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FINAL VALUES OF FUNCTORS

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For Guido Mislin, a mathematician who always
epitomized for me elegance, taste, and precision.

The conjecture (and associated speculation) that I would like to present below is motivated by ideas I learnt from Goodwillie and Weiss. I present it here because I think it is also very much in the spirit of Mislin's approach to mathematics.

All of my suggestions can be discussed for categories of a certain sort and covariant functors of appropriate kinds, and dualized to contravariant functors (changing the title of this note to "initial values" of functors) and can lead to interesting objects to compute and interpret. But it will still be too general, even when we restrict attention to groups and injections, a category I'll call G, and various of its full subcategories (which we'll then denote by adding on an adjective).

To get a feeling for this project, start with $G(\text{finite})$, the category of finite groups and injections. Consider first, the functor of group homology, H. Now we can form the limit of $H_*(\pi)$ over the category G. Of course this category has no final finite group (under injections !). If there were, the limit considered would be $H_*(\text{final object})$. Nevertheless, we can compute this limit easily :

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Note that every group embeds in a symmetric group, and then embedding S_n into S_{nm} diagonally, one quickly sees that above dimension 0, the limit is trivial. (We use the fact that conjugate maps induce the same homomorphism on group homology, and that Cayley embeddings of groups in symmetric groups are canonical up to conjugacy.) On all of G, the limit is also trivial, for a somewhat different reason: all groups embed in acyclic groups (as shown by Baumslag-Dyer-Miller [1]).

Consider now the functor $R(G)$ of real representations. Now we can form the limit of $R(\pi)$ over the category $G(\text{finite})$. The limit can be calculated and it's an infinite sum of $\mathbf{Q}$'s. (The n-th invariant is essentially the sum of the normalized trace of the character (remember the character is an algebraic number) on conjugacy classes of elements of order n.) This is an invariant we assign to arbitrary representations of arbitrary finite groups, and it tells us that although injections do not induce injections on $R(?)$, there is something that cannot be killed. This is called a "final value"; the terminology should suggest (1) the value on the nonexistent final object and (2) a universal a priori invariant that can be applied to any element of $R(\pi)$, for any π in the category. For infinite groups, one can consider the functors $K_0(\mathbf{Q}\pi)$ or $K_0(\mathbf{R}\pi)$ as substitutes for representation theory. The final value should be the same. It is a correct lower bound by the Hattori-Stallings trace.

I conjecture that the same happens for L-theory, $L(\mathbf{Z}\pi)$, in any of its decorations and for $K(C^*(\pi))$. In other words, for L-theory one gets an infinite sum of $\mathbf{Q}$'s in dimension 0 mod 4, and otherwise $L(e)$. For $K(C^*(\pi))$ the limit should vanish in odd dimensions and be the same sum of $\mathbf{Q}$'s in even ones. This is a correct lower bound for $G(\text{residually finite})$. It is also correct the category of groups that only have finitely many elements of finite order. It is also of interest to compute the limits for various subcategories, like amenable groups or solvable groups, and so on, or for the maximal C^*-algebra of a group where one can expect some infinite groups to also play a special role.

As mentioned, for group homology, one gets vanishing (above dimension 0), for G or $G(\text{finite})$. The elements of finite order, though, play a similar role for HC (cyclic homology) as they do for representation theory. It is reasonable to believe that the final value of cyclic homology is small above dimension 0 (by a strengthening of the Baumslag-Dyer-Heller construction). What about π-equivariant K-homology of the universal space of proper π-actions (see [5])? These would be relevant to the conjectures above under the usual isomorphism conjectures, but they are not completely trivial on their own.

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The following theorem is a converse to one aspect of the Borel conjecture [5]):

THEOREM 64.1. *If M^{4k+3} is a closed manifold of dimension > 3, and the fundamental group contains nontrivial torsion, then there are infinitely many other manifolds homotopy equivalent to it but not homeomorphic.*

Essentially the proof uses a nontrivial homomorphism from the final object of L-theory to detect the nonrigidity. But, one actually believes that there are more stringent lower bounds. (In other words, the "structure set" —in the sense of surgery theory— of such a manifold, which is known to be an abelian group, should have larger rank, if the fundamental group has elements of different orders). Moreover, one could also use such a calculation to show how torsion in a fundamental group gives rise to components of moduli of metrics of positive scalar curvature (on spin manifolds that have at least one such metric), see e.g. [4].

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[5]) The Borel conjecture asserts that aspherical manifolds are rigid. The simplest reason a manifold would be non-aspherical is that it has torsion in its fundamental group. The theorem says that that would cause non-rigidity. On the other hand, [3] studies some non-aspherical rigid manifolds.