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Group actions on homology spheres

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Introduction

Let π be a finite group and Σ^r a simply-connected closed manifold which is a $\mathbb{Z}/|\pi|$ homology sphere. An object of this paper is to prove:

Theorem A. *Assume r is odd and $r \neq 3$. Then π acts freely on the r-sphere if and only if π acts freely on Σ^r so that the induced action on $H_*(\Sigma^r)$ is trivial.*

This can be stated in a more symmetric manner. Let r be any positive integer not equal to 3. Then π acts freely and homologically trivially on Σ^r iff π acts freely and homologically trivially on S^r. In fact, there is a one-to-one correspondence between such actions on Σ^r and such actions on S^r. (The classification of such actions is discussed in §7.) In addition the actions constructed have the property that every group element is isotopic to the identity. Theorem A holds in the topological or piecewise-linear categories. Well-known counterexamples exist in the smooth category even for homotopy spheres.

Theorem A would be false without the assumption of homological triviality or without the assumption of simple-connectivity (see remark 8.8). In fact, there exist groups π and simply-connected $\mathbb{Z}/|\pi|$ homology spheres Σ^r, such that π acts freely on Σ^r, but not on S^r. Also there are groups π which act freely on S^r, but do not act freely on certain lens spaces L_p, with p prime to $|\pi|$.

If the group π is cyclic, a result similar to our main theorem was first proved by Löffler [L] by different methods. The second author independently established the cyclic case in [We1], by a method that extends to cyclic group actions on more general manifolds (see [C-W1]), and, via this paper, to general finite groups. However the difficulties in making this extension have application elsewhere.

The main theorem is motivated by the philosophy that all of the geometry of a group action is present at the order of the group. Thus, it is only natural to expect that any manifold "homologically resembling" a sphere admitting a

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π-action should admit a π-action as well. Nonetheless, there are obstructions to realizing this philosophy for general manifolds. The main technical burden of this paper is to avoid these obstructions in the cases discussed here.

It might be of interest to compare our method of proof with the complementary theorem of Madsen et al. [M-T-W] on the spherical space form problem, which constructs free π-actions on spheres. In very broad outline, they are similar. One first constructs Poincaré complexes whose covers have the appropriate homotopy type. Then it is necessary to compute a surgery obstruction and verify that the universal cover is a sphere (or the homology sphere Σ^r). However, for the two problems each of these steps is done differently.

For instance, in the space form problem one uses Swan's work on periodic resolutions to construct the desired finite Poincaré complexes. On the other hand, we use the methods of [Wel] which are more homotopy theoretic than algebraic to establish the same conclusion. Madsen-Thomas-Wall compute their surgery obstruction by hyperelementary induction taking advantage of "obvious" linear actions on the sphere. However, homology spheres have no obvious symmetries making a different approach necessary. We make essential use of the results of [D2] on numerical surgery invariants and construct explicit surgery problems (rather than ones which exist for homotopy theoretic reasons) so that we can compute these invariants. The last step, identifying the universal cover, in the space form case follows from the generalized Poincaré conjecture, while in our case this step requires further computations with the surgery exact sequence.

We now outline the paper. In §1 we use the ideas of propagation of group actions to show that the main theorem is true in the homotopy category of (finite) CW complexes. In §2 we make some surgery theoretic constructions which suffice to prove theorem A in simpler case where $r \equiv 1 \pmod 4$. In §3 we give invariants for deciding when a Poincaré complex has the homotopy type of a closed manifold, using the Quinn-Ranicki assembly map and the work of Taylor-Williams on the oozing problem. The key computational techniques are in §4. The idea is that given a surgery problem $f: M \to X$ such that the universal covering $\tilde{f}$ is a $\mathbb{Z}_{(\pi)}$-homology equivalence there are "numerical invariants" depending only on the $\mathbb{Z}\pi_1 X$-modules $H_i(\tilde{f})$ which determine the surgery obstruction. For odd dimensions this depends on reciprocity laws from number theory and Galois cohomology (see [D2] for the full account). The problem is then to construct *explicit* surgery problems so that we can use these ideas to obtain trivial surgery obstructions. In §5 we develop further technical material in preparation for the proof of theorem A given in §6.

In §7 the classification of free, homologically trivial group actions on a simply-connected $\mathbb{Z}/|\pi|$ homology sphere Σ^r is given. The classification of the possible homotopy types is determined by two invariants lying in $(\mathbb{Z}/|\pi|)^\times$, a torsion Euler characteristic and a local k-invariant. The question of which homotopy types are actually realized is reduced to algebraic questions in K and L-theory. Some of these results are new even in the classical spherical space form case $\Sigma^r = S^r$. In §8 we discuss extensions of theorem A and non-free actions. A future paper [D-W] will apply the results of this paper to give new homological conditions on the possible fixed sets of smooth semifree π-actions on spheres.

These "propagation" ideas have recently been powerful tools for the construction of group actions on manifolds, as evidenced by the works of many authors (see the references in §1 or the survey [We3]). Generally these techniques involve constructing Poincaré complexes and then evaluating surgery obstructions lying in the L-groups. The results of this paper (cf. [D2], [D-W]) give effective methods for evaluating the surgery obstructions arising from propagation.

§ 1. General constructions – homotopy theory

The following notation will be in force throughout this paper: π is a finite group, r is an odd number, and Σ is a simply-connected closed manifold with $H_*(\Sigma; \mathbb{Z}_{(\pi)}) = H_*(S^r; \mathbb{Z}_{(\pi)})$. We abbreviate $\mathbb{Z}_{(|\pi|)}$ and $\mathbb{Z}[1/|\pi|]$ by $\mathbb{Z}_{(\pi)}$ and $\mathbb{Z}[1/\pi]$ respectively. The notation $X \sim Y$ means X and Y have the same homotopy type.

In this section we apply the "propagation of group actions" techniques to show that theorem A is true in the category of (finite) CW complexes (up to homotopy type). The proposition below has been proved independently in varying degrees of generality in [J, Q, A-V, C-W1, L-R]. We review one argument for the service of the reader – the ideas will be useful later.

Proposition 1.1. *Let $f: X \to Y$ be a $\mathbb{Z}_{(\pi)}$-homology equivalence between simply-connected CW complexes and suppose π acts freely and homologically trivially on Y. Then there is a CW complex X' with a free homologically π-action such that*

$$\begin{array}{ccc} & X & \\ {\scriptstyle h} \nearrow & & \searrow {\scriptstyle f} \\ X' & \xrightarrow{\quad f' \quad} & Y \end{array}$$

homotopy commutes, h is a homotopy equivalence, and f' is equivariant. Furthermore, if X and Y/π are finitely dominated, then so is X'/π.

An action of π on a space Y is R-homologically trivial if the representation $\pi \to \operatorname{Aut} H_*(Y; R)$ is trivial. The above proposition is valid if $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/\pi]$.

The idea of the proof is to "mix" Y/π at the primes dividing $|\pi|$ with X at the remaining primes using localization theory. Let W be a CW complex and P a set of primes. Using the fibrewise localization functor of Bousfield and Kan [B-K], one can construct a complex $W_{(P)}$ and a map $W \to W_{(P)}$ inducing an isomorphism on π_1 and a localization of the higher homotopy groups.

Then Y/π is the homotopy pullback of

$$\begin{array}{ccc} & & (Y/\pi)[1/\pi] \\ & & \downarrow \\ (Y/\pi)_{(\pi)} & \longrightarrow & (Y/\pi)_{(0)}. \end{array}$$

Furthermore, the "plus" construction of [Wel, I, Lemma 3.1] shows that since π acts $\mathbb{Z}[1/\pi]$-homologically trivially on Y, there is a homotopy equivalence

$$(Y/\pi)[1/\pi] \xrightarrow{\simeq} Y[1/\pi] \times B\pi,$$

where $B\pi = K(\pi, 1)$. Then let X'/π be the pullback of

$$\begin{array}{ccc} & & X[1/\pi] \times B\pi \\ & & \downarrow \\ (Y/\pi)_{(\pi)} & \longrightarrow & X_{(0)} \times B\pi. \end{array}$$

Exactly the same reasoning shows:

Proposition 1.2. *One can change the roles of X and Y in* 1.1.

In other words one can push actions forward as well as pull them back. (In the terminology of [C-W1], actions propagate in both directions.) By applying this to the degree one collapse map $\Sigma \to S^r$, one obtains:

Corollary 1.3. *The main theorem is true in the category of finitely dominated CW complexes (up to homotopy type). That is, π acts freely and homologically trivially on a complex $\Sigma' \sim \Sigma$ iff π so acts on a complex $(S^r)' \sim S^r$.*

Thus any such π satisfies $H^{r+1}(\pi) = \mathbb{Z}/|\pi|$, has periodic cohomology, and has cyclic or generalized quaternionic 2-Sylow subgroup [C-E].

If M is a finitely generated $\mathbb{Z}\pi$-module such that $M \otimes \mathbb{Z}_{(\pi)} = 0$, then a theorem of [Rim] shows that there is an exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

where the P_i are projective $\mathbb{Z}\pi$-modules. Schanuel's lemma then shows that $\sigma(M) = [P_0] - [P_1] \in \tilde{K}_0(\mathbb{Z}\pi)$ depends only on the module M. The following proposition follows from this fact and the theory of the Wall finiteness obstruction $[W] \in \tilde{K}_0(\mathbb{Z}\pi_1 W)$ associated to a finitely dominated CW complex W. (See [Wal].)

Proposition 1.4. (Mislin [Ms]). *Let X and Y be finitely dominated CW complexes with fundamental group π. Let $f: \tilde{X} \to \tilde{Y}$ be π-equivariant and a $\mathbb{Z}_{(\pi)}$-homology equivalence. Then*

$$[Y] = [X] + \sum (-1)^i \sigma(H_i(\tilde{Y}, \tilde{X})) \in \tilde{K}_0(\mathbb{Z}\pi).$$

Corollary 1.5. *The main theorem is true in the category of finite complexes (up to homotopy type).*

Proof of 1.5. If one applies 1.1 or 1.2 to a $\mathbb{Z}_{(\pi)}$-homology equivalence $f: \Sigma \to S^r$ to obtain $f': \Sigma'/\pi \to S^{r'}/\pi$ then

$$[S^{r'}/\pi] = [\Sigma'/\pi] - \sigma(\mathbb{Z}/(\deg f)) - \sum_{i=1}^{r-1} (-1)^i \sigma(H_i(\Sigma)).$$

It is not difficult to show that

$$\sigma(\mathbb{Z}/s)+\sigma(\mathbb{Z}/t)=\sigma(\mathbb{Z}/st)$$
$$\sigma(\mathbb{Z}/(k|\pi|+1))=0.$$

Thus one can choose the degree of f so that $[S^{r'}/\pi]=[\Sigma'/\pi]$ and the corollary follows.

§ 2. General constructions – surgery theory

In this section we make general remarks which suffice to prove theorem A when $r\equiv 1 \pmod 4$. The methods in this section follow those of [Wel] in rough outline.

Recall that a Poincaré complex X has a Spivak normal bundle $\nu_X\colon X\to BG$. There are fibrations of infinite loop spaces

$$G/TOP\to BTOP\to BG$$
$$BTOP\to BG\to B(G/TOP).$$

A necessary and sufficient condition for the existence of a surgery problem (or degree one normal map) over X is that ν_X lifts to some map $\tilde{\nu}\colon X\to BTOP$, or equivalently, the map $\nu_X\colon X\to B(G/TOP)$ is trivial. Fixing a lift $\tilde{\nu}$, normal bordism classes of surgery problems over X are in one-to-one correspondence with elements of $[X, G/TOP]$. Such an element is called a normal invariant. (For background on the surgery classifying spaces we recommend [M-M]).

Proposition 2.1. *If π acts freely on S^r, and freely and homologically trivially on a complex $X\sim\Sigma$, then X/π is a Poincaré complex whose Spivak bundle lifts.*

Proof. X/π satisfies Poincaré duality since $(X/\pi)_{(p)}$ satisfies Poincaré duality for all primes p. We next need to check that the map $\nu_{X/\pi}\colon X/\pi\to B(G/TOP)$ is nullhomotopic, an issue which can be studied one prime at a time. It follows easily from Sullivan's analysis of p-local spherical fibrations that the local classifying map of the Spivak bundle of a Poincaré complex $Y\to BG_{(p)}$ depends only on the p-local homotopy type of Y. (See [Su2], also [K, Wel].) If $p\,|\,|\pi|$, $(X/\pi)_{(p)}\sim(S^r/\pi)_{(p)}$ and S^r/π is a manifold so the map to $B(G/TOP)_{(p)}$ is nullhomotopic. If p does not divide $|\pi|$, then $(X/\pi)_{(p)}\sim\Sigma_{(p)}\times B\pi$ so the map is again nullhomotopic.

Our next result only holds true for PL or topological actions.

Proposition 2.2. *If π acts freely and homologically trivially on a closed manifold Σ', homotopy equivalent to $\Sigma(=\Sigma^r,\ r>3)$, then π so acts on Σ.*

Proof. It suffices to show that the homotopy equivalence $(\Sigma\to\Sigma')\in\mathscr{S}(\Sigma')$ $(=\mathscr{S}^h_{TOP}(\Sigma'))$ is in the image of the transfer $\mathrm{tr}\colon\mathscr{S}(\Sigma'/\pi)\to\mathscr{S}(\Sigma')$. Consider the Sullivan-Wall surgery exact sequence:

$$\begin{array}{ccccccc}
L^h_{r+1}(\mathbb{Z}) & \xrightarrow{0} & \mathcal{S}(\Sigma^r) & \longrightarrow & [\Sigma^r, G/TOP] & \longrightarrow & 0 \\
\uparrow & & \uparrow \scriptstyle{\mathrm{tr}} & & \uparrow & & \uparrow \\
L^h_{r+1}(\mathbb{Z}\pi) & \longrightarrow & \mathcal{S}(\Sigma^r/\pi) & \longrightarrow & [\Sigma^r/\pi, G/TOP] & \longrightarrow & L^h_r(\mathbb{Z}\pi).
\end{array}$$

To complete the proof it is necessary to note three facts. First, $L^h_r(\mathbb{Z}\pi)$ is $|\pi|$-torsion. Indeed for r odd, $L^h_r(\mathbb{Z}\pi)$ is 2-torsion for $|\pi|$ even and 0 for $|\pi|$ odd (see [Wa3]). Second, $[\Sigma^r, \mathrm{G/TOP}]$ is torsion prime to $|\pi|$ (since $[\Sigma^r, G/TOP]\otimes\mathbb{Z}_{(\pi)} = [S^r, G/TOP]\otimes\mathbb{Z}_{(\pi)}=0$). Finally, we show the map

$$[\Sigma^r/\pi, G/TOP]\otimes\mathbb{Z}[1/\pi]\to[\Sigma^r, G/TOP]\otimes\mathbb{Z}[1/\pi]$$

is an isomorphism. Indeed the map $H^*(\Sigma^r/\pi)\otimes\mathbb{Z}[1/\pi]\to H^*(\Sigma^r)\otimes\mathbb{Z}[1/\pi]$ is an isomorphism and since G/TOP is an infinite loop space we can apply the Atiyah-Hirezbruch spectral sequence to deduce this.

Proof of theorem A for $r\equiv 1 \pmod 4$

Recall that if π acts freely and homologically trivially on Σ^r or S^r, then $H^{r+1}(\pi)=\mathbb{Z}/|\pi|$. The classification of groups of period congruent to 2 modulo 4 (see e.g. [D-M]) shows that π is a direct product of $\mathbb{Z}_{2^s}$ and a group of odd order. By [Wa3], $L'_1(\mathbb{Z}[\mathbb{Z}_{2^s}\times\pi_{\mathrm{odd}}])=0$ and $L'_1(\mathbb{Z}\pi)\to L^h_1(\mathbb{Z}\pi)$ is surjective. Thus $L^h_1(\mathbb{Z}\pi)=0$. Theorem A follows from 1.5, 2.1, the fact that $L^h_1(\mathbb{Z}\pi)=0$ (all surgery obstructions are zero!), and 2.2.

Remark. For $r\equiv 3 \pmod 4$, things are more difficult. The finite Poincaré complexes X/π constructed by propagation need not have the homotopy type of manifolds. Our proof in the general case will require choosing the homotopy type of X/π carefully, e.g. studying the degree of the maps $X/\pi\to S^r/\pi$ and the effect on surgery obstructions. It will be necessary to construct explicit surgery problems over X/π.

§ 3. Ooze and Poincaré complexes

A surgery problem over a Poincaré complex X is a degree one normal map $f\colon M\to X$. In this section we apply some results on the "oozing" problem, the determination of which elements of $L_n(\mathbb{Z}\pi)$ arise from surgery problems over closed manifolds. Define $C_n(\mathbb{Z}\pi)$ to be the subgroup of $L_n(\mathbb{Z}\pi)$ given by all surgery obstructions $\theta(f)$ where $f\colon M\to N$ is a surgery problem over some closed manifold N^n, with $\pi_1 N=\pi$. Just as with L-theory there are variant groups $C^s_n(\mathbb{Z}\pi)$, $C^h_n(\mathbb{Z}\pi)$, and $C^p_n(\mathbb{Z}\pi)$. Of course if X is an n-dimensional Poincaré complex and $f\colon M\to X$ is a degree one normal map with $\theta(f)\notin C_n(\mathbb{Z}\pi)$, then clearly X cannot have the homotopy type of a closed manifold. The main point of this section is that a converse is sometimes true.

Let X be an n-dimensional Poincaré complex whose Spivak bundle lifts. Define

$$\Sigma(X) \in L_n(\mathbf{Z}\pi_1 X)/C_n(\mathbf{Z}\pi_1 X)$$

to be the image of the surgery obstruction of *any* surgery problem over X. (This is a well-defined invariant by 3.2 and 3.3 below.) An interesting problem is to identify this invariant somehow intrinsically. For $\pi = \mathbf{Z}_2 \times \mathbf{Z}_2$ and $n \equiv 3 \pmod 4$, this is the Σ-invariant of [C-W2, I] and will be a sequel, expressed in terms of Poincaré transversality obstructions in which in particular define the invariant for all Poincaré complexes (with or without reduction of the Spivak bundle). Can this be done more generally?

After describing the work of Quinn-Ranicki and Taylor-Williams we show:

Theorem 3.1. *Let X^{4k+3} $(k>0)$ be a Poincaré complex whose Spivak bundle lifts. Assume that the 2-Sylow subgroup of $\pi = \pi_1 X$ is abelian, generalized quaternion, dihedral, or semidihedral. Then X has the homotopy type of a manifold iff $\Sigma^h(X) = 0$.*

Interesting special cases are $\pi = Q_8$ or $\mathbf{Z}_m$ for which $C^h_{4k+3}(\mathbf{Z}\pi) = L^h_{4k+3}(\mathbf{Z}\pi)$, so that X is homotopy equivalent to a closed manifold iff the Spivak bundle of X lifts to $BTOP$. This gives an alternate proof of theorem A when $r \equiv 3 \pmod 4$ and $\pi = Q_8$ or $\mathbf{Z}_m$.

Quinn and Ranicki have constructed an Ω-spectrum $\underline{L}_0$ whose 0-th space has the homotopy type of G/TOP. It comes equipped with natural "assembly" maps

$$\sigma_* : H_n(X : \underline{L}_0) \to L_n(\mathbf{Z}\pi_1 X).$$

The relation of the assembly map with ooze is given by:

Lemma 3.2. [Ra3, p. 557]. $C_n(\mathbf{Z}\pi) = \sigma_*(H_n(B\pi; \underline{L}_0))$.

If X is an n-dimensional Poincaré complex, then by duality one can identify $H_n(X; \underline{L}_0)$ with $H^0(X; \underline{L}_0) = [X, G/TOP]$ and conclude:

Lemma 3.3 [Ra1]. *If $f: M \to X$ is a degree one normal map, then the coset*

$$\sigma_*(H_n(X; \underline{L}_0)) + \theta(f)$$

is equal to the set of all elements of $L_n(\mathbf{Z}\pi_1 X)$ representing surgery obstructions of surgery problems over X.

Both lemmas hold with p, h, or s decorations.

Taylor and Williams [T-W1, 2] study a splitting of $(\underline{L}_0)_{(2)}$ as a wedge of Eilenberg-Maclane spectra. This splitting gives a decomposition

$$H_n(X; \underline{L}_{0(2)}) = \Big[\bigoplus_{(n/4) \geq i \geq 0} H_{n-4i}(X; \mathbf{Z}_{(2)})\Big] \otimes \Big[\bigoplus_{(n-2)/4 \geq i \geq 0} H_{n-4i-2}(X; \mathbf{Z}_2)\Big].$$

The induced homomorphisms

$$I_{n-4i}(X) : H_{n-4i}(X; \mathbf{Z}_{(2)}) \to L_n(\mathbf{Z}\pi_1 X) \otimes \mathbf{Z}_{(2)}$$
$$\kappa_{n-4i-2}(X) : H_{n-4i-2}(X; \mathbf{Z}_2) \to L_n(\mathbf{Z}\pi_1 X) \otimes \mathbf{Z}_{(2)}$$

depend only on the differences $n-4i$, $n-4i-2$. Since $L_*(\mathbb{Z}\pi)$ has no odd torsion for $\pi_1 X$ finite, no information is lost by tensoring with $\mathbb{Z}_{(2)}$. If $i: \pi_2 \to \pi_1 X$ is the inclusion of a 2-Sylow subgroup, transfer arguments show $I_n(X) = i_* I_n(\tilde{X}/\pi_2)$ and $\kappa_n(X) = i_* \kappa_n(\tilde{X}/\pi_2)$.

Theorem 3.4. [T-W2]. *Let π be finite.*

i) *$I_n^h(B\pi) = 0$ for $n > 0$ and $\kappa_n^p(B\pi) = 0$ for $n > 1$.*

ii) *If the 2-Sylow subgroup of π is abelian, generalized quaternion, dihedral, or semi-dihedral then $\kappa_n^h(B\pi) = 0$* for $n > 3$.

Proof of 3.1.

$$\begin{aligned} C^h_{4k+3}(\mathbb{Z}\pi) &= \sigma_*(H_{4k+3}(B\pi; \underline{L}_0)) && \text{(by 3.2)} \\ &= \operatorname{im}(\kappa_1^h(B\pi)) && \text{(by 3.4)} \\ &= \operatorname{im}(\kappa_1^h(X)) && (H_1(X;\mathbb{Z}_2) \to H_1(B\pi;\mathbb{Z}_2) \text{ is surjective}) \\ &= \sigma_*(H_{4k+3}(X; \underline{L}_0)) && \text{(by 3.4 and naturality).} \end{aligned}$$

The result then follows by 3.3.

Remark. The above can be accomplished by a specific geometric construction as follows. Let $f: M \to X^{4k+3}$ be a degree one normal map whose surgery obstruction is represented by $\kappa_1(X)(\alpha)$ where $\alpha \in H_1(X; \mathbb{Z}_2)$. Let $C \subset M$ be a circle such that $f_*[C] = \alpha$. C has a tubular neighborhood $C \times D^{4k+2}$. Let $(K^{4k+2}, \partial) \to (D^{4k+2}, \partial)$ be the Kervaire problem. Then the composite

$$M \# C \times \partial D^{4k+2}(C \times (K^{4k+2}, \partial)) \to M \to X$$

has vanishing surgery obstruction. This is "modifying by a codimension-one Arf invariant" and is used in the papers [Wel, 2]. This geometric construction can be done in *PL* category.

§4. Torsion surgery invariants

One way of detecting information about surgery obstructions is through torsion invariants. In this section we prove that if $f: M \to X^{4k+3}$ is a degree one normal map whose surgery kernels $K_i(M) = \ker(f_*: H_i(\tilde{M}) \to H_i(\tilde{X}))$ are torsion prime to the order of $\pi = \pi_1 X$, then the surgery obstruction $\theta(f) \in L_3^h(\mathbb{Z}\pi)$ is *completely determined* by the $\mathbb{Z}\pi$-modules $K_i(M)$.

Let π be a finite group. Let S denote the set of primes which do not divide the order of π. Thus $S^{-1}\mathbb{Z}\pi = \mathbb{Z}_{(\pi)}\pi$. Let $K_1(\mathbb{Z}\pi, S)$ denote the abelian group resulting from the Grothendieck construction on the category of finitely generated S-torsion $\mathbb{Z}\pi$-modules. (Note that all finitely generated S-torsion modules have homological dimension one by Rim's theorem [Rim].) Bass [Ba] gives an exact sequence

$$K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}_{(\pi)}\pi) \to K_1(\mathbb{Z}\pi, S) \xrightarrow{\sigma} K_0(\mathbb{Z}\pi) \to K_0(\mathbb{Z}_{(\pi)}\pi).$$

Let $\tilde{K}_1(\Lambda) = K_1(\Lambda)/\langle -1 \rangle$. Let $A = \operatorname{im}(\tilde{K}_1(\mathbb{Z}\pi) \to \tilde{K}_1(\mathbb{Z}_{(\pi)}\pi))$. The following theorem will be our major algebraic tool:

Theorem 4.1 (Davis [D2]). *The map $L_3^h(\mathbb{Z}\pi)\to L_3^A(\mathbb{Z}_{(\pi)}\pi)$ is injective if the involution on $\mathbb{Z}\pi$ is the "oriented" involution $g\mapsto g^{-1}$ for all $g\in\pi$.*

The proof involves using localization exact sequences and reciprocity laws to compare the L-theory of $\mathbb{Z}\pi$ and $\mathbb{Z}_{(\pi)}\pi$; the corresponding result with $\mathbb{Z}_{(\pi)}\pi$ replaced by the p-adic completion $\hat{\mathbb{Z}}_\pi\pi$ is false. Also the corresponding result in L_1^h is false for $\pi=Q(8)$, but it is true for many other groups (including $\pi=Q(2^n)$, $n>3$).

If M is a $\mathbb{Z}[\mathbb{Z}/2]$-module we write $H^n(M)$ for the Tate cohomology $\hat{H}^n(\mathbb{Z}/2; M)$. There is a Rothenberg exact sequence

$$\ldots\to L_{n+1}^h(\mathbb{Z}_{(\pi)}\pi)\to H^{n+1}(\ker\sigma)\to L_n^A(\mathbb{Z}_{(\pi)}\pi)\to L_n^h(\mathbb{Z}_{(\pi)}\pi)\to\ldots.$$

The existence of such sequences involving intermediate L-groups was an idea of Cappell. A proof of exactness is given in [Ra2, I, 9].

Theorem 4.2. *Let $f\colon M\to X^n$ be a degree one normal map with X finite and $\pi=\pi_1X$. Assume the surgery kernels are torsion prime to the order of π. Then if $\theta(f)\in L_n^h(\mathbb{Z}\pi)$ is the surgery obstruction,*

$$\operatorname{im}\theta(f)=\operatorname{im}\chi(K_*(M))\in L_n^A(\mathbb{Z}_{(\pi)}\pi).$$

Here $\chi(K_*(M))$ is the torsion characteristic $\sum(-1)^i[K_i(M)]\in H^{n+1}(\ker\sigma)$.

Combining Theorems 4.1 and 4.2, we see that if $n=4k+3$, then the surgery obstruction $\theta(f)$ is determined by the torsion $\mathbb{Z}\pi$-modules $K_*(M)$ – no information about the linking form is needed. This generalizes the "numerical invariants" of Pardon [P2] in three ways: from the highly connected case to the general case, from the 2-group case to the general finite group case, and most importantly from L^p to L^h.

We will prove 4.2 in greater generality than stated. Let T be a central multiplicative subset of a ring with involution Λ. By a (Λ, T)-module we mean a finitely generated T-torsion module of homological dimension one. Let $C=\{C_n\to\ldots\to C_0\}$ be a chain complex of finitely generated free Λ-modules such that C is T-acyclic (i.e. $T^{-1}C=T^{-1}\Lambda\otimes_\Lambda C$ has no homology). Then the Reidemeister torsion (see [Mi])

$$\Delta(C)\in K_1(T^{-1}\Lambda)/K_1(\Lambda)=\ker(\sigma\colon K_1(\Lambda, T)\to K_0(\Lambda))$$

is defined by giving C any Λ-base and computing the torsion of the based acyclic complex $T^{-1}C$. Now suppose further than $H_i(C)$ is a (Λ, T)-module for all i.

Proposition 4.3. *The Reidemeister torsion $\Delta(C)$ is equal to the Euler characteristic $\chi(C)=\sum(-1)^i[H_i(C)]\in\ker\sigma$.*

Proof. The proof is by induction on the dimension of C. If the dimension of C is 1 then there is a short exact sequence

$$0\to C_1\xrightarrow{f} C_0\to H_0\to 0.$$

By definition of the map $K_1(T^{-1}\Lambda)\to K_1(\Lambda, T)$ we have $[f]\to[H_0]$. Thus $\Delta(C_1\to C_0)=\operatorname{im}[f]=[H_0]$.

We next consider the case where $H_0(C)=0$. Then

$$C \cong \{C_n \to \ldots \to C_2 \to C_1'\} \oplus \{C_0 \xrightarrow{\sim} C_0\}$$

and the result is true for both summands by induction.

We now consider the case where $B_0 = \ker(C_0 \to H_0(C))$ is free over Λ. Then we have a chain map

$$\begin{array}{ccccccc} C_n & \longrightarrow & \ldots & \longrightarrow & C_1 & \longrightarrow & B_0 \\ \Big\| g_n & & & & \Big\| g_1 & & \Big\| g_0 \\ C_n & \longrightarrow & \ldots & \longrightarrow & C_1 & \longrightarrow & C_0 \end{array}$$

where g_0 becomes an isomorphism after localization. Thus

$$\begin{aligned} \Delta(C) &= \Delta(B_0 \to C_0) + \Delta(C_n \to \ldots \to C_1 \to B_0) \\ &= [H_0(C)] + \textstyle\sum_{i>0}(-1)^i[H_i(C)]. \end{aligned}$$

The last equality follows from the previous case.

For the general case we need a lemma.

Lemma 4.4. *Given a (Λ, T)-module M, there is a T-acyclic complex $D = \{D_2 \to D_1 \to D_0\}$ of finitely generated free Λ-modules such that $H_0(D) = M$ and such that $\Delta(D) = [H_0(D)] - [H_1(D)]$.*

Proof. Choose a projective resolution of M

$$0 \to P \to F \to M \to 0$$

where F is free. Let Q be a module such that $P \oplus Q$ is Λ-free and $T^{-1}Q$ is $T^{-1}\Lambda$-free. Let F' be a free submodule of Q such that there is a $t \in T^{-1}$ such that $F' \subset Q \subset tF'$. Let

$$D = \begin{bmatrix} F' \to Q & \\ \oplus & \\ P \to F \end{bmatrix}.$$

To compute $\Delta(D)$ we use

$$\begin{array}{ccccc} F' & \longrightarrow & Q \oplus P & \longrightarrow & F \\ \big\downarrow & & \big\downarrow & & \Big\| \\ F' & \longrightarrow & tF' \oplus F & \longrightarrow & F. \end{array}$$

Then

$$\begin{aligned} \Delta(D) &= -[tF'/F'] + [tF'/Q] + [F/P] \\ &= [F/P] - [Q/F'] \\ &= [H_0(D)] - [H_1(D)]. \end{aligned}$$

Now back to the general case of Proposition 4.3. Choose an M such that $[H_0(C)]+[M]\in\ker\sigma$. Use 4.4 to find a 2-dimensional complex D such that $\ker(C_0\oplus D_0\to H_0(C\oplus D))$ is free. The proposition then follows by applying the previous case to $C\oplus D$.

We now need a lemma which in the group ring case dates back to J. Shanenson's thesis [Sh].

Lemma 4.5. *Let $X\subset Y\subset \tilde{K}_1(\Lambda)$ where Λ is a ring with involution. Let (C,ψ) be an n-dimensional algebraic Poincaré complex in the sense of Ranicki* [Ra2, 3]. *If C is acyclic with torsion $\tau(C)\in Y$, then $[C,\psi]=\operatorname{im}\tau(C)$ in the Rothenberg sequence*

$$\dots\to H^{n+1}(Y/X)\to L_n^X(\Lambda)\to L_n^Y(\Lambda)\to\dots.$$

Proof. Let $(C',\psi')=\operatorname{im}\tau(C)$. Then C' is acyclic with $\tau(C')=\tau(C)$. According to Ranicki's formalism $((0,0),(C',\psi')+(C,-\psi))$ is a Poincaré pair so that $[C',\psi']=[C,\psi]\in L_n^X(\Lambda)$.

To deduce Theorem 4.2 from 4.3 and 4.5 we note that a degree one normal map $f\colon M\to X$ gives an algebraic Poincaré complex $[C,\psi]\in L_n^h(\mathbb{Z}\pi)$ where C is the algebraic mapping cone of $f_*\colon C_*(\tilde{M})\to C_*(\tilde{X})$. In our case $\mathbb{Z}_{(\pi)}\otimes C$ is acyclic with Reidemeister torsion $\sum(-1)^i[K_i(M)]$.

Motivated by Theorem 4.1 we make the following definitions:

Definition 4.6. The quadratic torsion group

$$QT(\pi)=\operatorname{coker}(L_0^h(\mathbb{Z}_{(\pi)}\pi)\to H^0(\ker\sigma))\cong\ker(L_3^A(\mathbb{Z}_{(\pi)}\pi)\to L_3^h(\mathbb{Z}_{(\pi)}\pi)).$$

By 4.1, $\ker(L_3^h(\mathbb{Z}\pi)\to L_3^h(\mathbb{Z}_{(\pi)}\pi)$ injects to $QT(\pi)$. If $f\colon M\to X$ is a $\mathbb{Z}_{(\pi)}\pi$-homology equivalence with $\pi=\pi_1X$, then the *quadratic torsion of f* is

$$q_\tau(f)=-\sum(-1)^i[H_i(f;\mathbb{Z}\pi)]\in\mathrm{QT}(\pi).$$

In particular if f is a degree one normal map the $q_\tau(f)=\sum(-1)^i[K_i(M)]$.

We now give criterion for the vanishing of the quadratic torsion. We suppose the order of π is even.

Lemma 4.7. (A) *If $(s,|\pi|)=1$, then $\mathbb{Z}\pi\otimes\mathbb{Z}_s$ represents the trivial element of $QT(\pi)$.*

(B) *If $t-1$ is a multiple of $LCM(8,2\cdot|\pi|)$ then $\mathbb{Z}_t$ (with trivial π-action) represents the trivial element of $QT(\pi)$.*

Proof. Let e be the identity of π. Let k be the integer so that $s=\pm(1+4k)$. The symmetric even matrix

$$\begin{bmatrix}2ke & e\\ e & -2e\end{bmatrix}$$

represents an element of $L_0^h(\mathbb{Z}_{(\pi)}\pi)$ with discriminant

$$[\pm s\cdot e]\in K_1(\mathbb{Z}_{(\pi)}\pi)/K_1(\mathbb{Z}\pi),$$

which corresponds to $\mathbb{Z}\pi\otimes\mathbb{Z}_s$ in $\ker\sigma$. This proves (A).

Let $N=\sum_{g\in\pi} g$. Then the symmetric even matrix

$$\begin{bmatrix} 2aN & e+bN \\ e+bN & -2e \end{bmatrix}$$

represents an element of $L_0^h(\mathbb{Z}_{(\pi)}\pi)$ with discriminant

$$[e+(4a+2b+b^2|\pi|)N]\in K_1(\mathbb{Z}_{(\pi)}\pi)/K_1(\mathbb{Z}\pi).$$

Its image in $\ker\sigma$ is $[\mathbb{Z}/1+(4a+2b+b^2|\pi|)|\pi|]$. Such elements generate the subgroup of $\ker\sigma$ given by $[\mathbb{Z}/t]$ with $t\equiv 1 \,(\mathrm{mod}\, LCM(8,2|\pi|))$.

Remark 4.8. If $[\mathbb{Z}_s]\in\ker\sigma$, we call the corresponding element of $QT(\pi)$ a *Swan formation.* The class of $[\mathbb{Z}_s]\in QT(\pi)$ depends only on the residue class of s modulo $LCM(8,2|\pi|)$.

§ 5. *BG*-maps and degree *n* maps

In the proof of theorem A, two generalizations of degree 1 normal maps arise; we weaken degree 1 to degree n and the notion of a normal map to a BG-map, a map of Spivak bundles. In this section we develop these concepts.

Definition 5.1. A map $f\colon X\to Y$ between Poincaré complexes is a *BG-map* if

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \nu_X}\searrow \quad \swarrow{\scriptstyle \nu_Y} & \\ & BG & \end{array}$$

commutes up to homotopy.

Remark. By Sullivan's analysis of p-local spherical fibrations, to see that the above diagram commutes, it suffices to check that for each prime p it commutes when localized at p.

Theorem 5.2. *Let X be a finite Poincaré complex of dimension $r=4k+3$ whose Spivak bundle lifts to BTOP and whose fundamental group π is finite. Let $f\colon M\to X$ be a degree one BG-map where M is a closed manifold and f is a $\mathbb{Z}_{(\pi)}\pi$-homology equivalence. Then there exists a degree one normal map $f'\colon M'\to X$ whose surgery obstruction $\theta(f')\in L_r^h(\mathbb{Z}\pi)$ maps to the quadratic torsion $q_\tau(f)\in L_r^A(\mathbb{Z}_{(\pi)}\pi)$. In particular if $q_\tau(f)=0$, then X has the homotopy type of a closed manifold.*

Proof. Let $\tilde{\nu}_X\colon X\to \widetilde{BTOP}(k)$ be a lift of the Spivak bundle ν_X to a topological block bundle. Then $\tilde{\nu}_M=f^*\tilde{\nu}_X$ is a lift of ν_M. Choose an element $\rho_M\in\pi_{r+n}(T(\nu_M))$ such that $h(\rho_M)\cap U=[M]$ where $h\colon\pi_{r+n}\to H_{r+n}$ is the Hurewicz homomorphism and $U\in H^n(T(\nu_M))$ is the Thom class. Then the Browder-Novikov transversality construction [Br] gives a degree one normal map $g\colon N\to M$, and a class $\rho_N\in\pi_{r+n}(T(\nu_N))$ such that $g_*\rho_N=\rho_M$. If we let $\rho_X=f_*\rho_M$

then we have a composite of normal maps in the sense of [Ra2, II]:

$$(N, \nu_N, \rho_N) \xrightarrow{g} (M, \nu_M, \rho_M) \xrightarrow{f} (X, \nu_X, \rho_X).$$

Hence by the composition formula of Ranicki [Ra2, II 4.3],

$$\theta(f \circ g) = \theta(g) + \theta(f) \in L_r^h(\mathbb{Z}\pi).$$

By the results of §4, $\theta(f)$ is given by $q_\tau(f)$ and hence does not depend on ρ_M and ρ_X. Note that $f \circ g$ and g are honest degree one normal maps. Hence

$$\theta(g) \in \mathrm{image}(\sigma_*: H_r(M; \underline{L}_0) \to L_r^h(\mathbb{Z}\pi)) \subset \sigma_*(H_r(X; \underline{L}_0)).$$

The last inclusion is true since $M \to B\pi$ factors through $X \to B\pi$. Then $\theta(f \circ g) = \theta(f) \in L_r^h(\mathbb{Z}\pi)/\sigma_*(H_r(X; \underline{L}_0))$. The theorem follows by applying Lemma 3.3.

We now require a result of Ian Hambleton and Ib Madsen which generalizes a result of Wall. Let π_2 be a 2-Sylow subgroup of a finite group π and $i: \pi_2 \to \pi$ the inclusion map. Given a π-cover (resp. π_2-cover) $\tilde{X}$ of X, let $i^* X = \tilde{X}/\pi_2$ (resp. $i_* X = \pi \times_{\pi a} \tilde{X}$). These induce functors between categories of spaces with reference maps to $B\pi$ and $B\pi_2$ and hence natural maps of L-groups.

Lemma 5.3. *Given $f: M \to N^k$, a degree n normal map between closed manifolds, and a π-cover of N, then the local surgery obstruction*

$$\theta(f) = |\pi: \pi_2| \cdot i_*(\theta(i^* f)) \in L_k(\mathbb{Z}[1/n]\pi).$$

Proof. In [H-M] a classifying space $(QS^0/TOP)_n$ for degree n maps between closed manifolds was constructed. There is a commutative diagram

$$\begin{array}{ccc}
\Omega_*(B\pi \times (QS^0/TOP)_n) & \xrightarrow{\theta} & L_k(\mathbb{Z}[1/n]\pi) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\Omega_*(B\pi_2 \times (QS^0/TOP)_n) & \xrightarrow{\theta} & L_k(\mathbb{Z}[1/n]\pi_2) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\Omega_*(B\pi \times (QS^0/TOP)_n) & \xrightarrow{\theta} & L_k(\mathbb{Z}[1/n]\pi).
\end{array}$$

For a finite group $L_*(\mathbb{Z}[1/n]\pi) \to L_*(\mathbb{Z}[1/n]\pi) \otimes \mathbb{Z}_{(2)}$ is injective. The result follows from tensoring the above diagram with $\mathbb{Z}_{(2)}$ and using the bordism theory fact (see e.g. [M-M]) that

$$i_* \circ i^*: \Omega_*(B\pi \times Y) \otimes \mathbb{Z}_{(2)} \to \Omega_*(B\pi \times Y) \otimes \mathbb{Z}_{(2)}$$

is multiplication by $|\pi: \pi_2|$ for any Y.

§6. Proof of theorem A

The aim of this section is to prove theorem A when $r \equiv 3 (\mathrm{mod}\, 4)$. In both the "if" and "only if" directions the 2-group case is easier and the general case involves a reduction to the 2-group case. Let $\Sigma = \Sigma^r$ be a simply-connected $\mathbb{Z}/|\pi|$ homology sphere. If $|\pi|$ is odd, $L_r^h(\mathbb{Z}\pi)$ is zero hence we assume $|\pi|$ is even.

Lemma 6.1. *There exists a BG-map $S^r \to \Sigma$ of degree prime to $|\pi|$.*

Proof. Recall $\pi_r(BG)$ is finite (see e.g. [M-M]). The desired map is a composite $S^r \xrightarrow{f} S^r \xrightarrow{g} \Sigma$ where $(\deg f)\cdot(\pi_r(BG)\otimes \mathbb{Z}[1/\pi])=0$ and $\deg g$ is relatively prime to $|\pi|$ (g exists by the mod-C Hurewicz theorem). One checks that $g\circ f$ is a BG-map by checking at each prime p.

We first assume π acts freely and homologically trivially on Σ and try to construct a free action on S^r. π has periodic cohomology so the 2-Sylow subgroup is cyclic or generalized quaterionic and hence π_2 acts freely on S^r.

Definition 6.2. For a space M of finite type define

$$\chi^{\mathrm{tor}}(M) = \prod_i |\mathrm{torsion}\, H_i(M)|^{(-1)^i}.$$

If M is a $\mathbb{Z}/|\pi|$ homology sphere then $\chi^{\mathrm{tor}}(M)\in \mathbb{Z}_{(\pi)}^{\times}$.

Choose a BG-map $\psi\colon S^r\to\Sigma$ so that $\deg\psi \equiv \chi^{\mathrm{tor}}(\Sigma) (\mathrm{mod}\, 4|\pi|\mathbb{Z}_{(\pi)})$. Then $\deg\psi = \chi^{\mathrm{tor}}(\Sigma)\cdot(a/b)$ where a and b are integers congruent to 1 modulo $4|\pi|$. Propagation (1.1) gives a BG-map $g\colon X/\pi \to \Sigma/\pi$ where $X \sim S^r$. One computes the finiteness obstruction by 1.4 and the proof of 1.5:

$$\begin{aligned}[X/\pi] &= [\Sigma/\pi] + \sigma(\mathbb{Z}/\deg\psi) - \textstyle\sum (-1)^i \sigma(\mathrm{torsion}\, H_i(\Sigma))\\ &= 0 + \sigma(\mathbb{Z}/a) - \sigma(\mathbb{Z}/b)\\ &= 0 \in \tilde{K}_0(\mathbb{Z}\pi).\end{aligned}$$

Thus we may assume X is a finite complex. By Lemma 4.7(B), the quadratic torsion $q_\tau(g)$ is zero. As in the proof of 5.2, we can choose bundle reductions $\tilde{v}_{\Sigma/\pi}$, $\tilde{v}_{X/\pi}$ and use the Browder-Novikov transversality construction to get

$$N \xrightarrow{f} X/\pi \xrightarrow{g} \Sigma/\pi$$

so that f and $g\circ f$ are degree one normal maps and

$$\theta(g\circ f) = \theta(f) + q_\tau(g) \in L_r^A(\mathbb{Z}_{(\pi)}\pi).$$

The surgery obstruction $\theta(f)$ can be evaluated as follows:

$$\begin{aligned}\theta(f) &= \theta(g\circ f) && (\text{since } q_\tau(g)=0)\\ &= |\pi\colon \pi_2|\, i_*(\theta(i^*(g\circ f))) && (\text{by } 5.3)\\ &= |\pi\colon \pi_2|\, i_*(\theta(i^*(f))) && (\text{since } q_\tau(i^* g)=0).\end{aligned}$$

The range of i^*f is X/π_2 and hence has the homotopy type of a closed manifold. (Indeed any *finite* complex Y with $\pi_1 Y$ a 2-group and $Y\sim S^r$ has the homotopy type of a spherical space form, cf. §7, Ex. B.) Thus $\theta(f)\in C_r^h(\mathbb{Z}\pi)$. By 3.1, X/π has the homotopy type of a closed manifold M. The generalized Poincaré conjecture implies M is homeomorphic to S^r/π.

We now prove the converse in theorem A. Assume π acts freely on S^r; we wish to construct a free π-action on Σ. The idea behind the proof is clear. First, propagation gives a Poincaré complex Σ'/π with $\Sigma \sim \Sigma'$. We then construct an explicit degree 1 BG-map f over Σ'/π with torsion kernels. If *correct choices* are made then the quadratic torsion will be trivial which implies π acts on a manifold having the homotopy type of Σ, thus by 2.2 on Σ as well. However there are several technical difficulties. We thank Jim Milgram for helping us overcome one of them.

There are two cases – depending on whether or not π is a 2-group. To unify the exposition we make the following hypothesis:

Hypothesis 6.3. *There exists a BG-map* $g: M \to S^r/\pi$ *where* g *is a* $\mathbb{Z}_{(\pi)}\pi$*-homology equivalence,* M *is a closed manifold,* $(\deg g) \cdot \chi^{\mathrm{tor}}(\Sigma) \equiv 1 (\mathrm{mod}\ 4|\pi|\, \mathbb{Z}_{(\pi)})$, *and*

(a) *if* π *is a 2-group and* $\tilde{M}$ *is the induced* π*-cover of* M, *then*

$$\sum_{i=1}^{r-1} (-1)^i ([H_i(\tilde{M})] - [H_i(\Sigma)]) = 0 \in QT(\pi),$$

(b) *if* π *is not a 2-group,* g *is actually a normal map.*

We postpone the construction of M.

Proposition 6.4. *Assuming hypothesis* 6.3, *if* π *acts freely on* S^r, *then* π *acts freely and homologically trivially on* Σ.

Proof. Apply 6.1 to construct a BG-map $\psi: S^r \to \Sigma$ so that $(\deg \psi) \equiv \chi^{\mathrm{tor}}(\Sigma) (\mathrm{mod}\ 4|\pi|\, \mathbb{Z}_{(\pi)})$. Apply propagation to get a BG-map $S^r/\pi \to \Sigma'/\pi$ where $\Sigma' \sim \Sigma$ and Σ' is a finite complex. Composing with the hypothesized map one obtains a BG-map $h: M \to \Sigma'/\pi$ whose degree is congruent to $1 (\mathrm{mod}\ 4|\pi|)$.

Define a degree one BG-map

$$M \amalg m\bar{\Sigma} \to \Sigma'/\pi$$

where $m\bar{\Sigma}$ is the disjoint union of $m = ((\deg h) - 1)/|\pi|$ copies of Σ with the orientation reversed. The map $\bar{\Sigma} \to \Sigma'/\pi$ comes from the covering map. By doing 0-dimensional surgery one obtains a BG-map

$$f: M \# \bar{\Sigma} \# \dots \# \bar{\Sigma} \to \Sigma'/\pi.$$

Then

$$\begin{aligned} q_\tau(f) &= \sum_{i=0}^{r} (-1)^i [K_i(M \# \bar{\Sigma} \# \dots \# \bar{\Sigma})] \\ &= \sum_{i=1}^{r-1} (-1)^i ([H_i(\tilde{M})] + \sum_{i=1}^{r-1} (-1)^i [\ker ((\mathbb{Z}\pi)^m \otimes H_i(\Sigma) \to H_i(\Sigma))] \\ &= \sum_{i=1}^{r-1} (-1)^i ([H_i(\tilde{M})] + \sum_{i=1}^{r-1} (-1)^i (m[\mathbb{Z}\pi \otimes H_i(\Sigma)] - [H_i(\Sigma)]) \\ &= \sum_{i=1}^{r-1} (-1)^i ([H_i(\tilde{M})] - [H_i(\Sigma)]) \quad \text{(by 4.7(A))}. \end{aligned}$$

Thus for π a 2-group, $q_\tau(f)=0$ by the hypothesis. By 5.2 and 3.1, Σ' has the homotopy type of a closed manifold so by 2.2, π acts freely and homologically trivially on Σ.

If π is not a 2-group, recall that $g\colon M \to S^r/\pi$ is a normal map. Thus by 5.3,

$$\theta(g)=|\pi:\pi_2|\, i_*(\theta(i^* g)) \in L_r^A(\mathbb{Z}_{(\pi)}\pi).$$

Computing quadratic torsions, one obtains

$$\sum_{i=1}^{r-1}(-1)^i([H_i(\tilde{M})]-[H_i(\Sigma)])=|\pi:\pi_2|\, i_* \circ i^* \sum_{i=1}^{r-1}(-1)^i([H_i(\tilde{M})]-[H_i(\Sigma)]) \in QT(\pi).$$

By the previous computation this yields

$$\theta(f)=|\pi:\pi_2|\, i_*(\theta(i^*(f))).$$

But Σ'/π_2 has the homotopy type of a closed manifold (by the 2-group case!) so by 5.2, $\theta(i^* f) \in C_r^h(\mathbb{Z}\pi_2)$. Thus $\theta(f) \in C_r^h(\mathbb{Z}\pi)$. Hence by 3.1, Σ'/π has the homotopy type of a closed manifold, and by 2.2 π acts on Σ.

Lemma 6.5. *If π is a 2-group, there exists a BG-map $g\colon M \to S^r/\pi$ satisfying hypothesis* 6.3.

Proof. Since π is a 2-group, there is a free action of $\pi \times \mathbb{Z}/p$ on S^r for any odd number p. Choose p so that $\chi^{\mathrm{tor}}(\Sigma)\cdot p^{(r-1)/2} \equiv 1 \pmod{4|\pi|\,\mathbb{Z}_{(\pi)}}$ and so that $S^r/(\mathbb{Z}/p)$ is stably-parallelizable [E-M-S-S]. Letting $M=S^r/\pi \times \mathbb{Z}/p$, we see that 6.3(a) is satisfied. By applying the ideas of § 1, we propagate across a map $S^r/(\mathbb{Z}/p) \to S^r$ of degree congruent to $\chi^{\mathrm{tor}}(\Sigma)^{-1} \pmod{4|\pi|\,\mathbb{Z}_{(\pi)}}$ to get a map $g'\colon M \to X/\pi$ with X a finite complex of the homotopy type of S^r. But then X/π has the homotopy type of a spherical space form S^r/π since π is a 2-group. This gives the desired BG-map g.

Lemma 6.6. *If π is not a 2-group, there exists a normal map $g\colon M \to S^r/\pi$ satisfying hypothesis* 6.3.

Proof. Choose $m \in \mathbb{Z}$ so that $m \equiv \chi^{\mathrm{tor}}(\Sigma)^{-1} \pmod{4|\pi|\,\mathbb{Z}_{(\pi)}}$. Consider the (trivial) covering map $f\colon m(S^r/\pi) \to S^r/\pi$. To construct a degree m normal map $g\colon M \to S^r/\pi$ which is a $\mathbb{Z}_{(\pi)}\pi$-homology equivalence it suffices to show the local surgery obstruction $\theta(f) \in L_r^h(\mathbb{Z}_{(\pi)}\pi)$ is zero. The techniques of Pardon [P2], show that $L_r^h(\mathbb{Z}_{(\pi)}\pi) \to \oplus_p L_r^h(\mathbb{F}_p\pi/\mathrm{rad})$ is an injection. According to [D1], the image of $\theta(f)$ under this map can be expressed as a difference of semi-characteristics. Hence

$$\operatorname{im}\theta(f)=(m-1)\,\chi_{1/2}(S^r;\mathbb{F}_p\pi/\mathrm{rad}) \in L_r^h(\mathbb{F}_p\pi/\mathrm{rad}).$$

But this is zero since $m-1$ is even and $L_r^h(\mathbb{F}_p\pi/\mathrm{rad})$ has exponent 2.

§ 7. Classification

We now discuss the classification of free, homologically trivial π-actions on $\mathbb{Z}/|\pi|$-homology spheres. Our starting point is this: Fix a free π-action on S^r (r

odd, $r>3$). Fix a closed, simply-connected manifold $\Sigma=\Sigma^r$ which is a $\mathbb{Z}/|\pi|$-homology sphere. Our classification proceeds in four steps. The first step is to classify the possible homotopy types. More precisely, the classification of (polarized) homotopy types of n-dimensional CW-complexes Σ'/π with $\Sigma'\sim\Sigma$ and trivial induced π-action on $H_*(\Sigma')$ is given by a k-invariant $k_{\Sigma'/\pi}\in\mathbb{Z}/|\pi|^\times$. The second step is to find which homotopy types contain *finite* complexes – this depends only on $k_{\Sigma'/\pi}$ and $\chi^{\mathrm{tor}}(\Sigma)\in\mathbb{Z}/|\pi|^\times$. The first two steps are motivated by Swan's paper [Sw], which deals with the case of the sphere. The next step is also motivated by [Sw], but is new even in the case $\Sigma=S^r$. This is to determine the set of homotopy types which contain *closed manifolds*. This depends on algebraically defined L-theoretic obstructions, which are determined by $k_{\Sigma'/\pi}$, $\chi^{\mathrm{tor}}(\Sigma)\in\mathbb{Z}/|\pi|^\times$. The final step is a formal argument which gives a 1–1 correspondence between free π-actions on S^r (within the original fixed homotopy type) and free, homologically trivial π-actions on Σ (within a specified homotopy type).

Definition 7.1. A *polarized* (π,Σ)*-complex* Σ'/π is an r-dimensional CW complex Σ' equipped with a free, cellular, homologically trivial π-action and a homotopy equivalence $\beta_{\Sigma'}\colon\Sigma'\to\Sigma$. Two polarized (π,Σ)-complexes Σ'/π, Σ''/π have the same *polarized homotopy type* if there is an equivariant homotopy equivalence $g\colon\Sigma'\to\Sigma''$ such that $\beta_{\Sigma'}\sim\beta_{\Sigma''}\circ g$.

Lemma 7.2. *Let Σ'/π be a polarized (π,Σ)-complex. There is a map $f\colon(\Sigma'/\pi)_{(\pi)}\to(S^r/\pi)_{(\pi)}$ inducing the identity on π_1. Any such map is a homotopy equivalence and the degree of any two such maps are equal modulo $|\pi|\,\mathbb{Z}_{(\pi)}$.*

Proof. The discussion of the Eilenberg-MacLane k-invariant in the space form case (see e.g. [T]) leads immediately to this result so we will be brief. The existence of a map f follows from obstruction theory. The last statement follows from the identification of the image of the fundamental class $[\Sigma'/\pi]\in\mathbb{Z}$ in $\mathbb{Z}/|\pi|=H^{r+1}(\pi;\mathbb{Z}_{(\pi)})$ with the Eilenberg-MacLane k-invariant of $(\Sigma'/\pi)_{(\pi)}$ and the fact that this k-invariant is an additive generator of $\mathbb{Z}/|\pi|$.

We define $k_{\Sigma'/\pi}\in\mathbb{Z}/|\pi|^\times$ to be the degree of any such f. Since the π-action is homologically trivial, Lemma 7.2 and the discussion in § 1 imply that Σ'/π has the homotopy type of the homotopy pullback of a diagram

$$\begin{array}{ccc} & & \Sigma[1/\pi]\times B\pi \\ & & \downarrow \\ & & \Sigma_{\mathbb{Q}}\times B\pi \\ & & \downarrow{\scriptstyle f_{\mathbb{Q}}} \\ (S^r/\pi)_{(\pi)} & \longrightarrow & S^r_{\mathbb{Q}}\times B\pi \end{array}$$

where the degree of the patching map $f_{\mathbb{Q}}$ is an element of $\mathbb{Z}_{(\pi)}^\times$ which equals $k_{\Sigma'/\pi}$ in $\mathbb{Z}/|\pi|^\times$. Conversely the pullback of any such diagram with $\deg f_{\mathbb{Q}}\in\mathbb{Z}_{(\pi)}^\times$ gives a polarized (π,Σ)-complex. Since $(S^r/\pi)_{(\pi)}$ has self homotopy equivalences (inducing the identity on π_1) of any degree $\equiv 1 \pmod{|\pi|}$ (see [T]) we see that if

$\deg f'_{\mathbb{Q}} \equiv \deg f''_{\mathbb{Q}}$ the two resulting pullbacks are homotopy equivalent. Summarizing:

Theorem 7.3. *The correspondence $\Sigma'/\pi \to k_{\Sigma'/\pi}$ is 1-1 between polarized homotopy types of (π, Σ)-complexes and elements of $\mathbb{Z}/|\pi|^{\times}$. Given any integer n such that $n = k_{\Sigma'/\pi} \in \mathbb{Z}/|\pi|^{\times}$, there is a map $f: \Sigma'/\pi \to S^r/\pi$ of degree n inducing the identity on π_1.*

Remark. Suppose $f: X \to Y$ induces an isomorphism on the fundamental group π, is a $\mathbb{Z}_{(\pi)}\pi$-homology equivalence, and π acts trivially on $H_*(\tilde{f}; \mathbb{Z}[1/\pi])$. Then if one of the two spaces is a finitely dominated Poincaré complex whose Spivak bundle lifts to $BTOP$, then so is the other.

The Swan map $\tau: \mathbb{Z}/|\pi|^{\times} \to \tilde{K}_0(\mathbb{Z}\pi)$ can be defined in terms of the Bass localization map $\sigma: K_1(\mathbb{Z}\pi, S) \to \tilde{K}_0(\mathbb{Z}\pi)$ by defining $\tau(k) = \sigma(\mathbb{Z}/k)$. The following is a mild generalization of [Sw, 7.4].

Lemma 7.4. *A (π, Σ)-complex Σ'/π has the homotopy type of a finite complex if and only if $\tau(\chi^{\mathrm{tor}}(\Sigma) \cdot k_{\Sigma'/\pi}) = 0$.*

Proof. Apply Mislin's result 1.4 to a map $\Sigma'/\pi \to S^r/\pi$ inducing the identity on π_1.

Remark. By 2.2, if a (π, Σ)-complex has the homotopy type of a closed manifold then it has the homotopy type of a orbit space of a free π-action on Σ. If $r \equiv 1 \pmod 4$, then any finite (π, Σ)-complex is homotopic to a closed manifold since $L^h_r(\mathbb{Z}\pi) = 0$. We thus assume $r \equiv 3 \pmod 4$. We assume $|\pi|$ is even for the same reason.

Lemma 7.5. *For any $a \in \mathbb{Z}/4|\pi|^{\times}$, there exists a simply-connected closed manifold T with the $\mathbb{Z}/|\pi|$-homology of S^r, with $\chi^{\mathrm{tor}}(T) \equiv a \pmod{4|\pi|}$, and with $\nu_T: T \to BG[1/\pi]$ trivial.*

Proof. Choose a prime p so that $p \cdot a \equiv 1 \pmod{4|\pi|}$ and the lens space $S^r/\mathbb{Z}_p$ is stably parallelizable [E-M-S-S]. Doing surgery to make $S^r/\mathbb{Z}_p$ $(r-1)/2$-connected gives T.

Corollary 7.6. *If π acts freely on S^r and $a \equiv 1 \pmod{|\pi|}$, then $[\mathbb{Z}/a] = 0 \in QT(\pi)/\mathrm{im}\, C^h_r(\mathbb{Z}\pi)$.*

Proof. Note that the free π-action on T constructed in the proof of theorem A satisfies $\chi^{\mathrm{tor}}(T) \cdot k_{T/\pi} = 1 \in \mathbb{Z}/|\pi|^{\times}$. Thus there is a degree one BG-map $f: T/\pi \to S^r/\pi$ with $q_\tau(f) = [\mathbb{Z}/a]$. By 5.2, there is a degree one normal map with the same obstruction.

Thus there is a map $\tau_r: \ker \tau \to QT(\pi)/\mathrm{im}\, C^h_r(\mathbb{Z}\pi)$ defined by $\tau_r(k) = [\mathbb{Z}/k]$. The major result of this section is:

Theorem 7.7. *A (π, Σ)-complex Σ'/π has the homotopy type of a quotient of a free π-action on Σ if and only if $\tau(\chi^{\mathrm{tor}}(\Sigma) \cdot k_{\Sigma'/\pi}) = 0 \in \tilde{K}_0(\mathbb{Z}\pi)$ and $\tau_r(\chi^{\mathrm{tor}}(\Sigma) \cdot k_{\Sigma'/\pi}) = 0 \in QT(\pi)/\mathrm{im}\, C^h_r(\mathbb{Z}\pi)$.*

Remark. For π a space form group the groups $C^h_r(\mathbb{Z}\pi)$ have been completely computed by Taylor and Williams in [T-W2]. Thus this conclusion of the theorem reduces to purely algebraic questions in K and L-theory.

Proof of 7.7. Equivalently we prove that a finite (π, Σ)-complex Σ'/π has the homotopy type of a closed manifold iff $\tau_r(\chi^{\mathrm{tor}}(\Sigma)\cdot k_{\Sigma'/\pi})=0$. Construct a BG-map $g\colon S^r/\pi \to \Sigma'/\pi$ inducing the identity on π_1. Apply 7.5 (with $a=k_{\Sigma'/\pi}^{-1}\in \mathbf{Z}/|\pi|^\times$ and theorem A to get a BG-map $h\colon T/\pi \to \Sigma'/\pi$ such that degree $h\equiv 1 (\operatorname{mod}|\pi|)$ and $\chi^{\mathrm{tor}}(T)=k_{\Sigma'/\pi}^{-1}\in \mathbf{Z}/|\pi|$. Then, as in the proof of 6.4,

$$f\colon T/\pi \# \bar{\Sigma} \# \ldots \# \bar{\Sigma} \to \Sigma'/\pi$$

gives a degree one BG-map with $q_\tau(f)=\chi^{\mathrm{tor}}(T)\cdot\chi^{\mathrm{tor}}(\Sigma)^{-1}=(\chi^{\mathrm{tor}}(\Sigma)\cdot k_{\Sigma'/\pi})^{-1}$. The result follows from 5.2 and 2.2.

Fix now a polarized homotopy type of a (π, Σ)-complex (resp. (π, S^r)-complex) which contains a quotient of a free π-action on Σ (resp. S^r). Our final classification result is:

Proposition 7.8. *For $r \neq 3$, free homologically trivial π-actions on Σ within the fixed polarized homotopy type are in one-to-one correspondence with such actions on S^r within its fixed polarized homotopy type.*

In our case this is essentially an exercise in the surgery exact sequence. It also follows from a general property of propagations proven in [C-W1].

Examples

(*A*) *Cyclic groups.* In this case all homotopy types of (π, S^r)-complexes are realized as lens spaces. If in addition $|\pi|$ is odd, fake lens spaces are classified by their Reidemeister torsion and ρ-invariant [Wa2]. Thus all homotopy types of (π, Σ)-complexes contain the quotient space of a free π-action on Σ. If $|\pi|$ is odd, free, homologically trivial π-actions on Σ are classified by Reidemeister torsions and ρ-invariants.

(*B*) *Generalized quaterion groups $Q(2^n)$.* By [F-K-W], $\tau(a)=0$ if and only if $a\equiv \pm 1 (\operatorname{mod} 8)$. Furthermore, any homotopy type of (π, S^r)-complex contains a linear space form provided the k-invariant is congruent to $\pm 1 (\operatorname{mod} 8)$. Thus τ_r is the trivial map. A (π, Σ)-complex contains the homotopy type of a quotient space of a free π-action on Σ if and only it contains the homotopy type of a finite complex.

(*C*) *Binary diherdral groups $Q(4p)$, p prime.* Computations show that τ is the zero map and $\tau_r(a)$ is zero if and only if a is a quadratic residue mod p (cf. *D* below). Thus every (π, Σ)-complex has the homotopy type of a finite complex. One-half of the homotopy types contain closed manifolds.

(*D*) *Metacyclic groups.* Using different techniques, Hambleton and Madsen [H-M] did extensive computations of the possible homotopy types of free π-actions on S^r for π metacyclic. Via 7.7, their results can be interpreted as computations of the maps τ and τ_r, and thus one can read off results on classification of free, homologically trivial π-actions on Σ.

(*E*) *General groups.* If $a\equiv b^2 (\operatorname{mod}|\pi|)$ where $\tau(b)=0$, then $\tau_r(a)=0$ (essentially by definition of the quadratic torsion). Since $\operatorname{im}\tau$ is relatively small (for example its exponent divides the Artin exponent of π) this provides many homotopy types of free homologically trivial π-actions. In particular this result can be used in many different cases to deduce the existence of manifolds whose universal cover is S^r which do not have the homotopy type of linear spherical space forms.

More generally, the results of [D2] show that if $a \equiv b^2 (\mathrm{mod}\,|\pi|)$, then $\tau_r(a)$ is in the image of $\tau(b)$ under the map $H^{r+1}(\tilde{K}_0(\mathbf{Z}\pi)) \to L_r^h(\mathbf{Z}\pi)$ in the Ranicki and Rothenberg $L^h - L^p$ exact sequence.

§ 8. Further results and applications

In this section we consider extensions and refinements of our main theorem and then give applications of the techniques developed here to certain other problems in the theory of transformation groups.

Theorem 8.1. *If π acts freely and homologically trivially on S^r $(r \neq 3)$, then π acts freely and homologically trivially on Σ in such a way that all elements are isotopic to the identity.*

Proof. We will show that the π-action constructed in the earlier sections automatically satisfies our conclusion. Since Σ is simply-connected it suffices to show that the action is pseudoisotopic to the identity because of Cerf's well known results in pseudoisotopy theory. The argument then follows Sullivan's proof that homotopy implies pseudoisotopy for homeomorphisms complex projective spaces [Su1].

Let M be a closed n-manifold with $n \geqq 5$ and $H(M)$ the group of homemorphisms of M homotopic to the identity modulo those pseudoisotopic to the identity. The structure set $\mathcal{S}(M \times I, \partial)$ can be interpreted, by the s-cobordism theorem, as the set of homeomorphisms with given homotopies to the identity modulo those which are homotopic (rel ∂) to a pseudoisotopy. Now let M be our simply-connected $\mathbf{Z}/|\pi|$ homology sphere Σ. The surgery exact sequence

$$\ldots \to L^s_{r+2}(\mathbf{Z}) - \mathcal{S}(M \times I, \partial) \to [\textstyle\sum M, G/TOP] \to L^s_{r+1}(\mathbf{Z}) \to \ldots$$

shows that $\mathcal{S}(M \times I, \partial)$ is a finite set whose cardinality is prime to that of π. This set $\mathcal{S}(M \times I, \partial)$ is a group by composition and the forgetful map

$$\mathcal{S}(M \times I, \partial) \to H(M)$$

in an epimorphism. Thus $H(M)$ is a finite group of order prime to $|\pi|$.

Note that the π action constructed on $M (= \Sigma)$ is, in fact, homotopic to the identity (this is true because the action on S^r has this property and the pullback diagrams used in propagation preserve this). This gives a homeomorphism $\pi \to H(M)$ which is necessarily trivial, hence the π-action is pseudoisotopic to the identity.

Remark 8.2. Although we have made statements of the type "the π-action is isotopic to the identity", all we mean is that the individual elements of the group are. No compatibility of the isotopy with the group action is asserted. In fact, for effective group actions it is easily seen that this is never possible.

We now turn to non-free actions and show the existence of *PL* or topologically-locally-linear actions on Σ quite generally. A sample result is:

Theorem 8.3. *Let $\rho: \pi \to SO(r+1)$ be a faithful representation with a fixed set (on S^r) of dimension at least 1 and codimension greater than 2. Then Σ admits a PL π-action such that $\Sigma^H \sim (S^r)^H$ for all $1 \neq H \subset \pi$ if and only if*

$$\sum_{i=1}^{r-1} \sigma(H_i(\Sigma)) = 0 \in \tilde{K}_0(\mathbb{Z}\pi).$$

In general, Σ always admits a topologically-locally-linear action with representation at a fixed point $\rho - 1$.

Proof. We first do the easier *PL* case. Puncture that problem by removing an invariant open ball around a fixed point on S^r, to obtain a G-disk D. Note that the singular set $U_{1 \neq H \subset \pi} D^H$ is contractible by a Mayer-Vietoris argument. (Remember: the action is linear.) Consider the problem of extending an action from the boundary of a regular neighborhood of the singular set to $D \# \Sigma$. This can be done if and only if the $\tilde{K}_0$ obstruction vanishes (by extension across homology collars [We1] or [A-V]). Once one has the action on $D \# \Sigma$, coning the boundary produces an action on Σ. Moreover, this argument did not depend on the normal structure around the singular set, so the $\tilde{K}_0$ condition is necessary for any *PL* action with that singular set. (Note: the action constructed above need not be *PL* locally-linear. This is a much harder problem, see [We3] and the papers cited there for more discussion.)

For the topological case, rather than removing an open disk and coning, one removes a fixed point and one-point compactifies. The details of such an argument, such as topological-local-linearity and the construction of an extension, are done in the semifree case in [We3, 6C].

Remark 8.4. Note that the proof works in greater generality than stated; for example, the action of π on S^r need not be linear, rather one could require that $U_{1 \neq H \subset \pi} D^H$ be mod $|\pi|$ acyclic.

Remark 8.5. If $\mathrm{Fix}(\rho)$ is 0-dimensional then the result is *PL* correct and topologically false – the $\tilde{K}_0$-condition is necessary at least for topological semifree actions. (The general case presumably will depend on forthcoming work of Steinberger and West on equivariant topological projective class and Whitehead groups.) If $\mathrm{Fix}(\rho)$ is empty, then the existence problem is quite difficult the major result known presently being the main result of this paper.

One can also vary the singular set using other techniques. Rather than discuss complicated singular data we record the semifree case for comparison:

Theorem 8.6. ([We3]). *Let $n > 0$, $k > 2$. Let $\Sigma_1^n \subset \Sigma_2^{n+k}$ be an inclusion of one $\mathbb{Z}_{(\pi)}$-homology sphere in another simply-connected one. Let $\rho: \pi \to SO(k)$ be a free representation. Then there is a topologically-locally-linear semifree π-action on Σ_2 with fixed set Σ_1 and normal representation ρ at any fixed point. There is a*

PL action iff

$$\sum(-1)^i \sigma(H_i(\Sigma_2 - pt, \Sigma_1 - pt)) = 0 \in \tilde{K}_0(\mathbb{Z}\pi).$$

Remark 8.7. The $n=0$ case depends on algebraic invariants. There is a *PL* action if $\sum(-1)^i \sigma(H_i(\Sigma_2 - pt)) = 0 \in \tilde{K}_0(\mathbb{Z}\pi)$. This is also necessary for a topological-locally-linear action. At least for Σ_2 with trivial Spivak bundle, a further necessary condition which together with the $\tilde{K}_0$-obstruction is sufficient is that the image of

$$\sum_{0<i</2} (-1)^i \sigma(H_i(\Sigma_2)) \in H^{n+1}(\tilde{K}_0(\mathbb{Z}\pi))$$

in $L_n^h(\mathbb{Z}\pi)/L_n^h(\mathbb{Z})$ vanishes.

Our final remark concerns the hypothesis of theorem A.

Remark 8.8. According to Milgram's computations of the Swan finiteness obstruction [M] the group $\pi = Q(24, 5, 1)$ (or in the notation adopted in his paper $Q(12, 5, 1)$) does not act freely on S^{11} (or even on a finite complex homotopic to S^{11}). According to Pardon [P1], this group acts freely on a simply-connected mod $|\pi|$ homology sphere Σ. Thus in our main theorem, homological triviality is necessary.

According to computations of S. Bentzen [Bn], the group $\pi = Q(24,313,1)$ *acts freely on* S^{11}, but $\pi \times \mathbb{Z}_7$ cannot act freely on any finite complex homotopic to S^{11}.In particular, π cannot act freely on $\Sigma = S^{11}/\mathbb{Z}_7$. Thus simple-connectivity is also necessary.

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