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Positive scalar curvature and a new index theory for noncompact manifolds ☆

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Dedicated to Alain Connes with great admiration

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ABSTRACT

In this article, we develop a new index theory for noncompact manifolds endowed with an admissible exhaustion by compact sets. This index theory allows us to provide examples of noncompact manifolds with exotic positive scalar curvature phenomena.

1. Introduction

If M is an n-dimensional manifold endowed with a Riemannian metric g, then its scalar curvature $\kappa : M \to \mathbb{R}$ satisfies the property that, at each point $p \in M$, there is an expansion

$$\mathrm{Vol}_M(B_\varepsilon(p)) = \mathrm{Vol}_{\mathbb{R}^n}(B_\varepsilon(0)) \left(1 - \frac{\kappa(p)}{6(n+2)}\,\varepsilon^2 + \cdots\right)$$

for all sufficiently small $\varepsilon > 0$. A complete Riemannian metric g on a manifold M is said to have uniformly positive scalar curvature if there is a fixed constant $\kappa_0 > 0$ such that $\kappa(p) \geq \kappa_0 > 0$ for all $p \in M$. For compact manifolds, obstructions to such metrics are largely achieved in one of two ways: (1) the minimal hypersurface techniques in dimensions at most 7 by Schoen–Yau [40] and in dimension 8 by Joachim and Schick [24]; (2) the Dirac index method for spin manifolds by Atiyah–Singer and its generalizations by Connes–Moscovici, Hitchin, Gromov, Lawson, Roe and Rosenberg, among others.

In the realm of noncompact manifolds it is now well recognized that the original approach by Gromov–Lawson [19] and Schoen–Yau [40], which proves that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature, is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds. Connes and Moscovici [10] develop a higher index theory that proves that any aspherical manifold whose fundamental group is hyperbolic does not have a metric of positive scalar curvature. Roe [33] subsequently introduces

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a coarse index theory to study positive scalar curvature problems for noncompact manifolds. Block and Weinberger [4] investigate the problem of complete metrics for noncompact symmetric spaces when no quasi-isometry conditions are imposed. They prove that, if G is a semisimple Lie group with maximal compact subgroup K and irreducible lattice Γ, then the double quotient $M = \Gamma \backslash G/K$ can be endowed with a complete metric of uniformly positive scalar curvature if and only if Γ is an arithmetic group with $\text{rank}_{\mathbb{Q}} \Gamma \geq 3$. This theorem includes, in light of the work of Borel and Harish-Chandra [5], previous results of Gromov–Lawson [19] in rational rank 0 and 1. In the case when the rational rank exceeds 2, Chang proves that any metric on M with uniformly positive scalar curvature fails to be coarsely equivalent to the natural one [6].

The Gromov–Lawson–Rosenberg conjecture states that a closed spin manifold M^n with $n \geq 5$ has a metric of positive scalar curvature if, and only if, its Dirac index vanishes in $KO_*(C_r^*\pi)$, where $\pi = \pi_1(M)$. While this conjecture is known to be false in general, it has been verified in number of cases. To study compact manifolds $(M, \partial M)$ with boundary with respect to a positive scalar curvature metric that is collared at the boundary, one would ideally like to produce a C^*-algebra that encodes information about both $\pi_1(M)$ and $\pi_1(\partial M)$. In this paper we show that such an algebra can be constructed with the appropriate properties, and apply it to obtain information about noncompact manifolds.

In the first section, we use the notion of localization algebras [47] and generalized asymptotic morphisms to define a relative group C^*-algebra $C^*_{max}(\pi_1(M), \pi_1(\partial M))$ along with a homomorphism

$$\mu_{max}: KO_*(M, \partial M) \to KO_*(C^*_{max}(\pi_1(M), \pi_1(\partial M)))$$

which we call the maximal relative Baum–Connes map. The usual Baum–Connes conjecture has many different guises, the simplest of which is that the Baum–Connes map $KO_*^{\Gamma}(\underline{E}\Gamma) \to KO_*(C_r^*\Gamma)$ is an isomorphism. The classical Strong Novikov conjecture states that the Baum–Connes map is injective. One may similarly hope that the map μ_{max} above is an injection if M and ∂M are both aspherical. In line with the compact case, we show that, if M has a metric of positive scalar curvature that is collared near the boundary, then the relative index of the Dirac operator in $KO_*(M, \partial M)$ belongs to the kernel of μ_{max}. In this section, we also formulate a relative Gromov–Lawson–Rosenberg conjecture for manifolds with boundary, which is a converse to the above statement. We prove that the relative Gromov–Lawson–Rosenberg conjecture holds for torsion-free amenable groups satisfying certain conditions on their cohomological dimensions.

In the next sections, we offer a new index theory for noncompact manifolds with so-called *admissible exhaustions*. We combine this theory with the machinery built in the first part of the paper to give various geometric applications: we first construct a noncompact manifold M with an exhaustion $\bigcup_{i=1}^{\infty}(M_i, \partial M_i)$ by compact submanifolds (of codimension 0) with boundary such that each $(M_i, \partial M_i)$ has a metric of positive scalar curvature collared at the boundary, but M itself has no metric of uniformly positive scalar curvature. Next, we construct a noncompact manifold N whose space $PS(N)$ of uniformly positive scalar curvature metrics has uncountably many connected components.

A companion paper [7] will use the techniques of this paper and more complicated topology to obtain a contractible manifold that has a positively curved exhaustion, but no metric of positive scalar curvature.

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2. The relative group C*-algebra and the relative Gromov–Lawson–Rosenberg conjecture

In this section, we introduce the concept of relative group C^*-algebras and formulate a relative version of the Gromov–Lawson–Rosenberg conjecture. The K-theory of the relative group C^*-algebras serves as the receptacle of the relative higher index of the Dirac operators.

In this paper all C^*-algebras are real. We deal only with metric spaces X that are locally compact and uniformly metrically locally simply connected; i.e. for all $\varepsilon > 0$ there is $\varepsilon' \leq \varepsilon$ such that every ball in X of radius ε' is simply connected.

If G is a discrete group, denote by $C_r^*(G)$ and $C^*_{max}(G)$ the usual reduced and maximal real C^*-algebras of G, respectively. Let $Y \subseteq X$ both be compact (metric) spaces. We wish to define a Baum–Connes map from the relative KO-homology group $KO_*^{lf}(X, Y)$ to the KO-theory of some relative C^*-algebra encoding the fundamental groups of both Y and X and the homomorphism between them. Here we assume that both X and Y are path connected. Let $\phi: C^*_{max}(\pi_1(Y)) \to C^*_{max}(\pi_1(X))$ be the map induced by the homomorphism $j_*: \pi_1(Y) \to \pi_1(X)$. Consider the mapping cone C^*-algebra of ϕ given by

$$C_{\phi,max} = \{(a, f): f \in C_0([0, 1), C^*_{max}(\pi_1(X))),\ \ a \in C^*_{max}(\pi_1(Y)), f(0) = \phi(a)\}.$$

Define $C^*_{max}(\pi_1(X), \pi_1(Y))$ to be the seventh suspension $S^7 C_{\phi,max}$ of $C_{\phi,max}$, i.e. $C_{\phi,max} \otimes C_0(\mathbb{R}^7)$, where $C_0(\mathbb{R}^7)$ is the C^*-algebra of continuous real-valued functions on $\mathbb{R}^7$ which vanish at infinity. The seventh suspension is chosen because KO-theory is eight-periodic. We call this algebra the *maximal relative group C^*-algebra of* $(\pi_1(X), \pi_1(Y))$. If in fact the homomorphism j_* is an injection, we can define a *reduced relative C^*-algebra* $C^*_{red}(\pi_1(X), \pi_1(Y))$ in the same way.

If M is a metric space, we say that a Hilbert space H is an *M-module* if there is a representation of the continuous functions $C_0(M)$ in H, that is, a C^*-homomorphism $C_0(M) \to B(H)$, the algebra of bounded operators on H. We will say that an operator $T : H \to H$ is *locally compact* if, for all $\varphi \in C_0(M)$, the operators $T\varphi$ and φT are compact on H. We define the *support*, $\mathrm{Supp}(\varphi)$, of $\varphi \in H$ as the complement of the largest open subset $U \subseteq M$ such that, if $f \in C_0(M)$ and f is supported on U, then $f\varphi = 0$. An operator $T: H \to H$ on an M-module H has *finite propagation* if there is $R > 0$ such that $\varphi T \psi = 0$ whenever $\varphi, \psi \in C_0(M)$ satisfy $d(\mathrm{Supp}(\varphi), \mathrm{Supp}(\psi)) > R$. The smallest such R is called the propagation of T, denoted by $\mathrm{prop}(T)$.

Recall that a locally compact metric space Z is said to have *bounded geometry* if there is a discrete subset $Y \subseteq Z$ such that (1) Y is c-dense for some $c \geq 0$, i.e. $d(z, Y) \leq c$ for all $z \in Z$; (2) for all $r > 0$ there is N in the natural numbers such that, for all $p \in Y$, we have $\#\{y \in Y: d(y, p) \leq r\} \leq N$. In the remainder of the article, we assume that all spaces have bounded geometry.

Definition 2.1. Let Z be a locally compact metric space. Let H be a Hilbert space and $B(H)$ the algebra of bounded operators on H.

(1) Denote by $\mathbb{R}(Z)$ the Roe algebra, i.e. the algebra of locally compact, finite propagation operators on some ample Z-module H. Here a Z-module is called ample if $\rho(f)$ is not a compact operator for any non-zero $f \in C_0(Z)$, where $\rho: C_0(Z) \to B(H)$ is the $*$-homomorphism in the definition of Z-module H (see Roe [33, Definition 4.5]).
(2) Denote by $C^*_{red}(Z)$ and $C^*_{max}(Z)$ the completions of $\mathbb{R}(Z)$ with respect to the reduced and maximal norm completions, respectively. Here we define the maximal norm in the following way. If $a \in \mathbb{R}(Z)$, then let $\|a\|_{max} = \sup_\psi \|\psi(a)\|$, where the supremum is taking over all $*$-homomorphisms $\psi: \mathbb{R}(Z) \to B(W)$, where W is real Hilbert space. By the bounded geometry assumption, the quantity $\|a\|_{max}$ is finite by Gong–Wang–Yu [17, Lemma 3.4]. Note that, if Z is compact, then the two completions are the same and coincide with $\mathcal{K}$, the C^*-algebra of compact operators, as $\mathbb{R}(Z)$ is already all of $\mathcal{K}$.
(3) Let $\pi_1(Z)$ act on $\widetilde{Z}$ by deck transformations and let $\mathbb{R}(\widetilde{Z})^{\pi_1(Z)}$ be the algebra of operators in $\mathbb{R}(\widetilde{Z})$ that are invariant under this action. We endow $\mathbb{R}(\widetilde{Z})^{\pi_1(Z)}$ with a maximal norm by defining $\|a\|_{max} = \sup_\psi \|\psi(a)\|$, where the supremum is taken over all $*$-homomorphisms $\psi: \mathbb{R}(\widetilde{Z})^{\pi_1(Z)} \to B(H)$, where H is a Hilbert space. Note that, although $\mathbb{R}(\widetilde{Z})^{\pi_1(Z)}$ is a subalgebra of $\mathbb{R}(\widetilde{Z})$, this maximal norm might be different from the one defined in (2) because the domain of ψ is different. We also mention that we are not assuming that the group $\pi_1(Z)$ is acting on the Hilbert space H and the algebras defined here are independent of the choice of the base point in the fundamental group (up to an isomorphism).
(4) Denote by $C^*_{red}(\widetilde{Z})^{\pi_1(Z)}$ and $C^*_{max}(\widetilde{Z})^{\pi_1(Z)}$ the closure of the algebra $\mathbb{R}(\widetilde{Z})^{\pi_1(Z)}$ with respect to the reduced and maximal norms, respectively. Here the maximal norm is taken as in (3).

Definition 2.2. For continuous bounded maps $g: [0, \infty) \to \mathbb{R}(Z)$, we define norms $\|g\|_{red} = \sup_{t \in [0,\infty)} \|g(t)\|_{red}$ and $\|g\|_{max} = \sup_{t \in [0,\infty)} \|g(t)\|_{max}$. Suppose that

(a) g is uniformly bounded and uniformly continuous, and
(b) the propagation of $g(t)$ tends to 0 as $t \to \infty$.

We define the following sets:

(1) Denote by $\mathbb{R}_L(Z)$ the collection of maps g satisfying (a) and (b).
(2) Denote by $C^*_{L,red}(Z)$ the closure of $\mathbb{R}_L(Z)$ with respect to $\|\cdot\|_{red}$, called the *reduced localization algebra of* X.
(3) Denote by $C^*_{L,max}(Z)$ the closure of $\mathbb{R}_L(Z)$ with respect to $\|\cdot\|_{max}$, called the *maximal localization algebra of* X. Here the maximal norm is taken as in (2) in the previous definition.
(4) Denote by $C^*_{L,red}(\widetilde{Z})^{\pi_1(Z)}$ and $C^*_{L,max}(\widetilde{Z})^{\pi_1(Z)}$ the closure of the algebra $\mathbb{R}_L(\widetilde{Z})^{\pi_1(Z)}$ with respect to the reduced and maximal norms, respectively. Here the maximal norm is taken as in (3) in the previous definition.

Remark 2.3. When Z is compact, then the two localization algebras in (2) and (3) coincide.

For the rest of this paper, we will simplify notation and simply write $C^*_L(Z)$ for either the reduced or maximal localization algebra.

Let X be a locally compact metric space. We shall briefly recall the local index map

$$\mathrm{ind}_L: KO^{lf}_*(X) \to KO_*(C^*_L(X)),$$

first introduced by Yu in [47]. We assume that $* \equiv 0 \bmod 8$. The other cases can be handled in a similar way with the help of suspensions. Here $KO^{lf}_*(X) \equiv KO^*(C_0(X))$.

Let (H, F) represent a cycle for $KO^{lf}_0(X)$, where H is a standard nondegenerate X-module and F is a bounded operator acting on H such that $F^*F - I$ and $FF^* - I$ are locally compact, and $\phi F - F\phi$ is compact for all $\phi \in C_0(X)$. For each positive integer n, let $\{U_{n,i}\}_i$ be a locally finite and uniformly bounded open cover of X such that $\mathrm{diam}(U_{n,i}) < \frac{1}{n}$. Let $\{\phi_{n,i}\}_i$ be a continuous partition of unity subordinate to the open cover. Define

$$F(t) = \sum_i ((n-t)\phi_{n,i}^{\frac{1}{2}} F \phi_{n,i}^{\frac{1}{2}} + (t-(n-1))\phi_{n+1,i}^{\frac{1}{2}} F \phi_{n+1,i}^{\frac{1}{2}})$$

for all positive integers n and $t \in [n-1, n]$, where the infinite sum converges in the strong topology. If prop denotes the propagation of an operator, then notice that $\mathrm{prop}(F(t)) \to 0$ as $t \to \infty$.

Observe that $F(t)$ is a multiplier of the localization algebra $C_L^*(X)$ and is invertible modulo the localization algebra. Hence the standard index construction in K-theory gives

$$\mathrm{ind}_L([(H,F)]) = [P_F] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in KO_0(C_L^*(X)),$$

where P_F is a certain idempotent in the matrix algebra of $C_L^*(X)^+$ constructed as follows. We call this class $\mathrm{ind}_L([(H,F)])$ the *local index of* F. We choose $P_F(t)$ to be the matrix

$$\begin{pmatrix} F(t)F^*(t) + (1-F(t)F^*(t))F(t)F^*(t) & F(t)(1-F^*(t)F(t)) + (1-F(t)F^*(t))F(t)(1-F^*(t)F(t)) \\ (1-F^*(t)F(t))F^*(t) & (1-F^*(t)F(t))^2 \end{pmatrix}.$$

See also Definition 4.2, page 1392, in Willett–Yu [45]. We write P_F for $P_F(t)$ for simplicity. For the rest of this paper, we also abbreviate $[(H,F)]$ as $[F]$ and $\mathrm{ind}_L[(H,F)]$ as $\mathrm{ind}_L[F]$.

The following isomorphism is proved in Yu [47, Theorem 3.2] in the case when X is a CW complex and for general metric space X in Qiao–Roe [30, Theorem 3.4].

Proposition 2.4. *The local index map* $\mathrm{ind}_L\colon KO_*(X) \to KO_*(C_L^*(X))$ *is an isomorphism.*

Definition 2.5. Let $Y \subseteq X$ be compact metric spaces. In the definitions of $C_L^*(Y)$ and $C_L^*(X)$, we choose the Y-module and X-module to be $\ell^2(Z_Y)\otimes H$ and $\ell^2(Z_X)\otimes H$ such that $Z_Y \subseteq Z_X$ are countable dense subsets of Y and X, respectively, and H is a separable and infinite-dimensional Hilbert space. The isometric inclusion from $\ell^2(Z_Y)\otimes H$ to $\ell^2(Z_X)\otimes H$ induces a homomorphism $i\colon C_L^*(Y) \to C_L^*(X)$.

Remark 2.6. The choices of X-module and Y-module are not canonical. However i induces a canonical KO-theory homomorphism.

Let C_i be the mapping cone of i given by

$$C_i = \{(a,f)\colon f \in C_0([0,1), C_L^*(X)),\ a \in C_L^*(Y),\ f(0) = i(a)\}.$$

Define the *relative KO-homology group of* (X,Y) to be $KO_*(X,Y) \equiv KO_*(S^7C_i)$.

This definition of relative KO-homology gives rise to a long exact pair sequence

$$\cdots \to KO_*(Y) \to KO_*(X) \to KO_*(X,Y) \to \cdots.$$

Lemma 2.7. *Let X be a compact space and let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space. Then there are isomorphisms*

$$C^*_{red}(\widetilde{X})^{\pi_1(X)} \cong C_r^*(\pi_1(X)) \otimes \mathcal{K}$$

and

$$C^*_{max}(\widetilde{X})^{\pi_1(X)} \cong C^*_{max}(\pi_1(X)) \otimes \mathcal{K}.$$

Proof. In Roe [34, Lemma 2.3] the $*$-isomorphism

$$(\mathbb{R}\widetilde{X})^{\pi_1(X)} \cong (\mathbb{R}\pi_1(X)) \otimes \mathcal{K}$$

is proved. This algebraic $*$-isomorphism extends to the required $*$-isomorphism in both the reduced and maximal case, since $\mathcal{K}$ is a nuclear C^*-algebra. □

Proposition 2.8. *Let X be a compact metric space with universal cover $\widetilde{X}$. There is $\varepsilon > 0$ depending only on X such that, if b is an operator in $\mathbb{R}(X)$ with propagation at most ε, then b lifts to a $\pi_1(X)$-invariant operator $\widetilde{b}$ of propagation at most ε in $\mathbb{R}(\widetilde{X})$ and the lifting is unique.*

Proof. In the definition of $\mathbb{R}(X)$, we choose the X-module to be $\ell^2(Z_X)\otimes H$ such that Z_X is a countable dense subset of X and H is a separable and infinite-dimensional Hilbert space. Let $p\colon \widetilde{X} \to X$ be the projection map. We define $Z_{\widetilde{X}} = p^{-1}(Z_X)$. We choose the $\widetilde{X}$-module to be $\ell^2(Z_{\widetilde{X}})\otimes H$ in the definition of $\mathbb{R}(\widetilde{X})$. Every operator $b \in \mathbb{R}(X)$ can be represented by a kernel $k(\cdot,\cdot)$ such that $k(x,y)$ belongs to $\mathcal{K}$ for all $(x,y) \in Z_X \times Z_X$ and $\mathrm{Supp}(k)$ is contained in $\{(x,y) \in X \times X\colon d(x,y) < r\}$ for some $r > 0$. The smallest such r is the propagation of b. Now let $k'(x',y') = k(p(x'),p(y'))$ for all $(x',y') \in Z_{\widetilde{X}} \times Z_{\widetilde{X}}$ satisfying $d(x',y') < r$ and $k'(x',y') = 0$ for all $(x',y') \in Z_{\widetilde{X}} \times Z_{\widetilde{X}}$ satisfying $d(x',y') \geq r$. By the compactness of X, there is $\varepsilon > 0$ such that, if b has propagation at most ε, then k' represents an element $\widetilde{b}$ of $\mathbb{R}(\widetilde{X})$ and $\widetilde{b}$ has the same propagation as b.

This discussion shows that there exists $\varepsilon > 0$ such that, if $b \in \mathbb{R}(X)$ and $\mathrm{prop}(b) < \varepsilon$, then there is a unique lifting of b in $\mathbb{R}(X)$ to $\phi(b)$ in $\mathbb{R}(\widetilde{X})$. □

Note that, if the propagations $\text{prop}(b_1), \text{prop}(b_2) < \varepsilon/2$, then this lifting respects multiplication and addition, i.e. $\phi(b_1b_2) = \phi(b_1)\phi(b_2)$ and $\phi(b_1 + b_2) = \phi(b_1) + \phi(b_2)$.

Definition 2.9. Let $s \in [0, \infty)$ and let X be a compact metric space. For all $b \in \mathbb{R}_L(X)$, denote by $b_s \in \mathbb{R}_L(X)$ the operator given by $b_s(t) = b(s + t)$ for all $t \in [0, \infty)$. Let ε be as in the above proposition. For each $b \in \mathbb{R}_L(X)$, there is $s_b > 0$ such that $\text{prop}(b_s) < \varepsilon$ when $s > s_b$. We define $\phi_s(b) = \widetilde{b_s} \in \mathbb{R}_L(\widetilde{X})^{\pi_1(X)}$ when $s > s_b$.

The next result indicates that ϕ_s is an asymptotic morphism in the following generalized sense.

Lemma 2.10. *Let X be a compact metric space. For all $b \in \mathbb{R}_L(X)$, let s_b be given as in the previous definition.*

(1) *There is $C > 0$ such that, for all $b \in \mathbb{R}_L(X)$, if $s > s_b$, then*

$$\|\phi_s(b)\|_{red} \le C\|b\|_{red} \quad \text{and} \quad \|\phi_s(b)\|_{max} \le C\|b\|_{max}.$$

(2) *For all $b \in \mathbb{R}_L(X)$, if $s > s_b$, then $\phi_s(b)^* = \phi_s(b^*)$.*

(3) *For all $b_1, b_2 \in \mathbb{R}_L(X)$, the operator*

$$\phi_s(b_1b_2) - \phi_s(b_1)\phi_s(b_2)$$

is zero for s bigger than a constant depending on b_1 and b_2.

Proof. Let $\{U_i\}_{i=1}^N$ be a finite open cover of X such that, for each i, the diameter of the union of all U_j satisfying $U_j \cap U_i \neq \emptyset$ is less than ε, where ε is as in Proposition 2.8. Let $\{\varphi_i\}_i$ be the continuous partition of unity subordinate to $\{U_i\}$. We have $\phi_s(b) = \sum_{i=1}^N \phi_s(\varphi_i b)$.

In the reduced case, by the definition of ϕ_s and the choice of φ_i, we have

$$\|\phi_s(\varphi_i b)\|_{red} = \|\varphi_i b\|_{red} \le \|b\|_{red}.$$

It follows that

$$\|\phi_s(b)\|_{red} \le N\|b\|_{red}$$

if $s > s_b$.

In the maximal case, we have the following natural $*$-isomorphism:

$$C^*(\widetilde{X})^{\pi_1(X)}_{max} \cong K \otimes C^*_{max}(\pi_1(X)),$$

where K is the C^*-algebra of all compact operators on $L^2(D)$ for some fundamental domain of $\widetilde{X}$. In the above isomorphism, $\phi_s(\varphi_i b)$ corresponds to $k \otimes 1$ for $k \in K$, where 1 is the identity element in the maximal group C^*-algebra $C^*_{max}(\pi_1(X))$. As a consequence, we have

$$\|\phi_s(\varphi_i b)\|_{max} = \|\varphi_i b\|_{max} \le \|b\|_{max}.$$

It follows that

$$\|\phi_s(b)\|_{max} \le N\|b\|_{max}$$

if $s > s_b$.

This proves (1). The proofs of (2) and (3) are straightforward. □

For any $\pi_1(X)$-invariant operator $a \in \mathbb{R}_L(\widetilde{X})^{\pi_1(X)}$, if the propagation of a is sufficiently small, then there exists a unique $b \in \mathbb{R}_L(X)$ such that $a = \tilde{b}$, where $\tilde{b}$ is as in Proposition 2.8. The map $\psi : a \to b$, gives a pushdown $\mathbb{R}_L(\widetilde{X})^{\pi_1(X)} \to \mathbb{R}_L(X)$ for operators with small propagation. Such a pushdown induces homomorphisms

$$KO_*(C^*_{L,max}(\widetilde{X})^{\pi_1(X)}) \to KO_*(C^*_L(X))$$

and

$$KO_*(C^*_{L,red}(\widetilde{X})^{\pi_1(X)}) \to KO_*(C^*_L(X)),$$

which are inverses to the homomorphisms induced by the liftings. These homomorphisms can be defined as follows. By an argument similar to the proof of Lemma 2.10, there exists a constant $c \ge 1$ such that $\|\psi(a)\| \le c\|a\|$ if the propagation of a is sufficiently small. For simplicity, we only describe the homomorphisms for KO_0. By an approximation, each element in $KO_*(C^*_L(\widetilde{X})^{\pi_1(X)})$ can be represented by a quasi-projection $q \in \mathbb{R}^*_L(\widetilde{X})^{\pi_1(X)}$ satisfying $q^* = q$ and $\|q^2 - q\| < \frac{1}{10c}$. Let $q_s \in \mathbb{R}^*_L(\widetilde{X})^{\pi_1(X)}$ be defined by $q_s(t) = q(t + s)$ for any non-negative number s. We choose s to be large enough so that q_s has sufficiently small propagation. We now define the homomorphism

$$KO_0(C^*_L(\widetilde{X})^{\pi_1(X)}) \to KO_*(C^*_L(X))$$

by mapping $[q]$ to $[\psi(q_s)]$, where ψ is the pushdown map and $[\psi(q_s)]$ is the K-theory element represented by the quasi-projection $\psi(q_s)$. Observe that $\psi(q_s)$ is a quasi-projection since the pushdown map ψ is norm-decreasing (up to the constant c) for operators with small propagations.

Lemma 2.10 implies that the liftings ϕ_s induce isomorphisms $KO_*(C_L^*(X)) \to KO_*(C_{L,max}^*(\widetilde{X})^{\pi_1(X)})$ and $KO_*(C_L^*(X)) \to KO_*(C_{L,red}^*(\widetilde{X})^{\pi_1(X)})$.

Definition 2.11. Let $j_*: \pi_1(Y) \to \pi_1(X)$ be the homomorphism induced by the inclusion $Y \to X$. Then j_* induces a map $\eta: \widetilde{Y} \to \widetilde{X}$ such that $\eta(gy) = i_*(g)\eta(y)$ for all $g \in \pi_1(Y)$ and $y \in \widetilde{Y}$.

Note that such η exists because X and Y are metrically locally simply connected.

Let p be the covering map $\widetilde{X} \to X$ and let $Y' = p^{-1}(Y)$. Let $p': \widetilde{Y} \to Y$ be the covering map from the universal cover $\widetilde{Y}$. Let Y'' be the Galois covering of Y corresponding to the subgroup $\ker(j_*)$ of $\pi_1(Y)$. The deck transformation group of Y'' is $\pi_1(Y)/\ker(j_*)$, isomorphic to $j_*\pi_1(Y)$. For simplicity, we will denote $\pi_1(Y)/\ker(j_*)$ by $j_*\pi_1(Y)$.

We have $Y' = \pi_1(X) \times_{j_*\pi_1(Y)} Y''$. This decomposition gives rise to a natural $*$-homomorphism

$$\psi': C_{max}^*(Y'')^{j_*\pi_1(Y)} \to C_{max}^*(Y')^{\pi_1(X)}.$$

Choose countable dense subsets Z_Y of Y and Z_X of X such that $Z_Y \subseteq Z_X$. Let H be a separable and infinite-dimensional Hilbert space. We use the modules $\ell^2(p^{-1}(Z_Y))\otimes H$, $\ell^2(p^{-1}(Z_X))\otimes H$, $\ell^2((p')^{-1}(Z_Y))\otimes H$, and $\ell^2((p'')^{-1}(Z_Y))\otimes H$, respectively, to define $C_{max}^*(Y')^{\pi_1(X)}$, $C_{max}^*(\widetilde{X})^{\pi_1(X)}$, $C_{max}^*(\widetilde{Y})^{\pi_1(Y)}$, and $C_{max}^*(Y'')^{j_*\pi_1(Y)}$.

Lemma 2.12. *There exists a $*$-homomorphism*

$$\psi'': C_{max}^*(\widetilde{Y})^{\pi_1(Y)} \to C_{max}^*(Y'')^{j_*\pi_1(Y)}$$

such that there is $\varepsilon > 0$ for which, if $k \in C^*(\widetilde{Y})^{\pi_1(Y)}$ is an operator with propagation at most ε and is represented as a kernel k on $(p')^{-1}(Z_Y)$ with values in $\mathcal{K}$, then there is a unique kernel k_Y on Z_Y with values in $\mathcal{K}$ such that $k(x,y) = k_Y(p(x), p(y))$ for all $x, y \in p^{-1}(Z_Y)$ satisfying $d(x,y) \le \varepsilon$ and $\psi''(k)$ is represented by a kernel k'' on $(p'')^{-1}(Z_Y)$ with values in $\mathcal{K}$ such that $k''(x,y) = k_Y(p''(x), p''(y))$ for all $x, y \in (p'')^{-1}(Z_Y)$ satisfying $d(x,y) \le \varepsilon$.

The homomorphism ψ'' in the above lemma can be considered as a folding construction. In the special case when $j_*\pi_1(Y)$ is trivial, we have $C_{max}^*(\widetilde{Y})^{\pi_1(Y)} \cong C_{max}^*(\pi_1(Y)) \otimes \mathcal{K}$ and $C_{max}^*(Y'')^{j_*\pi_1(Y)} \cong \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators. Then ψ'' is equivalent to the homomorphism induced by the canonical $*$-homomorphism from $C_{max}^*(\pi_1(Y))$ to $\mathbb{C}$ taking each finite sum $\sum_{g\in\pi_1(Y)} c_g g$ to $\sum_{g\in\pi_1(Y)} c_g$, where $c_g \in \mathbb{C}$ for all g.

Proof. Let H be the kernel of the homomorphism $j_*: \pi_1(Y) \to \pi_1(X)$. Let k be an operator in $\mathbb{R}(\widetilde{Y})^{\pi_1(Y)}$ represented by a kernel $k(x,y)$ on $(p')^{-1}(Z_Y)$. We define a kernel $k_a(x,y)$ on $(p')^{-1}(Z_Y)$ by the formula $k_a(x,y) = \sum_{h\in H} k(hx, y)$ for all $x, y \in (p')^{-1}(Z_Y)$. Note that the above sum is finite since k has finite propagation. We have $k_a(h_1x, h_2y) = k_a(x,y)$ for all $h_1, h_2 \in H$ and $x, y \in (p')^{-1}(Z_Y)$. For each $x, y \in (p')^{-1}(Z_Y)$, let $[x], [y]$ be the corresponding pair of equivalence classes in $(p'')^{-1}(Z_Y) = (p')^{-1}(Z_Y)/H$. We let $k''([x],[y]) = k_a(x,y)$. Note that k'' is well-defined. We now define a $*$-homomorphism $\psi'': \mathbb{R}(\widetilde{Y})^{\pi_1(Y)} \to \mathbb{R}(Y'')^{j_*\pi_1(Y)}$ given by $\psi''(k) = k''$. By maximality, this map ψ'' extends to a $*$-homomorphism $C_{max}^*(\widetilde{Y})^{\pi_1(Y)} \to C_{max}^*(Y'')^{j_*\pi_1(Y)}$. We choose $\varepsilon > 0$ small enough such that $d(hx, x) > 10\varepsilon$ for all $h \ne e$ in H and all $x \in \widetilde{Y}$. If $d([x],[y]) > \varepsilon$, then $d(hx, y) > \varepsilon$ for all $h \in H$. Therefore if k has propagation at most ε, then k'' has propagation at most ε. If ε is small enough, there is a unique kernel k_Y on Z_Y such that $k(x,y) = k_Y(p(x), p(y))$ for all $x, y \in p^{-1}(Z_Y)$ satisfying $d(x,y) \le \varepsilon$ and k_Y has propagation at most ε. If $d(x,y) \le \varepsilon$, then $d(hx, y) < \varepsilon$ for all $h \ne e$ in H and $x, y \in \widetilde{Y}$. Therefore $k_a(x,y) = k(x,y)$ if $d(x,y) \le \varepsilon$. It follows that $k''(x,y) = k_Y(p''(x), p''(y))$ for all $x, y \in (p'')^{-1}(Z_Y)$ satisfying $d(x,y) \le \varepsilon$. □

Let ψ'' be as in Lemma 2.12 above and let ψ' be as previously defined. We now define a $*$-homomorphism

$$\psi_{max} = \psi' \circ \psi'': C_{max}^*(\widetilde{Y})^{\pi_1(Y)} \to C_{max}^*(\widetilde{X})^{\pi_1(X)}. \tag{2.13}$$

This homomorphism can in turn be used to construct a $*$-homomorphism

$$\psi_{L,max}: \quad C_{L,max}^*(\widetilde{Y})^{\pi_1(Y)} \to C_{L,max}^*(\widetilde{X})^{\pi_1(X)}.$$

Let $C_{\psi_{L,max}}$ be the mapping cone of $\psi_{L,max}$ given by

$$\{(a,f): f \in C_0([0,1), C_{L,max}^*(\widetilde{X})^{\pi_1(X)}),\ a \in C_{L,max}^*(\widetilde{Y})^{\pi_1(Y)}, f(0) = \psi_{L,max}(a)\}.$$

Recall that $i: C_L^*(Y) \to C_L^*(X)$ is the homomorphism induced by the inclusion $Y \to X$, and C_i is its mapping cone. For each $(b,f) \in C_i$ with uniformly finite propagation, i.e. $\mathrm{prop}(b) < \infty$ and $\sup_{0\le t\le 1}(\mathrm{prop}(f(t))) < \infty$, there is $s_{(b,f)} > 0$ such that

$\text{prop}(b_s) < \varepsilon$ and $\text{prop}(f(t)) < \varepsilon$ for all $s > s_{(b,f)}$. We define

$$\chi_{s,max}(b,f) = (\phi_s(b_s), \phi_s(f(\cdot)_s)) \in C_{\psi_{L,max}}$$

for all $s > s_{(b,f)}$, where ϕ_s is as in Lemma 2.10. The map $\chi_{s,max}$ induces a homomorphism

$$(\chi_{s,max})_*\colon KO_*(S^7C_i) \to KO_*(S^7C_{\psi_{L,max}}).$$

This homomorphism can be defined as follows. For simplicity, we only consider the KO_0 case. Each element in $KO_0(S^7C_i)$ can be represented by a quasi-projection q in $(S^7C_i)^+$ with uniform finite propagation satisfying $q^* = q$ and $\|q^2-q\| < \frac{1}{10C}$, where C is as in Lemma 2.10 and $(S^7C_i)^+$ is obtained from S^7C_i by adjoining a unit. Lemma 2.10 implies that $q' = \chi_{s,max}(q)$ is a quasi-projection satisfying $(q')^* = q'$ and $\|(q')^2 - q'\| < \frac{1}{10}$. We now define $[\chi_{s,max}(q)]$ to be the K-theory element in $KO_*(S^7C_{\psi_{L,max}})$ represented by the quasi-projection q'.

Let e be the evaluation homomorphisms induced by the evaluation maps at 0 from $C^*_{L,max}(\widetilde{X})^{\pi_1(X)}$ to $C^*_{max}(\widetilde{X})^{\pi_1(X)}$ and from $C^*_{L,max}(\widetilde{Y})^{\pi_1(Y)}$ to $C^*_{max}(\widetilde{Y})^{\pi_1(Y)}$. These homomorphisms induce maps $e_*\colon KO_*(S^7C_{\psi_{L,max}}) \to KO_*(S^7C_{\psi_{max}})$ at the level of KO-theory.

Define μ_{max} to be the composition given by

$$KO_*(S^7C_i) \xrightarrow{(\chi_{s,max})_*} KO_*(S^7C_{\psi_{L,max}}) \xrightarrow{e_*} KO_*(S^7C_{\psi_{max}}).$$

By definition, μ_{max} is then a map

$$\mu_{max}\colon KO_*(X,Y) \to KO_*(C^*_{max}(\pi_1(X), \pi_1(Y)))$$

which we call the *maximal relative Baum–Connes map.* A *reduced relative Baum–Connes map*

$$\mu_{red}\colon KO_*(X,Y) \to KO_*(C^*_{red}(\pi_1(X), \pi_1(Y)))$$

can be similarly constructed if the homomorphism j from $\pi_1(Y)$ to $\pi_1(X)$ is injective.

Conjecture 2.14. *Let $Y \subseteq X$ and suppose that X and Y are both aspherical compact spaces.*

(1) *(Relative Novikov conjecture) The maximal relative Baum–Connes map*

$$\mu_{max}\colon KO_*(X,Y) \to KO_*(C^*_{max}(\pi_1(X), \pi_1(Y)))$$

is an injection.

(2) *(Relative Baum–Connes conjecture) If $j\colon \pi_1(Y) \to \pi_1(X)$ is an injection, then the reduced relative Baum–Connes map*

$$\mu_{red}\colon KO_*(X,Y) \to KO_*(C^*_{red}(\pi_1(X), \pi_1(Y)))$$

is an isomorphism.

Remark 2.15. If the classical Baum–Connes conjecture holds for $\pi_1(X)$ and $\pi_1(Y)$, then statement (2) is true for the pair $(\pi_1(X), \pi_1(Y))$. In general the maximal relative Baum–Connes conjecture may not be an isomorphism. The real version (KO) of the Baum–Connes conjecture follows from the classic (complex version) of the Baum–Connes conjecture (see Baum–Karoubi [2]). After inverting 2, even the injectivity of the complex Baum–Connes map implies the injectivity of the real Baum–Connes map (see Schick [38, Corollary 2.13]).

Recall that the notion of K-amenability was formulated by Cuntz [11, Definition 2.2]. This notion can be extended to the KO-setting.

Theorem 2.16. *Suppose that $Y \subseteq X$ are aspherical compact spaces such that $\pi_1(Y)$ and $\pi_1(X)$ are K-amenable and satisfy the Baum–Connes conjecture.*

(1) *Then μ_{max} is an isomorphism.*
(2) *Assume also that $\pi_1(Y) \to \pi_1(X)$ is an injection. Then μ_{red} is an isomorphism.*

Proof. By the definition of K-amenability, the natural homomorphisms $C^*_{max}(\pi_1(X)) \to C^*_r(\pi_1(X))$ and $C^*_{max}(\pi_1(Y)) \to C^*_r(\pi_1(Y))$ induce KK-equivalences. If $\pi_1(X)$ and $\pi_1(Y)$ are K-amenable and satisfy the Baum–Connes conjecture, and if $\pi_1(Y)$ injects into $\pi_1(X)$, then the KO-theory of the reduced relative group C^*-algebra coincides with the KO-theory of the maximal relative group C^*-algebra.

The theorem is proved from the following commutative diagram and the five-lemma.

$$\begin{array}{ccc}
KO_{n+1}(Y) & \longrightarrow & KO_{n+1}(C^*(\pi_1(Y))) \\
\downarrow & & \downarrow \\
KO_{n+1}(X) & \longrightarrow & KO_{n+1}(C^*(\pi_1(X))) \\
\downarrow & & \downarrow \\
KO_{n+1}(X,Y) & \longrightarrow & KO_{n+1}(C^*(\pi_1(X),\pi_1(Y))) \\
\downarrow & & \downarrow \\
KO_{n}(Y) & \longrightarrow & KO_{n}(C^*(\pi_1(Y))) \\
\downarrow & & \downarrow \\
KO_{n}(X) & \longrightarrow & KO_{n}(C^*(\pi_1(X)))
\end{array} \quad \square$$

We now prove that the existence of positive scalar curvature implies that a particular index vanishes in the KO-theory of the relative group C^*-algebra. For the rest of this section, the C^*-algebras involved are maximal. If the reduced relative group C^*-algebra is well defined, then the rest of this section extends to the reduced case as well. We will use $C^*(\pi_1(X),\pi_1(Y))$ to denote both the reduced and maximal relative group C^*-algebra when the use of such a notation does not cause confusion.

Let M be a spin manifold with boundary $N=\partial M$. We assume that the dimension of M is $0 \bmod 8$. The other cases can be handled in a similar way with the help of suspensions. More specifically, in dimension $k \bmod 8$ for some $0\le k<8$, we consider the manifold $M\times\mathbb{R}^{8-k}$. We can define a relative higher index of the Dirac operator associated to the space $M\times\mathbb{R}^{8-k}$ in $KO_0(C^*(\pi_1(M),\pi_1(\partial M))\otimes C_L^*(\mathbb{R}^{8-k}))$. We can apply the same argument below to show that this relative index vanishes in $KO_0(C^*(\pi_1(M),\pi_1(\partial M))\otimes C_L^*(\mathbb{R}^{8-k}))$ if $(M,\partial M)$ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary. This relative higher index corresponds to the relative index of the Dirac operator associated to M under the isomorphism

$$KO_0(C^*(\pi_1(M),\pi_1(\partial M))\otimes C_L^*(\mathbb{R}^{8-k}))\cong KO_k(C^*(\pi_1(M),\pi_1(\partial M))).$$

The above isomorphism can be implemented by the external product formula for the index of the Dirac operator on a product of two manifolds. As a consequence, the relative index of the Dirac operator associated to M vanishes in $KO_k(C^*(\pi_1(M),\pi_1(\partial M)))$ if $(M,\partial M)$ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary.

We extend the manifold by attaching a cylinder $W=N\times[0,\infty)$ to the boundary, forming a noncompact manifold Z. Let D be the Dirac operator on Z. Let f be an odd smooth real-valued chopping function in the sense of Roe on the real line satisfying the following conditions: (1) $|f(x)|\le 1$ for all x and $f(x)\to\pm1$ as $x\to\pm\infty$; (2) $g=f^2-1\in S(\mathbb{R})$, the space of Schwartz functions, (3) if $\widehat{f}$ and $\widehat{g}$ are the Fourier transforms of f and g, respectively, then $\mathrm{Supp}(\widehat{f})\subseteq[-1,1]$ and $\mathrm{Supp}(\widehat{g})\subseteq[-2,2]$. Such a chopping function exists (cf. Roe [32, Lemma 7.5]). We define

$$F_D=f(D)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(t)\exp(itD)dt \tag{2.17}$$

We remark that the above formula is well defined in our real Hilbert space setting since f is a real-valued function. By condition (3) above, it follows that the propagation of F_D is at most 1. Let

$$F_D=\begin{pmatrix}0 & F\\ F^* & 0\end{pmatrix}.$$

Let $[F]$ be its homology class in $KO_0^{lf}(Z)=KO^0(C_0(Z))$. We simplify the notation by replacing P_{F_D} with P_D. We write

$$\mathrm{ind}_L([F])=[P_D]-\left[\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}\right]\in KO_0(C_L^*(Z)),$$

where P_D is an idempotent in the matrix algebra of $C_L^*(Z)^+$ and ind_L is the local index map. The element $P_D-\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}$ belongs to the matrix algebra of the localization algebra $C_L^*(Z)$.

Let v be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing the generator in $KO_{-1}(C_0(\mathbb{R}^7))\cong KO_0(C_0(\mathbb{R}^8))$ (see Atiyah [1] or Schröder [41, Proposition 1.4.11]). Let $\tau_D=v\otimes P_D+I\otimes(I-P_D)$. Then we have $\tau_D^{-1}=v^{-1}\otimes P_D+I\otimes(I-P_D)$. If χ_M is the characteristic function on M, let $\tau_{D,M}=(1\otimes\chi_M)\tau_D(1\otimes\chi_M)$ and

$(\tau_D^{-1})_M = (1 \otimes \chi_M)\tau_D^{-1}(1 \otimes \chi_M)$. In the future pages, we will simply write χ_M for $1 \otimes \chi_M$. For all $s \in [0, 1]$, define $w_{D,M}(s)$ to be the product

$$\begin{pmatrix} I & (1-s)\tau_{D,M} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)(\tau_D^{-1})_M & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau_{D,M} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Define

$$q_{D,M}(s) = w_{D,M}(s) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} w_{D,M}^{-1}(s).$$

Now define $C_L^*(N \subseteq M)$ to be the closed two-sided ideal of $C_L^*(M)$ generated by $C_L^*(N)$ considered as a subalgebra of $C_L^*(M)$. Then $\tau_{D,M}$ and $(\tau_D^{-1})_M$ both lie in $C_L^*(M) \otimes C_0(\mathbb{R}^7)$. Both $\tau_{D,M}(\tau_D^{-1})_M - I$ and $(\tau_D^{-1})_M\tau_{D,M} - I$ lie in $C_L^*(N \subseteq M) \otimes C_0(\mathbb{R}^7)$. As a consequence $q_{D,M}(0)$ is an element in the matrix algebra of $(C_L^*(N \subseteq M) \otimes C_0(\mathbb{R}^7))^+$.

Let $P_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau = v \otimes P_0 + I \otimes (I - P_0)$. We have

$$\tau^{-1} = v^{-1} \otimes P_0 + I \otimes (I - P_0).$$

For all $s \in [0, 1]$, define $w(s)$ to be the product

$$\begin{pmatrix} I & (1-s)\tau \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)\tau^{-1} & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Let

$$q(s) = w(s) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} w^{-1}(s).$$

We define $[q_D]$ to be the KO-theory element

$$[(q_{D,M}(0), q_{D,M}(\cdot))] - [(q(0), q(\cdot))]$$

in $KO_0(S^7C_j)$, where C_j is the mapping cone associated with $j\colon C_L^*(N \subseteq M) \to C_L^*(M)$ and $S^7C_j = C_j \otimes C_0(\mathbb{R}^7)$. The inclusion map $i : C_L^*(N) \to C_L^*(N \subseteq M)$ induces an isomorphism

$$KO_*(C_L^*(N)) \cong KO_*(C_L^*(N \subseteq M)).$$

The above isomorphism can be proved as follows. Let N' be a closed subspace of M such that N' is diffeomorphic to $N \times [0, 1]$ and the diffeomorphism maps $N \times \{0\}$ to N. Let i_1 be the inclusion map from N to N' and let i_2 be the inclusion map from N' to M. The inclusion map i_1 induces a homomorphism

$$(i_1)_* : KO_*(C_L^*(N)) \to KO_*(C_L^*(N \subseteq N')).$$

By a Lipschitz homotopy argument, we know that the map $(i_1)_*$ is an isomorphism. The map i_2 induces a homomorphism:

$$(i_2)_* : KO_*(C_L^*(N \subseteq N')) \to KO_*(C_L^*(N \subseteq M)).$$

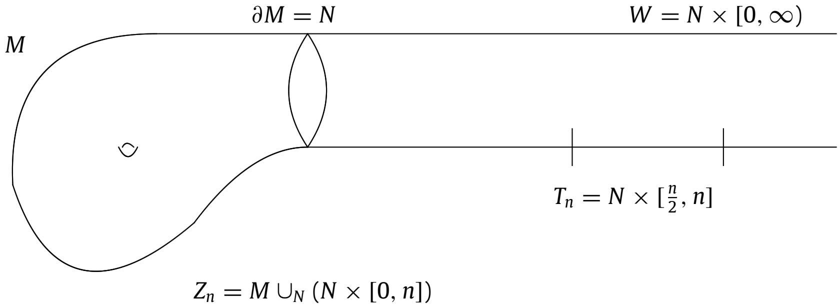
We can show that the map $(i_2)_*$ is an isomorphism by constructing the following inverse homomorphism π from $KO_*(C_L^*(N \subseteq M))$ to $KO_*(C_L^*(N \subseteq N'))$. For simplicity, we describe the construction of π when $* = 0$. By an approximation, each element in $KO_0(C_L^*(N \subseteq M))$ can be represented as a quasi-projection q with finite propagation in $C_L^*(N \subseteq M)$ satisfying $q^* = q$ and $\|q^2 - q\| < 1/10$. For any $s \in [0, \infty)$, let q_s be an element in $C_L^*(N \subseteq M)$ defined by $q_s(t) = q(t+s)$. $[q]$ is equivalent to $[q_s]$ in $KO_0(C_L^*(N \subseteq M))$. When s is sufficiently large, q_s is supported near N and is therefore an element in $C_L^*(N \subseteq N')$. We define $\pi([q]) = [q_s] \in KO_0(C_L^*(N \subseteq N'))$. Note that the K-theory class $[q_s] \in KO_0(C_L^*(N \subseteq N'))$ is independent of the choice of s for sufficiently large s. It is now straightforward to check that $(i_2)_*$ and π are inverses to each other. We have

$$i_* = (i_2)_* \cdot (i_1)_*.$$

It follows that the map i_* is an isomorphism.

As a consequence, we have the isomorphism $KO_0(S^7C_j) \cong KO_0(M, N)$.

We call the class $[q_D]$ the *relative KO-homology class of D.* We define the *relative higher index of D* to be $\mu(q_D) \in KO_0(C^*(\pi_1(M), \pi_1(N)))$.



Theorem 2.18. *If $(M, \partial M)$ is a compact spin manifold with boundary endowed with a metric of positive scalar curvature that is collared at the boundary, then the relative higher index of the Dirac operator is zero in $KO_*(C^*(\pi_1(M), \pi_1(\partial M)))$.*

Proof. As before, let $N = \partial M$ and $Z = M \cup_N (N \times [0, \infty))$. Denote by Z_n and Z'_n the truncations $Z_n = M \cup_N (N \times [0, n])$, $Z'_n = M \cup_N (N \times [0, \frac{n}{2}])$, and let T_n be the subset of Z_n given by $T_n = N \times \left[\frac{n}{2}, n\right]$. We assume that the dimension of Z is $0 \bmod 8$. The other cases can be handled in a similar way with the help of suspensions (refer back to the section after Theorem 2.16).

Let $u \in [1, \infty)$ and write

$$\mathrm{ind}_L(uD) = [P_{uD}] - \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right] \in KO_0(C^*_L(Z)).$$

We define $w_{D,Z_n}(s)$ and $q_{D,Z_n}(s)$ by replacing M with Z_n in the definitions of $w_{D,M}(s)$ and $q_{D,M}(s)$, respectively, before Theorem 2.18. By the propagation speed of the wave equations associated to D, we know that the propagation of $\exp(itD)$ is less than or equal to $|t|$. It follows that the propagation of P_{uD} is less than or equal to $100u$. This estimate is based on the matrix formula before Proposition 2.4 and the formula of F_D given by (2.17).

Claim 2.19. *For all $u > 0$, there exists $N_u > 0$ such that, for all $n > N_u$, we have*

$$\chi_{Z'_n}\left(q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\chi_{Z'_n} = 0,$$

$$\chi_{T_n}\left(q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\chi_{Z'_n} = 0,$$

$$\chi_{Z'_n}\left(q_{uD,Z_n}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right)\chi_{T_n} = 0.$$

Proof. Let $\alpha = \tau_{uD,Z_n}$ and $\beta = (\tau_{uD}^{-1})_{Z_n}$. We can compute

$$w_{uD,Z_n}(0)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} w_{uD,Z_n}^{-1}(0)$$
$$= \begin{pmatrix} (2\alpha - \alpha\beta\alpha)\beta & (2\alpha - \alpha\beta\alpha)(I - \beta\alpha) \\ (I - \beta\alpha)\beta & (I - \beta\alpha)^2 \end{pmatrix}.$$

We note that $(2\alpha - \alpha\beta\alpha)\beta = \alpha(I - \beta\alpha)\beta + (\alpha\beta - I) + I$. Let $N_u = 100u$. Let $p_i : Z \times Z \to Z$ be the projection onto the ith coordinate; i.e. $p_1 : (z_1, z_2) \mapsto z_1$ and $p_2 : (z_1, z_2) \mapsto z_2$. Using the formulas for α and β, and the fact that P_{uD} has propagation at most $100u$, we know that the images under p_i of the supports of the elements $\alpha\beta - I$, $\beta\alpha - I$, $(I - \beta\alpha)\beta$ and $\alpha(I - \alpha\beta)$ are all disjoint from Z'_n when $n > N_u$. As a consequence, the elements $\chi_{Z'_n}(\alpha\beta - I)$, $\chi_{Z'_n}(\beta\alpha - I)$, $(\alpha\beta - I)\chi_{Z'_n}$, $(\beta\alpha - I)\chi_{Z'_n}$, $(I - \beta\alpha)\beta\chi_{Z'_n}$ and $\chi_{Z'_n}\alpha(I - \alpha\beta)$ are all zero when $n > N_u$. Now our claim follows. □

Let $P_{uD}^{(n)} = \chi_{Z_n} P_{uD} \chi_{Z_n}$, where χ_{Z_n} is the characteristic function on Z_n. By the construction of P_{uD} we have $\|P_{uD}\| \le 10$. As a result, we have $\|P_{uD}^{(n)}\| \le 10$, giving an upper bound for $\|q_{uD,Z_n}\|$. Together with the above claim, this implies that, for all $u > 0$, $\left[\prod_n q_{uD,Z_n}(0)\right] \in \prod_n (S^7 C^*_L(Z_n))^+ / \bigoplus_n (S^7 C^*_L(Z_n))^+$ belongs to the image of the inclusion map

$$\prod_n (S^7 C^*_L(T_n))^+ / \bigoplus_n (S^7 C^*_L(T_n))^+ \to \prod_n (S^7 C^*_L(Z_n))^+ / \bigoplus_n (S^7 C^*_L(Z_n))^+,$$

where $(S^7C_L^*(Z_n))^+$ and $(S^7C_L^*(T_n))^+$ are respectively obtained from $S^7C_L^*(Z_n)$ and $S^7C_L^*(T_n)$ by adjoining a unit. We introduce this quotient of an infinite product by the direct sum to have a convenient place to encode vanishing for all sufficiently large n.

Identify $\left[\prod_n q_{uD,Z_n}(0)\right]$ with the corresponding element in

$$\prod_n (S^7C_L^*(T_n))^+ / \bigoplus_n (S^7C_L^*(T_n))^+.$$

Now $\left(\prod_n q_{uD,Z_n}(0), \prod_n q_{uD,Z_n}(s)\right)$ gives an element in the matrix algebra of

$$\prod_n (S^7C_{j_n})^+ / \bigoplus_n (S^7C_{j_n})^+,$$

where $s \in [0,1]$ is the variable and C_{j_n} is the mapping cone of the homomorphism $j_n : S^7C_L^*(T_n) \to S^7C_L^*(Z_n)$.

Recall that $W = N \times [0,\infty)$ and let $W' = N \times \mathbb{R}$ be the double of W. Let D' be the Dirac operator on W'. Let $\widetilde{W'}$ be the universal cover of W' and $\widetilde{D'}$ be the lifting of D' to $\widetilde{W'}$. Let $\widetilde{D}$ be the lifting of D to $\widetilde{Z}$, the universal cover of Z. Recall that $P_{u\widetilde{D'}}(0)$ and $P_{u\widetilde{D}}(0)$ can be expressed in terms of the wave operators $exp(it\widetilde{D'})$ and $exp(it\widetilde{D})$ (respectively $P_{uD'}(0)$ and $P_{uD}(0)$ can be expressed in terms of the wave operators $exp(itD')$ and $exp(itD)$). As a consequence, we know that $P_{u\widetilde{D'}}(0)$ and $P_{u\widetilde{D}}(0)$ are respectively liftings of $P_{uD'}(0)$ and $P_{uD}(0)$. Define

$$x_{n,u}(s) = q_{u\widetilde{D},\widetilde{Z}_n}(s),$$

where $P^{(n)}_{u\widetilde{D}} = \chi_{\widetilde{Z}_n} P_{u\widetilde{D}} \chi_{\widetilde{Z}_n}$. Note that $\widetilde{T}_n$ is a subset of $\widetilde{W'}$ and $\widetilde{Z}_n$ is a subset of $\widetilde{Z}$. Define

$$y_{n,u} = q_{u\widetilde{D'},\widetilde{W}'_n}(0),$$

where $W'_n = N \times (-\infty, n]$. By an argument similar to the proof of Claim 2.19, we know that $\left[\prod_n y_{n,u}\right]$ is an operator in the image of the inclusion map:

$$\prod_n (S^7C^*(\widetilde{T}_n)^{\pi_1(N)})^+ / \bigoplus_n (S^7C^*(\widetilde{T}_n)^{\pi_1(N)})^+ \to \prod_n (S^7C^*(\widetilde{W'})^{\pi_1(N)})^+ / \bigoplus_n (S^7C^*(\widetilde{W'})^{\pi_1(N)})^+.$$

We identify $\left[\prod_n y_{n,u}\right]$ with an element in $\prod_n (S^7C^*(\widetilde{T}_n)^{\pi_1(N)})^+ / \bigoplus_n (S^7C^*(\widetilde{T}_n)^{\pi_1(N)})^+$.

By Lemma 2.12 and Formula (2.13), there is a natural $*$-homomorphism

$$\phi_n : C^*(\widetilde{T}_n)^{\pi_1(N)} \to C^*(\widetilde{Z}_n)^{\pi_1(M)}.$$

Note that here it is crucial to use the maximal C^*-algebras.

For each n, the map ϕ_n induces a natural $*$-homomorphism

$$S^7C^*(\widetilde{T}_n)^{\pi_1(N)} \to S^7C^*(\widetilde{Z}_n)^{\pi_1(M)},$$

which we still denote by ϕ_n. We have

$$\left[\prod_n \phi_n(y_{n,u})\right] = \left[\prod_n x_{n,u}(0)\right]$$

in $\prod_n (S^7C^*(\widetilde{Z}_n)^{\pi_1(M)})^+ / \bigoplus_n (S^7C^*(\widetilde{Z}_n)^{\pi_1(M)})^+$. The above identity can be seen as follows. Let g be any real valued function in $S(\mathbb{R})$, the space of Schwartz functions, such that its Fourier transform $\hat{g}$ is compactly supported. Our desired identity would follow from

$$\phi_n(\chi_{\widetilde{T}_n} g(\widetilde{D'}) \chi_{\widetilde{T}_n}) = \chi_{\widetilde{Z}_n} g(\widetilde{D}) \chi_{\widetilde{Z}_n}$$

when n is sufficiently large relative to the size of the support of $\hat{g}$. The above formula follows from the fact that $exp(it\widetilde{D'})$ and $exp(it\widetilde{D})$ are respectively unique solutions to the heat equations associated to $\widetilde{D'}$ and $\widetilde{D}$ and the following identities:

$$g(\widetilde{D'}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(t) exp(it\widetilde{D'}) dt,$$
$$g(\widetilde{D}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(t) exp(it\widetilde{D}) dt.$$

Denote by C_{ϕ_n} the mapping cone of the map ϕ_n. The element $\prod_n (y_{n,u}, x_{n,u}(s))$ gives a KO-theory element

$$\left[\prod_n (y_{n,u}, x_{n,u}(s))\right]$$

in $KO_0\left(\prod_n S^7C_{\phi_n} / \bigoplus_n S^7C_{\phi_n}\right)$.

Let $V_{1,n}: L^2[0, n] \to L^2[0, 1]$ be the isometry given by $f(\cdot) \mapsto \frac{1}{\sqrt{n}} f(n\cdot)$ for all $f \in L^2[0, n]$. Let $V_{2,n}: L^2[\frac{n}{2}, n] \to L^2[\frac{1}{2}, 1]$ be the isometry given by $f(\cdot) \mapsto \frac{1}{\sqrt{n}} f(n\cdot)$ for all $f \in L^2[0, \frac{n}{2}]$. We can similarly construct isometries $V'_{1,n}: L^2(\widetilde{Z}_n) \to L^2(\widetilde{Z}_1)$ and $V'_{2,n}: L^2(\widetilde{T}_n) \to L^2(\widetilde{T}_1)$. Conjugation by $V'_{1,n}$ and $V'_{2,n}$ gives us a $*$-isomorphism $C_{\phi_m} \to C_{\phi_1}$. Recall that

$$C^*(\widetilde{T}_n)^{\pi_1(N)} \cong C^*_{max}(\pi_1(N)) \otimes K,$$
$$C^*(\widetilde{Z}_n)^{\pi_1(M)} \cong C^*_{max}(\pi_1(M)) \otimes K.$$

The map ϕ_1 can be naturally identified with $\phi \otimes Id$ via the above isomorphisms. Hence C_{ϕ_1} is naturally isomorphic to $C_\phi \otimes K$. Identifying $(S^7 C_{\phi_m})^+$ with $(S^7 C_\phi \otimes K)^+$ for sufficiently large n, we have the equation $[(y_{n,u}, x_{n,u}(s))] = \mu([q_D])$ in $KO_0(S^7 C_\phi)$, where K is the algebra of all compact operators. Note that here we are using the fact that the C^*-algebra $S^7 C_\phi$ is stable. The above equation can be seen as follows. We have a natural $*$-isomorphism: $S^7 C_j \cong S^7 C_{j_n}$, where C_{j_n} is defined as in the paragraphs after Claim 2.19. This algebra isomorphism induces an isomorphism at the K-theory level:

$$KO_0(S^7 C_j) \cong KO_0(S^7 S_{j_n}).$$

In the above isomorphism, when n is large enough, $[q_D]$ corresponds to $[(q_{uD,Z_n}(0), q_{uD,Z_n}(s))]$, where $[(q_{uD,Z_n}(0), q_{uD,Z_n}(s))]$ is defined as in the paragraphs after Claim 2.19. This implies that

$$\mu([q_D]) = \mu([(q_{uD,Z_n}(0), q_{uD,Z_n}(s))]).$$

By definition, we have

$$\mu([(q_{uD,Z_n}(0), q_{uD,Z_n}(s))]) = [(y_{n,u}, x_{n,u}(s))].$$

Combining the above equations, we have the desired equation.

It follows from the above paragraph that there is a natural isomorphism

$$\psi: KO_0\left(\prod_n S^7 C_{\phi_n} / \bigoplus_n S^7 C_{\phi_n}\right) \to KO_0\left(\prod_n S^7 C_\phi / \bigoplus_n S^7 C_\phi\right)$$

such that

$$\psi\left(\left[\prod_n (y_{n,u}, x_{n,u}(s))\right]\right) = \left[\prod_n \mu(q_D)\right],$$

where $\mu(q_D) \in KO_0(S^7 C_\phi) \cong KO_0(C^*(\pi_1(M), \pi_1(N)))$ was defined as the relative higher index of D before Theorem 2.18 and $KO_0(\prod_n S^7 C_\phi / \bigoplus_n S^7 C_\phi)$ is identified with

$$\prod_n KO_0(S^7 C_\phi) / \bigoplus_n KO_0(S^7 C_\phi).$$

When M has a metric of uniform positive scalar curvature, then by the Lichnerowicz formula we know that $P^{(n)}_{u\widetilde{D}}(0)$ converges to $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ in the operator norm as $u \to \infty$, $n \to \infty$ and $n \geq N_u$. More precisely, for any given $\epsilon > 0$, there exists $u_\epsilon \geq 1$ for which, given any $u > u_\epsilon$, there is a natural number N_u such that

$$\|P^{(n)}_{u\widetilde{D}}(0) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\| < \epsilon$$

for all $n \geq N_u$. We remark that here we are using the fact the operator $\widetilde{D}$ is a regular and essentially self-adjoint operator acting on the maximal Roe algebra viewed as a Hilbert module over itself (see [21]).

As a consequence, we know that

$$\tau_{u\widetilde{D},\widetilde{W}'_n} \to v \otimes \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + I \otimes \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and

$$y_{n,u} \to \exp\left(2\pi i \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) = I$$

in the operator norm as $u \to \infty$, $n \to \infty$ and $n \geq N_u$. Together with the formula for $x_{n,u}(s)$, we then have $\left[\prod_n (y_{n,u}, x_{n,u}(s))\right] = 0$ in $KO_0\left(\prod_n S^7 C_\phi / \bigoplus_n S^7 C_\phi\right)$. Therefore it follows that $\mu(q_D) = 0$. □

As mentioned in the introduction, the Gromov–Lawson–Rosenberg conjecture states that a closed spin manifold M^n with $n \geq 5$ has a metric of positive scalar curvature if, and only if, its Dirac index vanishes in $KO_*(C^*_r \pi)$, where $\pi = \pi_1(M)$. We formulate now a relative version of this conjecture.

Remark 2.20. A recent article of Schick–Seyedhosseini provides a different proof for a weaker version of the above vanishing theorem [39].

Conjecture 2.21 (*Relative Gromov–Lawson–Rosenberg*)**.** *Let* $(N, \partial N)$ *be a compact spin manifold with boundary. Let n be the dimension of N. If the relative higher index of the Dirac operator D is zero in* $KO_n(C^*(\pi_1(N), \pi_1(\partial N)))$*, then there is a metric of positive scalar curvature on N that is collared near* ∂N*.*

Remark 2.22.

(1) This conjecture is made by analogy to surgery theory, where obstructions to surgery for degree one normal maps have this formal structure.
(2) By the Gromov–Lawson surgery theorem [19, Theorem A] and Schoen–Yau [40, Corollary 6] as improved by Gajer [16], the π-π case of the conjecture is correct.
(3) Because of the failure of stability for the ordinary Gromov–Lawsonconjecture (Schick [37, Example 2.2] and Dwyer–Schick–Stolz [12]), we recognize that, in general, this statement cannot be true as stated. One should either interpret it as a stable conjecture (i.e. after crossing with some number of Bott manifolds, see Rosenberg–Stolz [35]) or as a guide to formulate the correct statement in the unstable situation. We hope to address this matter in a future paper.

In Rosenberg–Stolz [35], the index map $\alpha\colon \Omega_n^{spin}(B\pi) \to KO_n(C_r^*\pi)$ is factored in the following way:

$$\Omega_n^{spin}(B\pi) \xrightarrow{D_*} ko_n(B\pi) \xrightarrow{p} KO_n(B\pi) \xrightarrow{A} KO_n(C_r^*\pi)$$

where $p\colon ko_n(B\pi) \to KO_n(B\pi)$ is the canonical map from connective to periodic KO-homology and A is the standard assembly map. This sequence can be generalized to pairs. Let $(N, \partial N)$ be a manifold with boundary and let $\pi = \pi_1(N)$ and $\pi^\infty = \pi_1(\partial N)$. Then we have a composition

$$\Omega_n^{spin}(B\pi, B\pi^\infty) \xrightarrow{D_*} ko_n(B\pi, B\pi^\infty) \xrightarrow{p} KO_n(B\pi, B\pi^\infty) \xrightarrow{A} KO_n(C^*(\pi, \pi^\infty)).$$

Let $\mathrm{Pos}_n^{spin}(B\pi, B\pi^\infty)$ be the subgroup of $\Omega_n^{spin}(B\pi, B\pi^\infty)$ consisting of bordism classes represented by pairs $(M^n, \partial M^n, f)$ for which M admits a metric of positive scalar curvature that is collared near the boundary.

There is a map from ∂M to $B\pi^\infty$ classifying its universal cover $\widetilde{\partial M}$. By elementary homotopy theory, the composite map to $B\pi$ coincides up to homotopy with the map $M \to B\pi$ classifying its universal cover $\widetilde{M}$. The homotopy extension principle then implies that we have a map of pairs $(M, \partial M) \to (B\pi, B\pi^\infty)$.

Theorem 2.23. *Let* $(M, \partial M)$ *be a spin manifold with boundary of dimension* ≥ 6*. Let* $\pi = \pi_1(M)$ *and* $\pi^\infty = \pi_1(\partial M)$*. Let* $u\colon (M, \partial M) \to (B\pi, B\pi^\infty)$ *be the map described above. Then* $(M, \partial M)$ *has a positive scalar curvature metric which is collared near the boundary* ∂M *if, and only if, the index* $D_*[(M, \partial M), u]$ *lies in* $\mathrm{Pos}_n^{ko}(B\pi, B\pi^\infty)$*, where* $\mathrm{Pos}_n^{ko}(B\pi, B\pi^\infty)$ *is the image under* D_* *restricted to* $\mathrm{Pos}_n^{spin}(B\pi, B\pi^\infty)$*.*

Proof. First we will explain that the capacity of a spin manifold M^n for $n \geq 6$ to admit a positive scalar curvature metric depends only on its spin cobordism class. As in the closed case, this result follows from the Gromov–Lawson surgery theorem, or equivalently the reduction to spin cobordism. For manifolds $(M, \partial M)$ with boundary whose boundary is collared, there is a relative surgery theorem that follows from an improvement by Gajer [16] of the usual Gromov–Lawson surgery theorem. Gajer's theorem provides a positive scalar curvature metric which is a product on a collared neighborhood for the trace of the surgery. We remind the reader that the Gromov–Lawson theorem holds if the spin cobordism respects fundamental group and the dimension is at least 5. The proof of our theorem requires two applications of the Gajer/Gromov–Lawson Theorem, as we will now demonstrate. Let $(M, \partial M)$ and $(M', \partial M')$ be cobordant and suppose that $(M', \partial M')$ has a metric of positive scalar curvature that is collared near the boundary.

Gromov–Lawson surgery for the boundary allows to change the cobordism boundary W from boundary M to boundary M' by another one, cobordant to the first, which has a positive scalar curvature metric that is product near boundary M and boundary M', extending the positive scalar curvature metric on M'. We get a new cobordism W from M to M' which has a positive scalar curvature metric on M' and boundary W, and near boundary M' the metric is a product metric with a quadrant, which we can straighten (metrically) to a half space H. We can modify the interior bordism in such a way that Gromov–Lawson/Gajer surgery allows to extend the positive scalar curvature metric on M' over the new bordism. This construction is sufficiently local to leave the metric untouched on boundary W an M'.

As a next step, we need to show that all the elements of the kernel of the map from relative spin bordism to relative ko have positive scalar curvature, i.e. that $\ker D_* \subseteq \mathrm{Pos}_n^{spin}(B\pi, B\pi^\infty)$. Both away from the prime 2 and at the prime 2, the inclusion can be obtained from the relative versions of existing theorems. Away from 2, the result holds by readapting the result of Führing [15] on Baas–Sullivan theory. This result was stated in Rosenberg–Stolz [36] as unpublished work of Jung. Führing proves that a smooth spin closed manifold M of dimension $n \geq 5$ admits a metric of positive scalar curvature if its orientation class in $ko_n(B\pi)$ lies in the subgroup consisting of elements which contain positive representatives. At the prime 2, we can extend Theorem B (2) of Stolz [43]. Here he proves the following. Let X be a topological space. Suppose that $T_n(X)$ is the subgroup of $\Omega_n^{spin}(X)$ consisting of bordism classes $[E, f \circ p]$, where $p: E \to B$ is an $\mathbb{HP}^2$-bundle over a spin closed manifold B of dimension $n-8$ and f is a map $B \to X$. Then the map $\Omega_n^{spin}(X)/T_n(X) \to ko_n(X)$ is a 2-local isomorphism. In the papers of both Führing and Stolz it is effectively shown that the kernel of D_* is a homology theory. As such we can extend these results to pairs. □

Corollary 2.24. *Let* $p: ko_n(B\pi, B\pi^\infty) \to KO_n(B\pi, B\pi^\infty)$ *and*

$$A: KO_n(B\pi, B\pi^\infty) \to KO_n(C^*(\pi_1, \pi_1^\infty))$$

be as above, with $n \geq 6$. *The Relative Gromov–Lawson–Rosenberg conjecture holds if p and A are both injective.*

Theorem 2.25. *Let* $n \geq 6$. *Let* N^n *be a manifold with boundary such that* $\pi_1(N)$ *and* $\pi_1(\partial N)$ *are both amenable. Suppose that* $\pi_1(\partial N) \to \pi_1(N)$ *is an injection and that the cohomological dimensions of* $\pi_1(N)$ *and* $\pi_1(\partial N)$ *are less than n. If the classifying spaces* $B\pi_1(N)$ *and* $B\pi_1(\partial N)$ *are finite complexes, then the Relative Gromov–Lawson–Rosenberg conjecture holds for the pair* $(N, \partial N)$.

Proof. Let $A = \pi_1(N)$ and $B = \pi_1(\partial N)$. The E^2 term for the Atiyah–Hirzebruch spectral sequence for $KO_n(K(A, 1), K(B, 1))$ is $H_p(A, B; KO_q)$. Similarly the E^2 term for $ko_n(K(A, 1), K(B, 1))$ is $H_p(A, B; ko_q)$. The groups coincide when $q \geq 0$. There is a comparison map between the spectral sequences from the ko-sequence to the KO-sequence (for instance, see page 180 of [14]) which is an isomorphism on E^2 for $q \geq 0$. The reason that this map may fail to be an isomorphism on E^∞ is that there are differentials for the KO-sequence that can start in the fourth quadrant and end in the first. For this reason, a nonzero element in ko_n can vanish in KO_n. But if $n > \max\{cd(A), cd(B)\}$, differentials can only come from the line $p + q = n + 1$ with $p \leq \max\{cd(A), cd(B)\}$. But then q is positive and the map is therefore an isomorphism.

Using Higson–Kasparov [22, Theorem 1.1] extended into the KO setting, we see that the KO-theory groups of $C^*_{max}(\pi)$ and $C^*_{max}(\pi^\infty)$ are given by the KO-theories of their classifying spaces. Thus the relative assembly map $A: KO_n(B\pi, B\pi^\infty) \to KO_n(C^*(\pi_1, \pi_1^\infty))$ is an isomorphism. The rest of the proof is as the last paragraph of Theorem 2.23. □

Remark 2.26. This unstable version of Conjecture 2.21 for large n obviously implies the stable version of the conjecture for all n.

3. A new index theory for noncompact manifolds

In this section we will develop a new index theory for a noncompact manifold. Our index theory will depend on a choice of an exhaustion.

Definition 3.1. Let (Y, d) be a noncompact, complete metric space. Suppose that Y is also metrically locally simply connected, i.e. for all $\varepsilon > 0$ there is $\varepsilon' \leq \varepsilon$ such that every ball in X of radius ε' is simply connected. Let $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots$ be a sequence of connected compact subsets of Y. We say that $\{Y_i\}$ is an *admissible exhaustion* if

(1) $Y = \bigcup_{i=1}^\infty Y_i$;
(2) for each $j > i$, there is a connected compact subset $Y_{i,j} \subseteq Y$ such that $Y_j = Y_{i,j} \cup Y_i$ and $Y_{i,j} \cap Y_i = \partial Y_i$, where $\partial Y_i = Y_i - \mathring{Y}_i$ for all i and $\mathring{Y}_i$ denotes the interior of Y_i;
(3) $d(\partial Y_i, \partial Y_j) \to \infty$ as $|j - i| \to \infty$.

Often we will write $\{Y_i; Y_{i,j}\}$ for the exhaustion.

Let $\{Y_i; Y_{i,j}\}$ be an admissible exhaustion of Y. Define D_i^* to be the C^*-algebra inductive limit given by

$$D_i^* \equiv \lim_{j\to\infty, j>i} C^*_{max}(\pi_1(Y_j), \pi_1(Y_{i,j})) \otimes \mathcal{K},$$

where $\mathcal{K}$ is the C^*-algebra of compact operators. Let

$$\prod_{i=1}^\infty D_i^* = \left\{ (a_1, a_2, \ldots): a_i \in D_i^*, \ \sup_i \|a_i\| < \infty \right\}.$$

There is a natural homomorphism $\rho_{i+1}: D_{i+1}^* \to D_i^*$ induced by the group homomorphisms given by inclusions of the corresponding spaces. Let ρ be the homomorphism from $\prod_{i=1}^\infty D_i^*$ to $\prod_{i=1}^\infty D_i^*$ mapping $(a_1, a_2, \ldots)$ to $(\rho_2(a_2), \rho_3(a_3), \ldots)$.

We now define the C^*-algebra $A(Y)$ by:

$$A(Y) \equiv \left\{ a \in C\left([0, 1], \prod_{i=1}^\infty D_i^*\right) : \rho(a(0)) = a(1) \right\}.$$

Notice that $A(Y)$ is the C^*-algebra inverse limit of the D_i^* in a certain homotopical sense. In particular, this C^*-algebra encodes dynamical information about how the fundamental groups of the pieces of the exhaustion interact with each other. We emphasize that the definition of $A(Y)$ depends on the exhaustion $\{Y_i\}$ of Y. We will now define an index map $\sigma: KO_*^{lf}(Y) = KO^0(C_0(Y)) \to KO_*(A(Y))$.

There exists $\varepsilon_0 > 0$ such that, for any closed subspace Z of Y, any operator on a Z-module with propagation less than or equal to $\varepsilon_0 > 0$ can be lifted to the universal cover of Z. One can prove that the above constant ε_0 exists because Y is metrically locally simply connected (as defined in the beginning of Section 2). The proof is similar to that of Proposition 2.8.

If an operator F represents a class in $KO_0^{lf}(Y)$, for each $\varepsilon < \frac{\varepsilon_0}{100}$, we can choose another operator F_ε representing the same K-homology class such that the propagation of F_ε is smaller than ε. F_ε can be constructed as follows. Let $\{\phi_i\}_i$ be a continuous partition of unity subordinate to an open cover $\{U_i\}_i$ of Y satisfying $diameter(U_i) < \epsilon$. We define $F_\varepsilon = \sum_i (\phi_i)^{\frac{1}{2}} F(\phi_i)^{\frac{1}{2}}$, where the convergence is in strong operator norm. F_ε is equivalent to F in the K-homology group.

Let

$$\mathrm{ind}_L([F_\varepsilon]) = [P_{F_\varepsilon}] - \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right] \in KO_0(C_L^*(Y)),$$

where P_{F_ε} is the idempotent in the matrix algebra of $C_L^*(Y)^+$ as given in the definition of the local index such that the propagation of $P_{F_\varepsilon}(t)$ is less than $100\varepsilon < \varepsilon_0$ for all $t \geq 0$.

Let $P_{F_\varepsilon}^{(j)} = \chi_{Y_j} P_{F_\varepsilon} \chi_{Y_j}$ and let $\widetilde{P}_{F_\varepsilon}^{(j)}$ be the lifting of $P_{F_\varepsilon}^{(j)}$ to $\widetilde{Y}_j$, the universal cover of Y_j. Let v be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing the generator in $KO_{-1}(C_0(\mathbb{R}^7)) \cong KO_0(C_0(\mathbb{R}^8))$ (see Atiyah [1] or Schröder [41, Proposition 1.4.11]). Let

$$\tau_{F_\varepsilon}^{(j)} = v \otimes \widetilde{P}_{F_\varepsilon}^{(j)}(0) + I \otimes (I - \widetilde{P}_{F_\varepsilon}^{(j)}(0))$$

and let

$$(\tau_{F_\varepsilon}^{-1})^{(j)} = v^{-1} \otimes \widetilde{P}_{F_\varepsilon}^{(j)}(0) + I \otimes (I - \widetilde{P}_{F_\varepsilon}^{(j)}(0)).$$

For all $s \in [0, 1]$, define $w_{F_\varepsilon}^{(j)}(s)$ to be the product

$$\begin{pmatrix} I & (1-s)\tau_{F_\varepsilon}^{(j)} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)(\tau_{F_\varepsilon}^{-1})^{(j)} & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau_{F_\varepsilon}^{(j)} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

For each k, there exist $j_k > k$ and a sequence of positive numbers $\{\varepsilon_k\}$ converging to 0 such that $100\varepsilon_k < \varepsilon_0$ and $y_k = w_{F_{\varepsilon_k}}^{(j_k)}(0) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} (w_{F_{\varepsilon_k}}^{(j_k)}(0))^{-1}$ has propagation less than ε_0 for all k, and there is a unique z_k with propagation at most ε_0 in the matrix algebra of $(S^7 C_{max}^*(\widetilde{Y}_{k,j_k})^{\pi_1(Y_{k,j_k})})^+$ such that $y_k = \phi_{k,j_k}(z_k)$, where ϕ_{k,j_k} is the $*$-homomorphism from the matrix algebra of $(S^7 C_{max}^*(\widetilde{Y}_{k,j_k})^{\pi_1(\widetilde{Y}_{k,j_k})})^+$ to the matrix algebra of $(S^7 C_{max}^*(\widetilde{Y}_{j_k})^{\pi_1(Y_{j_k})})^+$. Note that the existence of such a $*$-homomorphism follows from Lemma 2.12 and Formula (2.13). The existence and uniqueness of such z_k is a result of the following claim, Proposition 2.8, and the assumption that y_k has small propagation and the requirement z_k has small propagation.

Claim 3.2. *Let $\widetilde{Y}_{j_k}$ be the universal cover of Y_{j_k} and let $\pi_k: \widetilde{Y}_{j_k} \to Y_{j_k}$ be the covering map. Then we have*

$$y_k = \chi_{\pi_k^{-1}(Y_{k,j_k})} y_k \chi_{\pi_k^{-1}(Y_{k,j_k})} \oplus (I - \chi_{\pi_k^{-1}(Y_{k,j_k})})$$

when k and j_k are sufficiently large.

Proof. This proof is identical to that of Claim 2.19. □

Let $\lambda \in [0, 1]$. We define $z_k(\lambda)$ by replacing $P_{F_{\varepsilon_k}}^{(j_k)}$ with $(1-\lambda)P_{F_{\varepsilon_k}}^{(j_k)} + \lambda P_{F_{\varepsilon_{k+1}}}^{(j_{k+1})}$ in the above definition of z_k. Define $y_k(\lambda) = \phi_{k,j_k}(z_k(\lambda))$. Let ψ_k be the natural homomorphism $\psi_k: S^7 C_{max}^*(\widetilde{Y}_{j_k})^{\pi_1(Y_{j_k})} \to S^7 C_{max}^*(\widetilde{Y}_{j_{k+1}})^{\pi_1(Y_{j_{k+1}})}$. Again, the existence of ψ_k follows from Lemma 2.12 and Formula (2.13). Let

$$\tau_k(\lambda) = v \otimes \left((1-\lambda)\psi_k(\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)}) + \lambda \widetilde{P}_{F_{\varepsilon_{k+1}}}^{(j_{k+1})}\right) + I \otimes I - \left((1-\lambda)\psi_k(\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)} + \lambda \widetilde{P}_{F_{\varepsilon_{k+1}}}^{(j_{k+1})})\right)$$

and

$$\tau_k'(\lambda) = v^{-1} \otimes \left((1-\lambda)\psi_k(\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)}) + \lambda \widetilde{P}_{F_{\varepsilon_{k+1}}}^{(j_{k+1})}\right) + I \otimes I - \left((1-\lambda)\psi_k(\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)} + \lambda \widetilde{P}_{F_{\varepsilon_{k+1}}}^{(j_{k+1})})\right)$$

for all $\lambda \in [0, 1]$. Recall that $\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)}$ is an element in the equivariant localization algebra and $\widetilde{P}_{F_{\varepsilon_k}}^{(j_k)}(0)$ is its evaluation at 0.

For all $s, \lambda \in [0, 1]$, define $(w_k(s))(\lambda)$ to be the product

$$\begin{pmatrix} I & (1-s)\tau_k(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)\tau_k'(\lambda) & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau_k(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Define

$$(c_k(s))(\lambda) = (w_k(s))(\lambda) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} ((w_k(s))(\lambda))^{-1}.$$

Note that $(1-\lambda)\psi_k(\widetilde{P}^{(j_k)}_{F_{\varepsilon_k}})+\lambda\widetilde{P}^{(j_{k+1})}_{F_{\varepsilon_{k+1}}}$ is an idempotent outside a small neighborhood of $\pi^{-1}_{k+1}(Y_{j_k,j_{k+1}})$, i.e. if χ is the characteristic function of the complement of the small neighborhood, then

$$\chi((1-\lambda)\psi_k(\widetilde{P}^{(j_k)}_{F_{\varepsilon_k}})+\lambda\widetilde{P}^{(j_{k+1})}_{F_{\varepsilon_{k+1}}})^2=\chi((1-\lambda)\psi_k(\widetilde{P}^{(j_k)}_{F_{\varepsilon_k}})+\lambda\widetilde{P}^{(j_{k+1})}_{F_{\varepsilon_{k+1}}})$$

and

$$\chi((1-\lambda)\psi_k(\widetilde{P}^{(j_k)}_{F_{\varepsilon_k}})+\lambda\widetilde{P}^{(j_{k+1})}_{F_{\varepsilon_{k+1}}})^2=\chi(1-\lambda)\psi_k(\widetilde{P}^{(j_k)}_{F_{\varepsilon_k}})+(\lambda\widetilde{P}^{(j_{k+1})}_{F_{\varepsilon_{k+1}}}).$$

As a consequence, the pair $(z_k(\lambda),(c_k(\cdot))(\lambda))$ lies in $(S^7D^*_k)^+$, where D^*_k is as in the definition of $A(Y)$. Let $a_k=(z_k(\cdot),c_k(\cdot))$.

Let $b=(q(0),q)$ be as in the definition of the relative K-homology class of D in Section 2. Let

$$p=(b,b,\dots,b,\dots)$$

viewed as an element of $(A(Y))^+$. We finally define the index of F in $KO_0(A(Y))$ to be

$$\sigma([F])=[(a_1,a_2,\dots)]-[(p(0),p)]\in KO_0(A(Y)).$$

One can similarly define the index map $\sigma\colon KO^{lf}_n(Y)\to KO_n(A(Y))$ when $n\not\equiv 0 \bmod 8$ with the help of suspensions (refer back to the section after Theorem 2.16).

The proof of the following vanishing theorem contains some of the same elements as are found in Section 2, but now in the context of a noncompact manifold M.

Theorem 3.3. *Let Y be a noncompact space with an admissible exhaustion $\{Y_i\}$. Let M be a noncompact manifold. Assume that there is a uniformly continuous proper coarse map $f\colon M\to Y$ with an admissible exhaustion $\{M_i;M_{i,j}\}$ of M such that each M_i is a compact manifold with boundary ∂M_i, $f^{-1}(Y_i)=M_i$, $f^{-1}(Y_{i,j})=M_{i,j}$ and $f^{-1}(\partial Y_i)=\partial M_i$. Suppose that M is spin and let D_M be the Dirac operator on M. If M admits a metric of uniform positive scalar curvature, then the index $\sigma(f_*[D_M])$ of D_M is zero in $KO_*(A(Y))$, where $f_*\colon KO^{lf}_*(M)\to KO^{lf}_*(Y)$ is the homomorphism induced by f.*

Proof. We assume that the dimension of M is $0 \bmod 8$. The other cases can be handled in a similar way with the help of suspensions (refer back to the section after Theorem 2.16).

Let Y_i, M_i, $Y_{i,j}$ and $M_{i,j}$ be given as in the statement of the theorem. In this proof, all C^*-algebras are the maximal ones.

Let f be an odd smooth chopping function on the real line satisfying the following conditions: (1) $f(x)\to\pm 1$ as $x\to\pm\infty$; (2) $g=f^2-1\in S(\mathbb{R})$, the space of Schwartz functions, (3) if $\widehat{f}$ and $\widehat{g}$ are the Fourier transforms of f and g, respectively, then $\mathrm{Supp}(\widehat{f})\subseteq[-1,1]$ and $\mathrm{Supp}(\widehat{g})\subseteq[-2,2]$. As stated earlier, such an odd chopping function exists (cf. Roe [32, Lemma 7.5]).

Let D_M be the Dirac operator on M. We define

$$F=f(D_M)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(t)\exp(itD_M)dt.$$

Let

$$\mathrm{ind}_L([F])=[P_F]-\left[\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\right]\in KO_0(C^*_L(M)),$$

where P_F is the idempotent in the matrix algebra of $(C^*_L(M))^+$ as given in the definition of the local index.

Recall that there exists $\varepsilon_0>0$ such that, for any closed subspace Z of Y, any operator on a Z-module with propagation less than or equal to ε_0 can be lifted to the universal cover of Z. Define $P^{(j)}_F=\chi_{M_j}P_F\chi_{M_j}$, where χ_{M_j} is the characteristic function of M_j. Let n_0 be the smallest natural number such that $n_0>\frac{10}{\varepsilon_0}$. We write

$$\exp(itD_M)=\underbrace{\exp\left(\frac{it}{n_0}D_M\right)\cdots\exp\left(\frac{it}{n_0}D_M\right)}_{n_0}. \qquad (*)$$

Let $j'>j$ be the smallest integer such that

$$d(M-M_{j'},M_j)>10n_0\varepsilon_0.$$

Here $n_0\varepsilon_0$ is roughly 10. We emphasize that the condition of admissible exhaustion implies the existence of such j'. Let $\widetilde{M}_{j'}$ be the universal cover of $M_{j'}$. Using the formula for P_F in terms of $\exp(itD_M)$, the identity $(*)$ and the fact that $\exp(\frac{it}{n_0}D_M)$ has propagation less than ε_0 for all $t\in[-2,2]$, we obtain a lifting of $P^{(j)}_F$ to $\widetilde{M}_{j'}$. We denote this lifting by $\widetilde{P}^{(j)}_F$. We claim that $P^{(j)}_F$ is an element in $(C^*_L(\widetilde{M}_{j'})^{\pi_1(M_{j'})})^+$. This follows from the formula for $P^{(j)}_F$ in terms of $\chi_{\pi^{-1}_{j'}(M_j)}exp(it\widetilde{D}_{\widetilde{M}_{j'}})\chi_{\pi^{-1}_{j'}(M_j)}$, where $\pi_{j'}$ is the covering map $\widetilde{M}_{j'}$ to $M_{j'}$ and $\chi_{\pi^{-1}_{j'}(M_j)}$ is the characteristic function of $\pi^{-1}_{j'}(M_j)$. The operator $exp(it\widetilde{D}_{\widetilde{M}_{j'}})\chi_{\pi^{-1}_{j'}(M_j)}$ is well defined for all $-2\le t\le 2$ using the unique local solution to the heat equation associated to $\widetilde{D}_{\widetilde{M}_{j'}}$.

For any $i<j$, let $m_{i,j}=i+[\frac{j-i}{2}]$ and $m'_{i,j}=i+[\frac{j-i}{4}]$, where $[\frac{j-i}{2}]$ and $[\frac{j-i}{4}]$ are respectively the integer parts of $\frac{j-i}{2}$ and $\frac{j-i}{4}$.

We define

$$P_F^{(i,j)}=\chi_{M_{m'_{i,j},j}}P_F\chi_{M_{m'_{i,j},j}},$$

where $\chi_{M_{m'_{i,j},j}}$ is the characteristic function of $M_{m'_{i,j},j}$. Let v be an invertible element in the matrix algebra of $C_0(\mathbb{R}^7)^+$ representing the generator in

$$KO_{-1}(C_0(\mathbb{R}^7))\cong KO_0(C_0(\mathbb{R}^8)).$$

Define

$$\tau_{i,j}=v\otimes P_F^{(i,j)}(0)+I\otimes(I-P_F^{(i,j)}(0)).$$

We have

$$\tau_{i,j}^{-1}=v^{-1}\otimes P_F^{(i,j)}(0)+I\otimes(I-P_F^{(i,j)}(0)).$$

Define

$$x_{i,j}=\begin{pmatrix} I & \tau_{i,j}\\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0\\ -\tau_{i,j}^{-1} & I\end{pmatrix}\begin{pmatrix} I & \tau_{i,j}\\ 0 & I\end{pmatrix}\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}$$

and

$$u_{i,j}=x_{i,j}\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}x_{i,j}^{-1}.$$

Let $|j-i|$ be large enough such that

$$d(M_i, M-M_{m_{i,j}})>10n_0\varepsilon_0.$$

We define $v_{i,j}\in(S^7C^*(M_{i,m_{i,j}}))^+$ and $w_{i,j}\in(S^7C^*(M_{m_{i,j},j}))^+$ by:

$$v_{i,j}=\chi_{M_{i,m_{i,j}}}u_{i,j}\chi_{M_{i,m_{i,j}}}+(I-\chi_{M_{i,m_{i,j}}}),$$
$$w_{i,j}=\chi_{M_{m_{i,j},j}}u_{i,j}\chi_{M_{m_{i,j},j}}+(I-\chi_{M_{m_{i,j},j}}).$$

By the propagation of P_F and the formula for $u_{i,j}$, we have

$$u_{i,j}=(v_{i,j}-I)\oplus(w_{i,j}-I)+I.$$

This equality is proved exactly in the same manner as Claims 2.19 and 3.2. From the definitions of $v_{i,j}$ and $w_{i,j}$, we then have $\text{prop}(v_{i,j})<100n_0\varepsilon_0$ and $\text{prop}(w_{i,j})<100n_0\varepsilon_0$.

Let j' be as in the construction of $\widetilde{P}_F^{(j)}$. Let $\widetilde{M}_{i,j'}$ be the universal cover of $M_{i,j'}$ and let $\pi_{i,j'}$ be the covering map from $\widetilde{M}_{i,j'}$ of $M_{i,j'}$. Again using the identity $(*)$ and the formula for P_F in terms of $\exp(itD_M)$ and the small propagation of $\exp(\frac{it}{n_0}D_M)$, we can lift $P_F^{(i,j)}$ to an element $\widetilde{P}_F^{(i,j)}$ in $(C^*(\widetilde{M}_{i,j'})^{\pi_1(M_{i,j'})})^+$, where j' is defined as in the construction of the lifting of $P_F^{(j)}$. Let $\widetilde{u}_{i,j}$ be the lifting of $u_{i,j}$ to $\widetilde{M}_{i,j'}$. Let $|j-i|$ be large enough such that

$$d(M_i, M-M_{m_{i,j}})>100n_0\varepsilon_0.$$

We define $\widetilde{v}_{i,j}\in(S^7C^*(\widehat{M}_{i,m_{i,j}})^{\pi_1(M_{i,j'})})^+$ and $\widetilde{w}_{i,j}\in(S^7C^*(\widehat{M}_{m_{i,j},j'})^{\pi_1(M_{i,j'})})^+$ to be the liftings of $v_{i,j}$ and $w_{i,j}$ respectively, where $\widehat{M}_{i,m_{i,j}}=\pi_{i,j'}^{-1}(M_{i,m_{i,j}})$ and $\widehat{M}_{m_{i,j},j'}=\pi_{i,j'}^{-1}(M_{m_{i,j},j'})$. We have

$$\widetilde{u}_{i,j}=(\widetilde{v}_{i,j}-I)\oplus(\widetilde{w}_{i,j}-I)+I.$$

Next we shall represent the index class $\sigma([D_M])$ as a KO-theory element explicitly constructed using the above liftings.

Let $\{j_k\}$ be a sequence of integers such that $j_k>k$ for each k and $j_k-k\to\infty$ as $k\to\infty$. Let z_k be the image of $\widetilde{w}_{k,j_k}$ under the inclusion map from $(S^7C^*(\widehat{M}_{m_{k,j_k},j'_k})^{\pi_1(M_{k,j'_k})})^+$ to $(S^7C^*(\widetilde{M}_{k,j'_k})^{\pi_1(M_{k,j'_k})})^+$. Let $\pi_{j'_k}$ be the covering map from the universal cover of $\widetilde{M}_{j'_k}$ to $M_{j'_k}$, where j'_k be as in the construction of $\widetilde{P}_F^{(j_k)}$. Let y_k be the element in the image of the inclusion map from $(S^7C^*(\pi_{j'_k}^{-1}(M_{k,j'_k}))^{\pi_1(M_{j'_k})})^+$ to $((S^7C^*(\widetilde{M}_{j'_k}))^{\pi_1(M_{j'_k})})^+$ defined by

$$y_k=\phi_{k,j'_k}(z_k),$$

where ϕ_{k,j'_k} is the homomorphism from $(S^7C^*(\widetilde{M}_{k,j'_k})^{\pi_1(M_{k,j'_k})})^+$ to $(S^7C^*(\widetilde{M}_{j'_k})^{\pi_1(M_{j'_k})})^+$. Note that this homomorphism is constructed in Lemma 2.12 and Formula (2.13).

As before, let ψ_k be the natural map $: S^7C^*(\widetilde{M}_{j'_k})^{\pi_1(M_{j'_k})} \to S^7C^*(\widetilde{M}_{j'_{k+1}})^{\pi_1(M_{j'_{k+1}})}$. We similarly define $z_k(\lambda)$ by replacing $P_F^{(j_k)}(0)$ with $(1-\lambda)P_F^{(j_k)}(0)+\lambda P_F^{(j_{k+1})}(0)$ in the definition of z_k. Note here that $P_F^{(j_k)}$ is an element in the localization algebra and $P_F^{(j_k)}(0)$ is its evaluation at 0. We define $y_k(\lambda)=\phi_{k,j_k}(z_k(\lambda))$.

Let

$$\tau_k(\lambda)=v\otimes\Big((1-\lambda)\psi_k(\widetilde{P}_F^{(j_k)}(0))+\lambda\widetilde{P}_F^{(j_{k+1})}(0)\Big)+I\otimes\Big(I-\Big((1-\lambda)\psi_k(\widetilde{P}_F^{(j_k)}(0))+\lambda\widetilde{P}_F^{(j_{k+1})}(0)\Big)\Big)$$

and

$$\tau'_k(\lambda)=v^{-1}\otimes\Big((1-\lambda)\psi_k(\widetilde{P}_F^{(j_k)}(0))+\lambda\widetilde{P}_F^{(j_{k+1})}(0)\Big)+I\otimes\Big(I-\Big((1-\lambda)\psi_k(\widetilde{P}_F^{(j_k)}(0))+\lambda\widetilde{P}_F^{(j_{k+1})}(0)\Big)\Big)$$

for all $\lambda\in[0,1]$. For all $s,\lambda\in[0,1]$, define $(w_k(s))(\lambda)$ to be the product

$$\begin{pmatrix} I & (1-s)\tau_k(\lambda) \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -(1-s)\tau'_k(\lambda) & I \end{pmatrix}\begin{pmatrix} I & (1-s)\tau_k(\lambda) \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Define

$$(c_k(s))(\lambda)=(w_k(s))(\lambda)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}((w_k(s))(\lambda))^{-1}.$$

Note that $(1-\lambda)\psi_k(\widetilde{P}^{(j_k)}(0))+\lambda\widetilde{P}_F^{(j_{k+1})}(0)$ is an idempotent outside a small neighborhood of $\pi^{-1}_{j'_{k+1}}(M_{j'_k,j'_{k+1}})$, where $\pi^{-1}_{j'_{k+1}}$ is the covering map from $\widetilde{M}_{j'_{k+1}}$ to $M_{j'_{k+1}}$.

As a consequence, for each $\lambda\in[0,1]$, the pair $(z_k(\lambda),(c_k(\cdot))(\lambda))$ lies in $(S^7D_k^*)^+$, where D_k^* is as in the definition of $A(Y)$.

Let $a_k=(z_k(\lambda),(c_k(\cdot))(\lambda))$. Let $b=(q(0),q)$ be as in the definition of the relative K-homology class of D in Section 2. Let

$$p=(b,b,\dots,b,\dots)$$

viewed as an element of $(A(Y))^+$. By a homotopy invariance argument, we have

$$\sigma([D_M])=[(a_1,a_2,\dots)]-[(p(0),p)]\in KO_0(A(Y)).$$

In the above construction, for each $\alpha\geq 1$, we can replace respectively the Dirac operator D_M by αD_M, n_0 by $[\alpha n_0]+1$, and j'_k by another natural number $j'_{k,\alpha}$ satisfying $d(M-M_{j'_{k,\alpha}})>10\alpha n_0\varepsilon_0$ to obtain the index of αD_M:

$$\sigma([\alpha D_M])=[(a_{1,\alpha},a_{2,\alpha},\dots)]-[(p(0),p)]\in KO_0(A(Y)).$$

Notice that the KO-theory class $\sigma([\alpha D_M])\in KO_0(A(Y))$ is independent of the choice of α.

For all k, we write $a_{k,\alpha}=(z_{k,\alpha},c_{k,\alpha})$. Let $\tau_{k,\alpha}$ be obtained by replacing D with αD in the definition of τ_k. By the assumption that M has uniform positive scalar curvature and the local nature of the Lichnerowicz formula, we have

$$\tau_{k,\alpha}\to v\otimes\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}+I\otimes\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

in the operator norm when $\alpha\to\infty$. This result implies the vanishing of $\sigma([D_M])$. □

Using the above notation D_i^*, we note that, by Guentner–Yu [20], there is a Milnor exact sequence given by

$$0\to\varprojlim{}^1 KO_*(D_i^*)\to KO_*(A(Y))\to\varprojlim KO_*(D_i^*)\to 0. \qquad (*)$$

This sequence gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim{}^1 KO_*(Y_i,\partial Y_i) & \longrightarrow & KO_*^{lf}(Y) & \longrightarrow & \varprojlim KO_*(Y_i,\partial Y_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim{}^1 KO_*(D_i^*) & \longrightarrow & KO_*(A(Y)) & \xrightarrow[\phi]{} & \varprojlim KO_*(D_i^*) & \longrightarrow & 0
\end{array}$$

where the map $\phi: KO_*(A(Y))\to\varprojlim KO_*(D_i^*)$ is induced by the $*$-homomorphism $\pi_i: A(Y)\to D_i^*$ from

$$A(Y)\equiv\left\{a\in C\left([0,1],\prod_{i=1}^{\infty}D_i^*\right):\rho(a(0))=a(1)\right\}$$

to D_i^* obtained from the ith component of the evaluation at 0. We will use this diagram in the next section.

4. A manifold with exotic positive scalar curvature behavior

We will now construct a noncompact manifold M endowed with a nested exhaustion of compact subsets M_i, such that the M_i can be endowed with positive scalar metrics which are in totality incompatible in the sense that M itself has no metric of uniformly positive scalar curvature.

In the last section we introduced a Milnor exact sequence with a $\varprojlim^1$ term. We quickly review some properties of this functor. If $\{G_i\}$ is an inverse system of abelian groups indexed by the positive integers together with a coherent family of maps $f_{j,i}\colon G_j \to G_i$ for all $j \geq i$, then $\varprojlim^1 G_i$ is defined in category theory to be the first derived functor of $\varprojlim$. Eilenberg–Moore [13] also provides a description as follows. If $\Psi\colon \prod G_i \to \prod G_i$ is defined by $\Psi(g_i) = (g_i - f_{i+1,i}(g_i))$, then $\varprojlim^1 G_i$ is defined by $\varprojlim^1 G_i \equiv \operatorname{coker}(\Psi)$. Gray [18] proves that, if each G_i is countable, then $\varprojlim^1 G_i$ is either zero or uncountable.

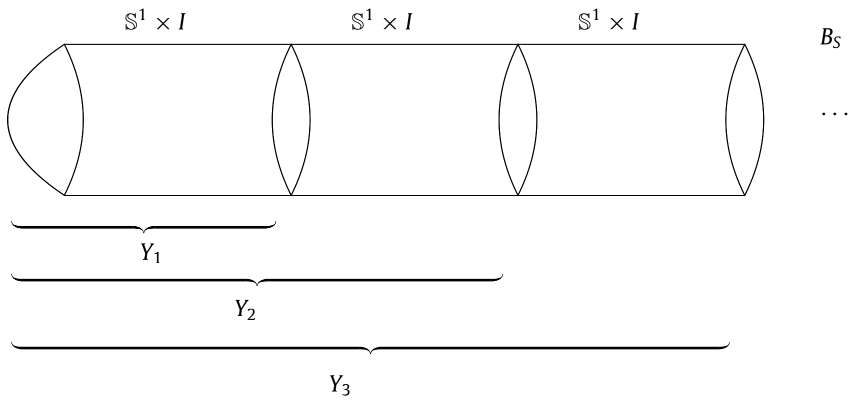
An example of an inverse system with a nontrivial $\varprojlim^1$ term is

$$S = \left\{ \mathbb{Z} \overset{\times 3}{\longleftarrow} \mathbb{Z} \overset{\times 3}{\longleftarrow} \cdots \right\}$$

in which case we have the uncountable group $\varprojlim^1 S = \widehat{\mathbb{Z}}_3/\mathbb{Z}$. Let $\mathbb{S}^1$ denote the standard circle. Consider the composite mapping cylinder B_S of the infinite composite

$$\mathbb{S}^1 \leftarrow \mathbb{S}^1 \leftarrow \mathbb{S}^1 \leftarrow \cdots$$

which is capped off at the left end (see picture below), where each map takes $z \in \mathbb{S}^1$ to z^3.



Let Y_j be the given exhaustion of B_S. For each i, let $\phi_i\colon (Y_{i+1}, \partial Y_{i+1}) \to (Y_i, \partial Y_i)$ be the obvious collapse map. Notice that Y_i is contractible and that ∂Y_j is a circle for all j, but the comparison map $\partial Y_a \to \partial Y_b$ has degree 3^{b-a}. Consider the sequence

$$0 \to \varprojlim{}^1 KO_n(Y_j, \partial Y_j) \to KO_n^{lf}(B_S) \to \varprojlim KO_n(Y_j, \partial Y_j) \to 0.$$

Proposition 4.1. *The group $\varprojlim^1 KO_2(Y_j, \partial Y_j)$ is nontrivial.*

Proof. We have an exact sequence

$$\widetilde{KO}_2(Y_i) \to \widetilde{KO}_2(Y_i/\partial Y_i) \cong KO_2(Y_i, \partial Y_i) \to \widetilde{KO}_1(\partial Y_i) \to \widetilde{KO}_1(Y_i)$$

By the contractibility of Y_i, we have $\widetilde{KO}_2(Y_i) = 0$ and $\widetilde{KO}_1(Y_i) = 0$. Therefore $KO_2(Y_i, \partial Y_i) \to KO_1(\partial Y_i)$ is an isomorphism. Consider the commutative square:

$$\begin{array}{ccc} KO_2(Y_i, \partial Y_i) & \xrightarrow[\cong]{\partial} & \widetilde{KO}_1(\partial Y_i) \\ \downarrow \phi_* & & \downarrow \times 3 \\ KO_2(Y_{i-1}, \partial Y_{i-1}) & \xrightarrow[\cong]{\partial} & \widetilde{KO}_1(\partial Y_{i-1}) \end{array}$$

It follows that ϕ_* is multiplication by 3. $\square$

Theorem 4.2. *Let $c \in KO_*^{lf}(B_S)$, where B_S is endowed with the exhaustion by compact sets $\{Y_i\}$ as above. There is $(M, f) \in \Omega^{spin,lf}(B_S)$ such that M is a non-compact spin manifold and f is a proper map from M to B_S satisfying*

(1) $f_*[D_M] = c$;
(2) *the inverse images* $(M_i, \partial M_i) = f^{-1}(Y_i, \partial Y_i)$ *are compact manifolds with boundary such that the induced maps* $\pi_1(M_i) \to \pi_1(Y_i)$ *and* $\pi_1(\partial M_i) \to \pi_1(\partial Y_i)$ *are all isomorphisms.*

Proof. We can replace B_S by a properly homotopy equivalent manifold with boundary by thickening an embedding of it in a high-dimensional Euclidean space. Consider a class (W^n, g) where $g: W^n \to B_S$ is surjective and proper. We also suppose that $n > \dim(B_S) + 4$. Note that, by Wall [44] section 1A, all compact spin manifolds of dimension at least 4 are spin cobordant to simply connected ones. Moreover, spin manifolds with boundary are cobordant rel boundary to simply connected ones. We can use this notion inductively over the skeleta of a triangulation of B_S to arrange (by a cobordism) that, for each simplex Δ, the transverse inverse image of Δ in W is simply connected. Recall that, after an arbitrarily small perturbation, maps can be made transverse to given sub-objects of manifolds; the inverse images of the perturbed maps are called the transverse inverse images. (See also the picture on page 122 of Wall [44] for the extension of solutions on a skeleton to the whole space).

Then one obtains a map $f: M \to B_S$ from a noncompact spin manifold M so that the inverse image of each simplex is simply connected, and therefore, for every subcomplex T of B_S, the inverse image $f^{-1}(T)$ has the same fundamental group as T. In particular, for the transverse inverse images A' of annular regions $A_s = Y_s - Y_{s+1}$ of B_S, we have $\pi_1(A') = \mathbb{Z}$, with inner and outer boundary components mapping in by the identity and $\times 3$, respectively. In other words, we have proven property (2). □

Theorem 4.3. *Let* ξ *be a nonzero class* $\varprojlim^1 KO_{n+1}(Y_j, \partial Y_j)$ *and consider* ξ *also as an element of* $KO_n^{lf}(B_S)$*. Let* M *be as given in the above theorem with the exhaustion* $(M_i, \partial M_i)$*. Then each* M_i *has a metric of positive scalar curvature which is collared at the boundary, but* M *itself does not have a metric of uniformly positive scalar curvature.*

Proof. We can choose a metric on $Y = B_S$ such that the map f in Theorem 4.2 is a uniformly continuous proper coarse map.

We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim^1 KO_{n+1}(M_i, \partial M_i) & \longrightarrow & KO_n^{lf}(M) & \longrightarrow & \varprojlim KO_n(M_i, \partial M_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim^1 KO_{n+1}(Y_i, \partial Y_i) & \longrightarrow & KO_n^{lf}(B_S) & \longrightarrow & \varprojlim KO_n(Y_i, \partial Y_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim^1 KO_{n+1}(D_i^*) & \longrightarrow & KO_n(A(B_S)) & \longrightarrow & \varprojlim KO_n(D_i^*) & \longrightarrow & 0.
\end{array}$$

By the definition of D_i^* and homotopy invariance of the fundamental group, we have

$$D_i^* \cong C^*_{max}(\pi_1(Y_i), \pi_1(\partial Y_i)) \otimes \mathcal{K}.$$

Since the Y_i are contractible and the ∂Y_i are circles, the outer vertical arrows from the second to third row are isomorphisms by Theorem 2.16, so the map $KO_n^{lf}(B_S) \to KO_n(A(B_S))$ is also an isomorphism. Note that by the choice of M, the element ξ in $KO_n^{lf}(B_S)$ will lift to the Dirac class $[D_M]$ in $KO_n^{lf}(M)$. By the commutativity of the diagram, the image of $[D_M]$ is zero in $\varprojlim KO_n(Y_i, \partial Y_i)$ so it is zero in $\varprojlim KO_n(D_i^*)$. Therefore it is zero in each $KO_n(D_i^*)$.

Finally note that the relative Gromov–Lawson–Rosenberg conjecture holds in our case. The relevant group at any point in the exhaustion is $\Omega_n^{spin}(e, \mathbb{Z})$, where e is the trivial group. However since

$$\Omega_{n-2}^{spin}(e) \xrightarrow{\times \mathbb{D}^2} \Omega_n^{spin}(e, \mathbb{Z})$$

is an isomorphism, and since the index-theoretic obstruction is not lost by crossing with $\mathbb{S}^1$, the conclusion follows from the Gromov–Lawson surgery theorem and Stolz's classification of high-dimensional simply connected spin manifolds with positive scalar curvature (Stolz [42]) for $n \geq 7$. Therefore there is a positive scalar curvature metric on each piece M_i of the exhaustion that is collared around the boundary. Moreover the image of $[D_M]$ is nonzero in $KO_n(A(B_S))$ so M itself has no metric of uniformly positive scalar curvature by Theorem 3.3. □

5. A manifold with uncountably many connected components of positive scalar curvature metrics

In this section we use the previously developed theory to identify a connected noncompact manifold M such that $PS(M)$, the space of complete positive scalar curvature metrics on M equipped with the C^∞-topology, has uncountably many connected components.

In various spin cases, it can be shown using index theory that $PS(M)$ has infinitely many concordance classes. In fact, one can prove that the 7-sphere $\mathbb{S}^7$ is such a manifold (see Gromov–Lawson [19] or Lawson–Michelsohn [3]). Because the set of positive scalar curvature metrics is open in the space of all Riemannian metrics, it is a point-set topological fact that $PS(M)$ has at most countably many components when M is compact. These properties may fail in the noncompact case. In the compact open topology, positivity is not necessarily an open condition. In the uniform topology, we do not have separability.

In the proof of the following theorem, we refer the reader to the paper of Xie–Yu [46, Theorem A], which develops the notion of a relative higher index $\operatorname{ind} D_{g_1,g_2}$ on a closed spin manifold N with two Riemannian metrics g_1 and g_2. This relative index is defined to be the higher index of the Dirac operator D_{g_1,g_2} on the infinite cylinder $N \times \mathbb{R}$, where the cross section $N \times \{x\}$ is endowed with g_1 if $x < -1$ and with g_2 if $x > 1$ and the metric in $N \times [-1, 1]$ can be chosen to be arbitrary. See Xie–Yu [46]. The nonvanishing of this relative index in $KO_*(C_r^*\pi_1(N))$ gives information about the concordance classes of positive scalar curvature metrics on N. In the case when the manifold M is not compact but has an admissible exhaustion by compact sets, a similar theory shows that a relative higher index can be constructed in $KO_*(A(M))$, where $A(M)$ is the algebra constructed in Section 3.

Prior to the theorem we also make the following observation. Let π be a fixed finitely presented group with generators $g_1, \ldots, g_s$ and relations $r_1, \ldots, r_t$. Let $n \geq 5$, Execute s successive 0-surgeries on $\mathbb{S}^n$ to produce a manifold K' with fundamental group F_s, the free group on s generators. The process of *surgery on maps* (see Wall [44] section 1A for explicit details) shows that one can then perform a 1-surgery on K' to produce a manifold with fundamental group $F_s/\langle r_1\rangle$, where $\langle r_1\rangle$ is the subgroup of F_s normally generated by r_1. After performing these 1-surgeries successively with respect to $r_2, \ldots, r_t$, we obtain a manifold K with fundamental group π. Since K is constructed from the sphere $\mathbb{S}^n$ from surgeries of codimension at least 3, it follows from the Gromov–Lawson surgery theorem [19, Theorem A] that K also has a metric of positive scalar curvature.

A trivial example of a manifold with uncountably many components of positive scalar curvature metrics is the disjoint union of countably many copies of $\mathbb{S}^7$. Here we present a connected example.

Theorem 5.1. *There is a connected noncompact manifold M for which the set $PS(M)$ of components of uniformly positive scalar curvature metrics on M is uncountable.*

Proof. Let β and β' be two non-concordant metrics of positive scalar curvature on $\mathbb{S}^7$ detected, as in Gromov–Lawson [19] by a relative index; they give rise to two non-concordant metrics α and α' of positive scalar curvature on $N = \mathbb{S}^1 \times \mathbb{S}^7$ detected by relative higher index. Consider the iterated connected sum given by

$$M = N\#N\#\cdots$$

with the obvious exhaustion $M_i = (\underbrace{N\#\cdots\#N}_{i}) - \mathbb{D}^n$. On each summand N we make a choice to endow N with either α or α'. Apply the Gromov–Lawson surgery theorem to modify the metric near each glueing so that M is positively curved at every point. Clearly the number of metrics on M constructed in this way is uncountable, and these metrics are all in different connected components of $PS(M)$ by an application of our relative higher index, which lies in $KO_*(A(M))$ as explained in the following. If β, β' are two distinct metrics on M defined in this way, let $D_{\beta,\beta'}$ be the Dirac operator on the product $M \times \mathbb{R}$. Here the metric on $M \times (-\infty, -1)$ is defined using the product metric of β on M and the standard metric on $(-\infty, -1)$, and the metric on $M \times (1, \infty)$ is defined using the product metric of β' on M and the standard metric on $(1, \infty)$. The metric on $M \times [-1, 1]$ can be an arbitrary complete metric. We can define a relative higher index $\operatorname{ind}(D_{\beta,\beta'})$ of $D_{\beta,\beta'}$ in $KO_*(A(M))$. By the relative higher index theorem in Xie–Yu [46, Theorem A], the relative higher index $\operatorname{ind}(D_{\alpha,\alpha'})$ does not lie in the image of the map $i_*: KO_*(\mathbb{R}) \to KO_*(C_r^*\pi_1(N))$, where $\mathbb{R}$ is the one-dimensional real C^*-algebra and $i: \mathbb{R} \to C_r^*\pi_1(N)$ is the inclusion map. The Pimsner Theorem (see [29, Theorem 18]) allows us to compute $KO_*(D_i^*)$. The above facts and the relative index theorem of [46] imply that if β, β' are two distinct metrics on M defined as above, then $(\pi_i)_*(\operatorname{ind}(D_{\beta,\beta'}))$ is nonzero in $KO_*(D_i^*)$ for i sufficiently large. Here $\pi_i: A(M) \to D_i^*$ is the $*$-homomorphism defined in Section 3. The restriction map to $\varprojlim$ in the Milnor sequence $(*)$ given after Theorem 3.3 tells us that the index is nonzero in $KO_*(A(M))$. Therefore β and β' are two metrics of M that lie in different connected components of $PS(M)$. Since β and β' are arbitrary, it follows that $PS(M)$ has uncountably many components. □

Remark 5.2. A more general construction that provides a host of examples can be given as follows. Let W be an $(n+1)$-dimensional spin manifold with nontrivial higher $\widehat{A}$-genus. For example, we may take W to be the torus T^{n+1}. Let $\pi = \pi_1(W)$. By the discussion above, we can produce a manifold N^n with a positive scalar curvature metric α and $\pi_1(N) = \pi$. We can perform a 0-surgery on the disjoint union of $N \times I$ and W to create a connected manifold X'. Execute additional surgeries (via surgery on maps) on X' to arrive at a manifold X with fundamental group π and two boundaries components both homeomorphic to N. (In other words, since $\pi' = \pi * \pi$, we kill each element of π' of the form $g_1g_2^{-1}$, where g_1 and g_2 represent the same element of π.)

Let α' be the positive scalar curvature metric on N as the other boundary component of X, as constructed by the Gromov–Lawson surgery theorem. Let $D_{\alpha,\alpha'}$ be the Dirac operator on $N \times \mathbb{R}$. Here the Riemannian metric on $N \times (-\infty, -1)$ is defined using the product metric of α on N and the standard metric on $(-\infty, -1)$, and the metric on $N \times (1, \infty)$ is defined using the product metric of α' on N and the standard metric on $(1, \infty)$. The metric on $N \times [-1, 1]$ can be arbitrary. From the nontriviality of the higher $\widehat{A}$-genus for W, we can infer that the relative higher index $\operatorname{ind} D_{\alpha,\alpha'}$ of N is nonzero in $KO_*(C_r^*\pi)$. We then form the iterated connected sum $M = N\#N\# \cdots$ and proceed as before.

In [7], we provide examples of noncompact contractible spaces with exotic positive scalar curvature behavior. In particular, for certain Davis manifolds, the universal cover has uncountably many nonconcordant positive scalar curvature metrics.

We refer the readers to following articles for useful general background information relevant to this article: [8,9,23, 25–28,31].

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