Fundamental group and covering spaces: the facts.

Intro.

I assume all spaces nice.
Details of all of the facts asserted here can be found in any of the standard references. They are collected here for your convenience.

Suppose Z is a space, and * a point of Z. We define $\pi_1(Z,*)$ as homotopy classes of maps f:[0,1] -> Z, such that f(0) = f(1) = *.

(Homotopy here means that one views as equivalent two functions which lie on a 1-parameter family of functions f_t:[0,1] ->Z, all of which satisfy the boundary condition $f_t(0) = f_t(1) = *$.)

These boundary conditions are absolutely critical for getting a nontrivial theory. $\pi_1(Z,*)$ is a group using concatenation of paths; the constant path is the identity and "going backwards is the inverse. $\pi_1(Z,*)$ is referred to as the fundamental group of Z. (If Z is path connected, the choice of * is irrelevant, in the sense that for a different choice of points, one gets an isomorphic group.)

Example: If Z is the circle $S^1 = \{u \text{ in } \mathbf{C} \mid |u| = 1\}$, we can define a map $\pi_1(S^1,1) \to \mathbf{Z}$ (the integers) by sending a map f to $(\log(f)(1) - \log(f)(0))/2\pi i$. This is referred to as the *winding number*.

You'll want to check that this makes sense, i.e. that one can define log(f) continuously on the interval, and that the above definition is independent of all choices, once one decides that $\log(f)(0) = 0$ – and that the winding number doesn't change during a homotopy.

If f and g have the same winding number then
$f_t(x) = \exp(t\log(f)(x) + (1-t)\log(g)(x))$
is a homotopy between g and f. (What goes wrong if they have different winding number?).

The maps $x \to \exp(2\pi inx)$ have winding numbers n, so this homomoprhism is an isomorphism.

By the way,we say that a space is simply connected if its fundamental group is trivial.

Categorical Nonsense

If f:(X,x) -> (Y,y) is a map, (this notation means that f(x) = y) then there is an induced homomorphism, defined in the most obvious way, $f_*: \pi_1(X,x) \to \pi_1(Y,y)$.

If g: (Y,y) -> (Z,z) is another map, then $(gf)_* = g_* f_*$. This is both obvious and useful. The first is clear and the second will be shown countless times in class.

Pointedly homotopic maps of pointed spaces, i.e. f,g :(X,x) -> (Y,y) homotopic through such maps, induce the same homorphisms on fundamental group, i.e. $f_* = g_*$. This implies a homotopy invariance property of π_1.

Covering Spaces

Fundamental group is important for at least two reasons. The first is the connection to covering spaces. We assume here that all spaces are connected (unless they arise in the middle of a proof, or something).

Definition. A map p: A -> B is a covering space, if:around each point b in B, there is a neighborhood N of b, so that $p^{-1}(N)$ is a disjoint union of sets A_i each of which is mapped homeomorphically onto N by p.

The map exp: **R** -> S^1 considered before is a good example.

Let us pick a point b in B and fix a point a in A which maps to B. In other words, let us refine our perspective and consider covering spaces as maps p:(A,a) -> (B,b). This way, to every cover one can unambiguously assign the subgroup $p_*(\pi_1(A,a))$ inside of $\pi_1(B,b)$.

If we did not pin down a basepoint in A, a covering space would only give a well defined conjugacy class of subgroup of $\pi_1(B,b)$.

Theorem (Classification of Covers): To every subgroup of $\pi_1(B,b)$ there is a covering space of B so that the induced subgroup is the given one.

Moreover if (A,a, p) and (A',a', p') are two different covers corresponding to the same subgroup, then there is a (unique) homeomorphism h:(A,a) -> (A',a') so that p = p'h.

Note that a cover of a cover is a cover, so that smaller subgroups correspond to "higher" covers.

The trivial subgroup corresponds to the "universal cover" of B. Theorems stated below imply that all of the maps p_* on fundamental groups are 1-1 (for covering maps!!! –not for maps in general!!). So the universal cover is the only simply connected cover that a space has.

Note too, that if p:(A,a) -> (B,b) is a universal cover, then for any a' in $p^{-1}(a)$, there is a unique homeomorphism $h_{a'}$:A -> A so that

- $h_{a'}(a) = a'$
- p h_a = p.

The set of homeomorphisms which satisfy the second condition form a group (sometimes called the <u>group of covering transformations</u> or the <u>group of deck transformations</u>), and we are essentially saying that one can determine the group element by seeing where a goes. It turns out that this group is isomorphic to the fundamental group of B.

Remark: There is a nice analogy to field theory. Let F be a field; covering spaces are like algerbaic extension fields. The universal cover is like the algebraic closure. The fundamental group is like the Galois group of the algebraic closure. The relation between covers and subgroups of the fundamental group is just like ordinary Galois theory. There are situations where this analogy is actually implemented geometrically, but let's not digress.....

Examples: The 2-sphere S^2 is simply connected. The projective plane $\mathbf{RP^2}$ has fundamental group $\mathbf{Z}/2\mathbf{Z}$ since it is the quotient of S^2 by making the identifications x = -x. The projection map is a covering map, and the group of covering transformations is just $\mathbf{Z}/2\mathbf{Z}$ = {id, x -> -x}. The nontrivial element in the fundamental group of $\mathbf{RP^2}$ can be thought of as the quotient of a great chord on S^2 that connects the north pole to the south pole.

The torus $\mathbf{R}^2/\mathbf{Z^2}$ has fundamental group $\mathbf{Z^2}$. Again, elements of the fundamental group can be thought of as the projections of chords connecting the origin to (m,n) in the lattice $\mathbf{Z^2}$.

An example that takes more work to think through is the universal cover of the figure 8. It is the unique tree where every vertex has 4 edges emanating from it. It is easy to see that this space is simply connected (indeed it is <u>contractible</u>, i.e. homotopy equivalent to a point). If one colors the edges blue and green, so that at each vertex two edges are blue and two green, then one can use this to define a covering map to 8, where the top loop is colored blue and the bottom green. What are the covering translations in this example?

In general, you should have little trouble using the the method of the calculation of $\pi_1(S^1)$ to show that if one has a simply connected space A and a group of homeomorphisms π of A so that A -> A/π is a covering map (this condition is called <u>proper discontinuity</u> of the action. It can be phrased as the condition that for each a in A there is a neighborhood N of a, so that for all g not the identity in π, gN and N are disjoint), then the quotient A/π has fundamental group π.

What requires more thought is why this is the general case, i.e. how does one construct universal covers and how does one get the group of deck transformations.

More on covering spaces.

Suppose that $p:(A,a) \to (B,b)$ is a covering map. Then for any $f: [0,1] \to (B,b)$ such that $f(0) = b$, there is a unique lifting $f^\wedge: [0,1] \to (A,a)$ such that $f^\wedge(0) = a$ and $pf^\wedge = f$.

Now suppose that $f: (X,x) \to (B,b)$ is an arbitrary map with $f(x) = b$, X path connected. We can try to use the previous remark to try to lift f to a map $f^\wedge: (X,x) \to (A,a)$ simply by defining $f^\wedge(x')$ to be the result of lifting the composite of f with any path connecting x to x'. Of course there will be many different paths connecting these two points, and this indeterminacy causes an obstruction. However, the analysis of this gives rise to the following critically important theorem:

<u>Lifting condition for Covering Spaces</u>: Suppose $p:(A,a) \to (B,b)$ is a covering map, and $f:(X,x) \to (B,b)$ is given. Then there is an $f^\wedge:(X,x) \to (A,a)$ so that $pf^\wedge = f$ (and, of course, $f^\wedge(x) = a$, as is included in this notation) if and only if $f_* \pi_1(Z,z)$ is a subgroup of $p_* \pi_1(A,a)$.

The proof is actually straightforward and it has many corollaries, which I leave to you.

Definition/Corollary: Say that a cover $p:(A,a) \to (B,b)$ is <u>normal</u> if there are deck transformation sending any element of $p^{-1}(b)$ to any other. The cover p is normal iff $p_* \pi_1(A,a)$ is a normal subgroup of $\pi_1(B,b)$.

Corollary: The uniqueness statement in the classification theorem holds.

Corollary: If Z is simply connected, then any map $f:(Z,z) \to (B,b)$ can be lifted to any cover.

Corollary: For any cover, $p:(A,a) \to (B,b)$, the induced map, p_* is 1-1.

Now for existence of covers.... we will only explain the universal cover, since other covers can be obtained as quotients of it by subgroups of $\pi_1(B,b)$.

Part of B being a nice space includes the fact that two nearby (in the C^0 topology) functions are homotopic. I leave it to you to decide if the spaces you know and love are nice this way. (I bet they all are.) We will assume that nearby paths conncting b to b' are homotopic through paths connecting these points. (Using an embedding in Euclidean space and the tubular neighborhood theorem, you should be able to check this for manifolds. For cell complexes, you might want to do an induction...)

Given (B,b) we will build a new space, called A. The points of A are equivalence classes of maps a:[0,1] -> B, so that a(0) = b. Two paths a,a' define the same point in A if

- a(1) = a'(1)
- a and a' are homotopic relative to their endpoints.

Note that without the second condition, the space obtained would just be B. With both of them, one can check that the map a -> a(1) is a covering map, and that a map from a circle to B lifts to A iff it is homotopic to a constant. By the lifting criterion, this implies that A is simply connected. So it is a universal cover.

Remark: This construction is not as bad as it looks. After a while you can really do it. As an exercise, do it to 8 to get the universal 4-valent tree as described above.

Of course, after a while, with experience you can "see" universal covers of lots of things. Practice on your friends and on household items.

Computation of fundamental group.

I had said that fundamental group was important for at least 2 reasons, the first being its connection to covering space theory. The second is that it is quite computable (in one sense anyway).

Example: If X is contractible then the fundamental group is trivial.

Example: If one sees the universal cover and group of deck transformations, then one also knows the fundamental group.

But actually, the key practical tool is Van Kampen's theorem. It describes the fundamental group of a union in terms of the fundamental groups of the pieces. I will describe a pretty useful, but not the most general version of it.

<u>Van Kampen's theorem</u>. Let Z denote the union of A and B, and X denote their intersection. We assume A, B, and X are all connected (and nonempty), and that the inclusions of X in A and B are "nice". Then $\pi_1(Z,x)$ is generated by $\pi_1(A,x)$ and

$\pi_1(B,x)$. The only relations among the elements of $\pi_1(A,x)$ and $\pi_1(B,x)$ are the ones forced by the fact that the elements of $\pi_1(C,x)$ can be thought of as elements of both of these groups.

Examples.

1. If A and B are simply connected, and their intesection is connected, then their union is simply connected. (Give an example where this fails when the intersection is disconnected.)

2. If X is simply connected, then $\pi_1(Z,x)$ is the <u>free product</u> $\pi_1(A,x)$* $\pi_1(B,x)$. The elements of the free products are just finite strings of elements of $\pi_1(A,x)$ and $\pi_1(B,x)$, and one multiplies strings by concatenting them, ignoring the identity, and combining contiguous elements of the same group.

As a special case, consider the free group on two generators F_2 = **Z*Z**. A typical element of that group looks like $xyx^{-3}y^2x$. Its inverse is $x^{-1}y^{-2}x^3y^{-1}x^{-1}$. Do you see it?

3. These things can be tricky if $\pi_1(X,x)$ is nontrivial. The group described is called a free product with amalgamation and is denoted by $\pi_1(A,x) *_{\pi 1(X,x)} \pi_1(B,x)$. There is a nice interpretation of what the elements of this look like when the induced maps of $\pi_1(X,x)$ into the other two pieces are injective, but without this it can get complicated. As a simple example suppose that X is a circle and that $\pi_1(A,x) = \mathbf{Z}/2\mathbf{Z}$ and $\pi_1(B,x) = \mathbf{Z}/3\mathbf{Z}$, so that the induced homomorphisms are the obvious surjections. (By the way, can you build such spaces?) Then, Van Kampen's theorem tells us that Z is simply connected. Do you see why?

The fundamental groups of both A and B are generated by that of the circle, i.e. there is one generator, say g. From A we learn that $g^2 = e$ and from B we learn that $g^3 = e$. So in the amalgamated free product (i.e. $\pi_1(Z,x)$) $g = e$, so the who group vanishes.

In principle any space that can be broken up into pieces can have its fundamental group described by generators and relations via Van Kampen's theorem.

For more, see the usual references.