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ON HYPERBOLIC GROUPS WITH SPHERES AS BOUNDARY

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Dedicated to Steve Ferry on the occasion of his 60th birthday

Abstract

Let G be a torsion-free hyperbolic group and let $n \geq 6$ be an integer. We prove that G is the fundamental group of a closed aspherical manifold if the boundary of G is homeomorphic to an $(n-1)$-dimensional sphere.

Introduction

If G is the fundamental group of an n-dimensional closed Riemannian manifold with negative sectional curvature, then G is a hyperbolic group in the sense of Gromov (see for instance [**6**, **7**, **21**, **22**]). Moreover, such a group is torsion-free and its boundary ∂G is homeomorphic to a sphere. This leads to the natural question whether a torsion-free hyperbolic group with a sphere as boundary occurs as a fundamental group of a closed aspherical manifold (see Gromov [**23**, page 192]). We settle this question if the dimension of the sphere is at least 5.

Theorem A. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6. The following statements are equivalent:*

(i) The boundary ∂G is homeomorphic to S^{n-1}.

(ii) There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.

The aspherical manifold M appearing in our result is unique up to homeomorphism. This is a consequence of the validity of the Borel Conjecture for hyperbolic groups [**2**]; see also Section 3.

The proof depends on the surgery theory for homology ANR-manifolds due to Bryant, Ferry, Mio, and Weinberger [**9**] and the validity of the K- and L-theoretic Farrell-Jones Conjecture for hyperbolic groups due to Bartels, Reich, and Lück [**4**] and Bartels-Lück [**2**]. It seems likely that this result holds also if $n = 5$. Our methods can be extended to this case if the surgery theory from [**9**] can be extended to the case of

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5-dimensional homology ANR-manifolds—such an extension has been announced by Ferry and Johnston. We also hope to give a treatment elsewhere by more algebraic methods.

We do not get information in dimensions $n \leq 4$ for the usual problems about surgery. For instance, our methods give no information in the case where the boundary is homeomorphic to S^3, since virtually cyclic groups are the only hyperbolic groups which are known to be good in the sense of Friedman [**19**]. In the case $n = 3$ there is the conjecture of Cannon [**11**] that a group G acts properly, isometrically, and cocompactly on the 3-dimensional hyperbolic plane $\mathbb{H}^3$ if and only if it is a hyperbolic group whose boundary is homeomorphic to S^2. Provided that the infinite hyperbolic group G occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina and Mess [**5**, Theorem 4.1] have shown that its universal cover is homeomorphic to $\mathbb{R}^3$ and its compactification by ∂G is homeomorphic to D^3, and the Geometrization Conjecture of Thurston implies that M is hyperbolic and G satisfies Cannon's conjecture. The problem is solved in the case $n = 2$, essentially as a consequence of Eckmann's theorem that 2-dimensional Poincare duality groups are surface groups (see [**16**]). Namely, for a hyperbolic group G, its boundary ∂G is homeomorphic to S^1 if and only if G is a Fuchsian group (see [**12**, **18**, **20**]).

In general, the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary ∂G of an infinite hyperbolic group G contains an open subset homeomorphic to Euclidean n-space, then it is homeomorphic to S^n. This is proved in [**25**, Theorem 4.4], where more information about the boundaries of hyperbolic groups can be found.

We also prove the following result.

Theorem B. *Let G and H be torsion-free hyperbolic groups such that $\partial G \cong \partial H$. Then G can be realized as the fundamental group of a closed aspherical manifold of dimension at least* 6 *if and only if H can be realized as the fundamental group of such a manifold.*

Moreover, even in case that neither can be realized by a closed aspherical manifold, they can both be realized by closed aspherical homology ANR-*manifolds, which both have the same Quinn obstruction* [**30**] *(see Theorem* 1.3 *for a review of this notion) provided that ∂G has the integral Čech cohomology of S^{n-1} for $n \geq 6$.*

In particular, if G is hyperbolic and realized as the fundamental group of a closed aspherical manifold of dimension at least 6, then any torsion-free group H that is quasi-isometric to G can also be realized as the fundamental group of such a manifold. This follows from Theorem B, because the homeomorphism type of the boundary of a hyperbolic group is invariant under quasi-isometry (and so is the property of being hyperbolic). The attentative reader will realize that most of the content of Theorem A can also be deduced from Theorem B, as every sphere appears as the boundary of the fundamental group of some closed hyperbolic manifold.

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The techniques and ideas of this paper are very closely related to the work of Steve Ferry; indeed, his unpublished work could have been used to simplify some parts of this work. It is a pleasure to dedicate this paper to him on the occasion of his 60th birthday.

1. Homology manifolds

A topological space X is called an *absolute neighborhood retract*, or ANR, if it is normal and for every normal space Z, every closed subset $Y \subseteq Z$ and every (continuous) map $f\colon Y \to X$ there exists an open neighborhood U of Y in Z together with an extension $F\colon U \to X$ of f to U.

Definition 1.1 (Homology ANR-manifold)**.** An *n-dimensional homology* ANR-*manifold* X is an absolute neighborhood retract satisfying:

- X has a countable base for its topology;
- the topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ the ith singular homology group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$.

Notice that a normal space with a countable basis for its topology is metrizable by the Urysohn Metrization Theorem (see [**29**, Theorem 4.1 in Chapter 4-4 on page 217]) and is separable, i.e., contains a countable dense subset [**29**, Theorem 4.1]. Notice furthermore that every metric space is normal (see [**29**, Theorem 2.3 in Chapter 4-4 on page 198]), and has a countable basis for its topology if and only if it is separable (see [**29**, Theorem 1.3 in Chapter 4-1 on page 191 and Exercise 7 in Chapter 4-1 on page 194]). Hence a homology ANR-manifold in the sense of Definition 1.1 is the same as a generalized manifold in the sense of Daverman [**14**, page 191]. A closed n-dimensional topological manifold is an example of a closed n-dimensional homology ANR-manifold (see [**14**, Corollary 1A in V.26, page 191]). A homology ANR-manifold M is said to have the *disjoint disk property (DDP)*, if for any $\varepsilon > 0$ and maps $f, g\colon D^2 \to M$, there are maps $f', g'\colon D^2 \to M$ so that f' is ε-close to f, g' is ε-close to g, and $f'(D^2) \cap g'(D^2) = \emptyset$; see for example [**9**, page 435]. We recall that a *Poincaré duality group* G is a finitely presented group satisfying the following two conditions: first, the $\mathbb{Z}G$-module $\mathbb{Z}$ (with the trivial G-action) admits a resolution of finite length by finitely generated projective $\mathbb{Z}G$-modules; second, there is n such that $H^i(G;\mathbb{Z}G) = 0$ for $n \neq i$ and $H^n(G;\mathbb{Z}G) \cong \mathbb{Z}$. In this case n is the formal dimension of the Poincaré duality group G.

Theorem 1.2. *Let G be a torsion-free group.*

(i) Assume that

- *the (non-connective) K-theory assembly map*
$$H_i\bigl(BG;\mathbf{K}_{\mathbb{Z}}\bigr) \to K_i(\mathbb{Z}G)$$
is an isomorphism for $i \leq 0$ and surjective for $i = 1$;
- *the (non-connective) L-theory assembly map*
$$H_i\bigl(BG;{}^w\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle}\bigr) \to L_i^{\langle -\infty\rangle}(\mathbb{Z}G,w)$$
is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w\colon G \to \{\pm 1\}$.

Then for $n \geq 6$ the following are equivalent:

a) G is a Poincaré duality group of formal dimension n;

b) there exists a closed ANR-*homology manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$;*

(ii) If the statements in assertion (i) *hold, then the homology* ANR-*manifold M appearing there can be arranged to have the DDP.*

(iii) If the statements in assertion (i) *hold, then the homology* ANR-*manifold M appearing there is unique up to s-cobordism of* ANR-*homology manifolds.*

Proof. (i) The assumption on the K-theory assembly map implies that $\mathrm{Wh}(G) = 0$, $\tilde{K}_0(\mathbb{Z}G) = 0$, and $K_i(\mathbb{Z}G) = 0$ for $i < 0$; compare [**27**, Conjecture 1.3 on page 653 and Remark 2.5 on page 679]. This implies that we can change the decoration in the above L-theory assembly map from $\langle -\infty\rangle$ to s (see [**27**, Proposition 1.5 on page 664]). Thus the assembly map A in the algebraic surgery exact sequence [**31**, Definition 14.6] (for $R = \mathbb{Z}$ and $K = BG$) is an isomorphism. This implies in particular that the quadratic structure groups $\mathbb{S}_i(\mathbb{Z},BG)$ are trivial for all $i \in \mathbb{Z}$.

Assume now that G is a Poincaré duality group of dimension $n \geq 3$. We conclude from Johnson and Wall [**24**, Theorem 1] that BG is a finitely dominated n-dimensional Poincaré complex in the sense of Wall [**35**]. Because $\tilde{K}_0(\mathbb{Z}G) = 0$, the finiteness obstruction vanishes and hence BG can be realized as a finite n-dimensional simplicial complex (see [**34**, Theorem F]). We will now use Ranicki's (4-periodic) total surgery obstruction $\overline{s}(BG) \in \overline{\mathbb{S}}_n(BG)$ of the Poincaré complex BG; see [**31**, Definition 25.6]. The main result of [**9**] asserts that this obstruction vanishes if and only if there is a closed n-dimensional homology ANR-manifold M homotopy equivalent to BG. The groups $\overline{\mathbb{S}}_k(BG)$ arise in a 0-connected version of the algebraic surgery sequence [**31**, Definition 15.10]. It is a consequence of [**31**, Proposition 15.11(iii)] (and the fact that $L_{-1}(\mathbb{Z}) = 0$) that $\overline{\mathbb{S}}_n(BG) = \mathbb{S}_n(\mathbb{Z}, BG)$. Since $\mathbb{S}_n(\mathbb{Z}, BG) = 0$, we conclude $\overline{s}(BG) = 0$. This shows that (i)a implies (i)b. (In this argument we ignored that the orientation homomorphism $w\colon G \to \{\pm 1\}$ may be non-trivial. The argument however extends to this case; compare [**31**, Appendix A].) Homology manifolds satisfy Poincaré duality and therefore (i)b implies (i)a.

(ii) It is explained in [**9**, Section 8] that this homology manifold M appearing above can be arranged to have the DDP. (Alternatively, we could appeal to [**10**] and resolve M by an n-dimensional homology ANR-manifold with the DDP.)

(iii) The uniqueness statement follows from Theorem 3.1(ii). q.e.d.

In order to replace homology ANR-manifolds by topological manifolds we will later use the following result that combines work of Edwards and Quinn; see [**14**, Theorems 3 and 4 on page 288], [**30**]).

Theorem 1.3. *There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ (known as the Quinn obstruction) for connected homology* ANR*-manifolds with the following properties:*

(i) If $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$.

(ii) Let M be a homology ANR*-manifold of dimension ≥ 5. Then the following are equivalent:*
- *M has the DDP and $\iota(M) = 1$;*
- *M is a topological manifold.*

Definition 1.4. An *n-dimensional homology* ANR*-manifold M with boundary ∂M* is an absolute neighborhood retract which is a disjoint union $M = \operatorname{int} M \cup \partial M$, where

- $\operatorname{int} M$ is an n-dimensional homology ANR-manifold;
- ∂M is an $(n-1)$-dimensional homology ANR-manifold;
- for every $z \in \partial M$ the singular homology group $H_i(M, M \setminus \{z\})$ vanishes for all i.

Lemma 1.5. *If M is an n-dimensional homology* ANR*-manifold with boundary, then $\widehat{M} := M \cup_{\partial M} \partial M \times [0,1)$ is an n-dimensional homology* ANR*-manifold.*

Proof. Suppose that Y is the union of two closed subsets Y_1 and Y_2 and set $Y_0 := Y_1 \cap Y_2$. If Y_0, Y_1, and Y_2 are ANRs, then Y is an ANR; see [**14**, Theorem 7 on page 117]. If Y_1 and Y_2 have countable bases $\mathcal{U}_1$ and $\mathcal{U}_2$ of the topology, then sets $U_1 \setminus Y_2$ with $U_1 \in \mathcal{U}_1$, $U_2 \setminus Y_1$ with $U_2 \in \mathcal{U}_2$ and $(U_1 \cup U_2)^\circ$ with $U_i \in \mathcal{U}_i$ form a countable basis of the topology of Y. (Here $(\ )^\circ$ is the operation of taking the interior in Y.) If Y_1 and Y_2 are both finite dimensional, then Y is finite dimensional [**29**, Theorem 9.2 on page 303]. If Y_1 and Y_2 are both locally compact, then Y is locally compact.

Thus the only non-trivial requirement is that for $x = (z, 0) \in \widehat{M}$ with $z \in \partial M$, we have $H_i(\widehat{M}, \widehat{M} \setminus \{x\}) = 0$ if $i \neq n$ and $\cong \mathbb{Z}$ if $i = n$. Let $I_z := \{z\} \times [0, 1/2)$. Because of homotopy invariance we can replace $\{x\}$ by I_z. Let $U_1 := M \cup_{\partial M} \partial M \times [0, 1/2) \subset \widehat{M}$ and $U_2 := \partial M \times (0, 1) \subset \widehat{M}$. Then $H_i(U_1, U_1 \setminus I_z) \cong H_i(M, M \setminus \{z\}) = 0$ and $H_i(U_2, U_2 \setminus I_z) = 0$. Because U_1 and U_2 are both open, we can use a Mayer-Vietoris sequence to deduce

$$H_i(\widehat{M}, \widehat{M} \setminus I_z) \cong H_{i-1}(U_1 \cap U_2, U_1 \cap U_2 \setminus I_z) \cong H_{i-1}(\partial M, \partial M \setminus \{z\}).$$

The result follows as ∂M is an $(n-1)$-dimensional homology ANR-manifold. q.e.d.

Corollary 1.6. *Let M be an homology* ANR*-manifold with boundary ∂M. If ∂M is a manifold, then $\iota(\operatorname{int} M) = 1$.*

Proof. We use $\widehat{M}$ from Lemma 1.5. If ∂M is a manifold, then so is $\partial M \times (0, 1)$. The result follows now from Theorem 1.3. q.e.d.

2. Hyperbolic groups and aspherical manifolds

For a hyperbolic group we write $\overline{G} := G \cup \partial G$ for the compactification of G by its boundary; compare [**7**, III.H.3.12], [**5**]. Left multiplication of G on G extends to a natural action of G on $\overline{G}$. We will use the following properties of the topology on $\overline{G}$.

Proposition 2.1. *Let G be a hyperbolic group. Then*

(i) $\overline{G}$ is compact;
(ii) $\overline{G}$ is finite dimensional;
(iii) ∂G has empty interior in $\overline{G}$;
(iv) the action of G on $\overline{G}$ is small at infinity: if $z \in \partial G$, $K \subset G$ is finite and $U \subset \overline{G}$ is a neighborhood of z, then there exists a neighborhood $V \subseteq \overline{G}$ of z with $V \subseteq U$ such that for any $g \in G$ with $gK \cap V \neq \emptyset$ we have $gK \subseteq U$;
(v) if $z \in \partial G$ and U is an open neighborhood of z in $\overline{G}$, then for every finite subset $K \subseteq G$ there is an open neighborhood V of z in $\overline{G}$ such that $V \subseteq U$ and $(V \cap G) \cdot K \subseteq U \cap G$.

Proof. (i) see for instance [**7**, III.H.3.7(4)].
(ii) see for instance [**3**, 9.3.(ii)].
(iii) is obvious from the definition of the topology in [**5**].
(iv) see for instance [**32**, page 531].
(v) follows from (iv): We may assume $1_G \in K$. Pick V as in (iv). If $g \in V \cap G$ and $k \in K$, then $g \in gK \cap V$. Thus $gK \subseteq U$. Therefore $gK \in U \cap G$. q.e.d.

Let X be a locally compact space with a cocompact and proper action of a hyperbolic group G. Then we equip $\overline{X} := X \cup \partial G$ with the topology

$\mathcal{O}_{\overline{X}}$ for which a typical open neighborhood of $x \in X$ is an open subset of X and a typical (not necessarily open) neighborhood of $z \in \partial G$ is of the form

$$(U \cap \partial G) \cup (U \cap G) \cdot K$$

where U is an open neighborhood of z in $\overline{G}$ and K is a compact subset of X such that $G \cdot K = X$. We observe that we could fix the choice of K in the definition of $\mathcal{O}_{\overline{X}}$: Let U, z, and K be as above and let K' be a further compact subset of X such that $G \cdot K' = X$. Because the G-action is proper, there is a finite subset L of G such that $K' \subseteq L \cdot K$. By Proposition 2.1(v) there is an open neighborhood $V \subseteq U$ of $z \in \overline{G}$ such that $(V \cap G) \cdot L \subseteq U \cap G$. Thus

$$(V \cap \partial G) \cup (V \cap G) \cdot K' \subseteq (U \cap \partial G) \cup (V \cap G) \cdot L \cdot K \subseteq (U \cap \partial G) \cup (U \cap G) \cdot K.$$

If $f \colon X \to Y$ is a G-equivariant continuous map where Y is also a locally compact space with a cocompact proper G-action, then we define $\overline{f} \colon \overline{X} \to \overline{Y}$ by $\overline{f}|_X := f$ and $\overline{f}|_{\partial G} := \mathrm{id}_{\partial G}$.

Lemma 2.2. *Let G be a hyperbolic group and X be a locally compact space with a cocompact and proper G-action.*

(i) $\overline{X}$ *is compact;*
(ii) ∂G *is closed in* $\overline{X}$ *and its interior in* $\overline{X}$ *is empty;*
(iii) if $\dim X$ *is finite, then* $\dim \overline{X}$ *is also finite;*
(iv) if $f \colon X \to Y$ *is a G-equivariant continuous map where Y is also a locally compact space with a cocompact proper G-action, then* $\overline{f}$ *is continuous.*

Proof. These claims are easily deduced from the observation following the definition of the topology $\mathcal{O}_{\overline{X}}$ and Proposition 2.1. q.e.d.

We recall that for a hyperbolic group G equipped with a (left invariant) word-metric d_G and a number $d > 0$, the Rips complex $P_d(G)$ is the simplicial complex whose vertices are the elements of G, and a collection $g_1, \ldots, g_k \in G$ spans a simplex if $d_G(g_i, g_j) \leq d$ for all i, j. The action of G on itself by left translation induces an action of G on $P_d(G)$. Recall that a closed subset Z in a compact ANR Y is a *Z-set* if for every open set U in Y the inclusion $U \setminus Z \to U$ is a homotopy equivalence. An important result of Bestvina and Mess [**5**] asserts that (for sufficiently large d) $\overline{P_d(G)}$ is an ANR such that $\partial G \subset \overline{P_d(G)}$ is Z-set. The proof uses the following criterion [**5**, Proposition 2.1]:

Proposition 2.3. *Let Z be a closed subspace of the compact space Y such that*

(i) the interior of Z in Y is empty;
(ii) $\dim Y < \infty$*;*

(iii) *for every* $k = 0, \dots, \dim Y$*, every* $z \in Z$ *and every neighborhood* U *of* z*, there is a neighborhood* V *of* z *such that every map* $\alpha\colon S^k \to V \setminus Z$ *extends to* $\tilde{\alpha}\colon D^{k+1} \to U \setminus Z$*;*

(iv) $Y \setminus Z$ *is an* ANR.

Then Y *is an* ANR *and* $Z \subset Y$ *is a* Z*-set.*

Condition (iii) is sometimes abbreviated by saying that Z is k-LCC in Y, where $k = \dim Y$.

Theorem 2.4. *Let* X *be a locally compact* ANR *with a cocompact and proper action of a hyperbolic group* G*. Assume that there is a* G*-equivariant homotopy equivalence* $X \to P_d(G)$*. If* d *is sufficiently large, then* $\overline{X}$ *is an* ANR*,* ∂G *is* Z*-set in* $\overline{X}$*, and* Z *is* k*-LCC in* X *for all* k*.*

Proof. Bestvina and Mess [**5**, page 473] show that (for sufficiently large d) $\overline{P_d(G)}$ satisfies the assumptions of Proposition 2.3. Moreover, they show that Z is k-LCC in $\overline{X}$ for all k. Using this, it is not hard to show, that $\overline{X}$ satisfies these assumptions as well: Assumptions (i) and (ii) hold because of Lemma 2.2. Assumption (iv) holds because X is an ANR. Because $f \mapsto \overline{f}$ is clearly functorial, the homotopy equivalence $X \to P_d(G)$ induces a homotopy equivalence $\overline{X} \to \overline{P_d(G)}$ that fixes ∂G. Using this homotopy equivalence, it is easy to check that ∂G is k-LCC in $\overline{X}$, because it is k-LCC in $\overline{P_d(G)}$. Thus Assumption (iii) holds. q.e.d.

Proposition 2.5. *Let* M *be a finite-dimensional locally compact* ANR *which is the disjoint union of an* n*-dimensional* ANR*-homology manifold* $\operatorname{int} M$ *and an* $(n-1)$*-dimensional* ANR*-homology manifold* ∂M *such that* ∂M *is a* Z*-set in* M*. Then* M *is an* ANR*-homology manifold with boundary* ∂M*.*

Proof. The Z-set condition implies that there exists a homotopy $H_t\colon M \to M$, $t \in [0,1]$ such that $H_0 = \mathrm{id}_M$ and $H_t(M) \subseteq \operatorname{int} M$ for all $t > 0$, see [**5**, page 470].

Let $z \in \partial M$. Then the restriction of H_1 to $M \setminus \{z\}$ is a homotopy inverse for the inclusion $M \setminus \{z\} \to M$. Thus $H_i(M, M \setminus \{z\}) = 0$ for all i. q.e.d.

There is the following (harder) manifold version of Proposition 2.5 due to Ferry and Seebeck [**17**, Theorem 5 on page 579].

Theorem 2.6. *Let* M *be a locally compact with a countable basis of the topology. Assume that* M *is the disjoint union of an* n*-dimensional manifold* $\operatorname{int} M$ *and an* $(n-1)$*-dimensional manifold* ∂M *such that* $\operatorname{int} M$ *is dense in* M *and* ∂M *is* $(n-1)$*-LCC in* M*. Then* M *is an* n*-manifold with boundary* ∂M*.*

Theorem 2.7. *Let* G *be a torsion-free word-hyperbolic group. Let* $n \geq 6$*.*

(i) The following statements are equivalent:
   a) The boundary ∂G has the integral Čech cohomology of S^{n-1}.
   b) G is a Poincaré duality group of formal dimension n.
   c) There exists a closed ANR*-homology manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$.*

(ii) If the statements in assertion (i) *hold, then the homology* ANR*-manifold M appearing there can be arranged to have the DDP;*

(iii) If the statements in assertion (i) *hold, then the homology* ANR*-manifold M appearing there is unique up to s-cobordism of* ANR*-homology manifolds.*

Proof. By [**21**, page 73], torsion-free hyperbolic groups admit a finite CW-model for BG. Thus the $\mathbb{Z}G$-module $\mathbb{Z}$ admits a resolution of finite length of finitely generated free $\mathbb{Z}G$ modules. By [**5**, Corollary 1.3], the $(i-1)$-th Čech cohomology of the boundary ∂G agrees with $H^i(G;\mathbb{Z}G)$. This shows that the statements (i)a and (i)b in assertion (i) are equivalent.

The Farrell-Jones Conjecture in K- and L-theory holds by [**2**, **4**]. This implies that the assumptions of Theorem 1.2 are satisfied; compare [**27**, Proposition 2.2 on page 685]. This finishes the proof of Theorem 2.7. q.e.d.

Proof of Theorem A. (i) Let G be a torsion-free hyperbolic group. Assume that $\partial G \cong S^{n-1}$ and $n \geq 6$. Theorem 2.7 implies that there is a closed n-dimensional homology ANR-manifold N homotopy equivalent to BG. Moreover, we can assume that N has the DDP. The universal cover M of N is an n-dimensional ANR-homology manifold with a proper and cocompact action of G. The homotopy equivalence $N \to BG$ lifts to a G-homotopy equivalence $M \to EG$. For sufficiently large d, $P_d(G)$ is a model for EG (see [**21**, page 73]). Thus there is a G-homotopy equivalence $M \to P_d(G)$. Theorem 2.4 implies that $\overline{M}$ is an ANR and ∂G is a Z-set in $\overline{M}$. We conclude from Lemma 2.2 that $\overline{M}$ is compact and has finite dimension. Thus we can apply Proposition 2.5 and deduce that $\overline{M}$ is a homology ANR-manifold with boundary. Its boundary is a sphere and in particular a manifold. Corollary 1.6 implies that $\iota(M) = 1$. By Theorem 1.3(i) this implies $\iota(N) = 1$. Using Theorem 1.3(ii) we deduce that N is a topological manifold. By Theorem 2.4 the boundary $\partial G \cong S^{n-1}$ is k-LCC in M for all k. Therefore we can apply Theorem 2.6 and deduce that $\overline{M}$ is a manifold with boundary S^{n-1}. The Z-condition implies that $\overline{M}$ is contractible, because M is contractible as the universal cover of the aspherical manifold N. The h-cobordism theorem for topological manifolds implies that $\overline{M} \cong D^n$. In particular, $M \cong \mathbb{R}^n$. This shows that (i) implies (ii). The converse is obvious. q.e.d.

3. Rigidity

The uniqueness question for the manifold appearing in our result from the introduction is a special case of the Borel Conjecture that asserts that aspherical manifolds are topological rigid: any isomorphism of fundamental groups of two closed aspherical manifolds should be realized (up to inner automorphism) by a homeomorphism. The connection of this rigidity question to assembly maps is well known and one of the main motivations for the Farrell-Jones Conjecture. For homology ANR-manifolds, the corresponding rigidity statement is (because of the lack of an s-cobordism theorem) somewhat weaker.

Theorem 3.1. *Let G be a torsion-free group. Assume that*

- *the (non-connective) K-theory assembly map*

$$H_i\big(BG;\mathbf{K}_{\mathbb{Z}}\big) \to K_i(\mathbb{Z}G)$$

is an isomorphism for $i \le 0$ and surjective for $i = 1$;

- *the (non-connective) L-theory assembly map*

$$H_i\big(BG;^w\mathbf{L}_{\mathbb{Z}}^{\langle -\infty\rangle}\big) \to L_i^{\langle -\infty\rangle}(\mathbb{Z}G,w)$$

is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w\colon G \to \{\pm 1\}$.

Then the following holds:

(i) Let M and N be two aspherical closed n-dimensional manifolds together with isomorphisms $\phi_M\colon \pi_1(M) \xrightarrow{\cong} G$ and $\phi_N\colon \pi_1(N) \xrightarrow{\cong} G$. Suppose $n \ge 5$.

Then there exists a homeomorphism $f\colon M \to N$ such that $\pi_1(f)$ agrees with $\phi_N \circ \phi_M^{-1}$ (up to inner automorphism);

(ii) Let M and N be two aspherical closed n-dimensional homology ANR-manifolds together with isomorphisms $\phi_M\colon \pi_1(M) \xrightarrow{\cong} G$ and $\phi_N\colon \pi_1(N) \xrightarrow{\cong} G$. Suppose $n \ge 6$.

Then there exists an s-cobordism of homology ANR-manifolds $W = (W, \partial_0 W, \partial_1 W)$, homeomorphisms $u_0\colon M_0 \to \partial_0 W$, and $u_1\colon M_1 \to \partial_1 W$ and an isomorphism $\phi_W\colon \pi_1(W) \to G$ such that $\phi_W \circ \pi_1(i_0 \circ u_0)$ and $\phi_W \circ \pi_1(i_1 \circ u_1)$ agree (up to inner automorphism), where $i_k\colon \partial_k W \to W$ is the inclusion for $k = 0, 1$.

Proof. (i) As discussed in the proof of Theorem 1.2, the assumptions imply that $\mathrm{Wh}(G) = 0$. Therefore it suffices to show that the structure set $\mathbb{S}^{TOP}(M)$ (see [**31**, Definition 18.1]) in the Sullivan-Wall geometric surgery exact sequence consists of precisely one element. This structure set is identified with the quadratic structure group $\mathbb{S}_{n+1}(M) = \mathbb{S}_{n+1}(BG)$ in [**31**, Theorem 18.5]. A discussion similar to the one in the proof of Theorem 1.2 shows that our assumptions imply that the quadratic structure group is trivial.

(ii) This follows from a similar argument that uses the surgery exact sequences for homology ANR-manifolds due to Bryant, Ferry, Mio, and Weinberger [**9**, Main Theorem on page 439]. q.e.d.

4. The Quinn obstruction depends only on the boundary

Let G be a torsion-free hyperbolic group. Assume that ∂G has the integral Čech cohomology of a sphere S^{n-1} with $n \geq 6$. By Theorem 2.7 there is a closed aspherical ANR-homology manifold N whose fundamental group is G.

Proposition 4.1. *In the above situation, the Quinn obstruction (see Theorem* 1.3*)* $\iota(N)$ *depends only on* ∂G.

Proof. Let H be a further torsion-free hyperbolic group such that $\partial H \cong \partial G$. Let N' be a closed aspherical ANR-homology manifold whose fundamental group is H. Then both the universal covers M of N and M' of N' can be compactified to $\overline{M}$ and $\overline{M'}$ such that $\partial G \cong \partial H$ is a Z-set in both; see Theorem 2.4. Now set $X := \overline{M} \cup_{\partial G} \overline{M'}$. We claim that X is a connected ANR-homology manifold. Thus

$$\iota(N) = \iota(M) = \iota(X) = \iota(M') = \iota(N')$$

by Theorem 1.3(i). To prove the claim we refer to [**1**], in particular pages 1270–1271. Both, M and M' are homology manifolds in the sense of this reference. By fact 6 of this reference, X is also a homology manifold. It remains to show that X is an ANR. This follows from an argument given during the proof of Theorem 9 of this reference. q.e.d.

Proof of Theorem B. Let G and H be torsion-free hyperbolic groups, such that $\partial G \cong \partial H$. Assume that G is the fundamental group of a closed aspherical manifold of dimension at least 6. Theorem 2.7(i) implies that $\partial G \cong \partial H$ has the integral Čech cohomology of a sphere S^{n-1} with $n \geq 6$ and that H is the fundamental group of a closed aspherical ANR-homology manifold M of dimension n. Because of Theorem 2.7(ii) this ANR-homology manifold can be arranged to have the DDP. Now by Proposition 4.1 (and Theorem 1.3(ii)) we have $\iota(M) = 1$. Using Theorem 1.3(ii) again, it follows that M is a manifold.

A similar argument works if G is the fundamental group of a closed aspherical homology ANR-manifold that is not necessarily a closed manifold. q.e.d.

5. Exotic examples

In light of the results of this paper one might be tempted to wonder if for a torsion-free hyperbolic group G, the condition $\partial G \cong S^n$ is equivalent to the existence of a closed aspherical manifold whose fundamental group is G. This is however not correct: Davis and Januszkiewicz, and

Charney and Davis constructed closed aspherical manifolds whose fundamental group is hyperbolic with boundary not homeomorphic to a sphere. We review these examples below.

Example 5.1. (i) For every $n \geq 5$ there exists an example of an aspherical closed topological manifold M of dimension n which is a piecewise flat, non-positively curved polyhedron such that the universal covering $\widetilde{M}$ is not homeomorphic to Euclidean space (see [**15**, Theorem 5b.1 on page 383]). There is a variation of this construction that uses the strict hyperbolization of Charney and Davis [**13**] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic.

(ii) For every $n \geq 5$ there exists a strictly negative curved polyhedron of dimension n whose fundamental group G is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to $\mathbb{R}^n$, but the boundary ∂G is not homeomorphic to S^{n-1}; see [**15**, Theorem 5c.1 on page 384 and Remark on page 386].

On the other hand, one might wonder if assertion (ii) in Theorem A can be strengthed to the existence of more structure on the aspherical manifold. Strict hyperbolization [**13**] can be used to show that in general there may be no smooth closed aspherical manifold in this situation.

Example 5.2. Let M be a closed oriented triangulated PL-manifold. It follows from [**13**, Theorem 7.6] that there is a hyperbolization $\mathcal{H}(M)$ of M has the following properties:

(i) $\mathcal{H}(M)$ is a closed oriented PL-manifold. (This uses properties (2) and (4) from [**13**, page 333].)

(ii) There is a degree 1-map $\mathcal{H}(M) \to M$ under which the rational Pontrjagin classes of M pull back to those of $\mathcal{H}(M)$. In particular, the Pontrjagin numbers of M and $\mathcal{H}(M)$ conincide. (See properties (5) and (6′) from [**13**, page 333].)

(iii) $\mathcal{H}(M)$ is a negatively curved piece-wise hyperbolic polyhedra. In particular, $G := \pi_1(\mathcal{H}(M))$ is hyperbolic. Moreover, by [**15**, page 348] the boundary of ∂G is a sphere.

Suppose that some Pontrjagin number of $\mathcal{H}(M)$ is not an integer. Then the same is true for $\mathcal{H}(M)$. In particular, $\mathcal{H}(M)$ does not carry the structure of a smooth manifold. If in addition $\dim \mathcal{H}(M) = \dim M \geq 5$, then by Theorem 3.1 (i) any other closed aspherical manifold N with $\pi_1(N) = G$ is homeomorphic to $\mathcal{H}(M)$ and does not carry a smooth structure either. Such manifolds M exist in all dimensions $4k$, $k \geq 2$; see Lemma 5.3. This shows that there are for all $k \geq 2$ torsion-free hyperbolic groups G with $\partial G \cong S^{4k-1}$ that are not fundamental groups of smooth closed aspherical manifolds. In particular, such a G is not the fundamental group of a closed Riemannian manifolds of non-positive curvature.

In the previous example we needed PL-manifolds that do not carry a smooth structure. Such manifolds are classically contructed using Hirzebruch's Signature Theorem.

Lemma 5.3. *Let $k \geq 2$. There is an oriented closed $4k$-dimensional PL-manifold M^{4k} whose top Pontrjagin number $\langle p_k(M^{4k}) \mid [M^{4k}]\rangle$ is not an integer.*

Proof. For all $k \geq 2$ there are smooth framed compact manifolds N^{4k} whose signature is 8 and whose boundary is a $(4k-1)$-homotopy sphere; see [**8**] and [**26**, Theorem 3.4]. By [**33**], this homotopy sphere is PL-isomorphic to a sphere. We can now cone off the boundary and obtain a PL-manifold M^{4k} (often called the Milnor manifold) whose only non-trivial Pontrjagin class is p_k and whose signature $\sigma(M^{2k})$ is 8. Hirzebruch's Signature Theorem implies that

$$8 = \sigma(M^{4k}) = \frac{2^{2k}(2^{2k-1}-1)B_k}{2k!}\langle p_k(M^{4k}) \mid [M^{4k}]\rangle$$

where B_k is the kth Bernoulli number; see [**26**, page 75]. For $k = 2, 3$ we have then

$$8 = \frac{7}{45}\langle p_2(M^8) \mid [M^8]\rangle = \frac{62}{945}\langle p_3(M^{12}) \mid [M^{12}]\rangle;$$

compare [**28**, page 225]. This yields examples for $k = 2, 3$. Taking products of these examples, we obtain examples for all $k \geq 2$. q.e.d.

6. Open questions

We conclude this paper with two open questions.

(i) Can the boundary of a hyperbolic group be an ANR-homology sphere that is not a sphere?

(ii) Can one give an example of a hyperbolic group (with torsion) whose boundary is a sphere, such that the group does not act properly discontinuously on some contractible manifold?

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