Algebraic Topology
Tu Th 9:30-10:50

Instructor: Shmuel Weinberger Eck 403
Shmuel@math.uchicago.edu

Graders: Reid Harris Eck 127
and Hana Kong TAAC (5607 S. Drexel)

HOMEWORKS ARE THE KEY TO THIS COURSE. I will try to assign problems by Tuesday of week n, and the homework will be due Tuesday of week n+1 by the start of class. Unless demanded by the class, there will be no in-class exams.

Homework may be worked on jointly, but each person should write their own solutions up. You should be prepared to explain orally what you did, either in class or privately in my office. Almost always problems will be much easier a week or two later: I hope that this way you anticipate later developments, or if not, appreciate them when you see them.

To the extent possible, I will try to explain concepts and theorems on Tuesdays and apply them and calculate etc. on Thursday. Whether this succeeds, time will tell. If it fails, then Tu and Th will asymptotically look the same.

I will be in my office on Wednesday from 11:30- 1:30. As I am on something like 13 different committees, things *do* come up often - so I *recommend* making an appointment, just to be sure. (I also know that this time might not work for everyone.) When possible I will make additional office hours on Monday or Wednesday.

The graders, Reid Harris will have office hours on Monday 10:30-11:30, and Hana Kong on Thursday from 12:30-1:30.

Recommended texts:

Hatcher, Algebraic Topology
May, A concise course in algebraic topology
Spanier, Algebraic Topology

Week 1 and 2 we discuss/review the main ideas of chapter 0 and 1 of Hatcher.

Homework 1. (Do the first seven of these during week 1. And the rest when you have time; try to think about all of them - **BUT DON'T GO CRAZY**[1] - what you don't see now will become obvious in time.)

---

[1] By definition, this means doing more than 10 hours of homework a week, unless you're in love with the subject.

1. Show that if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ is a quasi-polynomial in the sense that the a_i are bounded continuous functions on **C**, and so is $1/a_n$. Show that p has a root, i.e. there is a z_0 , such that $p(z_0) = 0$.

2. Consider S^3 as the boundary of $D^2 \times D^2$, so that it contains a canonical torus (the product of the two boundaries). Consider the curve parametrised by (u^p, u^q), where p and q, both >0 are integers, and u runs over unit complex numbers. Show that it is an embedding of the circle iff p and q are relatively prime. Compute the fundamental group of the complement of this circle. Compute the center of this group $Z(\pi)$, and show that different unordered pairs give different embeddings by showing that the quotient groups $\pi/Z(\pi)$ are different.

3. Let us consider finite 1-complexes with cellular involutions, i.e. actions of the group $\mathbf{Z}_2$ which map cells to cells. We then have equivariant homotopy category of these complexes. Show that a complex is contractible iff it is equivariantly contractible (i.e. equivariantly homtopy equivalent to a point). Give two different equivariant homotopy types on the homotopy type of a circle.

   Remark: Feel free to use an equivariant homotopy extension principle.

Let me elaborate. If G is a finite group, then we can consider the category of finite G-complexes. We just ask that G act on all the spaces, mapping cell to cell (and commuting automatically with the attaching maps). We only consider continuous maps that are "equivariant" i.e. commute with the G action. We similarly consider equivariant homotopy, i.e. homotopies between equivariant maps that are themselves equivariant. Two spaces with G action (i.e. "G-spaces") are equivariant homotopy equivalent if there are maps in both directions between these, so that the two composites are each equivariant homotopic to the identities of these spaces.

4. Look up the classification of surfaces and use it to show that if Γ is the fundamental group of a surface, then any two subgroups of the same finite index are isomorphic.

5. Which surfaces admit free actions of the dihedral group D_{2p} p odd $< u, v \mid u^2, v^2, (uv)^p >$?

6. Consider T^2 - a point and S^2- 3 points. Show that these spaces are homotopy equivalent. Are they homeomorphic?

7. Suppose M is a compact n-manifold (use your advanced calculus definition if you like) and f: $M \to T^n$ (the n-torus, say the product of n circles in $\mathbf{R}^{2n}$) has nowhere vanishing Jacobian, det Df. Show that M is homeomorphic (diffeomorphic) to a torus. (This requires "linear algebra over **Z**", or something equivalent.)

8. Suppose that X is a compact space. Show that the universal cover of X is compact iff the fundamental group of X is finite.

9. Recall that a metric space (X,d) is called a *path space*[2] if for each pair of points x and y there is a path $\gamma:[0,1] \rightarrow X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(t),x) = td(y,x)$. If (X, d) is a path space, show how to metrize any Y that covers X, so that the projection is locally an isometry (i.e. preserves distance).

10. Suppose (X,d) is a finite diameter path space, and we give it a measure μ, so that $\mu(X)$ is finite. Locally pull back the measure to a cover Y (so that the measure of an n-fold cover of X is $n\mu(X)$). Show that if $\pi_1(X)$ contains a free group then in the universal cover of X, the measure of the ball of radius r (around a basepoint) $\mu(B(r))$ grows exponentially in r, i.e. there is a c>1, such that for r sufficiently large $\mu(B(r))>c^r$.

11. Consider the function space $C^0(S^1: 8)$ where 8 is the graph that looks like the number 8 (and whose fundamental group is F_2, the free group on 2 generators).

    a. How many components does $C^0(S^1: 8)$ have?
    b. Give an example of two different elements of $\pi_1(8, v)$ v the vertex, which lie in the same component of the function space. (In general, how do you tell algebraically when two maps of a circle into a space X are homotopic, if you forget the basepoint?)
    c. Compute the fundamental group of the component of maps homotopic to a constant, and the fundamental group of some other component.

12. Say X is a *compromise space*, if there is a function f taking pairs of points of X into X, such that f(x,x') = f(x',x) and f(x,x) = x. Suppose X is a CW complex.

    a. Show that if every component of X is contractible, then X is a compromise space.
    b. Show the circle is not a compromise space.
    c. Show that a 1-complex is a compromise space iff all of its components are contractible (i.e. are trees)

13. A space X with base point e is an *H-space*, if it comes equipped with a function $f: X\times X \rightarrow X$, so that f(e,x) = f(x,e) = x.

    a. Use f to define pointwise multiplication. Show that $\pi_1(X, e)$ has the property that product in π_1 coincides with pointwise multiplication of the functions.
    b. Show that $\pi_1(X, e)$ is abelian.

14. Consider maps from the interval $f:[0,n] \rightarrow \{0,1,2, 3\}$ thought of as the vertices of a square, and we insist that

---

[2] We will have a different definition of "the path space of a space X" later in the course. Hopefully this will cause endless amounts of slapstick humor.

a. if $d(x,y) < 1$, then $f(x)$ and $f(y)$ coincide or are neighbors.
b. $f(0) = f(n)$.

Now introduce the equivalence relation *generated by* making f equivalent to any function g satisfying a and b, for which the value at any point p is either $f(p)$ or a neighbor of it. How many equivalence classes of such functions are there (more or less)?

Remark: Thus begins the subject of digital topology and its connection to algebraic topology.

15. Suppose $f,g: X \to S^n$ the unit sphere in $\mathbf{R}^{n+1}$ given the (chordal) metric as a subset of Euclidean space. If $\|f(x)-g(x)\| < 2$ for all x, then show that f and g are homotopic. If they agree on a subset A, show that this homotopy can be taken rel A. (For any reasonable compact space, something similar is true for some other constant replacing 2.)

Homework 2: (Do the first 8 and think about the rest).

1. If X is a CW complex, show that the homotopy classes of maps $X \to S^1$ is the product over the components X_k of X of $\mathrm{Hom}(\pi_1 X_k : \mathbf{Z})$ where Hom denotes the homomorphisms of $\pi_1 X_k$ to $\mathbf{Z}$. Explain why this is contravariantly functorial in the homotopy category of topological spaces (and not only in some pointed category).

2. Show that $\pi_i(S^n)$ is countable (using 1.15; the same is true for any countable complex).

3. Show the following are equivalent
   a. There is some space X so that "$\pi_n(X)$ is nonzero"
   b. "$\pi_n(S^n)$ is nonzero"
   c. "S^n is not contractible".
   d. "S^n is not a retract of D^{n+1}".
   e. The Brouwer fixed point theorem: Any $f: D^{n+1} \to D^{n+1}$ has a fixed point.

4. Show that an ascending union of contractible complexes is contractible. Show that a union of two contractible complexes whose intersection is contractible, is contractible.

5. An aspherical complex is a complex whose universal cover is contractible. Show that the union of two aspherical complexes along an aspherical subcomplex is aspherical, provided that the fundamental group of the subcomplex injects into that of the other pieces.

6. Show that $\pi_i(X\times Y) = \pi_i(X) \times \pi_i(Y)$.

7. Using $\pi_n(S^n) = \mathbf{Z}$, show that S^n is not a compromise space (see 1.12).

8. The map induced on π_i is an isomorphism for covering spaces for $i>1$.

9. Show that $RP^i \times S^n$ and $S^i \times RP^n$ are not homotopy equivalent if $i<n$. Show that they have isomorphic homotopy groups.

10. Using the fact that $\pi_4(S^4) \neq 0$, show that T^3 is not homotopy equivalent to a 2 dimensional complex, by show that its suspension has a nontrivial map to S^4. (Hint: use an embedding of T^3 to S^4 to build a nontrivial map from S^4 to ΣT^3.) Indeed, using π_4 in this way, show that T^3 has a nontrivial map to S^3 (although, this can't be detected by π_3. Why not?)

11. Classify pointed and unpointed maps from a complex X to $K(\pi,1)$.

12. What is $\pi_1(U(2))$? What is $\pi_2(U(2))$?

Homework 3. (Do the first 7, and think about the rest.) [Due Thursday of Week 4]

1. Show that if X is a simply connected finite CW complex, it is homotopy equivalent to a complex X' whose 1 skeleton is a single point. (Hint: Think about the proof of the Whitehead theorem.)

2. Show that if X is a finite n-dimensional CW complex, then $H_n(X)$ is a free abelian group.

3. Using the fact that π_1 is nonabelian and π_2 is abelian, give two different spaces X and Y which are not homotopy equivalent, but whose suspensions are homotopy equivalent. (Hint: You can let X be a wedge of circles union a 2-cell, and Y be another one but with a different attaching map of the 2 cell.)

4. Compute the homology (with **Z** coefficients) of a 2-dimensional torus. Of the Klein bottle.

5. Using the Mayer-Vietoris sequence, show that if S is a solid torus in the 3-sphere S^3, then $H_1(S^3\text{-}S) = \mathbf{Z}$.

6. Let $1 \in \mathbf{Z} = H_n(S^n;\mathbf{Z})$. For $f:S^n \to X$ define $H(f) \in H_n(X;\mathbf{Z})$ by $H(f) = f_*(1)$. Show that H defines a homomorphism $H: \pi_n(X) \to H_n(X;\mathbf{Z})$. Show that this homorphism is nontrivial.

7. Give an example showing that H is not always 1-1. Give an example where H is not onto.

8. Let $\mathbf{C}P^n$ = space of complex lines (i.e. one dimensional complex subspaces) in $\mathbf{C}^{n+1}$. Find a CW structure on CP^n and inductively compute its homology.

9. If π is a group, define $H_i(\pi ; A)$ to be $H_i(K(\pi,1); A)$. Show that the H_i define functors from countable groups to abelian groups. What is $H_i(Z; A)$? What is $H_i(Z/k; A)$ for A = **Z** or **Z/2** ?

10. Suppose X is an n-dimensional connected simplicial complex, so that every n-1 dimensional face is in the boundary of exactly 2 n faces. Show that $H_n(X)$ is either 0 or **Z**. Referring to exercise 5 above, can you guess when you are in each case?

Homework 4: (Due on **Thursday** of Week 5).

1. Hatcher p 155, problem 2
2. Problem 4
3. Problem 7
4. Problem 12
5. Problem 27
6. Suppose that C_* is a finite chain complex of *finite abelian groups***.** Show that the alternating product of the orders of the C_i is the same as the alternating product of the orders of $H_i(C)$.

Homework 5 (Do the first 6, but think about 7. Due **Tuesday** of Week 6)

1. Hatcher p. 131, problem 4
2. Problem 12
3. Problem 14
4. Problem 20
5. Problem 22
6. Problem 31
7. Show that $S^n \vee S^n \vee S^{2n}$ is not homotopy equivalent to $S^n x S^n$ although they have isomorphic homology groups. (Hint: Think about what the Hurewicz homomorphism can look like.) Deduce that $\pi_{2n-1}(S^n \vee S^n)$ is nonzero.

Homework 6. (Due Tuesday of Week 7)

1. Show that if X is a simply connected 2 dimensional space with $H_2(X)$ = **Z**, then X is homotopy equivalent to S^2.

2. Show that if X is a simply connected 3 dimensional space with $H_2(X) = 0$ and $H_3(X)$ = **Z**, then X is homotopy equivalent to S^3.

3. Suppose f maps $S^n \rightarrow S^n$. What is the relation of f_* on $H_n(S^n ;$ **Z**$)$ and on $H_n(S^n ;$ **Z/2**$)$?

4. Show that the homology of RP^n with coefficients in Z/2 is Z/2 in all dimensions between 0 and n. Show that RP^∞ is a K(Z/2, 1), and that for any map $f:RP^n \rightarrow RP^\infty$ that induces an isomorphism on π_1 induces a nontrivial map on $H_n(\ ;Z/2)$.

Deduce the Borsuk-Ulam theorem that any map g: $S^n \to S^n$ which is odd, i.e. g(-x) = -g(x) has odd degree.

5. Compute the homology of $RP^2 \times RP^2$ with Z/2 coefficients. ($RP^n \times RP^m$?)

6. Let C_* be a chain complex of free abelian groups. Show that if H(C) = 0 in all dimensions, then $H(C\otimes A) = 0$ for any abelian group A. Deduce that if f: $X \to Y$ induces an isomorphism in integral homology, then it does with any coefficients.

Homework 7. Due the Thursday after Thanksgiving (a week off!, but don't take a week off). Do the first 6.

1. Show that if f: $T^n \to T^n$ is a self-map of the n-torus $\mathbf{R}^n/\mathbf{Z}^n$, then deg(f) = det($f_*$) where f_* is the induced map on $H_1(T^n) = \mathbf{Z}^n$.
2. Hatcher p. 205 #7
3. #11
4. #13
5. p.229 #4
6. #7
7. #13
8. #15

Homework 8, due Thursday of Week 10. (Do the first six)

1. Show that is M is a compact 3-manifold, and $\pi_1(M)$ is finite, then the universal cover of M is a homotopy 3-sphere.

2. Show that if in #1, π_1 is a finite abelian group, then it is cyclic.

3. Suppose f:M $\to$ N is a degree one map between closed n-manifolds, show that f_* is surjective on integral homology.

4. If in #3, M is a sphere, show that f is a homotopy equivalence. (Hint: show that the map is automatically onto on π_1)

5. Show that any map f: $CP^3 \to CP^3$ which is degree 1 is homotopic to the identity. Why is this not true for CP^2?

6. Suppose M is an n-manifold and that E is an oriented S^k bundle over M, k>m; describe the homology of E in terms of the homology of M. (You can use field coefficients if you want to make your life a bit simpler.)

7. Suppose M^4 is a simply connected compact 4-manifold without boundary then M has positive Euler characteristic. Suppose M^4 is a compact 4-manifold without boundary and $\pi_1 M = \mathbf{Z}$ then M has nonnegative Euler characteristic. Can you prove the same is true for $\pi_1 M = \mathbf{Z}^2$**?**

8. Show that every presentation of $\mathbf{Z}^n$ has more relations than generators if n>3. (Hint: Think about building a 2-complex from a presentation, and then the process of building a $K(\pi,1)$ from it.)

9. Show that S^5 does not bound any compact oriented 6-manifold with even Euler characteristic. Show that S^3 does bound a compact oriented 4-manifold with even Euler characteristic. (Can you construct such 4-manifold with Euler characteristic = 0?)

The **take home final** is below It will be due by 11:59 pm central time on December 11. You should send me an email of a pdf version of your solution. You can use any source, written or electronic working on **your** solutions, but *please do not work on these together or ask someone for help*.

Do any 5 of the following.

1. Say that X is a compromise space, if there is a function f taking pairs of points of X into X, such that f(x,x') = f(x',x) and f(x,x) = x.
   a. Show that no sphere of any finite dimension is a compromise space.
   b. Moreover, show that this is also true with the analogous definition for any fixed number of points > 1, (and invariant under the symmetric group).

2. Suppose X is a finite complex with $\pi_1(X) \to \mathbf{Z}$ a homomorphism. Let X_N the cover induced from the subgroup N**Z** of **Z.** Let F be a field, show that if rank $H_i(X_N; F)$ is bounded in N, then it is a periodic function of N. (Hint: One approach, and there are others, is to show that X is homotopy equivalent to the mapping torus of the covering translate acting on the infinite cyclic cover -- i.e. the cover of X corresponding to 0 in **Z**.).

3. Consider the group Γ of upper triangular 3 x 3 matrices with coefficients in **Z,** and the subgroup Γ_N of matrices that are also congruent to 1 mod N. What is a $K(\Gamma_N, 1)$? What is the integral homology of this space? (Remark: When N =1 this group has a presentation $< a,b,t \mid tat^{-1} = ab, tbt^{-1} = b >$ Can you see why? You might see the presentation $<g, h \mid [g,[g,h]] = [h,[g,h]] = e>$ which comes from a different geometric picture. Either can be used, but the former is more elementary.)

4. Compute $\pi_i(U(2))$ for i=1,2,3. Here U(2) is the group of 2x2 unitary matrices (over **C**).

5. Show that $S^n \vee S^n \vee S^{2n}$ is not homotopy equivalent to $S^n \times S^n$ although they have isomorphic homology groups. Deduce that $\pi_{2n-1}(S^n \vee S^n)$ is

nonzero. If n is even, show that this element is nontrivial when pushed, using the fold map $S^n \vee S^n \to S^n$ into $\pi_{2n-1}(S^n)$.

6. Show that $RP^2 \times S^3$ and $S^2 \times RP^3$ have isomorphic homotopy groups, but different homology groups.

7. Show that every finite presentation of Z^n has more relations than generators if n>3. (Hint: Think about building a 2-complex from a presentation, and then the process of building a $K(\pi,1)$ from it.)

8. Show that there is no injective homomorphism $\mathbf{Z}^2 \to \pi_1$(Surface of genus 2).

9. Show that if G is a finite group that acts freely on a finite complex X, then $\Sigma(-1)^i[H_i(X; Q)]$ thought of as a virtual **Q** representation is a multiple of the regular representation.

10. Suppose that a circle is embedding in T^2, show that its homology class in $\mathbf{Z}^2$ is represented by either (0,0) or (a,b) where a and b are relatively prime. (You may assume that the circle is smoothly embedded, and even has a neighborhood homeomorphic to $S^1 \times [-\varepsilon,\varepsilon]$ if you wish.) Hint: Show that the complement of this circle must be connected.