietoris Sequence that $P$ is acyclic. To see that $\{G * G; F\}$ injects into $P$ choose $a \in A$, $a \neq 1$. It follows immediately from Lemma 2.3 that

$$\operatorname{gp}(a^{-1}Ga, G) = \{a^{-1}Ga * G; F\},$$

as required. □

For example, suppose $F$ has a finite $K(F, 1)$ of dimension $n$ and $G$ one of dimension $n+1$. Let $A$ denote the Higman group (Section 3), which has a finite $K(A, 1)$ of dimension 2. Then $A \times F$, $G$, and $F$ have gf dim equal to $n+2$, $n+1$, and $n$, respectively. Thus $P$ has a finite $K(P, 1)$ of dimension $n+2$, and $\Sigma F \to P$ geometrically reflects $F \to G$ one dimension higher.

As an illustration, let $F = \langle a \rangle$ be an infinite cyclic group and let $F \to G$ be an injection of $F$ into a copy of the Higman group $A$ (above). Then by iteration of the process described above we construct

$$F_n \to G_{n+1}$$

with $K(F_n, 1)$ a finite $n$-dimensional simplicial complex having the integral homology of an $n$-sphere and $K(G_{n+1}, 1)$ a finite $(n+1)$-dimensional simplicial complex having trivial integral homology.

We enlarge this class of construction considerably later in this paper.

## 7. On centers of acyclic groups

By a theorem of D. Gottlieb and J. Stallings [29] an acyclic geometrically finite group must have a trivial center. But the role of the center in homology remains somewhat unclear. Deferring its proof until Section 11, we are able to establish

**Theorem 7.1.** *Let $A$ be an abelian group. Then there is an acyclic group $G$ having $A$ as center.*

Our argument constructs $G$ with an infinite set of generators. In this section our object is to demonstrate the existence of a large class of finitely generated groups with large centers and with trivial $H_1(\ ; Z)$ and $H_2(\ ; Z)$.

We shall use the following lemma of O. Grun [10].

**Lemma 7.2.** *Let $G$ be a group and let $x$ be an element in the second center $\zeta_2(G)$ of $G$; i.e., if $g \in G$, then $[x, g] \in \zeta_1(G)$, the center of $G$. Then the map*

$$g \to [x, g], \quad g \in G,$$

*is a homomorphism of $G$ into $\zeta_1(G)$.*

Notice that the Grun homomorphism has abelian image and hence its kernel contains the derived group.

**Proposition 7.3.** *Let $G$ be a group and let*

$$G = \langle X ; R \rangle$$

*be a presentation of G. Furthermore, let*

$$W = \{w_j(\boldsymbol{x}) | j \in J\}$$

*be an indexed set of words $w_j(\boldsymbol{x})$ in the given generators $X$ of $G$, and let*

$$S = \{[w_j(\boldsymbol{x}), y] | w_j(\boldsymbol{x}) \in W, y \in X\}.$$

*Finally, let*

$$\tilde{G} = \langle X; R \cup S \rangle.$$

*and $\theta$ be the quotient map $G \twoheadrightarrow \tilde{G}$. Then if $H_1(G; Z)$ is trivial, $H_2(\theta; Z)$ is an epimorphism.*

**Proof.** The proof of this proposition uses the well-known formula of *H.* Hopf for $H_2(G; Z)$. To this end let us suppose

$$X = \{x_i | i \in I\}$$

and let $F$ be the free group on an equally indexed set

$$\Xi = \{\xi_i | i \in I\}$$

of generators. We obtain then in the obvious way the isomorphisms

$$F/K \approx G \quad \text{and} \quad F/L \approx \tilde{G}$$

with $K \leqslant L$. Notice that working modulo $[L, F]$ we have the congruence

$$L \equiv \mathrm{gp}(r(\boldsymbol{\xi}), [w_j(\boldsymbol{\xi}), \eta] | r(\boldsymbol{x}) \in R, j \in J, \eta \in \Xi).$$

Since $H_1(G; Z)$ is trivial, so is $H_1(\tilde{G}; Z)$. It follows that

$$[w_j(\boldsymbol{\xi}), \eta] = [w_j(\boldsymbol{\xi}), v_j(\boldsymbol{\xi})\lambda_j],$$

where $v_j \in [F, F]$ and $\lambda_j \in L$. Remembering that $L$ is central modulo $[L, F]$, we obtain from Grun's lemma, the congruences

$$[w_j(\boldsymbol{\xi}), v_j(\boldsymbol{\xi})\lambda_j] \equiv [w_j(\boldsymbol{\xi}), v_j(\boldsymbol{\xi})] \equiv 1$$

modulo $[L, F]$. Hence

$$L/[L, F] = \mathrm{gp}(r(\boldsymbol{\xi})[L, F] | r(\boldsymbol{x}) \in R). \tag{7.1}$$

Suppose that

$$w(\boldsymbol{\xi}) \in L \cap [F, F].$$

It follows from (7.1) that, modulo $[L, F]$, $w(\boldsymbol{\xi})$ is expressible as a word $v(r(\boldsymbol{\xi}))$ in the $r(\boldsymbol{\xi})$, with $r(\boldsymbol{x}) \in R$:

$$w(\boldsymbol{\xi}) \equiv v(r(\boldsymbol{\xi})).$$

Since $w(\boldsymbol{\xi}) \in [F, F]$, $v(r(\boldsymbol{\xi})) \in [F, F]$. This insures that $w(\boldsymbol{\xi})[L, F]$ is the image of $v(r(\boldsymbol{\xi}))[K, F]$ under $H_2(\theta; Z)$. $\square$

Of course, it follows from this proposition that if $H_2(G; Z)$ is finitely generated, then so is $H_2(\tilde{G}; Z)$, and if $H_2(G; Z) = 0$, then $H_2(\tilde{G}; Z) = 0$. This provides a way to obtain groups with trivial first and second integral homology which have large centers.

**Theorem 7.4.** *For each $n = 0, 1, \ldots, \infty$ there exist continuously many 14-generator groups $G$ with the properties*
(i) $H_1(G; Z) = H_2(G; Z) = 0$ *and*
(ii) $\zeta(G) \approx Z^n$.

**Proof.** In Corollary 5.8 we established the existence of continuously many 7-generator acyclic groups $\mathscr{A}(C_\sigma)$. For ease of notation put

$$G_\sigma = \mathscr{A}(C_\sigma).$$

Now each $G_\sigma$ contains a free subgroup of infinite rank. Let $c_1, c_2, \ldots$ freely generate a free subgroup of $G_\sigma$, and let $\bar{G}_\sigma$ be an isomorphic copy of $G_\sigma$.

Suppose that $G_\sigma$ is presented in the form

$$G_\sigma = \langle X; R \rangle.$$

Observe that

$$T_\sigma = G_\sigma * \bar{G}_\sigma$$

is acyclic. In particular $H_1(T_\sigma; Z) = H_2(T_\sigma; Z) = 0$. Put

$$\tilde{T}_\sigma = \langle X \cup \bar{X}; R \cup \bar{R} \cup \{[c_i\bar{c}_i^{-1}, z] \| 1 \leq i \leq n, z \in X \cup \bar{X}\} \rangle.$$

By Proposition 7.3

$$H_1(\tilde{T}_\sigma; Z) = H_2(\tilde{T}_\sigma; Z) = 0.$$

In order to complete the proof of this theorem, we still have to verify that the groups $\tilde{T}_\sigma$ are distinct and that they have the correct centers.

To this end put $P = G_\sigma \times A$, where $A$ is free abelian of rank $n$ on $a_1, \ldots, a_n$. Similarly put $\bar{P} = \bar{G}_\sigma \times \bar{A}$, where $\bar{A}$ is an isomorphic copy of $A$. Consider the subgroups

$$H = \mathrm{gp}(c_1a_1^{-1}, \ldots, c_na_n^{-1}, a_1, \ldots, a_n) \quad \text{and} \quad K = \mathrm{gp}(\bar{c}_1, \ldots, \bar{c}_n, \bar{a}_1, \ldots, \bar{a}_n).$$

The map from $H$ to $K$ defined by

$$c_ia_i^{-1} \to \bar{c}_i \quad \text{and} \quad a_i \to \bar{a}_i \quad \text{for } 1 \leq i \leq n$$

induces an isomorphism $\varphi : H \twoheadrightarrow K$. Hence we can form

$$T_\sigma^* = \{P * \bar{P}; H \overset{\varphi}{=} K\}.$$

It follows readily from the universal mapping property of free products with amalgamation and the given presentation of $\tilde{T}_\sigma$ in terms of generators and defining relations that

$$\tilde{T}_\sigma \approx T^*_\sigma.$$

This implies that the orders of the elements of finite order in $\tilde{T}_\sigma$ are the same as those of $G_\sigma$. Hence

$$\tilde{T}_\sigma \not\approx \tilde{T}_\tau \quad \text{if } \sigma \neq \tau.$$

Finally, observe that

$$\zeta(\tilde{T}_\sigma^{\cdot\cdot}) = \zeta(P) \cap H \cap \zeta(\bar{P}) = A$$

is free abelian of rank $n$. □

## 8. Realizing simplicial complexes

In this section we describe some general notions of "realizations" of simplicial complexes. This is preparatory to more explicit realizations leading to functors

$$lX : LX \to X$$

having properties analogous to the Kan–Thurston functor $t$, $T$, with certain strengthenings as mentioned in the introduction.

Roughly, the basic idea is to build a family of groups and injections corresponding to the faces of a simplex – this construction to be done in a symmetric way – and then to attach these by free products with amalgamation to build up groups corresponding to an arbitrary simplicial complex. Because of the need for symmetry the base-point presents a difficulty. Where is it to be?

Clearly, to make such constructions in a functorial manner we must dispense with the base point, and consequently also with the fundamental group. Of course, the way to do this is to use instead the fundamental groupoid, and to obtain fundamental groups by localizing at a point when needed. The group theoretic content is entirely in keeping with that of the previous sections, but the actual constructions have a different and more categorical flavor. Hence the language and flavor of this and subsequent sections change somewhat; see S. MacLane [19].

We write *Sets* for the category of sets and functions and $\Phi \subset$ *Sets* for the full subcategory of non-empty finite sets. We write ${}^{\mathrm{i}}\Phi \subset \Phi$ for the subcategory of injective maps. An *abstract simplicial complex* (in the classical terminology) is a subset $K \subset \operatorname{ob} \Phi$ such that $\emptyset \neq \tau \subset \sigma \in K$ implies that $\tau \in K$. We shall also regard such a $K$ as a subcategory of $\Phi$, or indeed of ${}^{\mathrm{i}}\Phi$, by supplying it with the inclusion maps $\tau \subset \sigma$ for $\sigma \in K$, and write

$$u_k : K \to {}^{\mathrm{i}}\Phi$$

for the inclusion of $K$ in ${}^{\mathrm{i}}\Phi$.

Let $\Phi^n$ denote the subcategory of $\Phi$ consisting of those sets having no more than $n+1$ elements, $n = 0, 1, 2, \ldots$, and let ${}^{\mathrm{i}}\Phi^n = {}^{\mathrm{i}}\Phi \cap \Phi^n$. Finally, we write ${}^{\mathrm{i}}\Phi_n$ for the subcategory of ${}^{\mathrm{i}}\Phi$ of sets having exactly $n+1$ elements; this is a connected groupoid.

If $K$ is an abstract simplicial complex, then $K^n = K \cap {}^{\mathrm{i}}\Phi^n$ is the *n-skeleton of K*, and $K \cap {}^{\mathrm{i}}\Phi_n$ is the set of *n-simplices of K*. The set of *vertices of K* is

$$vK = \bigcup_{\sigma \in K} \sigma.$$

A *simplicial map* $f: K \to L$ of abstract simplicial complexes is a function

$$vf: vK \to vL$$

such that if $\sigma \in K$, then $f\sigma \in L$. The abstract simplicial complexes and their simplicial maps form the category $\mathscr{K}$. The subcategory of injective simplicial maps is ${}^{\mathrm{i}}\mathscr{K}$. Evidently, $v$ is a functor $v: \mathscr{K} \to \mathit{Sets}$.

The functor $v$ has a left adjoint $\delta: \mathit{Sets} \to \mathscr{K}$ with $\delta x$ the set of singletons of elements of $X$ and it has a right adjoint $\Delta$ with $\Delta X$ the set of non-empty finite subsets of $X$. Also, $\delta$ has a left adjoint $\pi_0: \mathscr{K} \to \mathit{Sets}$ where $\pi_0 K$ is $vK$ modulo the equivalence relation generated by identifying points in the same simplex.

Although the category $\mathscr{K}$ is complete and cocomplete, the subcategory ${}^{\mathrm{i}}\mathscr{K}$ fails to be cocomplete. For obvious reasons most coproducts and pushouts are lacking. If the diagram

$$\begin{tikzcd} K_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & K_1 \arrow[d, "g_1"] \\ K_2 \arrow[r, "g_2"] & K \end{tikzcd} \tag{8.1}$$

is a pushout in $\mathscr{K}$ and $f_1$ and $f_2$ are injective, as also $g_1$ and $g_2$ so that (8.1) lies in ${}^{\mathrm{i}}\mathscr{K}$, we shall call it an *i-pushout*. A typical example is that in which $K_1$ and $K_2$ are subcomplexes of $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$.

A diagram in ${}^{\mathrm{i}}\mathscr{K}$ indexed by a directed set does have a colimit in ${}^{\mathrm{i}}\mathscr{K}$; it coincides with the colimit in $\mathscr{K}$. Here a typical example is the union of an *increasing family* of subcomplexes; i.e., of a family such that the union of any two is contained in a third.

These observations will be used in induction arguments below; the following *induction principle* holds.

**Proposition 8.1.** (i) *If $K$ is a finite abstract simplicial complex, then either $K = \Delta\sigma$ or $K = K' \cup K''$ where $K'$ and $K''$ are proper subcomplexes.*

(ii) *Any $K$ is the union of its finite subcomplexes, which form an increasing family.*

Our constructions involve functors from ${}^{\mathrm{i}}\mathscr{K}$ to various categories which respect i-pushouts and directed colimits. If $\mathscr{C}$ is a category, an *i-realization in* $\mathscr{C}$ is a functor

$$\mathcal{R} : {}^{\mathrm{i}}\mathcal{K} \to \mathcal{C}$$

such that i-pushouts go into pushouts and directed colimits are preserved.

In order to analyze these we observe that $\Delta({}^{\mathrm{i}}\Phi) \subset {}^{\mathrm{i}}\mathcal{K}$ and use $\Delta$ also for the restriction. If $\mathcal{R}$ is an i-realization, then

$$\mathcal{R}\Delta : {}^{\mathrm{i}}\Phi \to \mathcal{C},$$

and

$$\mathcal{R} \mapsto \mathcal{R}\Delta$$

defines a functor from the category of i-realizations in $\mathcal{C}$ to the functor category $\mathrm{cat}({}^{\mathrm{i}}\Phi, \mathcal{C})$.

**Proposition 8.2.** *If the category $\mathcal{C}$ is cocomplete, then*

$$\mathcal{R} \mapsto \mathcal{R}\Delta$$

*defines an equivalence of the category of i-realizations in $\mathcal{C}$ with* $\mathrm{cat}({}^{\mathrm{i}}\Phi, C)$.

**Proof.** The inverse is given as follows. For $F : {}^{\mathrm{i}}\Phi \to \mathcal{C}$, let $F^{\#}K = \mathrm{colim}\, Fu_K$, where $u_K : K \to {}^{\mathrm{i}}\Phi$ is the inclusion as above. Then $F^{\#}\Delta\sigma = F\sigma$, since $\sigma$ is terminal in $\Delta\sigma$. There is an evident natural transformation

$$(\mathcal{R}\Delta)^{\#} \to \mathcal{R}$$

for any i-realization $\mathcal{R}$. Induction over finite subcomplexes of $K$ (Proposition 8.1) shows $(\mathcal{R}\Delta)^{\#}K \to \mathcal{R}K$ is an isomorphism. □

If $x : {}^{\mathrm{i}}\mathcal{K} \to \mathcal{C}$ is any functor, then $X\Delta : {}^{\mathrm{i}}\Phi \to \mathcal{C}$, and if $\mathcal{C}$ is cocomplete, then $(X\Delta)^{\#}$ is an i-realization in $\mathcal{C}$. There is an evident natural transformation

$$(X\Delta)^{\#} \to X$$

such that $(X\Delta)^{\#}\Delta \to X\Delta$ is the identity, and this is universal for natural transformations $\mathcal{R} \to X$ of i-realizations into $X$.

**Proposition 8.3.** $X \mapsto (X\Delta)^{\#}$ *reflects functors* ${}^{\mathrm{i}}\mathcal{K} \to \mathcal{C}$ *into the category of i-realizations in the cocomplete category $\mathcal{C}$.*

The i-realizations in particular form a category because ${}^{\mathrm{i}}\Phi$ is (up to equivalence) a small category.

We refine this analysis further by describing a sequential construction on the skeletons of $F$; i.e., on the restrictions

$$F^n = F|{}^{\mathrm{i}}\Phi^n$$

of a functor $F : {}^{\mathrm{i}}\Phi \to \mathscr{C}$. For $\sigma \in \mathcal{O}b\,{}^{\mathrm{i}}\Phi$, set $\dot{\Delta}\sigma = \Delta\sigma - \{\sigma\}$; this defines a subfunctor $\dot{\Delta}$ of $\Delta$. If $\sigma \in {}^{\mathrm{i}}\Phi_n$, then $\dot{\Delta}\sigma \in {}^{\mathrm{i}}\Phi^{n-1}$. Now let

$$F^n : {}^{\mathrm{i}}\Phi^n \to \mathscr{C}$$

be a functor, with restrictions

$$F^{n-1} = F^n|{}^{\mathrm{i}}\Phi^{n-1} \quad \text{and } F_n = F^n|{}^{\mathrm{i}}\Phi_n.$$

Letting $F^{n-1}_{\#}\sigma = \operatorname{colim} F^{n-1}u_{\dot{\Delta}\sigma}$, $F^{n-1}_{\#} : {}^{\mathrm{i}}\Phi_n \to \mathscr{C}$ and $F^n$ defines a natural transformation

$$\varphi_n : F^{n-1}_{\#} \to F_n.$$

**Lemma 8.4.** *Extensions to ${}^{\mathrm{i}}\Phi^n$ of $F^{n-1} : {}^{\mathrm{i}}\Phi^{n-1} \to \mathscr{C}$ are in bijective correspondence with natural transformations $F^{n-1}_{\#} \to F_n$ of functors ${}^{\mathrm{i}}\Phi_n \to \mathscr{C}$.*

We shall refer to $\varphi_n$ as the *n-simplex span of F*, which can thus be described in terms of its $n$-simplex spans. We shall say also that it is the $n$-simplex span of $R = F^*$, the associated realization. Notice that in these terms

$$\varphi_n : R\dot{\Delta} \to R\Delta \quad \textit{on } {}^{\mathrm{i}}\Phi_n.$$

A commuting square

$$\begin{tikzcd} W_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & W_1 \arrow[d] \\ W_2 \arrow[r] & W \end{tikzcd} \tag{8.2}$$

in the category k $\mathscr{T}$op of Hausdorff compactly generated spaces is called a 2-*cofibration* if $f_1$ and $f_2$ are cofibrations and the canonical map of the pushout $V$ of $f_1$ and $f_2$ into $W$ is also a cofibration. If, in addition, the map $V \to W$ is a homotopy equivalence, then (8.2) is called a *homotopy pushout.*

A functor $X : {}^{\mathrm{i}}\mathscr{K} \to \mathrm{k}\,\mathscr{T}\mathrm{op}$ is *cofibered* if for every i-pushout (8.1) in $\mathscr{K}$, the image under $X$ is a 2-cofibration. Clearly, an i-realization ${}^{\mathrm{i}}\mathscr{K} \to \mathrm{k}\,\mathscr{T}\mathrm{op}$ is cofibered if it takes all morphisms in ${}^{\mathrm{i}}\mathscr{K}$ into cofibrations.

**Lemma 8.5.** *An i-realization $\mathscr{R} : {}^{\mathrm{i}}\mathscr{K} \to \mathrm{k}\,\mathscr{T}\mathrm{op}$ is cofibered if for all $\sigma$, $\mathscr{R}\dot{\Delta}\sigma \to \mathscr{R}\Delta\sigma$ is a cofibration.*

**Proof.** We need show only that if $K \subset L$, then $RK \to RL$ is a cofibration. The proof is by induction over finite subcomplexes $L'$ of $L$, showing that $\mathscr{R}K \to \mathscr{R}(K \cup L')$ is a cofibration. □

**Lemma 8.6.** *For any functor $X : {}^{\mathrm{i}}\mathscr{K} \to \mathrm{k}\,\mathscr{T}\mathrm{op}$ there exist a cofibred* i-*realization*

$X' : {}^{\mathrm{i}}\mathcal{K} \to \mathrm{k}\,\mathcal{T}\mathrm{op}$ *and a natural transformation* $\xi : X' \to X$ *such that for every non-empty finite set* $\sigma$, $\xi_{\Delta\sigma}$ *is a homotopy equivalence.*

**Proof.** We construct $X'$ and $\xi$ dimensionwise, dimension 0 presenting no problem. If both have been constructed through dimension $n-1$, and $\sigma$ has $n+1$ elements, let

$$X'\dot{\Delta}\sigma \to X'\Delta\sigma \xrightarrow{\xi_{\Delta\sigma}} X\Delta_\sigma$$

be a mapping cylinder for the composite

$$X'\dot{\Delta}\sigma \xrightarrow{\xi_{\dot{\Delta}\sigma}} X\dot{\Delta}\sigma \to X\Delta\sigma. \quad \square$$

We shall refer to such a pair $(X', \xi)$ as a *mapping cylinder* of $X$.

We shall say that $X : {}^{\mathrm{i}}\mathcal{K} \to \mathrm{k}\,\mathcal{T}\mathrm{op}$ is a *homotopy i-realization* if it takes i-pushouts in $\mathcal{K}$ into homotopy pushouts and preserves directed colimits.

**Proposition 8.7.** *If* $X$ *is a homotopy i-realization and* $(X', \xi)$ *is a mapping cylinder for* $X$, *then for all* $K \in \operatorname{ob} \mathcal{K}$, $\xi_K$ *is a homotopy equivalence.*

Once more the proof is by induction over the finite subcomplexes of $K$. And in an entirely analogous manner we obtain

**Proposition 8.8.** *If* $f : \mathcal{R} \to \mathcal{R}'$ *is a natural transformation of cofibered i-realizations such that for all* $\sigma$, $f_{\Delta\sigma}$ *is a homotopy equivalence, then* $f_K$ *is a homotopy equivalence for all* $K$.

By a *homology equivalence* in $\mathrm{k}\,\mathcal{T}\mathrm{op}$ we shall mean a map

$$f : V \to W$$

in $\mathrm{k}\,\mathcal{T}\mathrm{op}$ such that for every local coefficient system $A$ on $W$, the homomorphism

$$Hf : H(V; f^*A) \to H(W; A)$$

of homology with local coefficients is an isomorphism, where $f^*A$ is the local coefficient system on $V$ induced from $A$ by $f$. Notice that if $W$ has simply connected path components, then $f$ is a homology equivalence if

$$Hf : H(V; Z) \to H(W; Z)$$

is an isomorphism.

**Proposition 8.9.** *Let* $f : \mathcal{R} \to \mathcal{R}'$ *be a natural transformation of cofibered i-realizations such that for all finite* $\sigma$, *the map* $f_{\Delta\sigma}$ *is a homology equivalence. Then for all* $K$, $f_K$ *is a homology equivalence.*

The proof is by induction, using in the case $K = K' \cup K''$, the Mayer–Vietoris Sequence with local coefficients.

The *standard geometrical realization*

$$K \to |K|$$

is defined as

$$\mathrm{cnv}\, x^{\#} : {}^{\mathrm{i}}\mathcal{K} \to \mathrm{k}\,\mathcal{T}\mathrm{op},$$

where $\mathrm{cnv}\, x : {}^{\mathrm{i}}\Phi \to \mathrm{k}\,\mathcal{T}\mathrm{op}$ takes a non-empty finite set $\sigma$ into its convex hull in the $\mathbb{R}$ vector space with basis $\sigma$. It is well-known to be cofibered. In addition, it has the property of being weakly terminal among cofibered realizations:

**Proposition 8.10.** *If $\mathcal{R}$ is a cofibered i-realization, then there is a natural transformation $f : \mathcal{R} \to |\ \ |$. Any two such $f$ are homotopic.*

**Proof.** We proceed by induction on dimension. Since $|\Delta\{0\}|$ is a point, the induction starts in dimension 0. Suppose $f$ has been constructed through dimension $n-1$. Then it is enough to find a natural transformation

$$f_n : \mathcal{R}\Delta|{}^{\mathrm{i}}\Phi_n \to |(\Delta|{}^{\mathrm{i}}\Phi_n)|$$

such that for each $\sigma \in {}^{\mathrm{i}}\Phi_n$, the diagram

$$\begin{array}{ccc} \mathcal{R}\dot{\Delta}\sigma & \longrightarrow & \mathcal{R}\Delta\sigma \\ {\scriptstyle f_\sigma}\downarrow & & \downarrow{\scriptstyle f_{n,\sigma}} \\ |\dot{\Delta}\sigma| & \longrightarrow & |\Delta\sigma| \end{array} \tag{8.3}$$

commutes. Let us fix our attention on some $\sigma$, say $[n] = \{0, 1, \ldots, n\}$. Since $\mathcal{R}\dot{\Delta}[n] \to \mathcal{R}\Delta[n]$ is a cofibration and $|\Delta[n]|$ is contractible, there exists $g : \mathcal{R}\Delta[n] \to |\Delta[n]|$ such that in this instance (8.3) commutes. For any $\sigma$, we let

$$f_{n,\sigma} = \sum_{\theta : \sigma \cong [n]} \frac{1}{(n+1)!} |\Delta\theta|^{-1} g(\mathcal{R}\Delta\theta),$$

using the convex structure of $|\Delta\sigma|$ to define the sum.

The second statement is immediate from the fact that any $f : \mathcal{R} \to |\ \ |$ is of the form $(f\Delta)^{\#}$. If also $g : \mathcal{R} \to |\ \ |$, then

$$((1-t)f\Delta + tg\Delta)^{\#}, \quad t \in [0, 1],$$

provides the homotopy. $\square$

We make finally the following observations.

**Proposition 8.11.** *If $\mathcal{R}$ is a cofibered i-realization with $\mathcal{R}\Delta[0] \neq \emptyset$ and $\mathcal{R}\dot{\Delta}[1]$ pathwise connected, then for any $f : \mathcal{R} \to |\ \ |$, any $K$, and any $x \in \mathcal{R}K$,*

$$\pi_1 f_K : \pi_1(\mathcal{R}K, x) \to \pi_1(|K|, f_K x)$$

*is surjective.*

The hypothesis clearly implies that the standard generators in $|K|$ of $\pi_1(|K|, f_K x)$ can be lifted to $\mathcal{R}K$. $\square$

**Proposition 8.12.** *If $\mathcal{R}$ is a cofibered i-realization with $\mathcal{R}\Delta[n]$ contractible for all n, then for every K, $\mathcal{R}K \to |K|$ is a homotopy equivalence.*

We recall that a space $W$ is *acyclic* if $H(W; Z)$ is concentrated in dimension 0 and $H_0(W; Z) \approx Z$.

**Proposition 8.13.** *If $\mathcal{R}$ is a cofibered i-realization with $\mathcal{R}\Delta[n]$ acyclic for all n, then for every K, $\mathcal{R}K \to |K|$ is a homology equivalence.*

## 9. Examples of realizations

In this section we discuss some cofibered i-realizations which are composites of functors passing through the categories of groups or groupoids. The machinery related to this is more-or-less familiar, and we only sketch it.

We write Cat for the category of small categories and Gpd $\subset$ Cat for the full subcategory of groupoids. This inclusion has left adjoint $A \to A[A^{-1}]$, the category of fractions (see [8]), and right adjoint taking $A$ to the subcategory of isomorphisms in $A$.

The functor $\mathcal{O}$b : Cat $\to$ Sets has left adjoint disc : Sets $\to$ Cat, the discrete category functor, and right adjoint indisc : Sets $\to$ Cat, the indiscrete category functor with

$$(\text{indisc}\, E)(x, y) = \{(x, y)\} \quad \text{for } x, y \in E.$$

The functor disc has left adjoint $\pi_0$, with $\pi_0 A = \mathcal{O}\text{b}A/\sim$, where $f : a \to b$ implies $a \sim b$. Both disc and indisc have values in Gpd, and the adjunctions $\pi_0 \dashv \text{disc} \dashv \mathcal{O}\text{b} \dashv$ indisc holds also for the restricted functors between Gpd and Sets.

Cat is well-known to be complete and cocomplete; the same is evidently true of Gpd. We remark that a pushout of the form (8.1) in Gpd, for which $f_1$ and $f_2$ are injective, has $g_1$ and $g_2$ injective as well. We shall speak in this case of an *i-pushout in* Gpd, or denoting by ${}^{\text{i}}$Gpd the subcategory of injective morphisms, *in* ${}^{\text{i}}$Gpd.

We write $N$ for the "*nerve functor*" Cat $\to \mathcal{S}$, where $\mathcal{S}$ is the category of simplicial sets, and also for its restriction to Gpd. If $|\ |: \mathcal{S} \to \text{k}\,\mathcal{T}\text{op}$ is the (Milnor) *geometrical realization* ([21]; see also [26]), then the composite $A \to |NA|$ defines a functor

$$B : \text{Cat} \to \text{k}\,\mathcal{T}\text{op}.$$

Note that $N$, and hence $B$, preserves coproducts and directed colimits of injections, but not all colimits. In particular, $\pi_0 B = \pi_0$.

Also, note that if $A \to A'$ is an equivalence of categories, then $BA \to BA'$ is a homotopy equivalence.

If $\Gamma$ is a groupoid, then $B\Gamma$ is a CW-complex each of whose connected components is aspherical. If $\pi$ denotes the *fundamental groupoid functor*, then there is an obvious homomorphism

$$\Gamma \to \pi B\Gamma,$$

which is an equivalence of categories. Combining this observation with the Seifert–van Kampen Theorem and a theorem of J.H.C. Whitehead ([34], see Section 1), we obtain the folklore

**Theorem 9.1.** *The functor*

$$B : \mathrm{Gpd} \to \mathrm{k}\,\mathcal{T}\mathrm{op}$$

*takes i-pushouts into homotopy pushouts.*

We shall want to consider i-realizations $\mathcal{R} : {}^{\mathrm{i}}\mathcal{K} \to \mathrm{Gpd}$. In analogy with the topological case we call such a realization *cofibered* if it takes morphisms in ${}^{\mathrm{i}}\mathcal{K}$ into injective homomorphisms of groupoids; i.e., if it factors through ${}^{\mathrm{i}}\mathrm{Gpd}$.

**Lemma 9.2.** *An i-realization* $\mathcal{R} : {}^{\mathrm{i}}\mathcal{K} \to \mathrm{Gpd}$ *is cofibered if for all* $\sigma$, $\mathcal{R}\dot{\Delta}\sigma \to \mathcal{R}\Delta\sigma$ *is injective.*

The proof is completely analogous to Lemma 8.5.

We call a commutative square in ${}^{\mathrm{i}}\mathrm{Gpd}$ a 2-*cofibration* if the canonical morphism of the pushout into the terminal groupoid is injective.

**Lemma 9.3.** *In the commuting cubical diagram*

$$\begin{tikzcd}
\Gamma_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & \Gamma_1 \arrow[r] \arrow[d] & \Gamma \arrow[d] & \Gamma_2 \arrow[l] \arrow[d] \arrow[dr, phantom, "*"] & \Gamma_0 \arrow[l] \arrow[d] \\
\Gamma_0' \arrow[r] & \Gamma_1' \arrow[r] & \Gamma' & \Gamma_2' \arrow[l] & \Gamma_0' \arrow[l]
\end{tikzcd}$$

*in* Gpd *suppose that the rows are* i-*pushouts and that the starred squares are* 2-*cofibrations. Then the unstarred squares are also* 2-*cofibrations.*

We write $\mathrm{Gpd}^\circ$ for the category of pointed groupoids and ${}^{\mathrm{i}}\mathrm{Gpd}^\circ$ for the subcategory of injective morphisms. The forgetful functor $\varphi : \mathrm{Gpd}^\circ \to \mathrm{Gpd}$ preserves i-pushouts and directed colimits. The considerations above apply also to pointed groupoids, and in particular, composition with $\varphi$ preserves cofibered i-realizations.

The functor $\varphi$ takes ${}^{\mathrm{i}}\mathrm{Gpd}^\circ$ to ${}^{\mathrm{i}}\mathrm{Gpd}$, and its restriction has a subfunctor $\beta$ which takes a pointed groupoid into the full subcategory omitting the basepoint. Notice that if $\Gamma$ in $\mathrm{Gpd}^\circ$ is connected, then so is $\beta\Gamma$.

The category Gp of groups is identified in the obvious way with a full subcategory of $\mathrm{Gpd}^\circ$. The inclusion has the right adjoint $\Gamma \to \Gamma(x_0, x_0)$, where $x_0$ is the basepoint. If $\Gamma$ is connected, then $\Gamma(x_0, x_0) \to \Gamma$ is an equivalence of categories and

$$B\Gamma(x_0, x_0) \to B\Gamma$$

is a homotopy equivalence. Notice that on Gp the functor $B$ is the usual classifying space functor (bar construction).

A groupoid $\Gamma$ is *acyclic* if $B\Gamma$ is an acyclic space. In consequence of the remarks above, we have the following.

**Lemma 9.4.** *If $\Gamma$ is an acyclic pointed groupoid, then $\beta\Gamma$ is either acyclic or, precisely when $\Gamma$ is a group, empty*

**Lemma 9.5.** *If an i-pushout in* Gpd *or in* Gpd° *the other three terms are acyclic, then so also is the terminal one. Further, if the terms of a directed diagram in* ${}^{\mathrm{i}}$Gpd *or in* ${}^{\mathrm{i}}$Gpd° *are all acyclic, then so is the colimit.*

## 10. The basic realization

We next define i-realizations

$$L : {}^{\mathrm{i}}\mathcal{K} \to {}^{\mathrm{i}}\mathrm{Gpd} \quad \text{and} \quad \hat{L} : {}^{\mathrm{i}}\mathcal{K} \to {}^{\mathrm{i}}\mathrm{Gpd}^{\circ}$$

and a natural transformation

$$\lambda : L \to \beta\hat{L},$$

which we refer to collectively as the basic realization. They will depend on a group $H$ with an element $h \neq 1$, which remain fixed throughout the construction. (The construction can be thought of as a functorial variant of Theorem 6.1, with $1 \neq a \in A$ of that theorem having the role of $1 \neq h \in H$ here; see diagram (10.2) below).

They will also be subject to the following condition: if $K \to K'$ is in ${}^{\mathrm{i}}\mathcal{K}$, then

$$\begin{tikzcd} LK \arrow[r, "\bar{\lambda}_K"] \arrow[d] & \varphi\hat{L}K \arrow[d] \\ LK' \arrow[r, "\bar{\lambda}_{K'}"] & \varphi\hat{L}K' \end{tikzcd} \tag{10.1}$$

is a 2-cofibration, where $\bar{\lambda}_K$ is the composite

$$LK \to \beta\hat{L}K \to \varphi\hat{L}K.$$

The construction is inductive, starting in dimension 0 with

$$L\Delta\{x\} = \mathrm{disc}\{x\}$$

and

$$\hat{L}\Delta\{x\} = \mathrm{indisc}(\{x\} \cup \{x_0\}),$$

with basepoint at $x_0 \neq x$. Then

$$\beta\hat{L}\Delta\{x\} = L\Delta\{x\};$$

in dimension 0 we take $\lambda$ to be the identity, and (10.1) is trivially satisfied.

The inductive step is made by assigning the following $n$-simplex spans: if $\sigma$ has $n+1$ elements, then $L\Delta\sigma \equiv \beta\hat{L}\dot{\Delta}\sigma$ with $L\dot{\Delta}\sigma \to L\Delta\sigma$ given by $\lambda_{\dot{\Delta}\sigma}$, and $\hat{L}\Delta\sigma$ is defined by the pushout

$$\begin{tikzcd} L\dot{\Delta}\sigma \arrow[r] \arrow[d, "\langle 0\ 1\rangle"'] & \varphi\hat{L}\dot{\Delta}\sigma \arrow[d, "\varphi\hat{L}(\dot{\Delta}\sigma \to \Delta\sigma)"] \\ H \times L\dot{\Delta}\sigma \arrow[r] & \varphi\hat{L}\Delta\sigma \end{tikzcd} \tag{10.2}$$

$\varphi\hat{L}\Delta\sigma$ receiving its basepoint by the morphism $\varphi\hat{L}(\dot{\Delta}\sigma \to \Delta\sigma)$. Finally, the morphism

$$\lambda_{\Delta\sigma} : L\Delta\sigma \to \beta\hat{L}\Delta\sigma$$

is given by

$$\lambda u = (h, y)^{-1}[(\varphi\hat{L}(\dot{\Delta}\sigma \to \Delta\sigma))u](h, x)$$

for $u : x \to y$ in $\beta\hat{L}\dot{\Delta}\sigma = L\Delta\sigma$.

**Lemma 10.1.** *With the conventions above, the square*

$$\begin{tikzcd} L\dot{\Delta}\sigma \arrow[r] \arrow[d] & \varphi\hat{L}\dot{\Delta}\sigma \arrow[d] \\ L\Delta\sigma \arrow[r, "\lambda_{\Delta\sigma}"'] & \varphi\hat{L}\Delta\sigma \end{tikzcd}$$

*is a 2-cofibration.*

(This is of course a special case of condition (10.1).)

**Proof.** We must show that the canonical morphism of the pushout into $\varphi\hat{L}\Delta\sigma$ is injective. We do this by writing arrows in the pushout as reduced words. There are several cases, but it suffices to examine one. A typical word is of the form $f_1 g_1 \cdots f_k g_k$ with $f_i$ in $L\Delta\sigma$ and $g_i$ in $\varphi\hat{L}\dot{\Delta}\sigma$, none of them lying in $L\dot{\Delta}\sigma$. This goes into the word

$$(h, y_1)^{-1} f_1 (h, x_1) g_1 \cdots (h, y_k)^{-1} f_k (h, x_k) g_k,$$

where $f_i : x_i \to y_i$. But this is a reduced word in $\varphi\hat{L}\Delta\sigma$. $\square$

It follows almost immediately, using Lemma 9.3, that the $n$-skeletons of $L$ and $\hat{L}$ satisfy condition (10.1), and in particular, that for $\tau$ having $n+2$ elements, $L\dot{\Delta}\tau \to$

$\beta\hat{L}\dot{\Delta}\tau$ is injective; so the induction can proceed. A parallel argument shows that (10.1) holds in general.

Let $(L, \hat{L}, \lambda)$ be the basic realization constructed above, using the group $H$. From Theorem 9.1 it follows at once that $BL$ is a homotopy i-realization. By Lemma 8.6 it has a mapping cylinder $(Z, \xi)$, and by Proposition 8.7

$$\xi_K : ZK \to BLK$$

is a homotopy equivalence for all $K$.

**Lemma 10.2.** *If the group $H$ is acyclic, then for all $K$ so is $\hat{L}K$. Thus, also for each finite $\sigma$, $L\Delta\sigma = \beta\hat{L}\dot{\Delta}\sigma$ is acyclic.*

**Proof.** One verifies directly that for $\sigma$ having two elements, $\hat{L}\dot{\Delta}\sigma$ is the indiscrete groupoid on three objects and is thus acyclic. The general statement follows inductively, using the Mayer–Vietoris Sequence: Suppose in particular that the conclusion is known for $dimK < n$. Then, referring to (10.2), since $H\langle 0\ 1\rangle$ is an isomorphism, so also is $H\varphi\hat{L}(\dot{\Delta}\sigma \to \Delta\sigma)$; so that $\hat{L}\Delta\sigma$ is acyclic. □

It follows from Proposition 8.13 that the canonical maps $ZK \to |K|$ are all homology equivalences. We have accordingly proved the following version of the theorem of Kan and Thurston [14].

**Theorem 10.3.** *If $(L, \hat{L}, \lambda)$ is the basic realization corresponding to an acyclic group $H$, then there is a natural homotopy class of maps $\varphi : BL \to |\ \ |$ such that for each $K$, $\varphi_K$ is a homology equivalence.*

A groupoid $\Gamma$ is called *geometrically finite* if $B\Gamma$ has the homotopy type of a finite complex. It follows immediately from Theorem 9.1 that in an i-pushout of groupoids, if the other three are geometrically finite, then so is the terminal one.

A groupoid $\Gamma$ is of *geometrical dimension* $\leqslant n$ if $B\Gamma$ has the homotopy type of a CW-complex of dimension $\leqslant n$. From Theorem 9.1 again, we deduce that in an i-pushout of groupoids if the initial groupoid has geometrical dimension $\leqslant n-1$, and the other two $\leqslant n$, then the terminal one has geometrical dimension $\leqslant n$.

Finally, we say that $\Gamma$ is *geometrically finite of dimension* $\leqslant n$, gf dim $\Gamma \leqslant n$, if $B\Gamma$ has the homotopy type of a finite complex of dimension $\leqslant n$.

**Theorem 10.4.** *If $(L, \hat{L}, \lambda)$ is the basic realization corresponding to a group $H$ and* gf dim $H = 2$, *then for any finite complex $K$ of dimension $n$, the groupoids $LK$, $\hat{L}K$ are geometrically finite of dimensions $\leqslant n$, $\leqslant n+1$.*

This theorem is proved by a straightforward induction using the observation in (10.2) that if $\sigma$ has $n+1$ elements, then gf dim $L\dot{\Delta}\sigma \leqslant n-1$ and gf dim $H \times L\dot{\Delta}\sigma \leqslant n-1+2$.

We note that the Higman group $H$ of Section 3 satisfies the hypotheses of both Theorem 10.3 and 10.4.

## 11. Some equivalences of categories

The functors $|\ |$, $B$ and $L$ all preserve the obvious basepoints: $\Delta[0] \in {}^{\mathrm{i}}\mathcal{K}$, $\mathrm{disc}\{0\} \in \mathrm{Gpd}$, and $\{0\} \in \mathrm{k}\,\mathcal{T}\mathrm{op}$, and so they give rise to functors on the corresponding pointed categories. Furthermore, the maps $\varphi_K$ of Theorem 10.3 can be taken to be basepoint preserving as well and thus give rise to a natural transformation of the pointed functors $BL \to |\ |$.

Moreover, all of these functors preserve connectedness, and thus they restrict to the categories of pointed, connected objects. We shall denote these restrictions by

$$|\ |^\circ : {}^{\mathrm{i}}\mathcal{K}^\circ \to \mathrm{k}\,\mathcal{T}\mathrm{op}^\circ, \qquad B^\circ : \mathrm{Gpd}^\circ \to \mathrm{k}\,\mathcal{T}\mathrm{op}^\circ, \quad \text{and} \quad L^\circ : {}^{\mathrm{i}}\mathcal{K}^\circ \to \mathrm{Gpd}^\circ.$$

We have also the natural homotopy class $\varphi^\circ : B^\circ L^\circ \to |\ |^\circ$.

But now we can define the functor

$$L' : {}^{\mathrm{i}}\mathcal{K}^\circ \to \mathrm{Gp}$$

by $L'K = (L^\circ K)(x_0, x_0)$, where $x_0$ is the basepoint of $K$. The inclusion $L'K \to L^\circ K$ gives a natural transformation $u : L' \to L^\circ$ with each $u_K$ an equivalence of categories. Thus,

$$\varphi_K^\circ(B^\circ u_K) : B^\circ L'K \to |K|^\circ$$

is a homology equivalence for each $K$ in ${}^{\mathrm{i}}\mathcal{K}^\circ$. This is the pointed version of Theorem 10.3; we omit a formal restatement in favour of the amplification below.

But first we turn to proofs of two previously stated theorems.

**Proof of Theorem 1.9.** Suppose $\mathrm{gf\,dim}\, G \leq n$. Let $K$ be a connected, aspherical object of $\mathcal{K}^\circ$ of dimension $\leq n$ and with fundamental group $G$. Let

$$f : K \to cK$$

be the inclusion of $K$ in its cone. This is a ${}^{\mathrm{i}}\mathcal{K}^\circ$-morphism. We have then the commuting diagram

$$\begin{tikzcd} B^\circ L'K \arrow[r] \arrow[d] & {|K|^\circ} \arrow[d, "|f|"] \\ B^\circ L'cK \arrow[r] & {|cK|^\circ} \end{tikzcd}$$

in which the horizontal morphisms are homology isomorphisms. Let $G' = \pi_1(B^\circ L'K)$ and $H = \pi_1(B^\circ L'cK)$. Since $L$ is an i-realization, $G' \to H$ is an injection. Consider the pushout

$$\begin{tikzcd} G' \arrow[r, "a"] \arrow[d, "i"'] & G \times H \arrow[d] \\ H \arrow[r] & cG \end{tikzcd}$$

where $a$ is the diagonal. Since both $a$ and $i$ are injections, the Mayer–Vietoris Sequence applies and the group $cG$ is acyclic. Also, $\operatorname{gf\,dim} cG \leq 2n+1$. Since $G \times H \to cG$ is an injection, so is $G \times 1 \to cG$.

**Proof of Theorem 7.1.** Let $K = K(A, 2)$ be a pointed simplicial complex with second homotopy group isomorphic to $A$ and all others trivial. The path space $PK$ of $K$,

$$PK = \{\lambda : I \to K \mid \lambda 0 = * \in K\},$$

suitably topologized is a contractible space and end-point projection

$$\varepsilon_1 : PK \to K,$$

defined by $\varepsilon_1 \lambda = \lambda_1$, is a fibration. The fiber $\varepsilon_1^{-1} *$ is the loop space of $K$ and, by the homotopy exact sequence of a fibration, is a space $K(A, 1)$.

Consider the fibration

$$K(A, 1) \to T \xrightarrow{p} B^\circ L'K$$

over $B^\circ L'K$ induced from $\varepsilon_1$ by the homology equivalence

$$\varphi_K^\circ(B^\circ u_K) : B^\circ L'K \to K.$$

The morphism

$$E^r_{*,*}(p) \to E^r_{*,*}(\varepsilon_1)$$

of integral homology spectral sequences induced by $\varphi_K^\circ(B^\circ u_K)$ is an isomorphism at the $E^2$-level. Thus $H_*(T) \approx H_*(PK)$; the space $T$ is acyclic.

Since both fiber and base of $p$ are aspherical, so is the total space $T$. It is a space $K(G, 1)$ for some group $G$, and $G$ is acyclic by the previous paragraph.

We have thus an extension

$$0 \to A \to G \to G' \to 0,$$

where $G' = \pi_1(B^\circ L'K)$. The action of $G'$ on $A$ is induced through the action of $0 = \pi_1 K$ on $\pi_1 K(A, 1)$; thus it is trivial and the displayed extension is central.

To verify that $A$ is the center of $G$, we need show only that $G'$ has trivial center. But $G'$ is the directed colimit of centerless groups, each of them being formed by free products with amalgamation of centerless groups (see Corollary 4.5 of [20]). □

Continuing now with our analysis of these pointed functors, we note

**Lemma 11.1.** *If $K$ is a pointed simplicial complex, then $\pi_1 B^\circ L'K \to \pi_1 |K|^\circ$ is surjective and has perfect kernel.*

**Proof.** The first assertion follows from Proposition 8.11 and the second from the fact that $B^\circ L'K \to |K|^\circ$ is a homology equivalence. □

We shall call such a homomorphism of groups *perfect* and write $\mathrm{Gp}\mathcal{P}$ for the full subcategory of the morphism category of Gp whose objects are the perfect homomorphisms.

**Lemma 11.2.** *The morphism $\theta : G \to \pi$ of groups is perfect if and only if it is surjective, $H_1(\theta; Z)$ is a monomorphism, and $H_2(\theta; Z)$ is an epimorphism.*

There is an evident lifting of the functor $L'$ to

$$\tilde{L} : {}^{\mathrm{i}}\mathcal{K}^\circ \to \mathrm{Gp}\,\mathcal{P}$$

given by $\tilde{L}K = (L'K \to \pi_1|K|^\circ)$.

D. Quillen has defined ([25]) a functor

$$B^+ : \mathrm{Gp}\,\mathcal{P} \to \mathcal{H}\mathrm{o},$$

where $\mathcal{H}\mathrm{o}$ is the category of pointed, connected CW-complexes and pointed homotopy classes of maps, which is characterized by

**Theorem 11.3.** *There are a functor $B^+ : \mathrm{Gp}\,\mathcal{P} \to \mathcal{H}\mathrm{o}$ and a natural transformation*

$$b_\theta : B^\circ G \to B^+\theta,$$

*for $\theta : G \to \pi$ a perfect homomorphism, characterized uniquely to within isomorphism by either of the following equivalent conditions:*

(i) *$\pi_1 B^+\theta = \pi$ and $b_\theta$ is a homology equivalence with $\pi_1 b_\theta = \theta$, or*

(ii) *if $f : B^\circ G \to W$ and $\pi_1 f$ factors through $\theta$, then $f$ factors uniquely through $b_\theta$.*

We are allowing ourselves, here, to write $B^\circ : \mathrm{Gp} \to \mathcal{H}\mathrm{o}$ for the image of the classifying space functor in $\mathcal{H}\mathrm{o}$, having observed that it has values which are CW-complexes. The same convention applies also to $|\ |^\circ : {}^{\mathrm{i}}\mathcal{K}^\circ \to \mathcal{H}\mathrm{o}$.

**Proposition 11.4.** *There is a unique natural homotopy equivalence*

$$\Psi_K : B^+\tilde{L}K \to |K|^\circ$$

*such that in $\mathcal{H}\mathrm{o}$ the diagram*

$$\begin{tikzcd} B^\circ L'K \arrow[d, "b_{\tilde{L}K}"'] \arrow[dr, "\varphi_K^\circ(B^\circ u_K)"] & \\ B^+\tilde{L}K \arrow[r, "\Psi_K"'] & |K|^\circ \end{tikzcd}$$

*commutes.*

We shall denote by $\mathscr{E}$ the class of all morphisms $f$ in ${}^{\mathrm{i}}\mathscr{K}^{\circ}$ such that $|f|^{\circ}$ is a homotopy equivalence. Thus, $|\ |^{\circ}:{}^{\mathrm{i}}\mathscr{K}^{\circ}\to\mathscr{H}\mathrm{o}$ factors through a functor (which we also denote by $|\ |^{\circ}$)

$$|\ |^{\circ}:{}^{\mathrm{i}}\mathscr{K}^{\circ}[\mathscr{E}^{-1}]\to\mathscr{H}\mathrm{o},$$

where ${}^{\mathrm{i}}\mathscr{K}^{\circ}[\mathscr{E}^{-1}]$ denotes the category of fractions.

The following is a well-known folklore

**Theorem 11.5.** *$|\ |^{\circ}:{}^{\mathrm{i}}\mathscr{K}^{\circ}[\mathscr{E}^{-1}]\to\mathscr{H}\mathrm{o}$ is an equivalence of categories.*

Recall that a morphism in the category Gp $\mathscr{P}$ is a commuting square

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "\theta"'] & G' \arrow[d, "\theta'"] \\ \pi \arrow[r, "p"'] & \pi' \end{tikzcd}$$

with $\theta$ and $\theta'$ objects of Gp $\mathscr{P}$. We shall let $\mathscr{F}$ denote the class of those morphisms $(g, p)$ with $p$ an isomorphism and such that for every $\pi'$-module $A$,

$$H(g):H(G;(p\theta)^{*}A)\approx H(G';(\theta')^{*}A).$$

**Lemma 11.6.** *If $f:K\to K'$ is a ${}^{\mathrm{i}}\mathscr{K}^{\circ}$-morphism such that $|f|^{\circ}$ is a homotopy equivalence, then $\tilde{L}f\in\mathscr{F}$. If $(g, p)\in\mathscr{F}$, then $B^{+}(g, p)$ is a homotopy equivalence.*

**Proof.** $\tilde{L}f$ is the diagram obtained by applying $\pi_1$ to

$$\begin{tikzcd} B^{\circ}L'K \arrow[r] \arrow[d, "\varphi_K^{\circ}(B^{\circ}u_K)"'] & B^{\circ}L'K' \arrow[d, "\varphi_{K'}^{\circ}(B^{\circ}u_{K'})"] \\ |K|^{\circ} \arrow[r] & |K'|^{\circ} \end{tikzcd}$$

in which the vertical maps are homology equivalences. The second observation is a classical theorem of J.H.C. Whitehead: a map of pointed, connected CW-complexes which induces isomorphisms of fundamental groups and of homology with all local coefficients, is a homotopy equivalence. $\square$

Hence, $\tilde{L}:{}^{\mathrm{i}}\mathscr{K}^{\circ}\to\mathrm{Gp}\,\mathscr{P}$ and $B^{+}:\mathrm{Gp}\,\mathscr{P}\to\mathscr{H}\mathrm{o}$ induce functors (which we denote by the same symbols)

$$\tilde{L}:{}^{\mathrm{i}}\mathscr{K}^{\circ}[\mathscr{E}^{-1}]\to\mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}]\quad\text{and}\quad B^{+}:\mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}]\to\mathscr{H}\mathrm{o}.$$

One of our main results is

**Theorem 11.7.** *The functors* $\tilde{L}: {}^{i}\mathcal{K}^{\circ}[\mathscr{E}^{-1}] \to \mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}]$ *and* $B^{+}: \mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}] \to \mathscr{H}\mathrm{o}$ *are equivalences of categories.*

The proof is briefly deferred.

A special case is worth pointing out. The category $\mathscr{P}$ of perfect groups imbeds in $\mathrm{Gp}\,\mathscr{P}$ by $P \mapsto (P \to 0)$. A homomorphism $P \to P'$ of perfect groups is in $\mathscr{F}$ if it induces an isomorphism $H(P; Z) \approx H(P'; Z)$ of integral homology; i.e., $\mathscr{F} \cap \mathscr{P}$ consists precisely of such homomorphisms.

**Corollary 11.8.** *The functor* $B^{+}$ *induces an equivalence of the category* $\mathscr{P}[(\mathscr{F} \cap \mathscr{P})^{-1}]$ *with that of simply connected* CW*-complexes and homotopy classes of maps.*

The remainder of this paper is concerned solely with proving Theorem 11.7.

By Proposition 11.4 and Theorem 11.5., the composite

$$B^{+}\tilde{L}: {}^{i}\mathcal{K}^{\circ}[\mathscr{E}^{-1}] \to \mathscr{H}\mathrm{o}$$

is an equivalence of categories. Thus it is necessary to show only that

$$B^{+}: \mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}] \to \mathscr{H}\mathrm{o}$$

is faithful.

**Lemma 11.9.** *In the commuting diagram*

$$\begin{array}{ccccccccc}
G & \xrightarrow{g'} & G' & \xrightarrow{h'} & \tilde{G} & \xleftarrow{h''} & G'' & \xleftarrow{g''} & G \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\tilde{\theta}} & & \downarrow{\scriptstyle\theta''} & & \downarrow{\scriptstyle\theta} \\
\pi & \xrightarrow[p']{} & \pi' & \xrightarrow[q']{} & \tilde{\pi} & \xleftarrow[q'']{} & \pi'' & \xleftarrow[p'']{} & \pi
\end{array}$$

*in* Gp, *let the top and bottom rows be pushouts. Suppose that* $g'$ *and* $g''$ *are injective and that* $p'$ *is an isomorphism. Then if* $\theta$, $\theta'$ *and* $\theta''$ *are in* $\mathrm{Gp}\,\mathscr{P}$, *so also is* $\tilde{\theta}$. *If further,* $(g', p') \in \mathscr{F}$, *then so does* $(h'', q'')$.

Bother observations follow from consideration of the Mayer–Vietoris Sequence of the homology of the top square and consideration of the homology of the bottom square, with coefficients in an arbitrary $\tilde{\pi}$-module.

Although the category $\mathrm{Gp}\,\mathscr{P}$ does not have arbitrary pushouts, the previous lemma allows us to identify some useful ones.

We shall write $[g, p]$ for the image under $\mathrm{Gp}\,\mathscr{P} \to \mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}]$ of a morphism $(g, p)$ in $\mathrm{Gp}\,\mathscr{P}$.

**Lemma 11.10.** *Any morphism in* $\mathrm{Gp}\,\mathscr{P}[\mathscr{F}^{-1}]$ *can be written in the form* $[h, q]^{-1}[g, p]$.

**Proof.** It suffices to do this for one of the form $[\bar{g}, \bar{p}] \cdot [g', p']^{-1}$, where

$$\begin{tikzcd}
G' \arrow[d, "\theta'"] & G \arrow[l, "g'"'] \arrow[r, "\bar{g}"] \arrow[d, "\theta"] & \bar{G} \arrow[d, "\bar{\theta}"] \\
\pi' & \pi \arrow[l, "p'"] \arrow[r, "\bar{p}"'] & \bar{\pi}
\end{tikzcd}$$

and $(g', p') \in \mathcal{F}$. Let $cG$ be a cone on $G$ formed in any of the manners previously indicated and let $\tau_G : G \to cG$ be the injection.

Set $G'' = \bar{G} \times cG$, $\pi'' = \bar{\pi}$, $g'' = \langle \bar{g}\tau_G \rangle$, $\theta'' = \bar{\theta}\,\mathrm{proj}_{\bar{G}}$, and $p'' = \bar{p}$, and construct the pushout as in Lemma 11.9. Then

$$(\langle 1_{\bar{G}} \;\; 0 \rangle, 1_{\bar{\pi}}) \in \mathcal{F} \quad \text{and} \quad [\bar{g}, \bar{p}][g', p']^{-1} = [h''\langle 1_{\bar{G}} \;\; 0\rangle, q'']^{-1}[h', q'].$$

Suppose that $[h_i, q_i]^{-1}[g_i, p_i] : \theta' \to \bar{\theta}$, where

$$\begin{tikzcd}
G' \arrow[r, "g_i"] \arrow[d, "\theta'"] & G_i \arrow[d, "\theta_i"] & \bar{G} \arrow[l, "h_i"'] \arrow[d, "\bar{\theta}"] \\
\pi' \arrow[r, "p_i"'] & \pi_i & \bar{\pi} \arrow[l, "q_i"]
\end{tikzcd}$$

for $i = 0, 1$ and $(h_i, q_i) \in \mathcal{F}$. The pushout of $(h_0, q_0)$ and $(h_1, q_1)$ yields by Lemma 11.9, $\tilde{\theta} : \tilde{G} \to \tilde{\pi}$ in Gp $\mathcal{P}$ with $\theta_i \to \tilde{\theta}$ in $\mathcal{F}$.

**Lemma 11.11.** $[h_0, q_0]^{-1}[g_0, p_0] = [h_1, q_1]^{-1}[g_1, p_1]$ *if and only if the composites*

$$\theta' \xrightarrow{[g_i\, p_i]} \theta_i \to \tilde{\theta}$$

*are equal.*

It follows that the proof of Theorem 11.7 reduces to

**Lemma 11.12.** *Let* $(g_i, p_i) : \theta \to \bar{\theta}$, $i = 0, 1$, *be* Gp $\mathcal{P}$*-morphisms. If* $B^+(g_0, p_0) = B^+(g_1, p_1)$, *then* $[g_0, p_0] = [g_1, p_1]$.

**Proof.** The hypothesis clearly implies $p_0 = p_1$; we write $p$ for the common value. Consider the diagram

$$\begin{tikzcd}
G * G \arrow[rrr, "\langle (g_0\, g_1)\tau_{G*G} \rangle"] \arrow[dr, "\theta * \theta"] \arrow[ddd, "\langle (1\;1)\tau_{G*G} \rangle"'] & & & \bar{G} \times c(G * G) \arrow[dl, "\bar{\theta}\,\mathrm{proj}_{\bar{G}}"'] \arrow[ddd, "u"] \\
& \pi * \pi \arrow[r, "(p,p)"] \arrow[d, "(1\;1)"] & \bar{\pi} \arrow[d, "1"] & \\
& \pi \arrow[r, "p"'] & \bar{\pi} & \\
G \times c(G * G) \arrow[ur, "\theta\,\mathrm{proj}_G"] \arrow[rrr, "v"'] & & & E \arrow[ul, "\varphi"']
\end{tikzcd} \tag{11.1}$$

in Gp, in which the outer square is an i-pushout and $\varphi$ is defined by commutativity.

The inner square is also a pushout; in fact, it is universally so in the sense that its image under any coproduct preserving functor, e.g., $(H_q(-;Z)$, is again a pushout. Thus the sequences

$$H_q(\pi;Z)\oplus H_q(\pi;Z)\to H_q(\pi;Z)\oplus H_q(\bar\pi;Z)\to H_q(\bar\pi;Z)\to 0$$

are exact. Comparing these with the Mayer–Vietoris Sequence for the outer square, we see that $\varphi$ is perfect; i.e., $\varphi$ is in Gp $\mathscr{P}$.

Considering (11.1) as a square in Gp $\mathscr{P}$, we shall construct a diagram in the category of pointed CW-complexes which in $\mathscr{H}$o becomes the image of (11.1) under $B^+$.

Let $X$ represent $B^+\theta$; then $X\vee X$ evidently represents $B^+(\theta*\theta)$. Further, let $f:X\to Y$ represent $B^+(g_0,p)=B^+(g_1,p):B^+\theta\to B^+\bar\theta$. Since $c(G*G)$ is acyclic, $X$ represents $B^+(\theta\ \mathrm{proj}_G)$ and $Y$ represents $B^+(\bar\theta\ \mathrm{proj}_{\bar G})$. We claim that the pushout

$$\begin{tikzcd} X\vee X \arrow[r, "(ff)"] \arrow[d, "(i_0\, i_1)"'] & Y \arrow[d, "w"] \\ X\rtimes I \arrow[r] & W \end{tikzcd} \tag{11.2}$$

represents the effect of $B^+$ on (11.1).

We need to check only that $W$ does in fact represent $B^+\varphi$. The Seifert–van Kampen Theorem implies $\pi_1W\approx\bar\pi$. The canonical morphism $W\to B^+\varphi$ in $\mathscr{H}$o allows us to compare the Mayer–Vietoris Sequence of the homology of (11.1) and (11.2) with respect to arbitrary $\bar\pi$-modules. The conclusion then follows from Theorem 11.3(i).

But $w$ has an obvious left inverse. Since $B^+$ is full, there is a homomorphism

$$k:E\to G\times c(G*G)$$

such that $B^+(k,1)B^+(u,1)=1$. Thus, $[ku,1]$ is an isomorphism in Gp $\mathscr{P}[\mathscr{F}^{-1}]$. In this category of fractions, the commutativity of (11.1) becomes

$$[u,1][\langle g_0\,g_1)\tau\rangle,(p,p)]=[v,p][\langle(1\ 1)\tau\rangle,(1\ 1)],$$

from which we get

$$[u,1]\mathrm{proj}_{\bar G},1]^{-1}[(g_0\,g_1),(p\,p)]=[v,p][\mathrm{proj}_G,1]^{-1}[(1\ 1),(1\ 1)].$$

Composing with $[\mathrm{proj}_{\bar G},1][ku,1]^{-1}[k,1]$ on the left, we get

$$[(g_0\,g_1),(p\,p)]=\alpha[(1\ 1),(1\ 1)].$$

Finally, composing with the injections $i_0,i_1:\theta\to\theta*\theta$, we get

$$[g_0,p]=[g_1,p].\quad\square$$

## References


[1] G. Baumslag, A finitely generated, infinitely related group with trivial multiplicator, Bull. Australian Math. Soc. 5 (1971) 131-136.

[2] G. Baumslag, A remark on groups with trivial multiplicator, Amer. J. Math. 97 (1976) 863–864.

[3] G. Baumslag and K.W. Gruenberg. Some reflections on cohomological dimension and freeness, J. Algebra 6 (1967) 394–409.

[4] G. Baumslag and R. Strebel, Some finitely generated, infinitely related metabelian groups with trivial multiplicator, J. Algebra (to appear).

[5] W.H. Cockroft, On two-dimensional aspherical complexes, Proc. Lond. Math. Soc. 4(3) (1954) 375–384.

[6] E. Dyer and A.T. Vasquez, Some small aspherical spaces, J. Australian Math. Soc. 16 (1973) 332–352.

[7] S. Eilenberg and T. Ganea, On Lusternick–Schnirelman category of abstract groups, Annals of Math. 65 (1957) 517–518.

[8] P. Gabriel and M. Zisman, Calculus of Fractions and Homotopy Theory (Springer-Verlag, Berlin, 1967).

[9] K.W. Gruenberg, Cohomological Topics in Group Theory, Lecture Notes in Mathematics 143 (Springer-Verlag, Berlin, 1970).

[10] O. Grun, Beitrage zur Gruppentheorie, I., J. reine angew. Mat. 174 (1953) 1–14.

[11] G. Higman, A finitely generated infinite simple group, J. Lond. Math. Soc. 26 (1951) 61–64.

[12] G. Higman, B.H. Neumann and H. Neumann, Embedding theorems for groups, J. Lond. Math. Soc. 24 (1949) 247–254.

[13] F.E.A. Johnson and C.T.C. Wall, On groups satisfying Poincare duality, Annals of Math. 96 (1972) 592–598.

[14] D.M. Kan and W.P. Thurston, Every connected space has the homology of a $K(\pi, 1)$, Topology 15 (1976) 253–258.

[15] I. Kaplansky, Homological Dimension of Rings and Modules, Mimeo. Notes, Univ. of Chicago, 1959.

[16] O.H. Kegel and B.A.F. Wehrfritz, Locally Finite Groups (North Holland, Amsterdam, 1973).

[17] A. Kurosh, Local freie Gruppen, Dokl. Akad. Nauk. SSSR 24 (1939) 99–101.

[18] R.C. Lyndon, Cohomology of groups with a single defining relation, Annals of Math. 52 (1950) 650–665.

[19] S. MacLane, Categories for the Working Mathematician, (Springer-Verlag, Berlin 1971).

[20] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, second Ed. (Dover, New York, 1976).

[21] J. Milnor, Construction of universal bundles. II., Annals of Math. 63 (1956) 430–436.

[22] M. Nakaoka, Homology of the infinite symmetric group, Annals of Math. 73 (1961) 229–257.

[23] B.H. Neumann, An essay on free products with amalgamations, Phil. Trans. Royal Soc. Lond. 246 (1954) 503–554.

[24] B.H. Neumann, Some remarks on infinite groups, J. Lond. Math. Soc. 12 (1937) 195–198.

[25] D. Quillen, Cohomology of groups, Actes Congres Intern. 1970, v. 2, 47–51.

[26] G. Segal, Classifying spaces and spectral sequences, Pub. Math. I.H.E.S. 34 (1968) 105–112.

[27] J.P. Serre, Cohomologie Des Groupes Discrets, pp. 77–170, in: Prospects in Mathematics, Annals of Math. Studies, no, 70 (Princeton Press, NJ, 1971).

[28] J. Stallings, On torsion free groups with infinitely many ends, Annals of Math. 88 (1968) 312–334.

[29] J. Stallings, Centerless groups – an algebraic formulation of Gottlieb's Theorem, Topology 4 (1965) 129–134.

[30] R.G. Swan, Groups of cohomological dimension one, J. Algebra 12 (1969) 585–610.

[31] J. Stallings, The non-triviality of the restriction map in the cohomology of groups, Proc. Amer. Math. Soc. 11 (1960) 885–887.

[32] C.T.C. Wall, Finiteness conditions for CW-complexes. I., Annals of Math. 81 (1965) 56–69; and II. Proc. Royal Soc., A, 295 (1966) 129–139.

[33] C.T.C. Wall, Resolutions for extensions of groups, Proc. Camb. Phil. Soc. 57 (1961) 251–255.

[34] J.H.C. Whitehead, On the asphericity of regions in a 3-sphere, Fund. Math. 32 (1939) 149–166.