Algebraic topology (Autumn 2014)

The class hours are MWF 9-10:20 in Eck 206.
Homework is due Monday morning.
There will be a midterm and a final.

I will hold office hours on Mondays at 10:30, and occasional other times -- also by appointment in my office , Eck 403. My email is shmuel@math.uchicago.edu

Bena Tshishiku (tshishikub@gmail.com) and Nick Salter (nks@math.uchicago.edu) will be doing the grading and giving office hours (alternate weeks). They will have office hours on Friday 11-12. Bena will do odd weeks and Nick even. Bena's will be in Jones 205 and Nick's in Eckhart 11.

Recommended texts are by May, Hatcher and Spanier.

No class on Wednesdays November 12 and 19.

Homework 1. (Do 7 or as much as you can)

1. Show that any subgroup G of a free group F is free. Show that the rank, if the index is finite, only depends on [F: G]. Give a counterexample for infinite index.

2. Suppose f: $S^1 \to S^1$ is a continuous map, so that f(f(x)) = x, and f has no fixed points, show that there is a homeomorphism h: $S^1 \to S^1$ so that $hfh^{-1}(x) = -x$. (for all x)

3. Suppose that f: $\mathbf{C} \to \mathbf{C}$ is of the form $f(z) = z^n + h(z)$ and $\lim h(z)/||z||^n = 0$, show that f has a root (i.e. a solution to the equation f(x) = 0). Can it have infinitely many roots?

4. What is the universal cover of $RP^2 \vee RP^2$? Write down the covering map. What is $\pi_2(RP^2 \vee RP^2)$?

5. Suppose X is simply connected and f:X $\to$ **C** is a nowhere 0 function. Show that log(f) can be defined to be a continuous function. Is there a nonsimply connected X for which this is true?

6. Suppose f: M $\to$ N is a smooth map between compact smooth manifolds of the same dimension, and Jac(f) is nowhere vanishing, M connected and N is simply connected, show that f is a diffeomorphism. Show that if M is not compact, this need not be true.

7. Prove that the (p,q) torus knot is not isomorphic to the (r,s) torus knot unless {±p, ±q} = {±r, ±s}. Recall that the (a,b) torus knot is the embedding of S^1 in S^3 which

lies in the standard unknotted torus, via $z \to (z^a, z^b)$. (Here a and b are relatively prime integers > 1.)

8. Let $PU(n) = SU(n)/Z$ where Z is the center of SU(n) (the group of unitary n x n complex matrices with determinant = 1). What is $\pi_1(PU(n))$?

9. Consider $S^1 \vee S^2 \subset D^2 \vee S^2$ What is an element of π_2 that is nontrivial in the domain, but becomes trivial in the range? (There is an old conjecture that for 2 complexes $X \subset Y$, if $\pi_2 Y = 0$, then so is $\pi_2 X$. This exercise shows that the conjecture cannot be that inclusions of 2-complexes give injections of π_2.)

10. Suppose that X is a contractible polyhedron, show that for all k there is a function $f: X^k \to X$ that commutes with permuting the coordinates, and has $f(x,...x) = x$. Show that for $X = RP^2$ no such f exists for k=2.

Homework 2 . (Do 7 or as much as you can)

1. Give maps from S^1 into the figure 8 that are homotopic but not pointed homotopic.

2. Suppose X and Y are simply connected and homotopy equivalent, then they are pointed homotopy equivalent. (Assume X and Y are decent according to your own reasonable definition of decency). (*) Is this true without simple connectivity?

3. Let X be decent and connected. Consider the set of homotopy classes of maps from X to S^1 which send a base point to 1. Call this $\pi^1(X)$. Show that

   a. $\pi^1(X)$ is an abelian group using pointwise multiplication on the circle.
   b. If $f: X \to Y$ is a map, then there is an iduced map $f^*: \pi^1(Y) \to \pi^1(X)$. $(fg)^* = g^*f^*$.
   c. $\pi^1(X)$ is determined by the fundamental groups of the components of X. What's a formula?
   d. Show that there is a homotopy equivalence between Maps(X: S^1) and $\pi^1(X)$.

4. Definition: A line bundle over X is a pair (Z,f), where $f: Z \to X$ is a map so that X has a cover by open sets O_i, so that over O_i one has $f^{-1}(O_i) \cong O_i \times \mathbf{R}$ so that with this identification f is projection to the first factor. One also assumes that over $O_i \cap O_j$ the map $O_i \cap O_j \times \mathbf{R} \to O_j \cap O_i \times \mathbf{R}$ is *linear* over each point. Two line bundles are isomorphic if there is a map $h: Z \to Z'$ so that $f = f'h$ and h is linear on each "fiber" (inverse image of a point of X).

   Show that there is an invariant of line bundles taking values in $\mathrm{Hom}(\pi_1 X : \mathbf{Z}/2\mathbf{Z})$ obtained by sending a line bundle to the following (not quite) covering space. Send Z to Z -"0-section" and mod out by the multiplicative action of positive real numbers.

(Can you show that this invariant is complete?)

5. Suppose Z is a compact subset of Euclidean space and a neighborhood retract. Show that there is an $\varepsilon>0$, so that for any X, if $f,g:X\to Z$ have $d(f,g) < \varepsilon$, then f and g are homtopic and that the same is true rel A for any A in X. Deduce that the set of homotopy classes of maps from X to Z is always countable (for X a compact metric space).

6. (continuation) Same Z as in #5. Show that for any L, there are only finitely many homotopy classes of maps in $\pi_d(Z)$ with Lipschitz constant at most L. Indeed, show that the number grows at most like $\exp(L^d)$.

7. A space X is an H-space if there is a continuous function $\mu:X \times X \to X$, for which for a suitable $e\in X$, $\mu(e,x) = \mu(x,e) = x$. Show that $\pi_1(X,e)$ is abelian if X is an H-space.

8. Suppose G and H are groups, show that any finite subgroup of $G*H$ is conjugate to a subgroup of either G or of H.

Remark: A useful lemma is that no nontrivial free product is finite. If g is in G and h is in H, (both nontrivial), then gh has infinite order.

9. Show that $S^3\times\mathbf{C}P^2$ and $S^2\times S^5$ have isomorphic homotopy groups. (These manifolds are **not** homotopy equivalent. Why does this not contradict the Whitehead theorem?)

Hint: You might first think about the analogous problem for $S^3\times RP^2$ and $S^2\times RP^3$.

10. If M and N are connected n-manifolds, the result of removing little open n-balls from M and N and then glueing them together is called the connected sum of M and N and denoted by M#N. (Actually there are two isotopy classes of glueing maps, so if M and N are orientable, one must use the orientations to make a well defined construction.) Show that if $n>2$, $\pi_1(M\#N) = \pi_1 M * \pi_1 N$. Why is this not true for n=2? Give an example of a connected sum of 2-manifolds that is not a free product.

Remark: If n=3 a wonderful theorem of Stallings[1] asserts the converse of the first part of this problem: A closed 3 dimensional manifold is a connected sum iff its fundamental group is a nontrivial free product. Similarly, Stallings showed that a map $f:M^3\to S^1$ is homotopic to the projection of a fiber bundle iff f_* is nontrivial and its kernel is finitely generated. Do you see why this is necessary?

[1] In combination with the Poincare conjecture. (With PC, the statement is slightly more complicated.)

11. Show that no infinite group acts properly discontinuously on the Moebius strip. (Hint: Show that any homeomorphism h of the Moebius strip satisfies the condition $h(C) \cap C \neq \varnothing$.)

Homework 3 (Do 5 problems)

1. Suppose X is a 2-complex and $\pi_2 X = 0$, then show that $\pi_3 X = 0$.

2. Let $L_k^{2n-1}(a_1....a_n)$ and $L_k^{2n-1}(b_1....b_n)$ be two lens spaces (of the same dimension and same fundamental group). Show that there is a map f: $L_k^{2n-1}(a_1....a_n) \rightarrow^{-} L_k^{2n-1}(b_1....b_n)$ that induces an isomorphism on π_1.

3. If A, B and C are groups and i:C → A and j: C → B are injections induced by maps of spaces K(C,1) → K(A,1) and K(C,1) → K(B,1), then show that $K(A,1) \cup_{K(C,1)} K(B,1)$ is a $K(A*_C B, 1)$. Give a counterexample if they are not injections.

4. Show that any complex line bundle over S^n is trivial for n>2. (Change the definition from 2.4 from an **R** to a **C**.)

5. Show that if $f:S^1 \rightarrow S^1$ satisfies f(x) = -f(-x), then the element of $\pi_1(S^1) \approx$ **Z** is odd. If f(x) = f(-x) then it is even. (These are both true for S^n for all n; you will be able to prove this in a few weeks).

6. Show that $\Sigma(T^2)$ is homotopy equivalent to $S^2 \vee S^2 \vee S^3$ (Hint: use a cell decomposition; remember π_2 is abelian.)

7. Show that k and l are relatively prime iff every map from $L_k^{2n-1}(1,1,...1)$ to $L_l^{2n+1}(1,....1)$ is null homotopic.

8. What is $\pi_2(RP^2, RP^1)$?

9. Let SO(n) be the group of orthogonal matrices of determinant 1. Check that SO(2) is the circle. Observe that SO(n) acts on S^{n-1} transitively (i.e. with a single orbit) and with isotropy SO(n-1). Deduce SO(n) is connected for all n. Show that SO(n) → SO(n+1) induces an isomorphism on π_i for all i<n-1.

Remark: The limit groups $\pi_i(SO(n))$ as $n \rightarrow \infty$ form a periodic sequence in i, with period 8. They are **Z_2, Z_2, 0, Z**, 0, 0, 0, **Z** according to Bott periodicity. We will not prove this in this course. (Sadly)

Midterm (due October 29th at 9 am; you may consult books, but not people. Do 5)

1. Show that any subgroup of a free product of finite groups is a free product of a free group and a free product of finite groups.

2. How many isomorphism classes of covering spaces of a connected graph are there?

3. Let $X = (S^1)^n/S_n$ where S_n is the symmetric group on n-letters, which acts on the n-torus by permuting factors. What are $\pi_1(X)$ and $\pi_2(X)$? Let S^1 be mapped into X as the diagonal (i.e. $s \to (s,s,....s)$). What element of $\pi_1(X)$ is represented by this class?

4. Let X be a space with $\pi_i(X) = 0$ for i=1,2,3....k. Show that, in the notation of problem set #4 (you only need 4.4 for this!) $E^j(X) = 0$ for j at most k.

5. Suppose that $Z \to X$ is a vector bundle with fiber $\mathbf{R}^n$. Suppose that X is a k dimensional simplicial complex with k<n. Show that there is a map $f:X \to Z$ so that for all x, $f(x) \neq 0$ (note that 0 is a well defined vector in each fiber of a vector bundle).

6. Construct a 2 dimensional CW complex X with 1 0-cell, 2 1 cells and 2 2-cells, such that all maps from X to a circle are nullhomotopic, and X is not contractible.

Homework 4: (Due November 5 - do all of these, if you can)

1. Show that any map $T^2 \to 8$ (The figure 8) is homotopic to one factors through a circle. (Hint: Show that any abelian subgroup of a free group is cylic.) Show that the identity map $8 \to 8$ is not homotopic to a map that is not surjective.

2. Let X,x be a pointed space and let ΩX = the space of maps $f:[0,1] \to X$, such that $f(0) = f(1) = x$. Show that $\pi_i(\Omega X) \cong \pi_{i+1}(\Omega X)$. More generally, for any pointed Z, the homotopy classes of pointed maps $[\Sigma Z \to X]$ is isomorphic to the homotopy classes of pointed maps $[Z \to \Omega X]$

3. Show that if X is a CW complex with $\pi_i(X) = 0$ for $i \leqq k$, then X is homotopy equivalent to a CW complex Z, with Z^k = a point.

For the remaining problems you can assume that you know that $\pi_n(S^n) = \mathbf{Z}$.

4. Show that for every n there is a space $K(\mathbf{Z}, n)$ with the property that $\pi_i(K(\mathbf{Z}, n)) = 0$ unless i=n and $\pi_n(K(\mathbf{Z}, n)) = \mathbf{Z}$. Show that this space is uniquely determined up to homotopy type by n.

5. Show that for all n, the space $K(\mathbf{Z}, n)$ has a multiplication $\mu:K\times K \to K$ with an identity, i.e. $\mu(k, e) = \mu(e, k) = k$.

6. Show that $\Omega K(\mathbf{Z}, n) = K(\mathbf{Z}, n\text{-}1)$; using this show that for any X, the pointed maps $[X: K(\mathbf{Z}, n)]$ forms an abelian group. We shall denote this group $E^n(X)$.

7. Show that if X is a k-dimensional CW complex, then $E^n(X) = 0$ for $n>k$.

8. Show that $E^{k+1}(\Sigma X) \cong E^k(X)$

9. Show that if $A \subset X$ is an inclusion of CW complexes, then there is an exact sequence $\ldots \to E^k(X/A) \to E^k(X) \to E^k(A) \to E^{k+1}(X/A) \to \ldots$

The map $E^k(A) \to E^{k+1}(X/A)$ is induced by considering X/A as $X \cup_A cA$ and using any map $X \to cA$ that is the identity on A (why is there such a thing) to give a map $X/A \to \Sigma A$, and making use of problem 8.

10. Show that there is an exact sequence if $Z = X \cup Y$ and $A = X \cap Y$, all CW complexes:

$$\to E^k(Z) \to E^k(X) \oplus E^k(Y) \to E^k(A) \to E^{k+1}(Z) \to \ldots$$

11. Compute all $E^k(RP^n)$ (by induction on n). Deduce that there is no finite dimensional $K(\mathbf{Z}/2\mathbf{Z}, 1)$.

Homework 5. (Please do 6 of the following; due on Monday.)

1. Compute axiomatically the homology groups of $S^n \times S^m$.

2. Give spaces with the same homology but are not homotopy equivalent. (Hint: Use the previous problem and you'll need degree theory).

3. Prove that $\pi_5(S^3 \vee S^3)$ is not trivial. (Hint: This is related to the previous problems; think about using the space $S^3 \times S^3$.)

4. Compute the homology of complex projective space CP^n (by induction on n).

5. Using the Hurewicz homomorphism, but not the Hurewicz isomorphism theorem, compute $\pi_n(\Sigma^{n-1} RP^2)$.

6. Show that any subgroup of index k in the free group F_2 is isomorphic to the group F_{k+1}. What a coincidence! (It seems to be very rare for a group that has lots of subgroups of index k -- and many groups have many subgroups of finite index -- for their isomorphism class to just depend on the index.)

7. Show that $RP^2 \times S^3$ and $S^2 \times RP^3$ are not homotopy equivalent. (Why do they have the same homotopy groups?)

8. Using the Hurewicz theorem give a complex X which is not contractible, but so that ΣX is contractible. (Can you do it without Hurewicz?)

9. Using the Hurewicz theorem (for infinite complexes), show that if X is a finite simply connected complex, and f: $X \to X$ induces the trivial map on reduced homology, then for some n, f^n is nullhomotopic (i.e. homotopic to a constant). Give a map from $S^3 \vee S^2$ to itself that is trivial on homology, but not nullhomotopic.

10. Let A: $T^2 \to T^2$ be a diffeomorphism. From its induced map on H_1 compute the homology of the 3-manifold $(T^2\times\mathbf{R})/\mathbf{Z}$ where the generator of $\mathbf{Z}$ acts on $T^2\times\mathbf{R}$ by sending (t,r) to (t+1, Ar).

11. Suppose that f is a simplicial mapping of a finite complex onto itself. (It preserves a triangulation, so there is an n, so that $f^n = id$.) Show that $\Sigma(-1)^i tr(f_{*,i}) = \chi(F)$ where F = Fix(f), the fixed set of f.

12. (*, worth 3 problems) If X is a finite simplicial complex with a simplical map to S^1. Take covers of X associated to the subgroups n$\mathbf{Z}$ of $\mathbf{Z} = \pi_1(S^1)$. This defines a sequence of spaces X_n that all cover $X_1 = X$. Show that the dimension of $H_i(X_n; \mathbf{Q})$ is linear in n with a bounded nonnegative error. This error is periodic.

(Of course the hypothesis of the simplicial map is equivalent to being given a homomorphism $\pi_1X \to \mathbf{Z}$.)

Homework 6: (Please do 6.)

1. Show that if a fibration $F \to E \to B$ is over a contractible base, F is homotopy equivalent to E, i.e. there is a homotopy equivalence $E \to B\times F$ that commutes with the projection to B, i.e. the fibration is trivial. Show that the restriction of a trivial fibration to a subset of B is trivial.

2. Show that if $F\to E\to S^n$ is a fibration, then there is an exact sequence,

$$\ldots\to H_{m-n+1}(F) \to H_m(F) \to H_m(E) \to H_{m-n}(F) \to\ldots$$

3. Use #2 to compute $H_i(\Omega S^n)$.

4. Prove that there is an isomorphism $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ when i is less than 2n.

5. Suppose $f:X\to X$ and $g:Y\to Y$ are maps, and $A = X\cap Y$ is a subcomplex, and f=g restricted to A, and they map $A \to A$. Show that $L(f\cup g) = L(f) + L(g) - L(f\cap g)$.

6. Let #(X) be an invariant of finite simplicial complexes taking values in $\mathbf{Z}$ that is a homotopy invariant, and satisfies $\#(X\cup Y) = \#(X) + \#(Y) - \#(X\cap Y)$. Show that $\#(X) = (\#(*)-\#(\varnothing))\chi(X) + \#(\varnothing)$.

7. Prove that for a fibration with finite CW complex fibers, $\chi(E) = \chi(F)\chi(B)$. (Hint: Cover B by contractible subsets.

8. Consider a closed connected triangulated surface, Σ. Suppose f: $\Sigma \to \Sigma$ preserves the triangulation and is a homotopy equivalence. Show that if f is the identity on some 1-simplex, then either f_* on H_2 is multiplication by -1 or f is the identity. Deduce that if f is nontrivial, then the fixed set is a finite set of points.

9. (Look up the classification of compact connected oriented surfaces in some book if the following doesn't make sense to you.) Show that for every finite group G, there is a number n_G so that {genuses| Σ_g has an action by G that is trivial on H_2 but no element of G acts trivially} differs from the naural numbers that are 1 mod n_G in a finite set.

10. Prove that for any smoothly embedded circle in S^3 (you may assume that it has a neighborhood diffeomorphic to $S^1 \times \mathbf{R}^2$) the complement has the same homology as S^1. Deduce that there is a surjective homomorphism $\pi_1 \to \mathbf{Z}$. Give an example where the homology of the cover corresponding to the kernel of this homomorphism is not trivial (unlike the "unknot").

11. Compute $\pi_i(T^4 \# T^4)$ for i <4. (Here # denotes connected sum.)

Homework 7. (Please do 6)

1. Show that if a map $f: S^n \to S^n$ that is odd, i.e. if f(x) = f(-x), it has odd degree.

2. Show that if f: $\mathbf{R}^n \to \mathbf{R}^n$ has f^5 = fffff = id (5 times) then f has a fixed point. (This is not true if 5 is replaced by 6.)

3. Compute the cohomology of RP^n with $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{F_p}$ coefficients.

4. Do the same as #3 for the lens space $L^{2n-1}(k; a_1,...a_n)$. For which p is there nontrivial cohomology?

5. Let $G \times X \to X$ is a simplicial action a finite group G on a complex X. Show that the natural map $H^i(X/G ; \mathbf{Q}) \to H^i(X; \mathbf{Q})$ has image lying in the invariants $H^i(X; \mathbf{Q})^G$. Show that the map $H^i(X/G ; \mathbf{Q})$ x $H^i(X; \mathbf{Q})^G$ is an isomorphism.

6. Let $p_{\mathbf{F}}(X, t) = \Sigma \dim(H^i(X; \mathbf{F}))t^i$. Note that $p_{\mathbf{F}}(X, -1) = \chi(X)$. Show that $p_{\mathbf{F}}(X \times Y, t) = p_{\mathbf{F}}(X, t)p_{\mathbf{F}}(Y, t)$. Give a fibration where the analogous formula does not hold.

7. Show that the evaluation map of a cochain on chain descends to 'omology': i.e. there is a pairing of a cohomology class $\alpha \in H^i(X;\mathbf{F})$ on $H_i(X;\mathbf{F})$. This defines a homomorphism $H^i(X;\mathbf{F}) \to H_i(X;\mathbf{F})^*$ (or equivalently, there is a pairing $H^i(X;\mathbf{F}) \otimes H_i(X;\mathbf{F}) \to \mathbf{F}$). Show that this is an isomorphism whenever X is a finite CW complex (or equivalently, that the pairing is nonsingular). (Hint: Induct on cells.)

8. Show that for $X = \mathbf{R}P^2$ the previous exercise is false when $\mathbf{F}$ is replaced by $\mathbf{Z}$. (There is a pairing: it does not give rise to an isomorphism.)

9. For π a group, define $H_i(\pi)$ to be $H_i(K(\pi,1))$. Prove a Mayer-Vietoris sequence for amalgamated free products, where the subgroup injects.

10. Show that there is no presentation for $\mathbf{Z}^n$ with fewer than $n(n-1)/2$ relators. (Hint: Think about how one constructs $K(\pi,1)$ from a presentation.)

Note: There is also no presentation with fewer than n generators. Indeed, there is no CW structure on $K(\mathbf{Z}^n,1)$ with fewer than $n!/(n-k)!k!$ k-cells. Why not?)

11. Suppose that α is a homology class on which a nontrivial cup product evaluates nontrivially, show that α is not in the image of the Hurewicz homomorphism.

Homework 8: (Please do 7)

1. Show that $f\mathbf{:}\ \mathbf{CP^n} \to \mathbf{CP}^n$ is determined up to homotopy by its degree iff n is odd.
2. Show that every f: $\mathbf{CP^n} \to \mathbf{CP}^n$ has a fixed point if (and only if) n is even.
3. Show that every f: $\mathbf{RP^n} \to \mathbf{RP}^n$ has a fixed point if (and only if) n is even.
4. How many homotopy classes of maps are there from $S^2 \times S^2 \to \mathbf{CP}^2$? Which degrees occur?
5. Prove that if $f{:}T^n \to T^n$ induces A on π_1, then deg f = det A.
6. Give an example of a 4 x 4 matrix which does not occur as the induced map on H_1 for any self map of a surface of genus 2.
7. Show that if there is a degree one map $f\mathbf{:}\mathbf{S^n} \to M^n$ where M is a closed manifold, then f is a homotopy equivalence.
8. Show that if there is a map from $f\mathbf{:}\mathbf{S^n} \to M^n$ of nonzero degree, then $\pi_1(M^n)$ is finite and the universal cover has the same $H_*(; \mathbf{Q})$ as the sphere. (And conversely, if you are a budding algebraic topologist.)
9. Give an example of f: X → X, X a finite CW complex, with L(f) =0, but f is not homotopic to a map without fixed points. Hint: Let X be non-simply connected.
10. Let $V(d) \subset \mathbf{CP}^n$ be the solution of a degree d homogeneous polynomial equation. Show that $[V(d)] = d$ in $H_{2n-2}(\mathbf{CP}^n; \mathbf{Z}) = \mathbf{Z}$. Show that V(d) cannot be homotoped into the complement of V(d').

11. Suppose that M^n is a closed n-manifold, and π_1 is cyclic, and π_i $1<i<n-1$ trivial, then M is homotopy equivalent to a lens space. In particular if the fundamental group is not trivial nor Z_2 then the dimension is odd. (Prove this last point directly using the Lefshetz theorem.)

12. Prove that there is no simplicial $\mathbf{Z}_p$ action on a compact triangulated manifold that has exactly one fixed point.

13. Prove that if a surface of genus g has an effective $\mathbf{Z}_p \times \mathbf{Z}_p$ action, p odd, (preserving a triangulation), then $g = 1$ mod p.

THE FINAL

Prepare the solution to 5 problems that you will be willing to discuss with me for 10 minutes in an oral presentation. (We will not have time for all 5 problems, of course.)

1. Suppose f: $S^n \to S^n$ has degree a and g: $S^m \to S^m$ has degree b, what is the degree of the join $f*g$: : $S^{n+m+1} \to S^{n+m+1}$?
2. What are the degrees of self maps of S^1 that commute with the action of $\mathbf{Z}_k$ by rotation?
3. Build a $\mathbf{Z}_6$ action on S^3 that has no fixed points (for the whole group), and so that there is a nullhomotopic self map of S^3 that commutes with this self map. Use this to build a contractible 4-dimensional CW complex on which $\mathbf{Z}_6$ acts with no fixed points. Why is there no finite complex with this property?
4. Show that the only nontrivial group that can act freely on any CP^n is $\mathbf{Z}_2$.
5. What are the groups $H_i(D_{2p}; F)$ where D_{2p} is the nonabelian (Dihedral) group of order 2p (p a prime), and F is an arbitrary field? What about $H_i(D_{2p}; \mathbf{Z})$?
6. What is the rational cohomology of $SP^\infty(S^n)$ where $SP^\infty(S^n)$ = limit of $(S^n)^k/\Sigma_k$ is the infinite symmetric product. (One includes the k-th symmetric product in the k+1-st by inserting a base point as the last coordinate).
7. Show that any function $f: S^{2k}\times S^{2k} \to S^{2k}$ is nullhomotopic when restricted to the first or second coordinate.
8. Prove that all cup products of positive dimensional classes vanish in a suspension.
9. Show that CP^n is not a union of n-1 nullhomotopic closed subspaces. (Do you see why it is a union of n?)
10. Compute $H(\Omega\Sigma X; F)$ for a field F in terms of $H(X; F)$ (assume X is a connected finite complex). Hint: Use the idea of HW 6.3 (I do not ask for any multiplicative structures.)
11. Show that if X is an n-connected m complex, with $m<2n$, then X is homotopy equivalent to a suspension. Hint: Use the previous problem.
12. Prove that $CP^2\#-CP^2$ and $CP^2\#CP^2$ have the same homotopy groups and the same homology groups, but are not homotopy equivalent. Here $-CP^2$ is CP^2 but with the reversed orientation. In other words, one glues the two copies of CP^2 I a disk

together using different identifications of their boundary S^3's: either by the identity or by the orientation reversing diffeomorphism.

13. Suppose that X and Y are CW complexes with cells in only even dimensions, and f: $X \rightarrow Y$ is a map which induces isomorphisms on π_k for all even k. Show that f is a homotopy equivalence.
14. Let π be a group with a nontrivial homomorphism to SU(2). Suppose we add a generator and a relation to π, show that the new group still has a nontrivial homomorphism to SU(2) (and hence is nontrivial).
15. Give an example of a finitely presented group that is not the fundamental group of a closed 3-manifold.
16. Show if M^{2k} is a closed manifold which is the boundary of a compact manifold W^{2k+1}, then $\chi(W) = 2\chi(M)$. Deduce that the projective spaces RP^{2k} and CP^{2k} do not bound any compact manifold.
17. Show that for any X, the map $X \rightarrow K(\pi_1 X, 1)$ induces a surjection on H_2. Its kernel is the image of the Hurewicz homomorphism from π_2.
18. Show that any closed 4 manifold with fundamental group $\mathbf{Z}^3$ has Euler characteristic at least 2. (There are 4-manifolds with fundamental group $\mathbf{Z}^2$ and with fundamental group $\mathbf{Z}^4$ with χ=0.)
19. Consider orientable 3 manifolds M that fiber over the circle with fiber a torus T. Show that these are homotopy equivalent iff they are diffeomorphic. Let A be the monodromy automorphism of $H_1(T)$. Show that $H_1(M; \mathbf{Q}) = \mathbf{Q}$ iff $tr(A) \neq 2$. How does tr(A) relate to the structure of $H_1(M; \mathbf{Z})$? Deduce, if you can, that if $tr(A) \neq 2$, there are finitely many of these manifolds with a given H_1.
20. Show that $\Sigma(T)$ has the homotopy type of a wedge of spheres.
21. Show that for any Z, $[T^n : \Omega Z] = \oplus(n,i)\pi_i(\Omega Z)$ (where is the binomial coefficient).
22. Give an example of a space Y, where $[T^n : Y]$ is not identical to $\oplus(n,i)\pi_i(Y)$
23. Suppose G is a finite group acting simplicially on a finite simplicial complex X. Give a formula for the Euler characteristic of X/G in terms of the Lefshetz numbers L(g, X). (If the action is free, then this formula should boil down to the multiplicativity of Euler characteristic in covers, when one feeds in the Lefshetz fixed point theorem for each nonidentity element g of G.)
24. Discover something beautiful that you'd like to show me.