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# On the topological social choice model

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**Abstract**

The main goal of this paper is to show that if a finite connected CW complex admits a continuous, symmetric, and unanimous choice function for some number $n > 1$ of agents, then the choice space is contractible. On the other hand, if one removes the finiteness, we give a complete characterization of the possible spaces; in particular, noncontractible spaces are indeed possible. These results extend earlier well-known results of Chichilnisky and Heal.


*JEL classification:* D71

*Keywords:* Social choice; Topological methods; Pareto optimality

## 1. Introduction and statement of results

The goal of this paper is to give a new analysis of some aspects of the topological social choice model, studied by Chichilnisky [3], Chichilnisky–Heal [5] and others.

We imagine $n$ agents each picking elements out of $X$, the choice space. The problem is to give an aggregation of their choices $A(x_1 \ldots x_n)$ continuously in these variables and subject to two axioms:

1. $A(x \ldots x) = x$ (unanimity);
2. $A(x_1 \ldots\ x_n)$ is independent of the ordering (anonymity).

Note that the second condition is equivalent to asserting that the function $A$ is invariant with respect to an action of the symmetric group; much of our discussion

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would be unchanged if we instead demanded that $A$ be invariant under an action of any transitive group $G$ of permutations (see [4]).

Without further assumption, we shall assume that our choice space $X$ is connected and of the homotopy type of a CW complex (or, equivalently, a simplicial complex). The second assumption is a common one in algebraic topology and includes all manifolds, algebraic varieties, many function spaces (see [11]). Connectedness may not be such a natural assumption in economic applications, but one can readily see that under the hypothesis of being a CW complex, one can find such an $A$ for $X$ iff one can do so for every component. (However, the study of the set of such functions does not reduce to their study on the individual components.)

The main result of [5] is that if such an $A$ exists for every $n$, and $X$, in addition, has only finitely many cells in every dimension, then $X$ is contractible (i.e. homotopy equivalent to a point). A relation between these ideas and the Arrow impossibility theorem is given in [1]. In this paper, we give some initial results on what happens when these hypotheses fail, and give another contractibility result with weaker conditions on the aggregation function (combined with a tronger finiteness hypothesis on the space) and also give what seem to be the first examples of noncontractible spaces on which choices may be aggregated.

**Theorem 1.1.** *Suppose that $X$ is as above and is, in addition (of the homotopy type of) a finite complex, then if for some $n>1$, a continuous map $A$, satisfying* 1 *and* 2 *exists (for any transitive group of permutations), then $X$ is contractible.*

One can think of (homotopy) finiteness as a kind of compactness. A polyhedron is compact iff it has finitely many cells, i.e. iff it is finite. The theorem asserts that under such a hypothesis, social choice is impossible for noncontractible choice spaces, even if we restrict our attention to specific small-size populations.

If $X$ is contractible, then Chichilnisky–Heal [5] observed that such an $A$ exists, for every $n$, and invariant under the whole symmetric group. This is a special case of the following:

**Proposition 1.2.** *Let $n$ and a transitive group $G$ of permutations be given. Then the question of whether or not for a space $X$, a mapping $A$ satisfying unanimity and $G$-anonymity exists, just depends on the homotopy type of $X$.*

Recall that spaces are homotopy equivalent iff they are both deformation retracts of a common third space; informally, if they can be deformed into one another. The proposition is a very straightforward consequence of the homotopy extension principle (see [12]) and the result of reformulating condition 2 in terms of the quotient space, $X^n/G$, of the permutation $G$ action on the product $X^n$: we are asking that the identity map on the "diagonal" of $X^n/G$ extend to a map (to $X$) on the whole space. (See [12] for a discussion of why extension problems are, aside from pathology, homotopy invariant.) That the proposition is nonvacuous is a consequence of the following theorem:

**Theorem 1.3.** *Suppose that X is a connected CW complex, then if for all* $n>1$, *a continuous map A, satisfying* 1 *and* 2 *exists (for any transitive group of permutations), then X is of the homotopy type of a product of rational Eilenberg–MacLane spaces. Conversely, any such X admits such A for all n, invariant under the entire symmetric group.*

Eilenberg–MacLane spaces are spaces with at most a single nonvanishing homotopy group. Examples are the circle, hyperbolic manifolds, or the infinite-dimensional complex projective space. An Eilenberg–MacLane space is rational, if the homotopy group is actually an abelian group which is uniquely divisible by every natural number, i.e. a vector space over the rational numbers, $\mathbb{Q}$.

Below we will give examples, but, in any case, it is worth noting that any such $A$ on a noncontractible $X$ must have unusual properties; we will discuss these below. Suffice it to say that they have a kind of Pareto nonoptimality which we call "Solomonism". Thus, one can say that the overall institutional framework of continuity, unanimity, and anonymity (justice?) forces Pareto nonoptimality.

Finally, we will give several examples below, one of which is also worth citing here:

**Example 1.4.** The infinite-dimensional real projective space $RP^\infty$ admits a social choice function for any odd number of agents, but not for any even number.

In this example, the choice space does have finitely many cells in each dimension (as in the Chichilnisky–Heal theorem), but there are arbitrarily large populations for which choice is indeed possible. Moreover, it also shows that the problem of solving choice problems does not grow monotonically more difficult with population size.

## 2. Proofs of main theorems

All of our work depends on the theory of *H-spaces*. *H-spaces* are an algebraic topological analogue of the notion of a Lie group, or more generally, of a topological monoid, and it has been known from the very first paper to introduce them [9] that they have a very particular structure.

**Definition.** A space $Z$ is an *H-space*, if there is a function $\mu : Z \times Z \to Z$, and a point $p \in Z$ such that $\mu(z,p) = \mu(p,z) = z$ for all $z$ in $Z$.

In the definition, there is no gain in generality if we merely assume that $\mu(z,p)$ and $\mu(p,z)$ are both homotopy equivalences, i.e. induce isomorphisms on homotopy groups (see [12]). (More precisely, this definition allows no more *H-spaces*, but it does allow more "*H-multiplications*". As our interest is in the spaces, we will ignore this issue.) This variant has a number of advantages, one being that for connected complexes it only depends on the homotopy class of $\mu$, but not on which representative map chosen within the class, nor on the point $p$.

It follows from work of Hopf and Serre that an $H$-*space* whose homotopy groups are rational vector spaces is a product of rational Eilenberg–MacLane spaces, see e.g. [13]. The other main theorem about $H$-*spaces* that we will need is Browder's deep theorem [2] that a finite complex which is an $H$-*space* satisfies Poincare duality, and, in particular, has some reduced homology group isomorphic to the integers $\mathbb{Z}$.

**Theorem 2.1.** *Suppose that $X$ is a connected CW complex such that for some $n>1$ there is a map $A$ as above. Then $X$ is an $H$-space. Moreover, the fundamental group $\pi_1(X)$ is abelian and for all $i$, the homotopy groups $\pi_i(X)$ are uniquely $n$-divisible (i.e. multiplication by $n$ is an isomorphism on these groups).*

Before proving this theorem, we note that it implies the main theorems. To prove Theorem 1, note that Serre's mod $\mathscr{C}$ Hurewicz theorem (see [7,12]) asserts that the condition of divisibility on homotopy groups is equivalent to it holding on homology groups, but Browder's theorem prevents that. To prove Theorem 2, note simply that the divisibility of homotopy groups precisely implies the result in light of Hopf's theorem. (The construction of choice functions is deferred to the next section.)

Both parts of the theorem are proven simultaneously. Let $A$ be as in the theorem. We first consider $A_*: \pi_1(X^n) \to \pi_1(X)$. Note first that $\pi_1(X^n)$ is isomorphic to the direct sum of $n$ copies of $\pi_1(X)$, and on each factor $A_*$ is the same homomorphism, which we will call $\rho_*$ (by axiom 2). This notation is quite sensible, because in a moment we will study the map $\rho: X \to X$ defined by $\rho(x) = \mu(x, p, \ldots p)$, which induces $\rho_*$ on homotopy.

We first claim that $\rho_*$ is a surjection. This is immediate because we have the diagonal map $\triangle : X \to X^n$ defined by $\triangle(x) = (x, x, \ldots x)$ satisfies $A\triangle = I$, where $I$ is the identity map.

Since the individual summands of $\pi_1(X^n)$ commute with each other (if not, a priori, with themselves!) their images commute with each other, and thus $\pi_1(X)$ is abelian, as each of these images is the whole group.

Furthermore, we now apply the equation $n\rho_* = I$ (that comes from $A\triangle = I$, and the equality of each summand homomorphism) so that $\rho_*$ provides the unique $n$ divisibility.

The argument just given applies even more directly to the higher homotopy groups, so they are also all uniquely $n$ divisible. This conclusion directly implies as well that $\rho$ induces an isomorphism on homotopy groups, and is thus a homotopy equivalence. We shall let $\pi$ denote its homotopy inverse.

We claim that if we define $\mu(x, y) = A(\pi(x), \pi(y), \ldots, p)$, then $\mu$ gives us an $H$-*space* structure on $X$. This is clear, since on the set of points of the form $(x, p)$, by the definitions of $\rho$ and $\pi$, it is clearly a map homotopic to the identity, and the result for points of the form $(p, x)$ follows from the symmetry.

Our theorem is now proven.

**Remark 2.2.** Similar (but somewhat more elaborate at one technical point) reasoning shows that a connected space $X$ has a symmetric, unanimous, continuous choice function for two agents, iff $X$ is a "homotopy commutative" $H$-*space* and its homotopy groups are uniquely 2 divisible. For other values of $n$, it is harder to pin down exactly what kind of extra structure one needs on the *H-map*, $\mu$.

## 3. Examples and further discussion

While none of the algebraic topology necessary for our examples is at all nonstandard, we found it a bit difficult to find precise references in the literature.

As mentioned above, Eilenberg–MacLane spaces are spaces with at most a single nontrivial homotopy group. $X$ is said to be of type $K(G,n)$ if $\pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) = G$. One knows that for every group $G$ there is a space of type $K(G,1)$, and for all abelian $G$ there is a space of type $K(G,n)$ for $n > 1$. Moreover, these spaces are uniquely determined, up to homotopy type, by $G$ and $n$.

For any space $X$, the homotopy classes of maps from $X$ into a space of type $K(G,n)$ is isomorphic to the cohomology group $H^n(X;G)$. As a special case, if $A$ and $B$ are of types $K(G,n)$ and $K(H,n)$, respectively, then the homotopy classes of maps from $X$ to $Y$ are in a one-to-one correspondence with homomorphisms from $G$ to $H$. (See [12] for all of this.)

The simplest example of a rational Eilenberg–MacLane space is a two-dimensional space whose construction sheds some light on the phenomena present whenever aggregation is possible. We first recall the mapping cylinder construction. Let $f: X \to Y$ be a map, then $Cyl_f = X \times [0,1] \cup Y$, such that $(x,1) = f(x)$ in $Y$. Note that $Cyl_f$ always can be deformation retracted back to $Y$, so it has the same homotopy type as $Y$. As an interesting special case, the function $f: S^1 \to S^1$ given by squaring, $f(z) = z^2$ has as cylinder the Moebius strip.

Now if one considers maps $f: X \to Y$ and $g: Y \to Z$, then one can glue the cylinders $Cyl_f$ and $Cyl_g$ together along $Y$. We now do this with an infinite number of maps from the circle to itself of all positive degrees. This infinite-iterated mapping cylinder is of type $K(\mathbb{Q},1)$. In fact the same construction works for any odd-dimensional sphere $S^n$ to give a model for $K(\mathbb{Q},n)$. A construction for even $n$ can be obtained using loop spaces (see [12]). The general rational Eilenberg–MacLane space is a product of these spaces (up to homotopy type). For the verification of these assertions and for further details, the reader can consult [7].

Observe that for a product of any number of spaces, one can aggregate the choices of $n$ agents iff one can do so for the individual spaces. So, to complete our proof of the existence part of Theorem 2 we need only produce for all $K(\mathbb{Q},k)$ and all $n$ an aggregating function on the symmetric power $S^n(K(\mathbb{Q},k)) = (K(\mathbb{Q},k)^n)/\Sigma_n$. Since we are trying to build a map into an Eilenberg–MacLane space $K(\mathbb{Q},k)$, our task boils down to finding an element of the cohomology group $H^k(S^n(K(\mathbb{Q},k);\mathbb{Q})$ and its restriction to the diagonal $H^k(K(\mathbb{Q},k);\mathbb{Q}) = \mathbb{Q}$ corresponds to the identity map (equivalently the identity homomorphism).

We already know that $H^k(K(\mathbb{Q},k)^n;\mathbb{Q})$ is isomorphic to $Hom(\mathbb{Q}^n:\mathbb{Q})$ (by the result mentioned above about maps between Eilenberg–MacLane spaces. We easily have a homomorphism which restricts correctly to the diagonal on this space (i.e.

missing the symmetry), namely the average of all the coordinates. Moreover, this homomorphism is invariant under the action of the symmetric group $\Sigma_n$. Happily, this is enough by a general lemma of Groethendieck (see [8]) which identifies rational cohomology of the quotient space under a finite group action with the cohomology that is invariant under the action, completing our proof.

**Definition 3.1.** We say that an aggregating function on a metric space $X$ of many variables is *Solomonic* if there is a compact set $C$, such that for any real number $D$, one can find an $n$-tuple $(c_1 \dots c_n)$ of points of $C$, such that, $A(c_1 \dots c_n)$ is of distance at least $D$ from the set $C$.

The idea of Solomonicity is that it captures the "cut the baby in half" solution (see [10], although in the original case, this apparently was Pareto optimal). Justice then demands a solution which all agents would agree is inferior to choosing the choice any of the other agents suggest. (We assume that agents roughly prefer choices near their choice to ones far away in this interpretation.) In this view, Solomonicity goes strongly in the opposite direction of Pareto optimality.

**Proposition 3.2.** *If $X$ is a noncontractible connected CW complex with aggregation functions for all n, and is a proper metric space* (*i.e. all metric balls contain only finitely many cells*), *then the aggregation is necessarily Solomonic.*

This is a straightforward consequence of our proof of Theorem 2 and our understanding of rational Eilenberg–MacLane spaces. They all contain spheres, which can act the compact set $C$, whose homology classes are not divisible in any finite distance neighborhood. This failure tells us that some "tuple" of points from $C$ must aggregate outside of the neighborhood. In the model of $K(\mathbb{Q}, 1)$ given above, one must go $n$ stages down the iterated mapping cylinder to get divisibility by $n$ when $n$ is a prime number.

We close this paper with some more examples.

**Example 3.3.** If, instead of using maps of arbitrary degree, one only uses degree 2 maps (still infinitely often), then starting with any odd-dimensional sphere, one can show that the criteria of Remark 2.2 apply, and two agents can always aggregate on these spaces.

I do not know whether any power of 2 number of agents can be aggregated on this space. (They can be for a transitive group of permutations isomorphic to $((\mathbb{Z}/2)^n)$; I doubt whether they can be aggregated in a symmetric fashion with respect to the whole symmetric group.

**Example 3.4.** For a group $G$ one can ask whether there is a homomorphism from $G^k \to G$ satisfying the usual aggregation axioms, and, of course, the answer is that this is possible iff $G$ is abelian and uniquely $k$-divisible. By using a good model (such as the one given by [6]) one can see that for such $G$, $K(G, n)$ always has such an aggregation map, by building an explicit map that corresponds to "averaging on the group" that is explicitly symmetric.

More precisely, "averaging" is the composite of "sum" with "division by $k$". Thus, one needs a model for $K(G,n)$ for which addition is pointwise commutative and associative. In the Dold–Thom model points in the Eilenberg–MacLane space are finite subsets of some other space, and the addition corresponds to taking union, which clearly has the desired properties.

In particular taking $n=1$ and $G=\mathbb{Z}_2$ we see that some, and therefore any, $K(\mathbb{Z}_2,1)$ has aggregation for all odd numbers of agents. Since $RP^\infty$ is such a space (by covering space theory), it has such aggregation.

Of course, if we set, say, $G=\mathbb{Z}_{15}$ then we would get a space that admitted aggregation for any number of agents divisible by neither 3 nor 5.

## Note added in proof

It has been brought to my attention that Theorem 1.1 with essentially the same proof, as well as some of the nontrivial examples of spaces admitting social choice functions, were already published in the paper B. Eckmann, T. Ganea, and P. Hilton, Generalized means. 1962 Studies in mathematical analysis and related topics pp. 82–92 Stanford Univ. Press, Stanford, Calif. Their motivation was quite different. Another proof, when the group of symmetries of the aggregation procedure is the whole symmetric group (or at least contains an involution) can be obtained by replacing Browder's theorem in the proof below with J. Hubbuck's, On homotopy commutative H-spaces, Topology 8 (1969) 119–126.

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