Surgery Theoretic Methods in Group Actions

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This paper is intended to give a brief introduction to the applications of the ideas of surgery in transformation group theory; it is not intended to be any kind of survey of the latter theory, whose study requires many additional insights and methods. However, despite this disclaimer, there have been a number of signal achievements of the surgery theoretic viewpoint, notably in the directions of producing examples and, on occasion, giving complete classifications of particular sorts of actions.

We have divided this paper into three sections which deal with three different variants of classical surgery. The first deals with "CW surgery", i.e., surgery in the category of CW complexes. Then we discuss some *ad hoc* applications of classical surgery of closed manifolds in various problems, as opposed to the development of equivariant surgery theories. The latter point of view is discussed in the last section. We shall only briefly describe the theory of "pseudoequivalence" [Pe2] and its successes, which have tended to lie in the construction of actions with unusual properties, but instead concentrate on classification theorems for actions up to equivariant isomorphism.

With deep regret, we will deal exclusively with finite group actions in this paper because of space requirements. We also highly recommend the 15 year old conference survey [Sch3] for its many summaries of the state of the art at that turning point in its development.

This paper is dedicated to C.T.C. Wall, whose powerful and inspirational contributions to this story include the finiteness obstruction, non-simply connected surgery obstruction groups (their definition, application, and calculation), and his *tour de force* on free actions on the sphere: i.e., fake lens spaces and the space form problem.

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1 The CW Category

The analogue of the theory of surgery for CW complexes is the calculus of attaching handles to a given complex to produce a finite CW complex (weak) homotopy equivalent to another specified space. In the most classical context this is the combination of Wall's finiteness obstruction theory with Whitehead's simple homotopy theory. We begin our survey of equivariant surgery theory with a discussion of several of the high-points achieved by just CW theory.

1.1 The space form problem

This problem, to which we will return in the next section, is "Which groups act freely on some homotopy sphere or, in particular, on the standard sphere?" One can see ([Wo]) that there is a free action by (linear) isometries on some standard sphere (and the dimension can be computed) if and only if all subgroups of order pq, where p and q are not necessarily distinct primes, are cyclic.

The first nontrivial result on the purely topological problem is due to P. A. Smith who showed that if G acts freely on a finite dimensional complex homotopy equivalent to a sphere, then every subgroup of order p^2 is cyclic. (Cartan and Eilenberg refined this to the statement that if G acts freely on a homotopy S^d, then G has periodic cohomology with period $d+1$; we will see the reason momentarily.)

An early high point in "CW surgery" was Swan's proof [Sw] of a converse:

Theorem 1.1 *A group G acts freely on a finite dimensional complex X homotopy equivalent to S^{d-1} if and only if G has d periodic cohomology (which, according to Cartan and Eilenberg, is equivalent to all subgroups of order p^2 being cyclic, for p prime.) Moreover, X can be taken to be a finite complex if and only if an obstruction $w_d(G) \in \widetilde{K}_0(\mathbb{Z}G)/\mathrm{Sw}$ vanishes.*

Here, Sw denotes the subgroup of $\widetilde{K}_0(\mathbb{Z}G)$ represented by finite modules of order prime to $\#(G)$, with trivial action. (This is the image of a natural homomorphism, the Swan homomorphism $\mathrm{Sw}\colon (\mathbb{Z}/\#(G))^* \to \widetilde{K}_0(\mathbb{Z}G)$) which assigns, to an integer n, the Euler characteristic of a finite $\mathbb{Z}G$-projective resolution of $\mathbb{Z}/n\mathbb{Z}$ viewed as a module over $\mathbb{Z}G$ by giving it the trivial action.

Since $\widetilde{K}_0(\mathbb{Z}G)$ is a finite abelian group (another fundamental theorem of Swan) and, as will be apparent from the definition, $w_{kd}(G) = kw_d(G)$, it follows that there always exists a finite complex on which G acts freely, but of undetermined dimension.

[Note for instance that dihedral groups D_{2p} (indeed, all of the metacyclic groups, $M_{pq}, q \equiv 1 \bmod p$) satisfy the conditions of Swan's theorem and hence they act on finite complexes homotopy equivalent to a sphere. We will see in the next sections that the dihedral groups don't, however, act on a sphere, but the odd order metacyclic groups do.]

Wall [Wa2] had shown that $w_d(G)$ always has order 1 or 2 and Milgram [Mi] gave the first examples where $w_d(G) \neq 0$ (see also [Da] for different examples).

Essentially, Swan's method is to show that if G has periodic cohomology then $\mathbb{Z}$ has a periodic resolution over $\mathbb{Z}G$. More precisely, there are finitely generated projective modules P_i such that one has an exact sequence

$$0 \to \mathbb{Z} \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0.$$

This chain complex is chain equivalent to the equivariant chain complex for the G-action on X if there is such an action; and, conversely, from such a chain complex Swan observes one can build a well defined equivariant homotopy type. (If X is finite, one can, of course, use the cellular chain complex of X as a finite free resolution.)

Even in the case of cyclic groups, the "resolution" is not well defined, because the equivariant homotopy type can be varied. The indeterminacy is caught by the "k-invariant" in $(\mathbb{Z}/\#(G))^*$ (which is the degree of any equivariant map between these homotopy spheres; this is a number prime to $\#(G)$, by the Borsuk-Ulam theorem, and it is well defined up to $\#(G)$, by obstruction theory).

The periodicity of $H^*(G)$ follows from splicing together such resolutions to obtain:

$$\cdots \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to P_d \to P_{d-1} \to \cdots P_1 \to P_0 \to \mathbb{Z} \to 0$$

and using such a periodic resolution to compute group cohomology. Such a splicing trick also verifies the formula for $w_{kd}(G)$ mentioned above.

Swan's work has been greatly extended by tom Dieck [tD] who has studied (substantially in joint work with Petrie [tD-P]) the theory of "homotopy representations," which consists of $G-CW$ complexes where the the fixed sets of every subgroup are homotopy spheres. There are two kinds of invariants for these: dimension functions (like "d") and "generalized degrees" which are the analogues of the k-invariants.

Even for cyclic groups, there are values of dimension functions on the set of subgroups that are realized by homotopy representations, but which don't arise for geometric representations. The simplest is for $G = \mathbb{Z}_p$, $d(e) = d(G)$, which at least doesn't arise for effective linear actions. (The degree can be an arbitrary number prime to p.) More significantly, one can have for

$G = \mathbb{Z}_{pq}, d(G) = -1$ (no fixed points) $d(\mathbb{Z}_p) = \delta, d(\mathbb{Z}_q) = \delta'$ and $d(\mathbb{Z}_{pq}) = \delta''$ whenever δ, δ', and δ'' have the same parity and $\delta'' \geq \max(\delta, \delta')$; for linear representations $\delta'' \geq \delta + \delta'$.

Indeed, tom Dieck's work shows exactly when all dimension functions for a group are linear, and when this is true in the sense of a Grothendieck group. However, here the manifold realization problems are much less well understood than in the traditional space form problem; there are, in addition, a host of problems that arise when the "gap hypothesis" fails, i.e., for the situation where the dimensions of fixed point sets can be large relative to the dimension of the ambient complex (e.g., when $\delta'' \leq 2\max(\delta, \delta')$ in the $\mathbb{Z}_{pq}$ example). We will return to some of these issues in section 4.

1.2 Semifree actions on the disk

The first nonfree problem that commands study is, no doubt, the problem of $\mathbb{Z}_p$ actions on the disk, or for our present CW purposes, contractible CW complexes. We will discuss the case for general G. (Obviously, general G cannot act semifreely on a genuine disk – e.g., $G = \mathbb{Z}_p \times \mathbb{Z}_p$ cannot[1], but it can, however, act on a contractible complex, as G acts on the cone $c(G)$.)

For this problem one has a definitive solution, due to Jones [Jo1] for the cyclic case (where Sw is trivial), and Assadi [As] in general:

Theorem 1.2 *A finite CW complex F is homotopy equivalent to the fixed point set of a semifree G-action on a contractible finite complex if and only if*

(i) $\widetilde{H}_i(F;\mathbb{Z})$ *is a finite group of order prime to* $\#(G)$*, all* i*; and,*

(ii) $\Sigma(-1)^i \mathrm{Sw}(\#(\widetilde{H}_i(F)) = 0$ *in* $\widetilde{K}_0(\mathbb{Z}G)$*.*

Here Sw is the Swan homomorphism described above. The necessity of condition (i) is a conclusion of Smith theory. Bredon's book [Br] is a valuable textbook reference. Note that in this theorem the finiteness obstruction lies in precisely the subgroup that we threw away in the space form problem. Condition (i) alone suffices to make F the fixed point set on a G-ANR. (With no finiteness at all, one should reformulate (i) as being mod $\#(G)$ acyclicity, and then it is necessary and sufficient.)

The proof of this theorem comes about by trying to attach free cells equivariantly to obtain a finite contractible complex. After one has induc-

[1] This follows from the observation that the "p^2 condition" discussed above (in 1.1) is necessary even for free action on mod p homology spheres, e.g. to the complement of the fixed set in the disk. (See Theorem 1.2 (i).)

tively killed all homology through $\dim(F)$, there remains (under (i)) a projective module, which can be identified up to sign with $\Sigma(-1)^i \mathrm{Sw}(\#(\widetilde{H}_i(F))$. As in the Wall finiteness theory, this is stably free if and only if one can attach cells to remove this final bit of homology.

(Obtaining the G-ANR is a fairly simple "Eilenberg swindle" and will be discussed in the final section: see Theorem 3.2 and the surrounding discussion.)

1.3 Fixed sets for nonfree G-actions on the disk

If G is a p-group then an inductive Smith theory argument shows that the fixed set of a G-action on a finite dimensional mod p acyclic space (and, in particular, a contractible one) is mod p acyclic. Conversely, as G has a cyclic quotient, Jones's theorem from the previous subsection implies that any mod p acyclic finite complex is the fixed set of a G-action on some finite contractible CW complex.

On the other hand, for non-p-groups the situation is more complicated. For G cyclic, the Brouwer fixed point theorem implies that the fixed set is nonempty. Indeed, its Euler characteristic $e(F)$ must be 1, by a refinement of the Lefschetz fixed point theorem ($e(F) = L(f)$ for f a periodic map). Also, in general, for a p-group P acting on a space Y, $e(Y^p) \equiv e(Y) \mathrm{mod} p$. Combining these observations one can obtain necessary congruence conditions for fixed point sets. Oliver [O] showed that these are almost enough; occasionally, there are more refined congruences that hold, i.e., the primes that actually occur arise to a higher power than one would expect from just the preceding analysis:

Theorem 1.3 *If G is not a p-group, then there is a number $n(G)$ (which is readily computable from G) such that F is homotopy equivalent to the fixed point set of a simplicial G-action on a contractible finite complex if and only if $e(F) \equiv 1$ mod $n(G)$.*

For instance, $n(G) = 1$ if and only if the empty set can arise as a fixed point set, and then any F can arise. (Indeed, it turns out that by an explicit construction one can show that if $n(G) = 1$, then every polyhedron occurs as the fixed point set of some PL G-action on some disk!)

Quinn [QI, II] observed that for G-ANR's the same theorem holds with a possibly smaller value of $n(G)$, (i.e., there is another number $m(G)$ playing an analogous role.) This is closely related to the equivariant topological Whitehead theory discussed below.

Remark. Many of these theorems can be extended to more general "surgery theoretic" situations, and could then be viewed as analogues of a "Kervaire-

Milnor" type surgery theory. Only some of this is in the literature, to our knowledge. See, e.g. [OP] for the simply connected case of this perspective.

2 Ad Hoc Applications of Surgery

In this section we will describe several of the many applications of ordinary, as distinguished from distinctly equivariant, surgery theory. The next section deals with genuinely equivariant surgery.

2.1 The space form problem

Certainly one of the great areas of application (and motivations for the development) of non simply connected surgery was the "spherical space form problem", which, as noted above is the problem of classifying manifolds whose universal covers are the sphere. To this problem Wall made (at least) two great contributions:

Theorem 2.1 ([BPW]) *If G is an odd order cyclic group, then free PL (or topological) actions on an odd dimensional sphere (of dimension at least 5) are detected by their Reidemeister torsion and "rho" invariants. "Suspension" induces an isomorphism between G-actions on S^{2n-1} and on S^{2n+1}.*

The Reidemeister torsion is explained in [M3]. The "rho" invariant is the equivariant signature defect and can be defined as follows: If G is a finite group acting freely as an odd dimensional manifold M, some multiple $\kappa M = \delta W$ for some G-manifold W

$$\rho(M) = \frac{1}{\kappa}(G\text{-}\operatorname{sign}(W))$$

modulo the regular representation, which is the indeterminacy in G-sign (W) as one varies W among free G-manifolds with given boundary. (See [BPW], and also [AS] for a version allowing $W's$ with nonfree action.) It is essentially equivalent to the Atiyah-Patodi-Singer invariant [APS] associated to the various flat complex line bundles over the quotient manifold, associated to the different representations of G into $U(1)$.

Another point of view using intersection homology applied to the cone can be found in [CSW]. We note that Browder and Livesay had shown earlier [BL] that this is not at all true for $\mathbb{Z}_2$; indeed, they directly constructed obstructions to desuspension. Lopez de Medrano [LdM] and Wall [Wa3] gave a classification of the free involutions. (The second author, in [We 2], left as an exercise a proof of an incorrect description of the general free cyclic actions; we still leave the correct solution to this problem as an exercise![2])

Wall's other great theorem in this subject is:

Theorem 2.2 ([MTW]) *A finite group G acts freely on some (homotopy) sphere if and only if all subgroups of order $2p$ or p^2, for p a prime, are cyclic.*

The necessity of the p^2 condition is cohomological, as discussed in the previous section. The $2p$ condition is not necessary in the context of actions on finite CW complexes, but is necessary for the actions on manifolds, according to a theorem of Milnor, [M1], based on the Borsuk-Ulam theorem. Ronnie Lee [L] discovered more algebraic proof of this necessity via his theory of semicharacteristics; this was put on a surgery theoretic footing by Jim Davis in his thesis and published in [DA2].

The problem of figuring out exactly which groups act in which dimension, and in how many ways, has spawned a vast literature, much of which is surveyed in [DM]. In the final section, we will mention some results on nonfree actions that had this work as one of its main inspirations.

2.2 Semifree actions on the disk and sphere

Another historically important line of investigation that, at least initially, used essentially only conventional surgery theory is the study of semifree actions. These actions are ones where there are only two orbit types: the fixed point set (of the whole group, which is identical to the fixed set of any nontrivial element of the group), and the free part (where points are not fixed by any element of the group, i.e., on which the action is free).

Essentially the method most often used is this: study fixed point sets and their neighborhoods, and study the complements, and then study the possible ways of gluing these together. An early paper expounding this point of view is Browder's [BR]. (Another related approach has arisen, which is appropriate only to the topological category: study the manifold structures on the quotient of the free part that are controlled homotopy equivalent to the given complement, where the control is with respect to a map to the open cone of the fixed point set. See [Q1, We2, HuW].)

In the smooth category, the essential observations about classification of neighborhoods were first systematically exploited by Conner and Floyd [CF]. For instance, for a cyclic group of odd order the neighborhoods of fixed sets are essentially the same as a set of complex vector bundles, "the

[2] Such problems are incomparably easier now, given our complete knowledge of L-groups, assembly maps, their homotopical foundations and the connections between L-theory and topological K-theory.

eigenbundles of the action of the differential," whose sum must have underlying orthogonal bundle agreeing with the normal bundle of the fixed set in the ambient manifold. Many restrictions on the Chern classes of these eigenbundles can be read off from the Atiyah-Singer G-signature theorem [AS]. For instances, for semifree actions of groups of order p^r, p prime, on the sphere with fixed set of dimension 0, i.e., where there is a pair of fixed points, Atiyah and Bott (and Milnor) [AB] showed that the two representations at the fixed point sets must agree, for $p \neq 2$; and when the dimension of the fixed set is at least four, Ewing [E] showed that the Chern classes of these eigenbundles must all vanish!

In the PL case, one has equivariant block bundle neighborhoods whose classifying spaces must be analyzed. This is always done on the basis of Quinn's thesis [Q3] blocked surgery, see [BLR] and also [Ro] for a description of Casson's prior contributions to this circle of ideas. See [CW1, Jo2, Re] for how blocked surgery can be applied to the classification of equivariant PL regular neighborhoods.

Essentially, PL stratified surgery takes off from the fact that blocked surgery theory is not so different from ordinary surgery (especially if one "spacifies" the latter, i.e., forms an appropriate semi-simplicial space of manifolds homotopy equivalent to a given one, so that π_0 is the structure set usually studied in surgery theory; for blocked stratified surgery, with some computational methods, see [CW2]). Then both steps in the above outline can sometimes be completed simultaneously.

While a certain part of the discussion applies in great generality, the cases that attracted the most attention involve group actions on the disk or sphere. For reasons of space, we will concentrate on these cases.

A useful theorem for many of these investigations is the following, which itself doesn't depend on any surgery theory beyond the "π-π" vanishing theorem for surgery obstructions on manifolds with boundary:

Theorem 2.3 (**Extension across homology collars** [AsB, We3]) *Suppose given a manifold triad $(W^{n+1}; M^n, N^n)$ with W and M simply connected of dimension at least 5, and let G be a finite group acting freely and $\mathbb{Z}[1/|G|]$-homologically trivially on N, with $H_*(W, N; \mathbb{Z}(1/|G|)) \cong 0$. Then there is an extension of the action on N to one on W if and only if $\Sigma(-1)^i \mathrm{Sw}(\#(H_j(W,N)) = 0$ in $\widetilde{K}_0(\mathbb{Z}G)$. Moreover, such extensions are well defined up to an element of $Wh(G)$.*

The method of proof is to produce a Poincaré model for "W/G" and "M/G" using "Zabrodsky mixing" (mixing together different local homotopy types to produce an interesting global homotopy type), identifying the Wall finiteness obstruction with the quantity appearing in the theorem, checking that a normal invariant rel N/G exists, and doing surgery using the π-π theorem to obtain some extension on a manifold homotopy equivalent to W. Obtaining such an action on W itself, once one already has it on something homotopy equivalent, is another surgery calculation.

Remark. Assadi-Vogel [AsV] have a non simply connected extension of this result that uses an algebraic K-group that mixes finiteness obstructions with Whitehead torsions. Chase [Ch] has a version for homologically nontrivial actions, but which requires more hypotheses.

In the cyclic group case, the homomorphism Sw identically vanishes, and one obtains:

Corollary *A submanifold of the disk of codimension greater than* 2 *is the fixed point set of a smooth (orientation preserving)* $\mathbb{Z}_p$ *action if and only if*

1. *it is mod p acyclic, and*
2. *it is of even codimension, and*
3. *if p odd, the normal bundle has an almost complex structure.*

This corollary, when the codimension is very large, was first obtained by Jones, essentially by replacing cells by handles in his constructions described in (1.2) above.

In the PL locally linear case one can show that the analogous corollary remains valid without any condition analogous to (3), i.e., just assuming (1) and (2) (and a Swan condition for more general groups). See [CW1]. Remarkably, Jones [Jo2] developed a similar theory for PL actions without local linearity, although for odd order groups there is another interesting characteristic class obstruction. A stratified surgery theoretic approach to the results of that amazing paper is sketched in the exercises of [We1].

The only non-orientation preserving case is $\mathbb{Z}_2$, and was settled in PL by Chase [Ch].

There are a number of additional subtleties for actions on the sphere, quite different in the different categories. For instance, by making perspicacious use of the solution to the Segal conjecture for $\mathbb{Z}_p$, Schultz [Sch1] discovered that there is a p-local residue of the smooth structure of spheres that survives as the only obstruction to being the fixed point set of a smooth $\mathbb{Z}_p$ action on a very high dimensional sphere (besides having a normal bundle with a complex structure with torsion Chern classes). (See also [DW3] for some results on obstructions to being a smooth fixed point for some complicated groups.)

The PL locally linear case is solved in [We4] – there are L-theoretic analogues of the Swan condition that arise as well. (In the case of cyclic groups this obstruction vanishes, just like the Swan homomorphism does; an independent earlier proof for the prime power case appeared in [CW1].)

These L-theoretic "Swan homomorphisms" arose first in the work of Davis [Da3]. They also play an important role in the "homology propagation problem", namely, given a G-action on M and a $\mathbb{Z}/|G|$ homology equivalence $M' \to M$, when can one find a G-action on M' such that the map is homotopic to an equivariant map? (Extension across homology collars can be viewed as a special case of a variant of this problem – which itself has applications to more systematic converses to Smith theory.) See [CW4, DL, DW2] for information about homology propagation for closed manifolds, and the references [AsB, AsV, Jo4, Q4] for related material.

Finally, we should mention the general work of Hambleton and Madsen that classified semifree actions on $\mathbb{R}^{n+k}$ with $\mathbb{R}$ as fixed point set, by viewing it as an analogue of the classification of free G-actions on S^{k-1}, but replacing the usual L^h surgery groups by L^p for $n = 0$ (the L-group based on projective modules in place of free ones) and by L^{-n} for $n > 0$, where these L-groups are based on negative K-groups. These groups are related to L^{-n+1} in exactly the same way that L^p relates to L^h (or L^h relates to L^s), namely, via a generalized Rothenberg sequence, see [Sh1, R4].

2.3 Nonlinear similarity and the Smith problem

The problem of nonlinear similarity is that of deciding when two linear representations of a finite group are topologically conjugate. The PL version of this problem was solved by de Rham, who showed that PL equivariant representations are linearly equivalent by using Whitehead torsion ideas (see [Rt] for a modern treatment). R. Schultz [Sch2] and D. Sullivan proved that topological and linear similarity coincide in the topological category for odd p-groups, but the first examples of nonlinear similarities were constructed in [CS1]; these counterexamples were for all cyclic groups of orders a multiple of 4 and greater than 4. (Further examples for these groups were later constructed in [CSSW].[3])

All of these results are obtained using tools of classical surgery. The Schultz-Sullivan result can be simply seen, for instance, by realizing that a nonlinear similarity between two cyclic p-groups (by character theory, the critical case) immediately implies, by transversality, that the lens spaces associated to the eigenvalues of largest period are normally cobordant. Moreover, not only are those normally cobordant, but also the result of stabilizing them by adding on free representations remain normally cobordant (because such stabilized representations are, *a fortiori,* topologically conjugate). Using this, a quick application of Wall's rho-invariant criterion

[3] We should warn the reader, though, that there are some errors in some of the detailed calculations in that paper which are corrected in [HP1,2] which give a more complete picture, using a wide variety of controlled surgery theoretic ideas that were developed subsequently.

for normal cobordism implies that these eigenvalues are the same. Then one can "downward induct" to the remaining eigenvalues of lower period.

The [CS1] examples of nonlinearly similar representations essentially involve computing when non-trivial interval bundles over lens spaces are h-cobordant, which can be viewed as a transfer of the surgery obstruction of a normal cobordism[4] between lens spaces.[5] When this vanishes one modifies the construction to obtain that the unit spheres of certain representations are equivariantly h-cobordant. An "infinite process" $1 - 1 + 1 - 1 + \ldots$ argument, also known as the "Eilenberg swindle" (as in [St]), then would produce the equivariant homeomorphism.

By working carefully, similar methods produced some smooth h-cobordisms, which gave counterexamples to Smith's conjecture that a smooth cyclic group action on the sphere with two fixed points must have the same representations at the two fixed points; see [CS5]. These counterexamples included the case $G = \mathbb{Z}_{2r}, r \geq 4$. (Recall from the above cited result of [AB] this cannot occur for $G = \mathbb{Z}_{p^r}$, p odd.) Much further information can be found in [Sh2, PR, Sch3].

Remark. We should at least mention at this point the deep result of [HP] and [MR] that for odd order groups, nonlinear similarity is equivalent to linear similarity – but the techniques necessary for this result must wait till the next section.

2.4 Actions with one fixed point

At yet a further extreme in the theory of group actions are ones that are not at all modeled on linear ones. A beautiful example of one such is due to E. Stein [Stn], and it is probably the most complicated action, in terms of orbit structure, that has been produced by direct application of usual manifold surgery. It is an action of the dodecahedral group on the sphere with a single fixed point.

One could predict such an action from an observation of Floyd and Richardson [FR] that the dodecahedral group acts by isometries on the Poincaré dodecahedral homology M^3 with just a single fixed point; so one should just make a higher dimensional version of this and do an equivariant "plus" construction to get rid of fundamental group. However, the details are quite complicated, and we shall say nothing further about this problem here. Suffice it to say that all subsequent work on the construction of "one fixed point actions" depended rather on systematic exploitation of equivariant surgery, which we turn to in the next section.

[4] The paper also describes useful criteria for normal cobordism between linear lens spaces.

[5] This is done using the "generalized Browder-Livesay theory" of [CS4].

3 Equivariant Surgery

Suppose one has an equivariant normal map (whatever that is) and tries to surger it equivariantly to be a (perhaps equivariant) homotopy equivalence. Depending on the problem, one will inductively assume some level of success at the fixed point sets of smaller groups. At this point, though, one runs into trouble doing the surgery. Homotopy theory will tell you that there usually are spheres in your manifold that need to be killed, but they will often be embedded in a fashion that intersects some fixed point set. To do surgery equivariantly, one has to surger all of the translates under the group action of these spheres, and if these intersect each other, one is stuck. (If one can succeed, then one is lead into the algebra of equivariant intersection forms, and absorbing the perturbations of lower strata into the L-theory.)

There are essentially two approaches to dealing with this difficult problem. The first, emphasized by Petrie and his collaborators, is to assume a gap hypothesis; see [DP, DR, P1, PR] for this approach and a number of its applications. The point here is that the spheres one uses for surgery are always of at most half the dimension of the ambient manifold. Thus, if the dimension of the fixed set is strictly less than half that of the manifold, general position will enable the spheres to be moved so as to not intersect the fixed set, and aside from the middle dimensional case itself, not their own translates as well. The remaining middle dimensional difficulties are to be absorbed into the surgery obstruction group.

With this in hand, what happens under gap hypothesis might be summarized as follows: One defines a notion of normal invariants so that surgery is possible. The idea just described works to the extent of leading to a π-π vanishing theorem for relative surgery. (Rothenberg and Weinberger had shown that the π-π theorem fails for equivariant surgery when the gap hypothesis fails, see [DS] for a description.) The approach of [Wa] Chapter 9 shows how such vanishing theorems in surgery theories make possible the geometric definition of a surgery obstruction group. In such general settings, it is extremely rare to have a good algebraic definition of this group, but nevertheless the literature contains *ad hoc* calculations in many cases.

The normal invariants also actually pose a serious problem; that is, it's often quite hard to find them. Here the smooth category looks a lot more workable than the others, since G-transversality is in much better shape (see [CoWa] for a modern and elegant systematic development); but, in any case, this part of the subject still seems more art than technology.

Nevertheless, this approach has lead to the constructions of many quite interesting and exotic examples including one point actions on the sphere for many groups, (see, for instance [P2,3], and the recent very satisfying

[LM], beautiful theorems regarding possible dimension functions for smooth actions [DP], and varied examples of actions on the sphere with two fixed points but different representations at the fixed points (see [PR] and the surveys in [Sch3].)

A remarkable, systematic *tour de force* using this methodology can be found in the papers [MR] which extend Wall's classification theorem from free actions of odd order cyclic groups on the sphere to all odd order actions (assuming a gap hypothesis). This approach depends extensively on proving PL (and Top) G-transversality theorems, and this only works for odd order groups. The Browder-Livesay invariants discussed in the previous section obstruct G-transversality, even for G of order 2, and consequently this whole Madsen-Rothenberg edifice is not applicable to this case.

However, there is another approach to the key issue of equivariant surgery; while it has not been nearly as successful in constructing exotic actions with unusual orbit structures, etc., it has lead to more complete classification theories, especially in the PL and topological categories. It is to this topic that we turn now, leaving further discussion of equivariant surgery based on large gap hypotheses to Rothenberg's survey in this volume.

An idea of Browder and Quinn [BQ] is that we should look at isovariant maps, i.e., ones which map the free parts to the free parts (precisely: an equivariant map is isovariant if and only if $G_m = G_{f(m)}$ for all m). Then the homology kernel one studies is precisely the kernel of the free part (by excision), from which the homotopy classes of interest already have representatives there, as desired.

An unpublished theorem of Browder's (it's at least 10 years old; see also [Do] for a related result) asserts that any equivariant homotopy equivalence between G-manifolds satisfying a suitable gap hypothesis is equivariantly homotopic to an isovariant homotopy equivalence; and the inclusion of the space of isovariant homotopy equivalences in the space of equivariant equivalences is as highly connected as the excess that the dimensions satisfy beyond the gap hypothesis. One proof can be obtained by comparing the results of the two theories.

In their paper, Browder and Quinn [BQ] employed a very simple and calculable kind of normal invariants (indeed, the set of normal invariants is isomorphic to the familiar $[M/G, F/CAT]$), but at a great cost. They required a stratified transversality condition for all maps in their category. This is tantamount to having lots of bundle and framing data; objects then tend to have many self maps not homotopic to isomorphisms in this category, so many actions are counted more than once. Moreover, many equivariantly homotopy equivalent manifolds have no such transverse G-maps between them, thus severely restricting the kind of actions produced.

(See [HuW] for a discussion of [BQ].) Finally, in the topological category, the types of rigid structures their theory demands simply do not exist. (This nonexistence is manifest in the [CS] example of nonlinear similarity discussed above and in the equivariantly nonfinite G-ANR's of Quinn [QI, II] discussed below. A good local structure of the [BQ] sort would, in particular, enable one to remove open regular neighborhoods of lower strata, and thus yield a reasonable simple homotopy theory, such as exists in the PL category, for the complements.)

Section 4 of [HuW] in these proceedings gives a sketch of the Browder-Quinn theory. What is important for our purposes is not the details, but the basic idea; it is that one does surgery on a stratum, then using transversality extends the solution of the surgery problem (assuming 0 obstruction, of course) to a neighborhood of the stratum, and then deals with the new ordinary surgery obstruction that one is confronted with on the next stratum up.

Thus, the surgery obstruction groups in this theory are "built up" out of the surgery obstruction groups of all of the "pure strata", e.g. the sets of points with a given isotropy group (up to conjugation)[6]. We recommend that the reader consult [HuW] for a few examples which show how the strata can interact in forming the Browder-Quinn L-group. (For instance, in some cases of interest despite the fact that all the strata have nontrivial L- groups, the global L-group is trivial.)[7] [8]

For the rest of this paper we will focus our attention on the topological category; almost everything we say has a simpler analogue in the PL category – and works out very differently in the smooth category.[9]

[6] And for a general stratified space, one means the set of points that lie in a given stratum, but not in any lower one.

[7] Easy exercise: Let M be a manifold with no action. Compute $S^{\rm iso}$ $(M \times S^n)$, where the group action is constructed using the involution on S^n with fixed point set S^{n-1}, $n > 2$. Thought question: What do you make of the fact that isovariant homotopy equivalences can automatically be made transverse in this (usual) case?

[8] Deeper examples of how the strata can interact to cancel or almost cancel their L-theoretic contributions arise in the beautiful work of Davis, Hsiang, and Morgan [DH, DHM] on the concordance classification or certain smooth $U(n)$ and $O(n)$ actions (the so-called "multiaxial actions" on the sphere).

[9] The past fifteen years have seen a shift in emphasis from the smooth category to the topological in transformation group theory. The smooth category had seemed much more comprehensible than the topological, because of the slice theorem, equivariant tubular neighborhood theorem, and clear inductive techniques, as well as deep tools like the Atiyah-Singer G-signature formula. At this date, it seems that for finite groups the topological category is the better understood one: for instance, one still does not know exactly which submanifolds of S^n are the fixed point sets of smooth $\mathbb{Z}_p$ actions (even if we avoid the awkward codimension two case) (see [Sch1] for beautiful partial results which already display some fascinating phenomena) or the classification of actions with given fixed point set. In some sense, the key difference between the smooth and topological categories is that, in the latter, local and global issues are essentially the same. In the smooth category, global issues are largely surgery theoretic, while the more rigid local

The first topic that must be clarified is what we mean by the topological category. Of course arbitrary continuous actions can be quite wild: the fixed set need not have any manifold (or even homology manifold) points. It can be far from nice in any homological sense – every open subset can have infinitely generated rational homology!

Two choices have been commonly chosen. The first, introduced in [Br] and deeply investigated in [MR], is the locally linear category. One assumes that each orbit (G/G_x) has an invariant open neighborhood which is equivariantly homeomorphic to a neighborhood of some orbit within a linear G-representation. Smooth group actions all have this structure, and this condition forces all fixed point sets of all subgroups to be nice locally flat submanifolds of one another. This condition leads for instance, to fairly simple isotopy extension theorems.

The other choice is to assume just that all fixed point sets are locally flat submanifolds of one another. This is a wider category, called the tame category, but it fortunately turns out – quite non-trivially – that this category also has all the isotopy extension theorems one would want [Q2, Hu, see also HTWW and HuW], and remains suitable for classification theorems, as we will sketch. Quinn's isotopy extension theorem implies homogeneity of the strata; so if one has, for instance, a semifree action with connected locally flat fixed point set, then it is locally linear if and only if it is locally linear at a single fixed point.

Theorem 3.1 ([Q2]) *If G acts tamely on a topological manifold, then one can extend G-isotopies. In particular, any two points p and q in the same component of the submanifold consisting of points with G_p as their isotropy have equivariantly homeomorphic neighborhoods.*

Assadi's obstruction is of course just the Wall finiteness obstruction for the quotient of the complement of the fixed set, which must vanish in the PL category because of regular neighborhood theory. The above theorem shows that one cannot find such neighborhoods in the topological setting. (The first examples of this phenomenon were discovered by Quinn [QI, II].) The idea of the proof is to remove a point from a stratum, work noncompactly, then one point compactify, and use the magic of their being no isolated topological singularities.

By this point, it does not really matter much which of these two topological categories the reader keeps in mind. When we develop precise classification theorems, it will turn out that the second category is somewhat more convenient; but in principle all one has to do at the end of any discussion is to examine what is occurring at several points in order to check whether one has succeeded in working in the locally linear category.

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ones involve via linearization, the unstable homotopy theory of classical groups.

The first interesting point is that locally linear actions need not have equivariant handle decompositions. The first example of this was Siebenmann's locally triangulable nontriangulable sphere [Si1] whose key feature was nontriangulability for algebraic K-theoretic reasons rather than Kirby-Siebenmann invariant familiar from the unequivariant theory of topological manifolds. Other examples appear in [Q1 II, DR, We1]. Needless-to-say, once existence is called into question, then there shouldn't be uniqueness, and this had been forcefully shown by the examples [CS1] of nonlinear similarity mentioned above, i.e., of nonequivalent orthogonal representations that cannot be PL equivalent because of torsion considerations (de Rham's theorem), but which are nevertheless topologically conjugate.

Non-uniqueness of handle structure can also be seen quite easily using Milnor's [M2] counterexamples to the Hauptvermutung. The nonexistence is also quite easy using Siebenmann's proper h-cobordism theorem [Si2], if one grants the homogeneity of tame actions.

Example. Suppose that one starts with any smooth $\mathbb{Z}_p$ action on a manifold where the fixed set is nonempty and has codimension at least 4. Then, for $p > 3$, $Wh(\mathbb{Z}_p) \neq 0$, and so one can erect an equivariant h-cobordism with nontrivial torsion (working in the quotient of the complement). However, this smoothly nontrivial h-cobordism is topologically trivial, by an Eilenberg swindle (similar to the YG construction in [St]).

Example. Suppose that the fixed set has positive dimension and is disconnected. Instead of realizing an element of the usual Whitehead group of the closed complement, remove two points from the fixed set from different components. Attach to $M \times I$ the realization of an element of the proper Whitehead group of the new closed complement (which is $\widetilde{K}_0(\mathbb{Z}_p)$), and then end-point compactify. One still has a locally linear action by the isotopy homogeneity theorem. However, one can easily check that this space does not have a closed regular neighborhood of its singular set.

A striking application of this overall technique geometrically can be found in [We1]:

Theorem 3.2 *A submanifold Y^n of S^{n+r}, for $r > 2$ and $n + r > 4$, is the fixed set of a semifree locally linear orientation preserving G-action on the sphere if and only if G acts freely and linearly on S^{r-1} and Y is a mod $|G|$ homology sphere.*

One should compare this with the conditions that arose in Assadi's theorem mentioned in section 1. In that theorem there are restrictions on the orders of the homology modules of Y, depending on G (and they are indeed nontrivial). Here the topological category turns out to be easier to analyze than the PL locally linear category, because in the former there can be no isolated singularities by Quinn's theorem. (See [We4] for the analogous theorem in the PL locally linear setting.)

In short, the proof goes like this: One produces an action on $\mathbb{R}^{n+r}$ with fixed point set $\widetilde{Y}$ ($= Y$ minus a point) using local bundle information [CW1] to get the action on a neighborhood and extension across homology collars [AsB, We3] to extend outwards. Then, as above, one one-point compactifies and is guaranteed local linearity.

Examples like these very naturally suggested bringing to bear the technology of controlled topology [ChF, AH, Q1, Y, FP, ACFP, We2]. Steinberger and West [Stnb] and Quinn [Q2], in the more general setting of stratified spaces (with a missing realization result provided in [HTWW, Hu]) gave an s-cobordism theorem.[10]

Theorem 3.3 *If G acts tamely on a manifold M, with no fixed point set of codimension one or two in another, then*

$$Wh_G^{top}(M) \cong \bigoplus Wh_{NH/H}^{top}(M^H, \mathit{rel\ sing}).$$

These latter groups can be computed via an exact sequence:

$$H_*(M/G; \mathbf{Wh}(G_m)) \to Wh(G) \to Wh_G^{top}(M, \mathit{rel\ sing}) \\ \to H_*(M/G; \mathbf{K}_0(G_m)) \to K_0(G).$$

These "*rel sing*" theories can be interpreted as being the theory of h-cobordisms that are trivial on the lower strata. Because they are not assumed to be trivialized in a neighborhood of these strata (as would be automatic in the PL and smooth categories), we have the nontrivial assembly maps ($H_*(M/G; \mathbf{Wh}(G_m)) \to Wh(G)$ and $H_*(M/G; \mathbf{K}_0(G_m)) \to K_0(G)$) which control the "leaking" of obstructions through lower strata. The map $Wh_G^{top}(M, \mathit{rel\ sing}) \to H_*(M/G; \mathbf{K}_0(G_m))$ measures the obstruction to putting a mapping cylinder around singular set of the h-cobordism (assuming we had one on the bottom of the h-cobordism to begin with). And the map $H_*(M/G; \mathbf{Wh}(G_m)) \to Wh(G)$ measures the effect of changing mapping cylinder structures. (We recommend the reader rethink the above two examples in light of this theorem in order to appreciate it.)

Corollary *If M is G-simply connected, i.e., if M^H is simply connected for all $H \subset G$, and M satisfies the "no low codimension" gap condition, then $Wh_G^{top}(M) = 0$.*

This holds for the sphere of a representation whenever the fixed point set is two or more dimensional and there are no one or two dimensional eigenspaces. (These can be dealt with, of course, but with a bit more calculation.) Thus, one can see that once one has two trivial summands, the

[10] In the locally linear setting, one has to be careful because the realization of h-cobordisms when the fixed set consists of isolated points might leave the category.

question of the existence of a nonlinear similarity is unaffected by stabilization with a third or more.

Another key feature that distinguishes the smooth category from the PL and topological locally linear categories, not to mention the tame category, is the issue (mentioned above) of equivariant transversality. In the smooth category, there is a stable equivariant transversality theorem (see e.g. [Pe2, PR, Co]). However, for $G = \mathbb{Z}_2$ the Browder-Livesay examples of nondesuspendable involutions on the sphere give rise to counterexamples to transversality, even for maps into $\mathbb{R}$ with its nontrivial involution. (See [MR]; a simple nonlocally linear example arises from the natural map from the open cone of a non-desuspendable free involution on the sphere mapping to $\mathbb{R}$ the transverse inverse image of 0 would have around its fixed point a desuspension of the action.) For odd order groups, one would have similar counterexamples in the tame category, because Wall's desuspension theorem only covers desuspension with respect to the linear $\mathbb{Z}_p$ actions. However, as noted above, the remarkable theorem of Madsen and Rothenberg shows that for locally linear odd order group actions, transversality holds. This enabled them to extend Wall's theorem to nonfree actions, give a classifying space for topological locally linear equivariant normal invariants (assuming a "gap hypothesis") and an associated surgery theory. Furthermore, they gave an extension of Sullivan $KO_*[1/2]$ orientation of topological manifolds to a $KO^G_*[1/2]$ orientation for odd order locally linear G-manifolds. (It was this invariant which they used to prove the result on odd order nonlinear similarity mentioned above.)

Soon after, Rothenberg and Weinberger gave an analytic approach to the $KO^G_*[1/2]$ class, which works for all G, (see [RsW,RtW]) and in the tame category. A general and direct controlled topological approach for group actions on topological pseudomanifolds was given by the present authors and Shaneson in [CSW], using intersection chain sheaves. This class fails to be an orientation in general[11]. For smooth G-manifolds, this class is the equivariant symbol of the signature operator. This takes us a certain way towards explaining the following theorem:

Theorem 3.4 (i) (Combination of [MR, RtW, CSW, RsW, We2])
Suppose G is a finite group acting tamely on a manifold M. Then one can define a KO^G-homology fundamental class

$$\triangle(M) \in KO^G_*(M) \otimes \mathbb{Z}[1/2]$$

[11] A similar thing occurs in the theory of the usual Sullivan class. It is definable as an element of the more refined homology theory $H^*(\_; L^*(e))$ where it is an orientation for topological manifolds. Actually it can be defined even for ANR homology manifolds, but in that generality the class need not be an orientation. In fact it is an orientation away from 2 if and only if the homology manifold is resolvable (i.e., a cell-like image of a topological manifold). For more of a discussion of the theory of homology manifolds and its analogies to group actions, see [We5].

which is a topological invariant. Its image under the natural map $M \to pt$ sends $\triangle(M)$ to the G-signature(M).
(ii) ([CWY] using [We2])
Suppose that there are no low codimensional fixed point sets; then

$$S^{\text{iso}}(M) \otimes \mathbb{Z}[1/2] \;\cong\; \bigoplus S^{\text{iso}}(M^H, \textit{rel sing}\;) \otimes \mathbb{Z}[1/2].$$

In fact, $S^{\text{iso}}(M, \text{relsing})$ is a covariantly equivariantly functorial abelian group, the elements of which for M G-simply connected are determined away from 2 by $\triangle$, and all values of $\triangle$ are realized subject to the augmentation condition.

Remark. S^{iso} is the structure set defined using isovariant homotopy equivalences. Recall that by Browder's theorem this is the same as the equivariant structure set when a strong gap hypothesis applies.

Remark. The splitting of the structure set here into pieces defined by the fixed point sets means (ignoring the issue of the prime 2) that any equivariant structure on a fixed point set (of any subgroup) can be extended to one on the whole G-manifold. In the case where the subgroup involved is G itself, this is the phenomenon studied earlier by "replacement theorems" [CW2], wherein any manifold homotopy equivalent to the fixed point set is actually the fixed set of an equivariantly homotopy equivalent group action. A typical statement of that theory is the following:

Theorem 3.5 *Suppose G is an odd order abelian group acting locally linearly on a G-manifold M (with no codimension two fixed point sets), smoothly in a neighborhood of a 1-skeleton. If F' is a manifold homotopy equivalent to the fixed point set F, then there is an equivariantly homotopy equivalent G-action with fixed point set F'.*

Unfortunately that statement is false for even order groups, but it's true away from the prime 2! (The theorem in [CW2] is, in fact, stronger in that one also sees that the new G-action is on the original manifold M.)

In any case, it's not even entirely obvious that such a homotopy equivalent manifold even embeds in the manifold under discussion. Indeed, in the smooth category, characteristic class theory easily shows that this false. Implicit in this theorem is the homotopy invariance of PL or topological embedding in codimension larger than 2 (Browder, Casson, Haefliger, Sullivan and Wall), which, of course, involves systematically changing the putative normal bundles to our submanifolds. In other words, this type of phenomenon is exactly the type of thing not studiable by the original theory of [BQ]. Nevertheless, as we will see, calculations of sheaves of Browder-Quinn surgery groups, constructed as noted above for use in a much more constrained setting, play an important role in the topological category.

We can only give a short sketch of a proof here. The first part is to simply recognize that isovariant structures on M are the same thing as "stratified structures on M/G", where one works in the category of homotopically stratified spaces [Q2, We2, Hu, HuW]. [We2] extended [BQ] to (i) apply to these spaces, and (ii) to maps which are stratified but not necessarily transverse. (Surveys of this are [We5,HuW]; an early precursor to this general theory is [CW3].)

The problem then is to compute what the theory actually says. The surgery obstruction groups away from 2 can be analyzed by a trick (see [CSW, LM, DS]). To any G-manifold (G finite) one can define a symmetric signature (see [R1]) $\delta^*_G(M) \in L^*(\mathbb{Q}\Gamma)$, where Γ is the "orbifold fundamental group" (= fundamental group of the Borel construction) which fits into an exact sequence $1 \to \pi_1(M) \to \Gamma \to G \to 1$. This can be done in the same way as symmetric signatures are defined in general, simply observing that the $\mathbb{Q}$-chain complex of a G-manifold, or better, the Γ-manifold that is the universal cover of M, is made up out of projective chain complexes, since all isotropy is finite. Now we use the fact that, according to Ranicki [R], changing coefficients from $\mathbb{Z}\pi$ to $\mathbb{Q}\pi$ only affects L-theory at the prime 2 (as does allowing projective rather than free modules in the definition of L-groups).

If one assembles all of the equivariant signatures of all the strata together, one has an *a priori* method of detecting all of the surgery obstructions that will be inductively arising in the Browder-Quinn process, i.e., many secondary, tertiary, etc., obstructions are, in fact, primary. This kind of argument yields the collapse of some spectral sequence and this gives a splitting away from 2 of the L^{BQ} into ordinary L-groups, just like the splitting asserted on structure sets.

Remark. Results of [LM] can be used to see that for many odd order group actions that there is an integral splitting, as well.

It turns out that this method is sufficiently canonical to apply not just to individual L-groups, but to "cosheaves of L-spectra" as well. As a result, as noted by the present authors with Min Yan, here the normal invariants [We2], which turn out to be a cosheaf homology theory also break up into pieces defined on the various strata. This is enough to get the splitting of S^{iso} into pieces corresponding to the various strata.

Now, to compute these pieces, one shows that the normal invariants (of a stratum *rel sing*) essentially boils down to equivariant K-groups. This is an extension of Sullivan's "characteristic variety" theorem for the structure of G/PL at odd primes and can be proven in a couple of ways, by now. For example, [CSW] explained how to produce the equivariant signature class via the calculation such a cosheaf homology group. On the other hand, [RsW][We2] uses just the PL version of the equivariant

Teleman signature operator to give a calculation of the cosheaf homology (which still is an improvement on the approach of [RtW] in that it does not require the construction of any Lipschitz structures, nor of the harder Lipschitz signature operator.)

Remark. The remaining statement regarding equivariant functoriality can be found in [CWY], where we, together with Min Yan, analyze these *rel sing* structure sets, as well as the issue of integral splitting. (It does not split in general; there is a spectral sequence whose differentials are related to assembly maps for finite groups.) That work also deals with the calculation of normal invariants at the prime 2, and their relation to Bredon homology. Earlier work on that problem can be found in [Na].

Remark. The non simply connected case can be dealt with in a similar fashion; essentially one just has to use non simply connected L-groups on occasion to replace the $KO^G(*)$.

Needless to say, this theorem has implications for nonlinear similarity, in terms of giving necessary conditions, such as the the fact that for odd order groups nonlinear and linear similarity coincide ([HP] and [MR]) or the more general topological invariance of generalized Atiyah-Bott numbers of [CSSWW]; it also quickly leads to the "topological rationality principle" for representations of general finite groups of [CS2]. The reader should also consult [HaP1,2] for a great deal of information regarding nonlinear similarity for cyclic groups.

We would like to close this survey with describing some relatively recent ideas on equivariant structure sets when the gap hypothesis does not hold.

One key to this is the work of [Y] and [WY] (which extends earlier work of [DS] on equivariant L-groups) which shows that there are isovariant periodicity theorems for structure sets, as suggested by the form of the normal invariants away from 2 and equivariant Bott periodicity. Unfortunately, one does not have a general statement, so we will leave precise statements to the references. However, a payoff from this is the following: one can, as a consequence, often show that existence and classification problems do not actually depend on all of the stratified data, but only depend on the equivariant homotopy data.

The model for this is the following. An easy application of the total surgery obstruction theory of [R3], modified in [BFMW], is that a Poincaré complex X is homotopy equivalent to an ANR homology manifold if and only if $X \times \mathbb{CP}^2$ is. Ordinarily this is not too useful because it is hard to verify directly that $X \times \mathbb{CP}^2$ is a manifold. However, in the situation of group actions, say for odd order groups, the substitute is a product of $\mathbb{CP}^2$'s with permutation action, so that after crossing some number of times, the gap hypothesis becomes valid.

Now, using Browder's theorem one sees that the difference between equivariance and isovariance goes away by this process. It follows that if an equivariantly equivalent Poincaré space can be geometrically constructed, then stably and isovariantly one can realize geometrically the Poincaré space, and thus, by Yan's work, so can the original Poincaré complex.

This then leads (under good conditions) to a decomposition $S^{equi} \cong S^{\mathrm{iso}} \times ?$ where ? is a purely homotopy theoretic object. It consists of the isovariant homotopy types of Poincaré objects within the given equivariant homotopy type. While this does not look particularly computable, it turns out to be closely related to the set of PL (or topologically locally flat) embeddings of the fixed point set homotopic to the given embedding.

This strangely reduces the geometrical problem to homotopy theory, and then relates the homotopy theory back to a different geometrical problem rather than deal with it homotopy theoretically. This process leads to, for instance, a result conjectured by the first author:

Theorem 3.6 ([We7]) *A semifree action on a simply connected manifold with simply connected fixed set of codimension greater than 2 is determined within its equivariant homotopy type, up to finite ambiguity, by the equivariant signature class and the underlying topological types of the fixed point set and its embedding in the ambient manifold.*

As a sample application regarding the equivariant version of the Borel rigidity conjecture, i.e., the conjecture that aspherical manifolds are determined up to homeomorphism by their fundamental groups, we have:

Theorem 3.7 ([We7], see [Shi] for more refined information): *The cyclic group $\mathbb{Z}_p$, p a prime, $p > 3$, acts affinely (via a permutation representation on the coordinates) on T^{2p-3} in such a way that S^{equi} (T^{2p-3}) is infinite. Moreover, all of the elements are smoothable and they remain distinct when crossing with any tours with trivial action, but they all do become equivalent on taking a suitable finite sheeted cover.*

One can actually put a group structure on many of these structure sets and give actual calculations. (This last is due to *positive* results on the equivariant Borel conjecture, see [CK]; more results can be obtained by combining [FJ] with [We2].) [12]

[12] [CK] gives a direct geometric proof that, in the presence of the gap hypothesis. the equivariant rigidity conjecture holds for odd order affine action on the torus. [We2] gives an analysis of the rigidity problem (again, assuming isovariance or the gao hypothesis) in terms of assembly maps, which are quite similar to the ones arising in Farrell and Jones' "isomorphism conjecture" for the computation of L-groups. They announced that the isomorphism conjecture is true for $L^{-\infty}$ of uniform lattices in real connected line groups and for the algebraic K-groups of these groups. Combining the ingredients

All but the smoothability comes from the type of analysis discussed above: one constructs embeddings of the torus $T^{p-1} < T^{2p-3}$ that are isotropic to their images under the linear $\mathbb{Z}_p$ action and then using the connections between embeddings and Poincaré embeddings and isovariant Poincaré complexes, one gets a homotopy model that can be geometrically realized.

The smoothability is explained in [FRW].

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yields a calculation of the isovariant structure set in many cases, although the answer is not always 0! See [Ck2] for an example (which they attribute to the second author of this paper) of a nonrigid affine involution on the torus, whose construction is based on the UNil group of [Ca] and another example of a nonrigid affine torus whose construction is based on a nonvanishing Nil group. Unfortunately, we cannot explain any more about this fascinating problem here.

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