ARITHMETIC MANIFOLDS OF POSITIVE SCALAR CURVATURE

JONATHAN BLOCK AND SHMUEL WEINBERGER

1. Introduction

Gromov and Lawson [GL2] and Schoen and Yau [ScY] have shown that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature. As is very well recognized by now, their approach is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds of positive scalar curvature. Our goal in this paper is to explore the situation if we study the problem of complete metrics with no quasiisometry conditions at all.

Let M be an irreducible locally symmetric space of noncompact type of finite volume. It is the double coset space $\Gamma\backslash G/K$ associated to a lattice Γ in a semisimple Lie group G. Our main theorem is the following:

Theorem 1.1. *Let $M = \Gamma\backslash G/K$, G semisimple and Γ an irreducible lattice. M can be given a complete metric of uniformly positive scalar curvature $\kappa \geq \epsilon > 0$ if and only if Γ is an arithmetic group of $\mathbb{Q}$-rank at least 3.*

Amplification 1.2.
1. *In the $\mathbb{Q}$-rank 1 and $\mathbb{Q}$-rank 2 cases, one cannot have a metric of uniformly positive scalar curvature even in the complement of a compact set.*
2. *On the other hand, in the cases where we do construct these metrics, they can be chosen to have finite volume or bounded geometry in the sense of having bounded curvatures and injectivity radius bounded away from zero (but of course not both).*

We will not address here the natural question of whether the positive scalar curvature metric can be chosen coarse quasi- isometric to the natural metric on M. The metrics constructed here in section 2

[1]AMS 1991 Subject Classification: 53C20, 05B45, 58G10
The first author was partially supported by an NSF-YI award. The second author was partially supported by an NSF grant. The authors would like to thank Miriam and Bud Diamond for their hospitality while this paper was written.

have the coarse quasiisometry type of a ray. The classical methods discussed above show that the metrics cannot be chosen quasiisometric (i.e. uniformly biLipschitz) to the natural metric on M. Another natural question we do not address here is whether the complete positive scalar curvature metrics can be chosen to be bounded geometry in the sense of having bounded curvatures and volume.

Our argument in the last section obstructing the existence of complete positive scalar curvature metrics for quotients where the Lie group has low rank or the arithmetic group has low $\mathbb{Q}$-rank is based on the picture of the ends of these manifolds given by Borel and Serre [S]. According to the compactness criterion of Borel and Harish-Chandra, $\mathbb{Q}$-rank $= 0$ implies compactness and necessity in this case is the theorem of Gromov-Lawson and Schoen-Yau mentioned above. For $\mathbb{R}$-rank $= 1$ one has the situation of cusps and there are infranilmanifold "sections" of the cusps at the various ends. These are examples of what Gromov and Lawson call "bad ends", as their universal covers have contracting maps to appropriate Euclidean spaces, and the results of Gromov and Lawson cover this case as well. Thus one is in the situation of $\mathbb{R}$-rank at least 2 and, by Margulis's work, arithmetic groups. For $\mathbb{Q}$-rank $= 2$, there is still a beautiful aspherical, but rather mysterious, manifold which is a cross section of infinity. We do not know whether it has the appropriate "hypersphericality" to obtain the nonexistence result from [GL2]. However, instead we prove an appropriate Novikov conjecture type result and Bochner type argument to still show that we have a "quite bad end", namely the kind of end that does not support a metric whose negative scalar curvature set has compact closure.

Our results should be understood within the context of the (even stable) Gromov-Lawson-Rosenberg conjecture and its analogue, the Novikov conjecture. The topological analogue of characteristic class obstructions to positive scalar curvature are characteristic class obstructions to proper homotopy equivalence. The analogue of Theorem (1.1) in this setting, the rigidity aspects of which are essentially due to Farrell and Jones, [FJ3], is:

Theorem 1.3. *Let $M = \Gamma\backslash G/K$, G semisimple and Γ an irreducible lattice. There exists a non-resolvable homology manifold proper homotopically equivalent to M if Γ is an arithmetic group of $\mathbb{Q}$-rank at least 3. If Γ is arithmetic of $\mathbb{Q}$-rank* 0 *or* 1 *then M is properly rigid. If $\mathbb{Q}$-rank* 2*, then if one knows the Borel conjecture for the fundamental group at infinity, then M is properly rigid.*

This theorem answers a conjecture of Quinn's, Conjecture 4.1 (Topological Rigidity), p. 603, [Q], and suggests that it be reformulated to take into account the metric structure at infinity.

The general problem can be reformulated (in our relative situation) by surgery theory into a statement about a relative assembly map:

$$A : H_*(B\pi, B\pi^\infty; \mathbb{L}) \to L(\pi, \pi^\infty)$$

Here π and π^∞, are the fundamental group of a manifold $M$and its end respectively. If A is injective, then the Poincare dual of the L-class pushed into this relative group homology is a proper homotopy invariant. It is natural to expect that the same thing is true as an obstruction for complete positive scalar curvature metrics on a spin manifold M, except that one uses the image of the Dirac operator $f_*[\not{D}_M] \in KO_n(B\pi, B\pi^\infty)$ rather than a signature class in $H_*(B\pi, B\pi^\infty; \mathbb{L})$. For instance, a conjectural obstruction should be the rational vanishing of this class. More precisely, one could formulate:

Conjecture 1.4. *If M is a complete spin manifold with uniformly positive scalar curvature, and for which $\pi_1(M)$ and $\pi_1^\infty(M)$ are torsion free, then $f_*[\not{D}_M] = 0$ in $KO_n(B\pi_1(M), B\pi_1^\infty(M))$.*

For π_1 and π_1^∞ with torsion, one must use the relative term associated to the functoriality of the "left hand side of the Baum-Connes conjecture" proven in [BB1].

Thus, unlike the case of closed manifolds discussed by Rosenberg in [RS], the main problem in completing the picture analogous to the signature operator case (and by analogy to the closed manifold situation) is that there is no assembly map from $KO_n(B\pi_1, B\pi_1^\infty)$ to the appropriate K-theory of some C^*-algebra that could be the relative K-theory, and which might reasonably be an isomophism for torsion free groups. The Baum-Connes conjecture makes predictions about the reduced C^* algebra, and for such C^* algebras one only has induced maps for injections of groups, or at least homomorphisms where the kernels are amenable. In the $\mathbb{Q}$-rank 2 case we consider, the relevant kernel is free of infinite rank, so this heuristic cannot be directly applied. It is thus a first interesting case where a new method is needed and that is what is provided here. The same reasoning we employ verifies the reduction of the vanishing of the higher relative $\hat{A}$-genus to the injectivity of an assembly map for e.g. any simply connected manifold, even if the fundamental group at infinity is not amenable. What we have verified in essence is the vanishing of $f_*[\not{D}_M] \in KO_{n-1}(B\pi_1^\infty)$ assuming suitable Novikov type conjectures for the group π_1^∞.

In a subsequent paper we give further evidence for this conjecture showing that in the (rather complementary) case where π_1^∞ injects into π_1, one can reduce the conjecture to the injectivity of the appropriate assembly map. This excludes many manifolds which do not have "bad ends" at all, e.g. which can support metrics whose nonpositive scalar curvature set is compact. For instance, a punctured n-torus or punctured hyperbolic manifold have "good ends" but as suggested by this conjecture, they have no complete metrics of positive scalar curvature.

We hope to eventually combine the methods of the special cases considered here with the approach of that forthcoming paper to give a systematic treatment of obstructions to complete metrics of positive scalar curvature under plausible Novikov type conjectures, that are hopefully verifiable in interesting cases.

2. Constructing metrics on non-compact manifolds

Our goal in this section is to construct manifolds of uniformly positive scalar curvature on locally symmetric spaces of $\mathbb{Q}$-rank at least 3. We will in fact prove a rather general theorem about existence of metrics of uniformly positive scalar curvature on locally compact manifolds. (This will by no means be the most general theorem on the subject.) It is a generalization of the bordism criteria of Gromov and Lawson [GL1] and Rosenberg and Stolz, [RS1].

Now we will apply the Schoen-Yau-Gromov-Lawson surgery theorem and its proof which assert that if we do surgery on in codimension ≥ 3 on a manifold of positive scalar curvature, then we can put on the new manifold a metric of positive scalar curvature and that this procedure is local. Furthermore, by examination of their proof, one can achieve that the scalar curvature κ on the new manifold is greater than or equal to $\lambda\kappa_0$ where κ_0 is the scalar curvature on the original manifold and $\lambda < 1$ is any fixed constant. When there are an infinite, but locally finite sequence of surgeries we can achieve:

Theorem 2.1. *Let M be a locally compact manifold with a complete metric of uniformly positive scalar curvature $\kappa \geq \epsilon > 0$. Suppose that M' is another manifold obtained from M by a locally finite sequence of surgeries of codimension at least 3. Then M' also has a complete metric of scalar curvature $\kappa \geq \epsilon' > 0$. Moreover, if M has finite volume then M' can be made with this property too.*

Proof. According to Gromov and Lawson [GL1] we can equip M' with a metric of positive scalar curvature . We want to see that we can perform these surgeries with their adaptation of the metric to achieve three things:

1. Completeness
2. Uniform positivity of the scalar curvature and
3. finite volume.

Recall that the surgery process replaces $\partial(D^{i+1}) \times D^{n-i}$ with $D^{i+1} \times \partial(D^{n-i})$. The only way to confound completeness is to make the metric on D^{i+1} small so that a sequence of points that was not Cauchy in the original metric is Cauchy in the new metric, by using "short cuts" through the disks. To avoid this we require that the metric on D^{i+1} (this is from $D^{i+1} \times \partial(D^{n-i})$) have the property that for $x, y \in \partial(D^{i+1})$ that $d_{D^{i+1}}(x, y) \geq md_{\partial(D^{i+1})}(x, y)$. Having arranged this, order the sequence of surgeries. Perform them sequentially in such a manner that the metric in the Nth surgery have $\partial(D^{i+1}) \times D^{n-i}$ replaced by $D^{i+1} \times \partial(D^{n-i})$ where the metric on D^{i+1} is big as above and the sphere $\partial(D^{n-i})$ is chosen so small that $\kappa \geq ck_0$ and so that the volume of $D^{i+1} \times \partial(D^{n-i})$ is $\leq \frac{1}{2^N}$; this last is achieved by using very thin tubular neighborhoods of the spheres in the original construction of [GL1]. Note, that if the manifold M started with finite volume, the metric we construct on M' also has finite volume. This completes the proof. ∎

In light of the surgery theorem 2.1, to construct manifolds of positive scalar curvature, we need a criterion that implies that we can construct a manifold from a given one by a locally finite sequence of surgeries of codimension at least 3. We now make an analogous definition to [RS1]. Consider a map $B \to BO$, which we can assume is a fibration by standard homotopy theory. We will assume for simplicity that B has locally finite n-skelata, for $n \leq 3$. (This condition simplifies certain arguments below.) A proper B-manifold is a locally compact manifold M with a proper map $\hat{\nu} : M \to B$ such that the composition $M \to B \to BO$ classifies the stable normal bundle. A proper B-cobordism between proper B-manifolds M_1 and M_2 is a proper B-manifold W such that

$$\partial W = M_1 \coprod M_2$$

and such that the structure map for W restricts to the structure maps of M_1 and M_2 on the boundary. Finally, let us recall that a proper map $f : X \to Y$ of locally compact spaces is called *properly k-connected* if for all (P, Q), a locally finite k-dimensional CW pair and all diagrams of proper maps

$$\begin{array}{ccc} P & \rightarrow & Y \\ \uparrow & & \uparrow f \\ Q & \rightarrow & X \end{array}$$

there exists a proper map $g : P \to X$ such that $P \to X \to Y$ is properly homotopic to $P \to Y$.

Theorem 2.2. *Let M be a proper B-manifold of dimension at least 5, such that the structure map $\hat{\nu} : M \to B$ is properly 2-connected. Assume that M is properly B-cobordant to a proper B-manifold X, a complete manifold with uniformly positive scalar curvature. Then M has a complete metric of uniformly positive scalar curvature. Moreover, if X has finite volume then M can also be made with finite volume.*

Proof. Let W be a proper B-cobordism between M and X, dimension W is at least 6. By performing a locally finite sequence of surgeries that do not change M and X in dimensions ≤ 2 we can make $W \to B$ properly 3-connected. This uses the assumption that B has a locally finite 3-skeleton. ([T] is the standard, but unpublished reference for proper surgery. The same kind of arguments but in the slightly different context of bounded surgery are carried out in [FP], Theorem 5.3) Since $M \to B$ is properly 2-connected and $W \to B$ is properly 3-connected, this implies from the definition of properly 2-connected that $M \to W$ is properly 2-connected. Hence we see from the proof of the proper s-cobordism theorem, [sieb], that W has a handle decomposition from M to X without handles of dimension $0, 1$ or 2. Thus the dual decomposition from X to M has a presentation without handles of codimension $0, 1$ or 2. Hence we can go from X to M by a locally finite sequence of surgeries of codimension at least 3. Thus by the surgery theorem, 2.1, the conclusion of the theorem follows. □

Now we construct the metrics on the locally symmetric spaces.

Proof. (of sufficiency in Theorem 1.1)

Let $M = \Gamma\backslash G_{\mathbb{R}}/K$, G semisimple algebraic and Γ a lattice and an arithmetic group of $\mathbb{Q}$-rank at least 3. Let us recall what this means. A subgroup $T \subset G$ defined over $\mathbb{Q}$ isomorphic to $G_m^{\times q}$ is called a split torus. Let q be the dimension of a maximal split torus. By definition, this number q is called the $\mathbb{Q}$-rank of Γ. Then M is compactified à la Borel and Serre [S] as a manifold with corners. This is accomplished by adding to $\widetilde{M} = G_{\mathbb{R}}/K$ certain Euclidean spaces to form $\overline{\widetilde{M}}$ in such a way that $G_{\mathbb{Q}}$ acts on $\overline{\widetilde{M}}$ and so that $\overline{M} = \Gamma\backslash\overline{\widetilde{M}}$ is compact. Some basic facts about this compactification are:

1. $\partial\overline{\widetilde{M}}$ has the homotopy type of its Tits building $\mathcal{B}$ of $\mathbb{Q}$-parabolic subgroups of G.
2. By the theorem of Solomon and Tits, $\partial\overline{\widetilde{M}}$ has the homotopy type of a bouquet of $q-1$-spheres.
3. Not only is M a $K(\Gamma, 1)$, but so too is $\overline{M}$.

From these facts we deduce

Proposition 2.3. *If $\Gamma \subset G_{\mathbb{R}}$ is as above, then*

1. *If $q = 0$ then M is compact.*
2. *If $q = 1$, then $\pi_1^\infty(M) \to \pi_1(M)$ is injective. (Of course $\pi_1(M) = \Gamma$.)*
3. *If $q = 2$, there is an exact sequence of groups*
$$1 \to \mathbb{F}_\infty \to \pi_1^\infty(M) \to \pi_1(M) \to 1$$
where $\mathbb{F}_\infty$ is the free group on an infinite number of generators.
4. *If $q \geq 3$, then $\pi_1^\infty(M) = \pi_1(M)$.*

Remark 2.4. *If there is more than one end, (which can happen only if $q = 1$), we mean to just take one of them.*

Set $B = \overline{M} \times [0, \infty)$ and let $B \to BO$ be the map classifying the normal bundle. Note that B has a locally finite 3-skeleton since $\overline{M}$ is a finite complex. Consider M as a proper B-manifold,
$$\hat{\nu} : M \to B$$
by $\hat{\nu}(m) = (m, \rho(m))$ where $\rho : M \to [0, \infty)$ is distance from some fixed point, which is proper since M is complete. Now note that as a homotopy type $\overline{M}$ is obtained from $\partial\overline{M}$ by attaching cells of dimension q and larger. So
$$\partial\overline{M} \to \overline{M}$$
is 2-connected if $q > 2$. This implies that $\hat{\nu} : M \to B$ is properly 2-connected. Now consider $(M, \hat{\nu}) \in \Omega_n^B([0, \infty))$ the cobordism group of dimension n proper B-cobordism classes of proper B-manifolds. For our particular $B = \overline{M} \times [0, \infty)$ this bordism group clearly vanishes. Hence $(M, \hat{\nu})$ is properly B-cobordant to any other proper B-manifold. Hence it suffices to show that there exists a proper B-manifold with a complete metric of uniformly positive scalar curvature and finite volume.

To do this, we start with M. Consider inside $M \times [0,\infty)$ the 2-skeleton, K, and let $N(K)$ be a regular neighborhood of K in $M \times [0,\infty)$. Now we may assume that $N(K)$ does not intersect $\{0\} \times M$ by taking only the part of the 2-skeleton that does not touch $\{0\} \times M$ and taking the second barycentric subdivision before taking the regular neighborhood (or just by existence of collars). Then $X = \partial N(K)$ is a manifold with the following three properties:

1. $\pi_1(X) = \pi_1^\infty(X) = \pi_1(M)$
2. The normal bundle of $[0,\infty) \times M$ restricts to the stable normal bundle of X, since X is a hypersurface in $[0,\infty) \times M$
3. Furthermore, X has a B-structure given by the composite

$$\begin{array}{ccc} X & \subset & M \times [0,\infty) \\ & & \downarrow (\mathrm{Id} \times \rho, \mathrm{Id}) \\ & & \overline{M} \times [0,\infty) \times [0,\infty) \\ & & \downarrow (\mathrm{Id}, +) \\ & & \overline{M} \times [0,\infty) \end{array} \tag{1}$$

Now, X being the boundary of a regular neighborhood of the 2-skeleton means that X can be constructed as follows: Take an n-sphere for every 0-simplex. Clearly, this has a complete metric of uniformly positive scalar curvature and finite volume. Then do a 0-surgery for every 1-simplex, a 1-surgery for each 2-simplex. By 2.1 X has a complete metric of uniformly positive scalar curvature and finite volume. Hence X is a proper B-manifold with uniformly positive scalar curvature and finite volume.

We have thus completed the proof of sufficiency in Theorem 1.1 and its amplification, except for the assertion about bounded geometry. Now let us describe the modifications necessary to produce bounded geometry instead of finite volume. To do this, we will be more explicit about the proper cobordism between M and X. The idea here is that since M has a collared end, (and we will see that X can also be so constructed), the proper cobordism between them can be arranged to also have a collared end. Thus we can perform the surgeries from X to M in a periodic manner, in particular, ensuring bounded geometry. We begin with the following topological lemma:

Lemma 2.5. *Let Y and Z be two locally compact manifolds with collared ends, i.e.*

$$Y = cY \cup_{\partial cY} (\partial cY \times [0,\infty))$$

for some core compact codimension 0 submanifold cY with smooth boundary. Assume that W is a proper cobordism between Y and Z.

Then there is a new cobordism W' such that W' has a collared end, i.e.
$$W' = cW' \cup_{\partial cW'} (\partial cW' \times [0, \infty))$$
where here $\partial cW'$ means the frontier of a core submanifold cW'.

Proof. Given W as in the hypothesis of the lemma, choose a proper smooth function f on W. Assume t is some large number such that t is a regular value and $\{f \leq t\} \cap Y$ and $\{f \leq t\} \cap Z$ are both core submanifolds for a collaring of Y and Z. Then set $cW' = \{f \leq t\}$ and finally let
$$W' = cW' \cup_{\partial cW'} (\partial cW' \times [0, \infty))$$
This W' does the job. □

Now we construct X similarly to the finite volume case. Except we begin with a smooth triangulation of $\overline{M}$ and consider in it the interior 2-skeleton K, that is the union of simplices of dimension less than or equal to 2 that do not intersect the boundary. Since K does not intersect the boundary, the regular neighborhood $N(K)$ is a manifold with boundary lying completely in the interior of $\overline{M}$. Then as above, $N(K)$ can be constructed by taking a 0-handle for each 0-simplex, attach a 1-handle for every 1-simplex, and a 2-handle for every 2-simplex. We thus arrive at a compact manifold, with boundary $\partial N(K)$, which is therefore constructed by surgeries as in the finite volume case. $\partial N(K)$ is collared in $N(K)$, i.e. there is a neighborhood of $\partial N(K)$ diffeomorphic to $\partial N(K) \times (0, 1]$. Extend $N(K)$ by an infinite collar, that is, add to $\partial N(K) \times (0, 1]$, $\partial N(K) \times [1, \infty)$. This is X. This comes equipped with a $B = \overline{M} \times [0, \infty)$ structure. And since $\Omega_n^B([0, \infty))$ vanishes, M and X are properly B-cobordant.

Finally, we appeal to a result of Gajer's, [Gj], that says: if N is a compact manifold curvature (not necessarily connected) with a metric ds_N^2, of positive scalar, and let N' be obtained from N by a surgery of codimension ≥ 3. Let W be the trace of this surgery. Then W can be given a metric of positive scalar curvature ds_W^2 which is a product metric $ds_N^2 + dt^2$ in a collared neighborhood of N and in a collared neighborhood of N'. Recall the construction of $N(K)$ and $\partial N(K)$ by handle attachments and surgeries, respectively. We started by taking a 0 handle for each 0-simplex. We can put on the boundary of these 0-handles a metric of positive scalar curvature and extend it to the handle to retain positive scalar curvature and so that it is a product in a collared neighborhood of the boundary. Now Gajer's theorem lets us do the same for the 1 and 2 dimensional handle attachments as well. Thus we end up with a metric of positive scalar curvature on $N(K)$

which is a product on a collared neighborhhod of $\partial N(K)$. Since the metric has such a product structure, it extends to the infinite collar and thus we arrive at our B-manifold X with a complete metric of uniformly positive scalar curvature and bounded geometry, together with the fact that it properly B-cobordant to M.

Now use the previous lemma to ensure that the cobordism from M to X has a collared end as well. This also implies that the cross sections of infinity $\partial\overline{M}$ and ∂X are B-cobordant.

Let's look at the structure of the surgeries that must be done to go from X to M by virtue of the fact that the end of the cobordism W is collared. Since the cross sections of infinity are B-cobordant one can go from X to M by a finite sequence of surgeries of dimension of codimension ≥ 3. After that any surgeries that should be done at infinity should first be done outside a compact set, and in a translation invariant fashion. (This can be done without self intersection among the handles by the standard construction that goes from a handlebody on the cobordism from ∂X to $\partial\overline{M}$ to one on the product with a ray – wherein each i-handle becomes replaced by an infinite translation invariant collection of i and $i+1$ handles.) Having done this one has a complete bounded geometry uniformly positively scalar curvature metric manifold V which has the same end as M, and is B-cobordant to it by a cobordism with compact support (i.e. which is a product outside a compact set). The compact Gromov-Lawson surgery theorem then gives the metric on M. It clearly has bounded geometry, completing our proof. ☐

Remark 2.6. 1. *One could use* $B = B\pi_1(M)\times Bspin\times[0,\infty)$ *in the case where* M *is* $\pi-\pi$ *and Spin-Spin (i.e. spin and spin at infinity) and similar constructions for the other possible configurations of* π_1*'s and spin or non-spin conditions as in* [RS1]. *The use of* $B = \overline{M}\times[0,\infty)$ *is in our case a slick way to deal with the Spin or non-spin issues that normally arise,* [RS1].

2. *As an example of how the spin issues can effect things consider the following. Hitchin,* [H] *shows that some of the exotic spheres have no metric of positive scalar curvature. Let* Σ *be one such exotic sphere. This sphere bounds a manifold* W*, which can not however be spin. Let* M *be the non-compact manifold* $W-\Sigma$*. This manifold is non-spin but is spin at infinity. This manifold is* $\pi-\pi$ *where* π *is the trivial group, since* W *can be taken to be simply connected. So it might seem that the same argument as above would imply that* M *has a metric of uniformly positive scalar curvature . However, the map* $M\to\overline{M}\times[0,\infty)$ *is not*

properly 2-connected, which is derived from the fact that M is (non-spin,spin). We will remark in the last section that M in fact has no metric of uniformly positive scalar curvature .

3. *As another quick application of the above cobordism theorem, the univerisal cover of a manifold M^n with $\pi_1(M) = \mathbb{Z}^k$ always has a complete metric of positive scalar curvature for $k > 2$. For $k = 0, 1$ or 2 and M spin, combining Stolz's theorem,* [St] *with the Bochner argument of the next section, then if dim$M > k + 4$, M has a metric of uniformly positive scalar curvature if and only if the higher $\hat{\mathcal{A}}$-invariant associated to $M \to T^k$ the k-torus vanishes; by* [GL1], *if M is not spin and is high dimensional, then there is no obstruction to positive scalar curvature on $\tilde{M}$.*

3. The index theorem

We extend the calculus of correspondences defined in [CS] slightly to handle the following situation. Let A be a C^*-algebra, X and Y locally compact Hausdorff spaces, Y a manifold (see [BB] for the case where Y is a stratified space). There are certain geometrically defined elements of $KK_*(C_0(X), C_0(Y) \otimes A)$ defined by the following data which is called an A-correspondence a la [CS]: A diagram

$$
\begin{tikzcd}
 & (Z, \mathcal{E}) \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}
\tag{2}
$$

1. where Z is a smooth manifold,
2. $\mathcal{E}$ is a vector bundle over Z whose fibers are projective A-modules,
3. $f : Z \to X$ is a continuous and proper map,
4. $g : Z \to Y$ is continuous and Spin^c, which means that the Euclidean vector bundle $T_f^* = T^*Z \oplus f^*T^*Y$ is endowed with a Spin^c structure, and thus a bundle of spinors $\mathcal{S}$. So $\mathcal{S}$ is a complex Hermitian bundle. Each $\xi \in T_f^*|_x$ defines an endomorphism $c(\xi)$ of $\mathcal{S}_x$ such that
   (a) $\xi \mapsto c(\xi)$ is linear,
   (b) $c(\xi) = c(\xi)^*$ and $c(\xi)^2 = ||\xi||^2$,
   (c) $\mathcal{S}_x$ is irreducible as a module over $\mathrm{Cliff}_{\mathbb{C}}(T_f^*|_x)$,
   (d) if $j = \mathrm{dim}Z - \mathrm{dim}Y$ is even, the bundle $\mathcal{S}$ is $\mathbb{Z}/2$-graded and $c(\xi)$ is of degree one.

Connes and Skandalis show how to associate to $g : Z \to Y$ a Spin^c map an element $g! \in KK_j(C_0(Z), C_0(Y))$. So if we are given an A-correspondence $\mathcal{C}$ as above (2), we form the following element

$$[C] = f_*(\mathcal{E} \otimes g!) \in KK_j(C_0(X), C_0(Y) \otimes A)$$

We single out the following special case for future reference.

Proposition 3.1. *The $KK_n(C_0(M), C_0(pt.))$ element corresponding to the following correspondence under the correspondence between correspondences and elements of KK*

(3)
$$\begin{tikzcd} & M \arrow[dl, "Id"'] \arrow[dr] & \\ M & & pt. \end{tikzcd}$$

where M^n is a Spin^c-manifold is equal to the class defined by the Kasparov bimodule (H, F_0) where H is the Hilbert space of L^2 spinors with the usual grading operator, [K] *and $F_0 = \not{D}(1 + \not{D}^2)^{-\frac{1}{2}}$. Here $\not{D}$ is the Dirac operator of M.*

Proof. This follows from the symbol calculus of [FM]. □

The Kasparov product of two elements defined by correspondences can be accomplished geometrically as follows.

Theorem 3.2. *(Connes and Skandalis,* [CS]*) Let*

$$\mathcal{C}_1 = (Z_1, \mathcal{E}_1, f_1, g_1)$$

be an A_1-correspondence from X to Y of dimension j_1 and

$$\mathcal{C}_2 = (Z_2, \mathcal{E}_2, f_2, g_2)$$

an A_2 correspondence from Y to V of dimension j_2, Y and V manifolds, such that g_1 and f_2 are transverse. Then their Kasparov Product $\mathcal{C}_1 \otimes_{C_0(Y)} \mathcal{C}_2 \in KK_{j_1+j_2}(C_0(X), C_0(V) \otimes A_1 \otimes A_2)$ is represented by the $A_1 \otimes A_2$-correspondence $[\mathcal{C}_1 \otimes \mathcal{C}_2] = (Z, \mathcal{E}, f, g)$ described in the diagram

(4)
$$\begin{tikzcd} & & (Z = Z_1 \times_Y Z_2, p_1^*\mathcal{E}_1 \otimes p_2^*\mathcal{E}_2) \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ & (Z_1, \mathcal{E}_1) \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & (Z_2, \mathcal{E}_2) \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ X & & Y & & V \end{tikzcd}$$

where

1. $Z = Z_1 \times_Y Z_2$

2. *$\mathcal{E} = p_1^*\mathcal{E}_1 \otimes p_2^*\mathcal{E}_2$ where p_1 and p_2 are the natural projections*
3. *f and g are clear.*

Finally, we recall the definition of cobordant correspondences. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two A- correspondences from X to Y. They are called *cobordant* if there exists $\mathcal{C} = (Z, \mathcal{E}, f, g)$ such that

1. Z is a manifold with boundary $\partial Z = Z_1 \coprod Z_2$
2. $\mathcal{E}|_{Z_k} = \mathcal{E}_k$
3. $f|_{Z_k} = f_k$, and $g|_{Z_k} = (-1)^{k+1} g_k$

for $k = 1, 2$. Here $g|_{\partial Z}$ is given a Spin^c structure by requiring the Spin^c structure on $T_g^*|_{\partial Z}$ to be isomorphic to the product Spin^c structure on $T_{g|_{\partial Z}}^* \times \nu$ where ν is the inward normal bundle.

Theorem 3.3. *(Connes and Skandalis) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two cobordant A correspondences, then*

$$[\mathcal{C}_1] = [\mathcal{C}_2]$$

in $KK_j(C_0(X), C_0(Y) \otimes A)$

Now using this calculus, we prove a simple index theorem which is a generalization of Roe's partitioned index theorem, [Rp].

Let M be our not necessarily compact manifold, with $\pi_1 M = \Gamma$. Let M be Spin^c and $f : M \to \mathbb{R}$ a proper differentiable map. As above, let $\mathcal{V}$ be the canonical flat $C_r^*\Gamma$- bundle over M. We then form the following correspondence $\mathcal{C}_M$:

$$\begin{tikzcd} & (M,\mathcal{V}) \arrow[dl, "\mathrm{Id}"'] \arrow[dr] & \\ M & & \mathrm{pt.} \end{tikzcd} \tag{5}$$

So $[\mathcal{C}_M] \in_n (C_0(M), C_r^*\Gamma)$. Define $f* \in C_0(C_0(\mathbb{R}), C_0(M))$ by

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathbb{R} & & M \end{tikzcd} \tag{6}$$

Form the product $f * \otimes_M \mathcal{C}_M \in KK_n(C_0(\mathbb{R}), C_0(\mathrm{pt}))$ which is represented by

$$\begin{tikzcd} & (M,\mathcal{V}) \arrow[dl, "f"'] \arrow[dr] & \\ \mathbb{R} & & \mathrm{pt.} \end{tikzcd} \tag{7}$$

Finally, if $t \in \mathbb{R}$ is a regular value for f, define $\mathcal{C}_t$ to be the correspondence

$$\begin{tikzcd} & \{t\} \arrow[dl] \arrow[dr] & \\ \text{pt.} & & \mathbb{R} \end{tikzcd} \tag{8}$$

$\mathcal{C}_t \in KK_1(C_0(\text{pt.}), C_0(\mathbb{R}))$. Then $[\mathcal{C}_t \otimes f * \otimes \mathcal{C}_M] \in KK_{n-1}(C_0(\text{pt.}), C_r^*\Gamma)$ is given by the correspondence

$$\begin{tikzcd} & (N, \mathcal{V}|_N) \arrow[dl] \arrow[dr] & \\ \text{pt.} & & \text{pt.} \end{tikzcd} \tag{9}$$

where $N = f^{-1}(t)$ and is equipped with its natural Spin^c structure. This last correspondence corresponds to the higher index class defined by Rosenberg $\text{Ind}\not{D}_N \in KK_{n-1}(C_0(\text{pt.}), C_r^*\Gamma)$ and $\not{D}_N$ is the Dirac operator corresponding to the Spin^c- structure on N. So we have

Theorem 3.4. *The class in $KK_{n-1}(C_0(pt.), C_r^*\Gamma)$ defined by the product $[\mathcal{C}_t \otimes f * \otimes \mathcal{C}_M] = Ind\not{D}_N$*

When $\Gamma = \{e\}$ is the trivial group, one can check that product $[\mathcal{C}_t \otimes f * \otimes \mathcal{C}_M]$ is the partitioned index of M corresponding to the partition N and the theorem above is Roe's partitioned index theorem.

4. The Bochner argument

Let M be a complete Riemannian manifold with $\pi_1 M = \Gamma$, and let $\widetilde{M}$ denotes its universal cover. Let $H = L^2(\widetilde{M})$ and $\langle \cdot, \cdot \rangle_0$ its inner product. We describe the basic Hilbert $C_r^*\Gamma$-bundle over M. Let $\mathcal{V} = \widetilde{M} \times_\Gamma C_r^*\Gamma$ where $C_r^*\Gamma$ acts on the right of $\mathcal{V}$; $\mathcal{V}$ is a flat bundle of projective $C_r^*\Gamma$-modules. So $\Gamma_c(M, \mathcal{V})$, the space of compactly supported continuous sections is canonically isomorphic to

$$\{s : \widetilde{M} \to C_r^*\Gamma \mid s(m\gamma) = \gamma \cdot s(m) \text{ which are compactly supported mod } \Gamma\}$$

Define a $C_r^*\Gamma$-valued inner product on $\Gamma_c(M, \mathcal{V})$ by

$$\langle s_1, s_2 \rangle = \int_F s_1(\widetilde{m})^* s_2(\widetilde{m}) d\widetilde{m}$$

where F is a fundamental domain of the Γ-action on $\widetilde{M}$. The canonical flat connection on $\mathcal{V}$

$$\nabla : \Gamma^\infty(M; \mathcal{V}) \to \Gamma^\infty(M; \wedge^1 T^* M \otimes \mathcal{V})$$

is given by $(\nabla s) = ds$ where s is viewed as a function from $\widetilde{M} \to C_r^*\Gamma$. One also checks that

$$\langle s_2, s_1\rangle = \int_F s_2(\widetilde{m})^* s_1(\widetilde{m}) d\widetilde{m} = \int_F (s_1(\widetilde{m})^* s_2(\widetilde{m}))^* d\widetilde{m} = \langle s_1, s_2\rangle^*$$

and

$$\langle s_1, s_2 \cdot \varphi\rangle = \langle s_1, s_2\rangle\varphi$$

where $\varphi \in C_r^*\Gamma$. Thus $\Gamma_c(M;\mathcal{V})$ is a pre Hilbert $C_r^*\Gamma$-module. Let $\mathcal{H}(M;\mathcal{V})$ denote its completion to a Hilbert module.

Another useful description of $\mathcal{H}(M;\mathcal{V})$ is as follows: Let $\mathcal{E}_0 = C_c(\widetilde{M})$. Define $\langle\cdot,\cdot\rangle : \mathcal{E}_0 \times \mathcal{E}_0 \to \mathbb{C}\Gamma$ by

$$\begin{aligned}\langle s_1, s_2\rangle &= \sum_\gamma (\int_{\widetilde{M}} \langle \overline{s_1(m)}, s_2(m\gamma)\rangle dm)\gamma \\ &= \sum (\langle s_1, \gamma s_2\rangle_0)\gamma\end{aligned}$$

Then

$$\begin{aligned}\langle s_2, s_1\rangle &= \sum_\gamma (\langle s_2, \gamma s_1\rangle_0)\gamma \\ &= \sum_\gamma \langle \gamma^{-1} s_2, s_1\rangle_0 \gamma \\ &= \sum_\gamma \overline{\langle s_1, \gamma^{-1} s_2\rangle_0}\gamma \\ &= \sum_\gamma \overline{\langle s_1, \gamma s_2\rangle_0}\gamma^{-1} \\ &= \langle s_1, s_2\rangle^* \qquad (10)\end{aligned}$$

$\mathbb{C}\Gamma$ acts on $\mathcal{E}_0$ by $(s \cdot \varphi) = \sum_\gamma \gamma^{-1} s \varphi(\gamma)$. Again one easily checks that this action with $\langle\cdot,\cdot\rangle$ turns $\mathcal{E}_0$ into a $C_r^*\Gamma$ pre-Hilbert module.

Now define an homomorphism $\Phi : \mathcal{E}_0 \to \Gamma_c(M;\mathcal{V})$ by $(\Phi s)(\widetilde{m}) = \sum(\gamma \cdot s)\gamma$.

Proposition 4.1. Φ *defines a* $C_r^*\Gamma$ *Hilbert module isomorphism from the completion of* $\mathcal{E}_0$ *with respect to the* $C_r^*\Gamma$*-valued Hermitian product defined above to the* $C_r^*\Gamma$ *Hilbert module* $\mathcal{H}(M;\mathcal{V})$.

Proof. It is straightforward to check that Φ commutes with the $C_r^*\Gamma$ actions. For $s_1, s_2 \in \mathcal{E}_0$ we have

$$\langle s_1, s_2\rangle = \sum (\langle s_1, \gamma s_2\rangle_0)\gamma$$

while

$$\begin{aligned}
\langle \Phi s_1, \Phi s_2\rangle &= \int_F (\Phi s_1)(\widetilde{m})^*(\Phi s_2)(\widetilde{m})d\widetilde{m} \\
&= \int_F (\sum_{\gamma_1} s_1(\widetilde{m}\gamma_1)\gamma_1)^*(\sum_{\gamma_2} s_2(\widetilde{m}\gamma_2)\gamma_2)d\widetilde{m} \\
&= \sum_{\gamma_1,\gamma_2}(\int_F \overline{s_1(\widetilde{m}\gamma_1)}s_2(\widetilde{m}\gamma_2)d\widetilde{m})\gamma_1^{-1}\gamma_2 \\
&= \sum_{\gamma_1,\gamma}(\int_F \overline{s_1(\widetilde{m}\gamma_1)}s_2(\widetilde{m}\gamma_1\gamma)d\widetilde{m})\gamma \\
&= \sum_{\gamma}(\int_{\widetilde{M}} \overline{s_1(\widetilde{m})}s_2(\widetilde{m}\gamma)d\widetilde{m})\gamma_1 \\
&= \langle s_1, s_2\rangle
\end{aligned}$$

So Φ is an isometry. Now define Ψ by $(\Psi s)(\widetilde{m}) = s(\widetilde{m})(e)$. One checks directly that Ψ is also a map of Hilbert modules and is inverse to Φ. $\square$

So $\mathcal{H} = \mathcal{H}(M;\mathcal{V})$ can be also be viewed as the Hilbert $C_r^*\Gamma$-module obtained by completing $\mathcal{E}_0$. If $\mathcal{S}$ is a vector bundle on M (with a Hermitian metric) then we form $\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S})$ in the obvious way . Then, as we've seen, the $C_r^*\Gamma$-valued inner product on $\mathcal{H}$ is given by

$$\langle s_1, s_2\rangle_{C_r^*\Gamma} = \sum_\gamma \langle s_1, \gamma\cdot s_2\rangle_0\cdot\gamma.$$

Note that $\langle s_1, s_2\rangle_0 = \langle s_1, s_2\rangle(e)$. So clearly $||\langle s,s\rangle||_{C_r^*\Gamma} \geq |\langle s,s\rangle_0|$. Hence $\mathcal{H}\hookrightarrow H$.

Now let M be spin and $\not{D}$ the Dirac operator acting on the spinors of M. We may couple $\not{D}$ to the flat bundle $\mathcal{V}$ using the canonical flat connection on $\mathcal{V}$ to obtain

$$\not{D}_{\mathcal{V}} : \Gamma_c^\infty(M;\mathcal{S}\otimes\mathcal{V}) \to \Gamma_c^\infty(M;\mathcal{S}\otimes\mathcal{V})$$

In terms of our other description, we can define $\widetilde{\not{D}}$ as acting on $\Gamma_c^\infty(\widetilde{M};\mathcal{S})$. Φ can be extended to a map

$$\Phi : \Gamma_c^\infty(\widetilde{M};\mathcal{S}) \to \Gamma_c^\infty(M;\mathcal{V}\otimes\mathcal{S})$$

Then we have that

$$\not{D}_{\mathcal{V}}\Phi = \Phi\widetilde{\not{D}}$$

This depends on our canonical choice of flat connection on $\mathcal{V}$ and is left to the reader.

For $k = 0, 1, \ldots,$ let

$$\langle s_1, s_2\rangle_k = \sum_{j=0}^{k} \langle \not{D}^j s_1,\ \not{D}^j s_2\rangle_0$$

be the standard Sobelov norm on $\Gamma_c^\infty(\widetilde{M}; \mathcal{S})$ and $H^k(\widetilde{M}; \mathcal{S}) = H^k(\mathcal{S})$ its completion. Similarly, let

$$\langle s_1, s_2\rangle_{C_r^*\Gamma, k} = \sum_\gamma \langle s_1, \gamma s_2\rangle_k \gamma$$

and $\mathcal{H}^k(M; \mathcal{V} \otimes \mathcal{S}) = \mathcal{H}^k(\mathcal{V} \otimes \mathcal{S})$ the completion as a Hilbert $C_r^*\Gamma$-module. And as above we have $\mathcal{H}^k(M; \mathcal{V} \otimes \mathcal{S}) \hookrightarrow H^k(\widetilde{M}; \mathcal{S})$.

Lemma 4.2. *$\mathcal{H}^k(M; \mathcal{V} \otimes \mathcal{S}) = \mathcal{H}^0(M; \mathcal{V} \otimes \mathcal{S}) \cap H^k(\widetilde{M}; \mathcal{S})$.*

Some standard facts about $\mathcal{H}^k$ from [FM]

1. $\not{D}_{\mathcal{V}} :\ \mathcal{H}^k(\mathcal{V} \otimes \mathcal{S}) \to \mathcal{H}^{k-1}(\mathcal{V} \otimes \mathcal{S})$ is a bounded operator
2. For $\varphi \in C_0(M)$, the multiplication operator $M_\varphi :\ \mathcal{H}^k(\mathcal{V} \otimes \mathcal{S}) \to \mathcal{H}^l(\mathcal{V} \otimes \mathcal{S})$ is $C_r^*\Gamma$-compact for $l < k$.

According to the discussion above $\Gamma_c^\infty(\widetilde{M}; \mathcal{S})$ is a common domain for $\widetilde{\not{D}}$ acting on $H(\widetilde{M}, \mathcal{S})$ and for $\not{D}_{\mathcal{V}}$ acting on $\mathcal{H}(M; \mathcal{S} \otimes \mathcal{V})$. According to [GL2], $\widetilde{\not{D}}$ is selfadjoint on $\widetilde{M}$. Hence we may apply the spectral theorem. We now show how the spectral calculus can be extended to the Hilbert module $\mathcal{H}(M; \mathcal{V} \otimes \mathcal{S})$.

Let ψ be a bounded function on $\mathbb{R}$ such that its Fourier transform $\hat{\psi}$ is compactly supported. We now can apply ψ to $\widetilde{\not{D}}$ to get a bounded operator $\psi(\widetilde{\not{D}})$ on $L^2(\widetilde{M}; \mathcal{S}) = H(M; \mathcal{S})$. Since $\mathcal{H}(M; \mathcal{S}) \subseteq H(\mathcal{S})$ we can apply $\psi(\widetilde{\not{D}})$ to elements of $\mathcal{H}(M; \mathcal{S})$.

Proposition 4.3. *$\psi(\widetilde{\not{D}})|_{\mathcal{H}(M:\mathcal{S})}$ has image in $\mathcal{H}(M; \mathcal{S})$, and defines a bounded Hilbert module operator in $\mathcal{L}(\mathcal{H}(M; \mathcal{S}))$. Moreover,*

$$||\psi(\widetilde{\not{D}})||_{\mathcal{L}(\mathcal{H}(M;\mathcal{S})} \leq ||\psi(\widetilde{\not{D}})||_{\mathcal{L}(H(\widetilde{M};\mathcal{S}))}$$

Proof. Since $\Gamma_c^\infty(\widetilde{M}; \mathcal{S})$ is dense in $\mathcal{H}(M; \mathcal{S})$ (and in $H(\widetilde{M}; \mathcal{S})$) we need to check that for $s \in \Gamma_c^\infty(\widetilde{M}; \mathcal{S})$ that

$$\psi(\widetilde{\not{D}})s \in \mathcal{H}(M; \mathcal{S})$$

and that

$$||\psi(\widetilde{\not{D}})s||_{\mathcal{H}(M;\mathcal{S})} \leq ||\psi(\widetilde{\not{D}})||_{\mathcal{L}(H(M;\mathcal{S}))}||s||_{\mathcal{H}(M,\mathcal{S})}$$

According to the ordinary spectral theorem, if $s \in \Gamma_c^\infty(\widetilde{M};\mathcal{S})$ then $\psi(\widetilde{\not{D}})s \in H^0(\widetilde{M};\mathcal{S})$ while by the condition on supp $\hat{\psi}$ it follows that supp$\psi(\widetilde{\not{D}})s$ is compact by a finite propagation speed argument (see Roe, [R]). Hence $\psi(\widetilde{\not{D}})s \in \mathcal{H}(M;\mathcal{V}\otimes\mathcal{S})$.

Let $N = ||\psi(\widetilde{\not{D}})^2||_{\mathcal{L}(H(\widetilde{M};\mathcal{S}))}$. Then $N - \psi(\widetilde{\not{D}})^2$ is an invariant and positive bounded operator on $H(\widetilde{M},\mathcal{S})$. As such, for any $\xi \in H(\widetilde{M},\mathcal{S})$ the function on Γ defined by

$$\begin{aligned}\gamma &\mapsto \langle (N-\psi(\widetilde{\not{D}})^2)\xi,\, \gamma\xi\rangle_0 \\ &= \langle (N-\psi(\widetilde{\not{D}})^2)^{\frac{1}{2}}\xi, (N-\psi(\widetilde{\not{D}})^2)^{\frac{1}{2}}\gamma\xi\rangle_0 \\ &= \langle (N-\psi(\widetilde{\not{D}})^2)^{\frac{1}{2}}\xi,\, \gamma\cdot(N-\psi(\widetilde{\not{D}})^2)^{\frac{1}{2}}\xi\rangle_0\end{aligned}$$

is positive definite. So finally we have for $\xi \in \Gamma_c^\infty(\widetilde{M};\mathcal{S})$ that

$$\sum \langle (N-\psi(\widetilde{\not{D}})^2)\cdot\xi, \gamma\xi\rangle_0\gamma$$

is a positive element of $C_r^*\Gamma$. Therefore, in $C_r^*\Gamma$ we have the inequality

$$\sum \langle \psi(\widetilde{\not{D}})^2\xi, \gamma\xi\rangle_0\gamma \le N\sum_\gamma \langle \xi, \gamma\xi\rangle_0\gamma$$

So

$$\begin{aligned}||\psi(\widetilde{\not{D}})\xi||^2_{\mathcal{H}(M,\mathcal{V}\otimes\mathcal{S})} &= ||\sum_\gamma \langle \psi(\widetilde{\not{D}})\xi, \gamma\psi(\widetilde{\not{D}})\xi\rangle_0\gamma||_{C_r^*\Gamma} \\ &= ||\sum_\gamma \langle \psi(\widetilde{\not{D}})^2\xi, , \gamma\xi\rangle_0\gamma||_{C_r^*\Gamma}\end{aligned}$$

(again by the Γ-invariance of $\widetilde{\not{D}}$)

$$\le \quad N||\sum_\gamma \langle \xi, \gamma\xi\rangle_0\gamma||_{C_r^*\Gamma} = N||\xi||^2_{\mathcal{H}(M,\mathcal{V}\otimes\mathcal{S})}$$

Hence, $\psi(\widetilde{\not{D}})$ is a bounded operator in the Hilbert module norm.

Finally, since $\widetilde{\not{D}}$ is Γ invariant and $\psi(\widetilde{\not{D}})$ is self adjoint, one checks that $\psi(\widetilde{\not{D}})$ satisfies

$$\langle \widetilde{\not{D}}s_1, s_2\rangle_{C_r^*\Gamma} = \langle s_1, \widetilde{\not{D}}s_2\rangle_{C_r^*\Gamma}$$

for $s_1, s_2 \in \Gamma_c^\infty(\widetilde{M};\mathcal{S})$. Therefore $\psi(\widetilde{\not{D}})$ extends to an adjoinable bounded Hilbert module operator, finishing the proof of the proposition. □

Let $\varepsilon : \mathcal{S} \to \mathcal{S}$ be the grading operator on the spinors. From now on we write $\not{D}$ for the operator $\widetilde{\not{D}}$ or $\not{D}_\mathcal{V}$ which as we've seen can be identified.

Proposition 4.4. *The operator* $\not{D}+\lambda\varepsilon$ *has a bounded inverse* Q_λ : $\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S})\to\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S})$ *for all* $\lambda>0$ *and* $||Q_\lambda||_{\mathcal{L}(\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S}))}<\lambda^{-1}$

Proof. For $s\in\Gamma_c^\infty(\widetilde{M};\mathcal{S})$ we have

$$\begin{aligned}(11)\qquad ||(\not{D}+\lambda\varepsilon)s||_0^2 &= \langle(\not{D}+\lambda\varepsilon)s,(\not{D}+\lambda\varepsilon)s\rangle_0\\ &= \langle(\not{D}^2+\lambda^2)s,s\rangle_0\\ &= \langle\not{D}^2 s,s\rangle+\lambda^2\langle s,s\rangle\\ &\geq \lambda^2||s||^2\end{aligned}$$

Hence there exists c with $\mathrm{Spec}(\not{D}+\lambda\varepsilon)\cap(-c,c)=\emptyset$.

Let ψ be a continuous function on $\mathbb{R}$ with $\psi(t)=\frac{1}{t}$ for $|t|\geq c$. ψ can be approximated by ψ_n with $||\psi-\psi_n||<\frac{1}{n}$ and $\hat{\psi}_n$ having compact support. Then $\hat{\psi}_n(\not{D}+\lambda\varepsilon)\in\mathcal{L}(\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S}))$ by the previous proposition and

$$||\psi_n(\not{D}+\lambda\varepsilon)||_{\mathcal{L}(\mathcal{H}(M;\mathcal{V}\otimes\mathcal{S}))}\leq||\psi_n(\not{D}+\lambda\varepsilon)||_{\mathcal{L}(H(\widetilde{M};\mathcal{S}))}$$

hence since $\psi_n(\not{D}+\lambda\varepsilon)$ is Cauchy in $\mathcal{L}(H(\widetilde{M};\mathcal{S}))$ it's Cauchy in $\mathcal{L}(\mathcal{H}(M,\mathcal{V}\otimes\mathcal{S}))$ so

$$\psi(\not{D}+\lambda\epsilon)\in\mathcal{L}(\mathcal{H}(M,\mathcal{V}\otimes\mathcal{S}))$$

The statement about the norm is clear. □

Corollary 4.5. Q_λ *in fact defines a bounded operator* $Q_\lambda:\ \mathcal{H}^0(M,\mathcal{S})\to\mathcal{H}^1(M;\mathcal{S})$

Proof. The estimate (11) can be refined to

$$(12)\qquad ||(\not{D}+\lambda\varepsilon)s||_0^2\geq\langle\not{D}s,\not{D}s\rangle+\lambda^2\langle s,s\rangle\geq c^2||s||_1^2$$

So we have $||(\not{D}+\lambda\varepsilon)s||_0\geq c||s||_1$ hence $(\not{D}+\lambda\varepsilon)^{-1}:\ H^0\to H^1$.

$$\begin{aligned}(13)\qquad ||\not{D}+\lambda\varepsilon)s||_{C_r^*\Gamma}^2 &= ||\langle(\not{D}+\lambda\varepsilon)^2 s,(\not{D}+\lambda\varepsilon)s\rangle_{C_r^*\Gamma}||_{C_r^*\Gamma}\\ (14)\qquad &= ||\sum_\gamma\langle(\not{D}+\lambda\varepsilon)s,\ \gamma(\not{D}+\lambda\varepsilon)s\rangle_0\gamma||_{C_r^*\Gamma}\\ (15)\qquad &= ||\sum_\gamma(\langle\not{D}^2,\gamma s\rangle+\lambda^2\langle s,\ \gamma s\rangle)_0\gamma||_{C_r^*\Gamma}\\ (16)\qquad &\geq \lambda^2||s||_{C_r^*\Gamma,1}\end{aligned}$$

So we have

$$Q_\lambda:\ \mathcal{H}^0(M;\mathcal{V}\otimes\mathcal{S})\to\mathcal{H}^1(M;\mathcal{V}\otimes\mathcal{S})$$

is bounded. □

The rest of what follows in this section is influenced heavily by Bunke, [Bu]. Let $\varphi : M \to \mathbb{R}$ where $\varphi \in C_c^\infty(M)$ and $\varphi + \frac{\kappa}{4} \geq c$ on all of M. (Recall that κ is the scalar curvature function.)

Then we have

$$\langle(\not{D}^2 + \lambda^2 + \varphi)s,\, s\rangle_0 = \langle(\triangle + \lambda^2 + \frac{\kappa}{4} + \varphi)s, s\rangle \geq (\lambda^2 + c)\langle s, s\rangle_0 \tag{17}$$

for all λ.

We have $\not{D}^2 + \lambda^2$ is invertible with inverse Q_λ^2 and has norm $||Q_\lambda^2|| \leq \lambda^{-2}$. Now

$$\not{D}^2 + \lambda^2 + \varphi = (\not{D}^2 + \lambda^2)(1 + Q_\lambda^2\varphi)$$

therefore

$$(\not{D}^2 + \lambda^2 + \varphi)^{-1} = (\sum_{i=0}^{\infty}(-1)^i(Q_\lambda^2\varphi)^i)Q_\lambda^2$$

exists for large λ.

Now we want to see that $(Q^\varphi)^2 = (\not{D}^2 + \varphi)^{-1}$ exists and $||Q^\varphi|| \leq c^{-1}$.

Set $(Q_\lambda^\varphi)^2 = (\not{D}^2 + \varphi + \lambda^2)^{-1}$. We've seen this exists for large λ. For $\mu \in \mathbb{R}$

$$(\not{D}^2+\varphi+\mu^2) = \not{D}^2+\varphi+\lambda^2-(\lambda^2-\mu^2) = (\not{D}^2+\varphi+\lambda^2)(1-(\lambda^2-\mu^2)(Q_\lambda^\varphi)^2)$$

So

$$(\not{D}^2 + \varphi + \mu^2)^{-1} = (\sum_{i=0}^{\infty}(\lambda^2 - \mu^2)^i(Q_\lambda^\varphi)^{2i})(Q_\lambda^\varphi)^2$$

In order for this sum to converge we need $1 > ||(\lambda^2 - \mu^2)(Q_\lambda^\varphi)^2||$. But $||(\lambda^2 - \mu^2)(Q_\lambda^\varphi)^2|| \leq (\lambda^2 - \mu^2)(\lambda^2 + c)^{-1}$ so we need $(\lambda^2 - \mu^2) < \lambda^2 + c$ or $\mu^2 > -c$. Hence $(Q_\lambda^\varphi)^2 = (\not{D}^2 + \varphi + \lambda^2)^{-1}$ exists for $\forall\lambda$. So $(Q^\varphi)^2 = (\not{D}^2 + \varphi)^{-1} : \mathcal{H}^0 \to \mathcal{H}^2$ is a bounded operator.

Now define $F = \not{D}(Q^\varphi)$. Then $F : \mathcal{H}^0 \to \mathcal{H}^0$ is a bounded operator. We need to see that its adjoinable. We'd like to see that $F^* = (Q^\varphi)\not{D}$ but this is a priori not defined on $\mathcal{H}^0$. We will see that it extends. So let $s \in \mathcal{H}^1$. Then

$$\begin{aligned} ||(Q^\varphi)\not{D}s||^2 &= ||\langle(Q^\varphi)\not{D}s, (Q^\varphi)\not{D}s\rangle|| \\ &= ||\langle(Q^\varphi)^2\not{D}s, \not{D}s\rangle|| \\ &= ||\langle\not{D}(Q^\varphi)^2 s, \not{D}s\rangle + \langle(Q^\varphi)^2(\nabla\varphi)(Q^\varphi)^2 s, \not{D}s\rangle|| \end{aligned}$$

(since $[\not{D}, Q] = -Q\nabla\varphi Q$)

$$= ||\langle\not{D}^2(Q^\varphi)^2 s, s\rangle|| + ||\langle\not{D}(Q^\varphi)^2\nabla\varphi(Q^\varphi)^2 s, s\rangle|| \leq A||s||^2 \tag{18}$$

for some $A > 0$ since $\not{D}^2(Q^\varphi)^2$ and $\not{D}(Q^\varphi)^2\nabla\varphi(Q^\varphi)^2$ are bounded operators. So $Q^\varphi\not{D}$ extends to a bounded operator from $\mathcal{H}^0 \to \mathcal{H}^0$.

Now we calculate a few things:

$F = \not{D} Q^\varphi$ and $F^* = Q^\varphi \not{D}$.

First we see that $Q^\varphi = \dfrac{2}{\pi}\displaystyle\int_0^\infty (Q^\varphi_\lambda)^2 d\lambda$

So

$$\begin{aligned} \not{D} Q^\varphi &= \frac{2}{\pi}\int_0^\infty \not{D}(Q^\varphi_\lambda)^2 d\lambda \\ &= \frac{2}{\pi}\int_0^\infty (Q^\varphi_\lambda)^2 \not{D} - (Q^\varphi_\lambda)^2 \nabla\varphi (Q^\varphi_\lambda)^2 d\lambda \\ &= Q^\varphi \not{D} - \frac{2}{\pi}\int_0^\infty (Q^\varphi_\lambda)^2 \nabla\varphi (Q^\varphi_\lambda)^2 d\lambda. \end{aligned}$$

So we see that $F - F^* = \frac{2}{\pi}\int_0^\infty (Q^\varphi_\lambda)^2 \nabla\varphi (Q^\varphi_\lambda)^2 d\lambda$ The intergrad is norm convergent and compact since $(Q^\varphi_\lambda)^2 \nabla\varphi (Q^\varphi_\lambda)^2$ is the composition $\mathcal{H}^0 \overset{(Q^\varphi_\lambda)^2}{\to} \mathcal{H}^2 \overset{\nabla\varphi}{\to} \mathcal{H}^0 \overset{(Q^\varphi_\lambda)^2}{\to} \mathcal{H}^0$ and the middle operator is $C^*_r\Gamma$ compact by Rellich's lemma, (4). So $F - F^* \in \mathcal{K}(\mathcal{H}^0)$

Now

$F^2 - 1 = F^2 - FF^* + FF^* - 1 = F(F - F^*) + FF^* - 1 = FF^* - 1 + A$

where $A \in \mathcal{K}(\mathcal{H}^0)$ and

$$\begin{aligned} FF^* - 1 &= \not{D} Q^\varphi Q^\varphi \not{D} - 1 \\ &= \not{D}(Q^\varphi)^2 \not{D} - 1 \\ &= \not{D}^2 (Q^\varphi)^2 - \not{D}(Q^\varphi)^2 \nabla\varphi (Q^\varphi)^2 - 1 \\ &= 1 - \varphi(Q^\varphi)^2 - \not{D}(Q^\varphi)^2 \nabla\varphi (Q^\varphi)^2 1 \\ &= \varphi(Q^\varphi)^2 - \not{D}(Q^\varphi)^2 \nabla\varphi (Q^\varphi)^2 \end{aligned}$$

Again by Rellich's lemma $\mathcal{H}^0 \overset{(Q^\varphi)^2}{\to} \mathcal{H}^2 \overset{\varphi}{\to} \mathcal{H}^0$ is compact. Also $\mathcal{H}^0 \overset{(Q^\varphi)^2}{\to} \mathcal{H}^2 \overset{\nabla\varphi}{\to} \mathcal{H}^0 \overset{(Q^\varphi)^2}{\to} \mathcal{H}^2 \overset{\not{D}}{\to} \mathcal{H}^0$ is compact, since $\nabla\varphi : \mathcal{H}^2 \to \mathcal{H}^0$ is compact and all the other operators are bounded. So $F^2 - 1 \in \mathcal{K}(\mathcal{H}^0)$. That $[F, a] \in \mathcal{K}(\mathcal{H}^0)$ $\forall a \in C_0(M)$ follows as in Bunke. We have thus verified the following

Corollary 4.6. *The pair $(\mathcal{H}^0, F)$ defines an element in $KK_*(C_0(M), C^*_r\Gamma)$ and in fact, pulls back to define an element in*

$$KK_*(C_0(M)^+, C^*_r\Gamma) = KK_*(C(M^+), C^*_r\Gamma)$$

(Here, $C_0(M)^+$ denotes the algebra $C_0(M)$ with a unit adjoined and is therefore the same as the continuous functions on the one-point compactification M^+.) We now show that the element so defined in

$KK_*(C_0(M), C_r^*\Gamma)$ equals the element usually associated to a Dirac operator on the manifold M.

Proposition 4.7. *As elements of $KK_*(C_0(M), C_r^*\Gamma)$, $(\mathcal{H}^0, F)$ and $(\mathcal{H}^0, F_0)$ are equal, where $F_0 = \not{D}(1 + \not{D}^2)^{-\frac{1}{2}}$.*

Proof We show that $a(F - F_0) \in \mathcal{K}(\mathcal{H}^0)$ for all $a \in C_0(M)$. Let $(Q_\lambda^0)^2 = (\not{D}^2 + \lambda^2 + 1)^{-1}$. Then

$$\begin{aligned} a(F - F_0) &= \frac{2}{\pi}\int_0^\infty a\not{D}((Q_\lambda^\varphi)^2 - (Q_\lambda^0)^2)d\lambda \\ &= \frac{2}{\pi}\int_0^\infty (-[\not{D}, a] + \not{D}a)((Q_\lambda^\varphi)^2 - (Q_\lambda^0)^2)d\lambda \\ &= \frac{2}{\pi}\int_0^\infty (-\nabla a + \not{D}a)((Q_\lambda^\varphi)^2 - (Q_\lambda^0)^2)d\lambda \\ &= \frac{2}{\pi}\int_0^\infty (-\nabla a + \not{D}a)((Q_\lambda^\varphi)^2(\varphi - 1)(Q_\lambda^0)^2)d\lambda \end{aligned}$$

which is compact for $a \in C_c^\infty(M)$ since the integrand is the sum of the two compositions

$$\mathcal{H}^0 \overset{(Q_\lambda^0)^2}{\to} \mathcal{H}^2 \overset{\varphi-1}{\to} \mathcal{H}^2 \overset{(Q_\lambda^\varphi)^2}{\to} \mathcal{H}^4 \overset{a}{\to} \mathcal{H}^3 \overset{\not{D}}{\to} \mathcal{H}^0$$

and

$$\mathcal{H}^0 \overset{(Q_\lambda^0)^2}{\to} \mathcal{H}^2 \overset{\varphi-1}{\to} \mathcal{H}^2 \overset{(Q_\lambda^\varphi)^2}{\to} \mathcal{H}^4 \overset{\nabla a}{\to} \mathcal{H}^0$$

and the operators multiplication by a and ∇a are compact on the appropriate Sobelev spaces. Hence for $a \in C_0(M)$ and $a_i \in C_c^\infty(M)$ with $a_i \to a$ uniformly we have $a_i(F - F_0) \to a(F - F_0)$ and so $a(F - F_0)$ is compact for all $a \in C_0(M)$.

That $(F - F_0)a \in \mathcal{K}(\mathcal{H}^0)$ follows from the fact that both $[F, a]$ and $[F_0, a] \in \mathcal{K}$.

Hence the two elements are equal in $KK(C_0(M), C_r^*\Gamma)$ □

We can now state our Bochner-Lichnerowicz theorem. Suppose M is a Spin manifold with fundamental group Γ. Let $f : M \to \mathbb{R}$ be a proper differentiable function. Then there is a map

$$KK_*(C_0(M), C_r^*\Gamma) \overset{f_*}{\to} KK_*(C_0(\mathbb{R}), C_r^*\Gamma) \overset{C_t}{\to} KK_{*-1}(C_0(\text{pt.}), C_r^*\Gamma)$$

where C_t is the correspondence described by (8).

Theorem 4.8. *Let M be a complete Riemannian spin manifold, with fundamental group Γ and scalar curvature κ satisfying $\kappa \geq \epsilon > 0$ off of a compact set. Then the image of $(\mathcal{H}^0, F_0) \in KK_n(C_0(M), C_r^*\Gamma)$ in $KK_{n-1}(C_0(pt.), C_r^*\Gamma)$ is 0.*

Proof. By the previous proposition, the elements $(\mathcal{H}^0, F)$ and $(\mathcal{H}^0, F_0)$ define the same class in $KK_n(C_0(M), C_r^*\Gamma)$. So by Corollary (4.6), the class of $(\mathcal{H}^0, F_0)$ pulls back to $KK_n(C_0(M)^+, C_r^*\Gamma)$. From the sequence

$$0 \to C_0(M) \to C_0(M)^+ \to C_0(pt) \to 0$$

we get a long exact sequence in KK theory which is the first column of the following commutative diagram

$$\begin{array}{ccc} KK_n(C_0(M)^+, C_r^*\Gamma) & \to & KK_n(C_0(\mathbb{R})^+, C_r^*\Gamma) \\ \downarrow & & \downarrow \\ KK_n(C_0(M), C_r^*\Gamma) & \to & KK_n(C_0(\mathbb{R}), C_r^*\Gamma) \\ \downarrow & & \downarrow \\ KK_{n-1}(C_0(\mathrm{pt.}, C_r^*\Gamma) & = & KK_{n-1}(C_0(\mathrm{pt.}, C_r^*\Gamma) \end{array}$$

Now the result follows from the fact that the compostition of the two left vertical arrows is zero. $\square$

5. The Novikov conjecture for fundamental groups of ends

In this section we verify the Novikov conjecture for the relevant end groups.

Theorem 5.1. *Let M be an inrreducible locally symmetric space of finite volume and non-positive curvature. Then the assembly map*

$$KK_n(C_0(B\pi_1^\infty(M)), \mathbb{C}) \to K_n(C_r^*(\pi_1^\infty(M)))$$

is split injective, where $\pi_1^\infty(M)$ is the fundamental group of the end of M.

Proof. In the $\mathbb{Q}-$ rank ≥ 3 the fundamental group of the end is the same as the fundamental group of M, (2.3) and the result therefore follows from Kasparov's theorem, [K]. In the $\mathbb{Q}-$ rank 1 case, $\pi_1^\infty(M)$ injects into $\pi_1(M)$, (2.3) and the result again follows from Kasparov's theorem. In the only interesting case, $\mathbb{Q}-$rank 2, we have an extension, (2.3)

$$1 \to \mathbb{F}_\infty \to \pi_1^\infty(M) \to \pi_1(M) \to 1$$

As we shall see, the Novikov conjecture for $\pi_1^\infty(M)$ follows from Kasparov's theorem for $\pi_1(M)$ together with Pimsner and Voiculsescu's theorem, [PV] for $\mathbb{F}_\infty$.

Now recall from section 2 that the Borel-Serre compactification $\overline{M}$ of M is formed by adding to the universal cover $\widetilde{M}$ a boundary $\partial\widetilde{M}$ such that $\pi_1(M)$ acts on $\overline{\widetilde{M}} = \widetilde{M} \cup \partial\widetilde{M}$ and $\overline{\widetilde{M}}/\pi_1(M)$ is compact. Therefore $\partial\overline{M}$ is $\partial\widetilde{M}/\pi_1(M)$ and $\pi_1^\infty(M) = \pi_1(\partial\overline{M})$. Also, $\partial\widetilde{M}$ has the homotopy type of a bouquet of circles ($\mathbb{Q}-$ rank is 2). $\pi_1(M)$ does not act on $\pi_1(\partial\overline{M}) = \mathbb{F}_\infty$ but it does act on the fundamental groupoid $\pi(\partial\overline{M})$.

Lemma 5.2. 1. *$C_r^*(\pi(\partial\overline{M}))$ is Morita equivalent to $C_r^*(\mathbb{F}_\infty)$*
2. *$C_r^*(\pi_1^\infty(M))$ is Morita equivalent to the crossed-product algebra $C_r^*(\pi_1(M), C_r^*(\pi(\partial\overline{M})))$.*

Proof. The first part is a standard fact. As for the second, form the topological semi-direct product groupoid

$$\Pi = \{(q,\gamma) | q \in \pi_1(M), \gamma \in \pi(\partial\overline{M})\}$$

Define the source and target maps $s, r : \Pi \to \partial\overline{M}$ by $s(q,\gamma) = s(\gamma)$ and $r(q,\gamma) = q \cdot r(\gamma)$. Multiplication is defined by

$$(q_1,\gamma_1)\cdot(q_2,\gamma_2) = (q_1 q_2, (q_2^{-1}\gamma_1)\gamma_2)$$

when $s(q_1,\gamma_1) = r(q_2,\gamma_2)$. It is easy to see that $C_r^*(\pi_1 M, C_r^*(\pi\partial\overline{M})) \cong C_r^*(\Pi)$. Now we see that Π is equivalent as a topological groupoid to $\pi_1^\infty(M)$. Indeed, since Π is connected (i.e. there is a morphism between every object of Π), Π is equivalent as topological groupoids to the group

$$\{(q,\gamma) \in \Pi | s(q,\gamma) = r(q,\gamma) = x\}$$

for any object x. It follows from the homotopy lifting property that this group is isomorphic to $\pi_1(\partial\overline{M}/\pi_1(M,x)) = \pi_1^\infty(M)$. So $C_r^*(\Pi)$ is Morita equivalent to $C_r^*\pi_1^\infty(M)$, [MRW]. $\square$(the lemma)

Now notice that $B\pi_1^\infty(M) = \partial\overline{M} \times_{\pi_1(M)} E\pi_1 M$. We have the following composition:

$$\begin{array}{ccl}
KK_n(C_0(B\pi_1^\infty(M)),\mathbb{C}) & = & KK_n(C_0(\partial\overline{M} \times_{\pi_1(M)} E\pi_1 M),\mathbb{C}) \\
 & \overset{(1)}{\cong} & KK_n^{\pi_1(M)}(C_0(\partial\overline{M} \times E\pi_1 M),\mathbb{C}) \\
 & \overset{(2)}{\cong} & KK_n^{\pi_1 M}(C_0(E\pi_1 M), C_r^*(\pi(\partial\overline{M})) \\
 & \overset{\mu}{\to} & KK_n(\mathbb{C}, C_r^*(\pi_1(M), C_r^*(\pi(\partial\overline{M})) \\
 & \overset{\text{Morita}}{\cong} & KK_n(\mathbb{C}, C_r^*(\pi_1^\infty(M)))
\end{array}$$

The isomorphism (1) follows since the action of $\pi_1(M)$ is free and proper. The isomorphism (2) follows from the Pimsner-Voiculescu theorem, [PV], together with a Mayer-Vietoris argument on the quotient $E\pi_1(M)/\pi_1(M) = B\pi_1 M$. The map μ is the Baum-Connes assembly map for $\pi_1(M)$ with coefficients in $C_r^*\pi(\partial\overline{M})$, which is split injective, according to [BCH]. This is implicit in Kasparov, [K]. This completes the proof of the theorem. □.

The preceding proof can be greatly generalized and seems to be useful in many cases where one as an extension of a group for which one knows Novikov with coefficients by a group for which one knows Baum-Connes. For example, the same method proves the following result.

Proposition 5.3. 1. *Consider the class of discrete groups $\mathcal{P}$ that contains the abelian groups and is closed under amalgamated free products and HNN extensions. Let $\Gamma \in \mathcal{P}$. Then the Baum-Connes assembly map,* [BCH]

$$\mu : KK_*^{\Gamma}(\underline{E\Gamma}, \mathbb{C}) \to K_*(C_r^*(\Gamma))$$

is an isomorphism.

2. *If $1 \to \Gamma \to \Delta \to K \to 1$ is an extension where $\Gamma \in \mathcal{P}$ and K is a discrete subgroup of a Lie group then the standard assembly map*

$$K_*(B\Delta) \to K_*(C_r^*(\Delta))$$

is split injective.

Proof. The first statement follows from [P], while the second statement follows from the first, Kasparov's theorem and the technique used above. □

6. Completion of the proofs of Theorem (1.1) and Theorem (1.3)

We need to show that if Γ is not an arithmetic lattice or even if it is arithmetic, if its $\mathbb{Q}-$rank is less than 3, then M has no uniformly positive scalar curvature metric. If the Lie group G has $\mathbb{R}-$rank below 2, then all associated homogenous manifolds are either compact and eliminated by the classical results cited in the introduction or, by the Margulis lemma, have an almost flat manifold cross section at infinity, and are thus Λ^2 enlargable in the sense of [GL2], and thus have no metrics of positive scalar curvature . If the $\mathbb{R}-$rank is at least 2, then by Margulis's arithmeticity theorem the lattice Γ is arithmetic. We can thus use the analysis of Borel-Serre, [S]. If the $\mathbb{Q}-$rank is 0, we are again in the compact case. If $\mathbb{Q}-$rank is 1 or 2, then there is an aspherical manifold at infinity who fundamental group satisfies the strong Novikov conjecture as we saw in the previous section. (Note that in the $\mathbb{Q}-$rank at least 3 case, the manifold at infinity is not aspherical.) Suppose that M has a metric which is uniformly positive scalar curvature outside of a compact set. One can cut M open along a cusp and double the resulting cusp using the given Riemannian structure near the two ends and any Riemannian structure in the interior (there is no canonical metric doubling for a manifold whose boundary is not metrically collared, although away from the a neighborhood of the former boundary, one can use the original metric), to obtain a new manifold, V, diffeomorphic to the slice cross $\mathbb{R}$, and which is uniformly positive scalar curvature outside of a compact set. If V is spin then one can directly apply the Bochner-Lichnerowicz argument of section 4 with the index theorem of section 3 to get vanishing of the K-theoretic higher $\hat{A}$-genus, which contradicts the strong Novikov conjecture, since for an aspherical manifold, the homology A-genus does not vanish. If V is not spin, it still has a spin universal cover (by asphericity), so one can modify this argument in exactly the same way that Rosenberg does in [R]; one uses the Dirac operator on the universal cover thought of as being equivariant with respect to a $\mathbb{Z}/2\mathbb{Z}$ group extension of $\pi_1(V)$. This completes the proof of Theorem (1.1).

Remark 6.1. *Consider the manifold M constructed in Remark 2.6, 2., from a Hitchin sphere. We can use the same doubling trick to show that M has no metric of uniformly positive scalar curvature . For if M had such a metric, we cut M at a cross-section of infinity, and double the end as above, putting any metric on the section where it is glued. We this arrive at a manifold with a metric of uniformly positive scalar curvature off a compact set. Then a generalization of the index theorem above to real K-theory shows that this is obstructed by the same element that obstructs in Hitchin's argument.*

Let us now turn to the issue of proper rigidity of $K\backslash G/\Gamma$. We will only make several remarks here, postponing a more detailed discussion to another paper. We will break down Theorem 1.3 into several propositions.

Definition 6.2. *A manifold X is called* properly rigid *if any ANR homology manifold with the disjoint disk property proper homotopy equivalent to it is homeomorphic to it. This is equivalent to a slightly stabilized version of the more naive version of rigidity using manifolds proper homotopy equivalent to X in light of* [BFMW]*. In particular all of the compact rigidity results of e.g.* [HsS], [FH], [FJ1], [FJ2] *apply equally in this more general context.*

Proposition 6.3. *If $\mathbb{Q}$-rank of Γ is at least 3, then $K\backslash G/\Gamma$ is never properly rigid in the setting of homology manifolds. (It will be properly rigid in the topological category if and only if $[K\backslash G/\Gamma : F/Top] = 0$, which seems to be rather infrequent.)*

Proof. Since the fundamental group at infinity is Γ, the proper surgery obstruction group vanishes, so the structure sequence for for homology manifolds [BFMW] implies that there is always a proper homotopy equivalent homology manifold with any given resolution obstruction. The parenthetical remark follows from ordinary proper surgery, [T].

Proposition 6.4. *If the real rank of G is 1 or the $\mathbb{Q}$-rank of Γ is 0 or 1, then $K\backslash G/\Gamma$ is properly rigid (even in the setting of homology manifolds).*

Proof. This is established in [FJ3] aside from the issue of homology manifolds. (Essentially, the proper homotopy equivalence is first made into a homeomorphism at infinity, using the Borel conjecture for the group at infinity as well as the vanishing of Whitehead and projective class groups. Then one has an essentially compact situation, so the Borel conjecture for Γ makes this rel infinity homotopy equivalence homotopic, rel infinity, to a homeomorphism.) However, the Novikov conjecture for Γ or the fundamental group at infinity (for the noncompact case) implies that the resolution obstruction of Quinn [BFMW] is homotopy invariant for aspherical homology manifolds, so one is reduced to the case of manifolds, after all. $\square$

Remark 6.5. *Note that the same proof works in the $\mathbb{Q}$-rank 2 case if the Borel conjecture for the fundamental group at infinity were known.*

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Jonathan Block, University of Pennsylvania, Philadelphia, PA, 19104

Shmuel Weinberger, University of Chicago, Chicago, IL, 60637