Degree theory and intersections.

Sard's theorem, and its relative, transversality are powerful tools for understanding manifolds. One can give a whole course on this. This is not that. I just want to review here some of the key points.

Good references for this material are:
Milnor, Topology from the differentiable viewpoint.
Hirsch, Differential topology.
Guillemin and Pollack, Differential topology.

These contain a number of nice applications, like the Brouwer fixed point theorem, the hairy ball theorem, invariance of domain, the Poincare-Hopf theorem. We will give others in class. Yet others are suggested in the problems: http://www.math.uchicago.edu/~shmuel/ATProbs1.pdf .

Definition. Suppose f: M --> W is a smooth map, and that V is a smooth submanifold of W. We say that f is transverse to V if $Df_*(TM)+TV = TW$ for all points of $f^{-1}V$. (Here T denotes the tangent space, and Df denotes the differential of f.)

If f is transverse to N, then the implicit function theorem implies that $f^{-1}V$ is a smooth submanifold of M; its codimension equals that of V in W.

If everything is oriented, then $f^{-1}V$ inherits an orientation, as well.

The transversality theorem says that functions can be perturbed slightly to be transverse to any given submanifold.

An important special case is where f is an inclusion map of a submanifold. Then $f^{-1}V$ is called the transverse intersection of M and V. Its codimension in W is the sum of the codimensions of M and V. (In this case, one can actually move f to be transverse to V by a small smooth isotopy.) By the way, one should beware that intersection is not symmetric! In the oriented setting, the two orientations differ by a sign $(-)^{cd(M)cd(V)}$, where cd denotes the codimension.

Another important special case is when V is a point. Then the condition that f is transverse to V means that V is a regular value of f. The movement of f is easily guaranteed by Sard's theorem, which guarantees that arbitrary smooth (enough) maps have a plethora of regular values.

If one further assumes that M and W are compact, that dim M = dim W, and one has orientations, then $f^{-1}V$ is a union of points with signs (determined by the signs of the Jacobians of f at the inverse image points). Adding these up gives us an integer, by definition, the degree of f.

That the degree is independent of which regular value one chooses is proven simultaneously with (smooth) homotopy invariance. Applying the same construction to a homotopy gives a one dimensional manifold with boundary that is bounded by the inverse images of the different points (for the different maps). Since compact 1 manifolds are unions of intervals and circles, and the boundary of an iterval is 2 points with opposite signs (– remember we make once and for all a convention about how to orient boundaries using, say an outward normal) we get the equality of these signs.

Without orientations, one still has a mod 2 degree in **Z**/2**Z** which can be quite useful.

The same considerations apply to intersections. For instance if M and V have <u>complementary dimension</u>, ie dim M + dim V = dim W, then the intersections are points (generically) and, in the presence of an orientation, we get an integer, called the <u>intersection number</u>.

As a simple application of this, note that if f: Z -> Z is a self map of a manifold, then if f has no fixed points (i.e. f(z) = z has no solutions) the diagonal and the graph of f, both submanifolds diffeomorphic to Z inside of ZxZ have empty intersection. If for a particular f one sees this intersection is nonzero, in the sense of intersection number, then every map homotopic to f has a fixed point.

Maybe an even simpler application is this. Let W be the Moebius strip and let M = V = the center circle. When we make M transverse to V, we move it off itself slightly. Since it is not hard to do this an produce just a single intersection point, there is no way to homtop this circle to not intersect