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Large scale geometry, by P. Nowak and G. Yu, EMS Textbooks in Mathematics, European Mathematical Society, Zürich, 2012, xiv+189 pp., ISBN 978-3-03719-112-5, €38.00, $41.80

Everyone knows that many phenomena are best studied at one scale or another, and that deep problems often involve the interactions of several scales. From the broad conception of our understanding of physics, based on the atomic hypothesis to the germ theory of disease to the success of calculus with its move to the infinitesimal, the value of the study of phenomena at small scales and the techniques of integrating such information to the macroscopic level are now the most basic intuition in the scientific world view—with attendant backlashes against "blind reductionism".

In the subject of global geometry, the important general direction of "understanding manifolds under some condition of curvature" is of this sort. Curvature is the infinitesimal measure of how the space differs from Euclidean space, and one tries to obtain global conclusions from uniform assumptions on this type of local hypothesis.

There is also another extreme: going from the cosmic, the large scale, back to the local. In retrospect, at least, one can see this trend in, say, Liouville's theorem that bounded analytic functions are constant: boundedness is a large scale condition; surely, if one looks from a very large scale, a bounded function should not be viewed as different than a constant, and Liouville's theorem tells us that under a condition of analyticity, the large scale information tells all.

The realization that one can learn a great deal about a space, say a manifold, by considering the large scale geometry of associated spaces is a more recent, yet highly appreciated, insight. It has ramifications in global analysis, random walks, differential geometry, geometric topology, representation theory, geometric group theory, but surely transcends mathematics, as we ponder, for example, connections between cosmology and particle physics.

Let me begin by sketching a few of such connections—although, as Persi Diaconis [Dia09] said in another context, "To someone working in my part of the world, asking about applications... is a little like asking about applications of the quadratic formula." I sincerely apologize to the literally hundreds of mathematicians—including most of my friends—that I am unavoidably slighting by my idiosyncratic choice of examples and references.

From the large-scale point of view, we should view a compact space as being essentially a point. We should view $\mathbf{Z}^n$ and $\mathbf{R}^n$ as being essentially equivalent (the technical word is coarsely quasi-isometric), although they surely are not homeomorphic (or even isomorphic as sets). However, in the old days of dot matrix printers we routinely represented objects by drawing some samples from them. Let us try to make sense of the notion of isomorphism relevant to this way of thinking.

If π is a finitely generated group, say π has a finite generating set S, then we can make π into a metric space by first making it into the vertex set of a graph: list all the group elements and, by connecting g to $s^{\pm 1}g$ one makes π into a graph.

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The metric structure comes by declaring all edges to have length one, and then declaring the distance between two group elements to be the length of the shortest path between them.

Note that the metric on π does depend on S. However, if T is another generating set for π, then the distance d_S defined using S and the distance d_T defined using T are interrelated: there is a C (related to the sizes of elements of S viewed from the point of view of T and vice versa) so that $C^{-1}d_T \leq d_S \leq Cd_T$.

What can we do using this? We can define the growth of a group to be the rough size, i.e., number of elements in a ball of radius R in this metric as $R \to \infty$. For the free abelian groups $\mathbf{Z}^n$ these grow roughly like R^n (a conclusion independent of generating set). Which other groups have polynomial growth? An amazing theorem of Gromov [Gro81] asserts that it is exactly the groups containing nilpotent groups as subgroups of finite index.

Note that the growth of $\mathbf{Z}^n$ is the same as the volume growth of balls in $\mathbf{R}^n$. This can be generalized, as observed by Milnor [Mil68] and A. Schwartz. If M is a compact manifold, then the growth of the volume of large balls in the universal cover of M is equivalent to the volume growth of the group. Essentially, the fundamental domain of the universal cover of M should be thought of as just a variant of a generating set of π. (Who needs finiteness anyway? S should be allowed to be a compact set of generators to allow for locally compact groups rather than merely finitely generated ones.)

Technically, this changes the equivalence relation on metrics to have another constant K, so that $C^{-1}d_T - K \leq d_S \leq Cd_T + K$ which gives us the notion of coarse quasi-isometry. The coarse volume growth of a space is the number of balls of radius 1 that can be packed in a ball of radius R. For a complete simply connected manifold with negative curvature, one can show that quantity grows at least exponentially in R, and therefore the fundamental groups of compact manifolds with this property must be quite different from abelian groups.

Another thing one can do with this relation is to discuss the ends of π:[1] consider the complements of balls of arbitrary radius in the Cayley graph. These form an inverse system: if $S > R$, then the complement of $B(R)$ includes into the complement of $B(S)$. We can consider the inverse limit of the sets of components of these complements. This is easily seen to be independent of generating set (or manifold acted upon properly discontinuously and cocompactly by π). It is classical that groups have $0, 1, 2$ or infinitely many ends, 0 iff finite, 2 iff the group contains $\mathbf{Z}$ as a subgroup of finite index, and ∞ (for torsion-free groups) iff it is a nontrivial free product, by a beautiful and important theorem of Stallings [Sta68].

Another question one can ask is what happens to a random walk as one goes around the universal cover? Does one get close to every point? This, too, turns out to be a coarse problem. If the fundamental group contains $\mathbf{Z}^d$ for $d = 0, 1, 2$ with finite index, then it turns out that the walk does this (i.e., is recurrent) and otherwise not! In the case of $\mathbf{R}^d$ for $d > 2$ (the universal cover of the d-torus), this is a well-known theorem of Polya. For general groups this requires the theorem of Gromov above, together with an understanding of how volume growth connects to random walks (see [VSCC92]).

A similar question can be asked about the spectrum of the Laplacian on universal covers. Is $0 \in \mathrm{Spec}(\Delta)$? Again, the answer to this is coarse (according to a

[1] The ends of a group were introduced first by Freudenthal.

theorem of Brooks [Bro81] that depends on the seminal work of Kesten [Kes59]): it only depends on the fundamental group, and the condition is that the group Γ is amenable.[2] Alternatively (and equivalently), for nonamenable groups there is an analytic inequality valid for L^2-functions on the *universal* cover of any manifold with that fundamental group

$$\|\Delta f\|_2 \geq c\|f\}_2$$

for some $c > 0$.

Amenability of groups was introduced by von Neumann in order to get to the heart of the Banach–Tarski paradox (see, e.g., [Lub94]) and is itself a subject of great interest. It is the condition on Γ that one can average bounded functions on it, i.e., construct an invariant mean on the bounded function on Γ, $M : L^\infty\Gamma \to \mathbf{R}$. Folner [Føl55] characterized this in terms of the existence of exhaustions by subsets with large volume whose boundaries have small surface area.

Interestingly enough, all of the above phenomena can be expressed nicely within a framework of large-scale homology theories (with appropriate growth conditions on simplices); see [BW97] (see also [Roe93]). Doing so, for example, enables one to prove for groups (with certain finiteness properties) that cohomological dimension is a geometric property (i.e., preserved by coarse quasi-isometry) [Ger93], as is satisfying Poincaré duality. For some additional deeper results in this spirit, see [Sau06].

Most groups are not amenable. Any group with a free subgroup is not.[3] Among its characterizations is that the trivial representation is weakly contained in the regular representation. The opposite extreme is when the trivial representation is separated (in the Fell topology) from the regular representation. This is called Kazhdan's Property (T); see [Kaž67]. (For an excellent book on this condition, its equivalents, and some of its applications, see [BdlHV].) Concretely, it means that if Γ acts linearly isometrically on a Hilbert space, then unless the action has a fixed vector, there is a predictable amount that each vector must be moved by some generator. (In other words, if there are almost fixed vectors, then there are fixed vectors.)

Originally, Property (T) groups were constructed via Lie theory, but now there are many other sources of such groups (see [BdlHV], [Val04], [Sha06] for more information). In particular, random groups in some models (see, e.g., [Żuk03]) have Property (T). Unlike amenability, Property (T) is not geometric. As first observed by Gersten, a lattice in the universal cover of $\mathrm{SO}(2, n)$ has Property (T) but is quasi-isometric to the product of a lattice in $\mathrm{SO}(2, n)$ with $\mathbf{Z}$ which does not have Property (T) because of one-dimensional unitary representations that factor through $\mathbf{Z}$.

Nevertheless, this property does have a geometric feel, and it has many geometric consequences. One breathtaking consequence was an observation by Margulis [Mar73]. The Cayley graphs of finite quotients of a Property (T) group, such as,

[2] I'll explain this in a moment.

[3] Von Neumann had asked whether this was the only obstruction. The first example showing that this is not true was due to Olshanski (although (see [Why99]) this conjecture is true in some geometric sense!) In recent years, many fascinating amenable and nonamenable groups were constructed and analyzed by a variety of methods; see, e.g., [OS02], [BV05].

e.g., $\mathrm{SL}_n(\mathbf{Z}), n > 2$, form a sequence of expanders,[4] that is, graphs of uniformly bounded vertex degree for which the random walk is rapidly mixing. These graphs form networks that are hard to separate, i.e., that no separation in these graphs can separate two large subsets from each other without itself being large (i.e., linearly bounded in the size of the subsets). The survey [LLR95] gives many reasons why such graphs are valuable.

This consequence is clearly "opposite" to the Folner condition that characterizes amenability: *that* condition asserts there are many efficient separators. A very short argument (see, e.g., [Roe03]) shows that expander graphs cannot be efficiently embedded in any Euclidean space, or even in Hilbert space. In other words, if Γ_n is a sequence of expander graphs, then thinking of them as metric spaces with the path metric, there is no sequence of 1-Lipschitz maps f_n into Hilbert space with the property that $d(f_n v, f_n w) > F(d(v, w))$ for some function F that goes to infinity (like logloglog), i.e., the map cannot be effectively proper. The argument, roughly speaking, goes like this: Being Lipschitz with a small Lipschitz constant gives an upper bound on the Laplacian of the function, which gives an upper bound on the average distance square from the origin (assuming expansion)[5] which would say that in the graph on average all points are near some fixed one. However, this would contradict the sequence of graphs having unbounded diameter (which follows from unbounded size and bounded degree).

This conclusion should be contrasted with Bourgain's [Bou85] general result that n point metric spaces can be embedded in Euclidean space with $O(\log(n))$ distortion. (I would be committing a crime if I didn't mention at this point that there are deep sources of bi-Lipschitz distortion in embedding finite metric spaces that occur even in the amenable case, starting with Semmes's observation that Pansu's work [Pan89] shows that the Heisenberg group does not bi-Lipschitz embed in Hilbert space. Some extensions will be mentioned below.)

Property (T) is closely related to ideas of rigidity. It implies that certain representations of the group cannot be deformed—while, say, Mostow rigidity asserts the uniqueness of a discrete faithful representation of a lattice in a semisimple Lie group G (with no SL_2 factors). Indeed, Mostow rigidity and its cousins are other important motivations for geometric group theorists. Mostow's proof of his rigidity theorem in rank one [Mos68] already involved the understanding of some basic large scale geometric structures that followed from acting properly discontinuously on a rank one symmetric space, and in particular, how one constructs and exploits a "boundary at infinity".[6]

(One such spin off, the theory of (word) hyperbolic groups comes from the attempt to make large-scale the key features of compact hyperbolic manifolds—spaces whose nature is determined by an infinitesimal hypothesis.)

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[4] This "expander of quotient" property for linear groups has over the past decade been liberated from Property (T) via deep work in additive combinatorics, with very significant applications in analytic number theory and beyond. See [HLW06] for a general introduction and [Lub12], [Kon13], [Kow08] for references and a discussion of these newer ideas.

[5] This is called a Poincaré inequality.

[6] There are many "boundary" theories now for infinite groups beyond generalizations of Mostow's. Just to mention one more, there are Poisson boundaries that reflect random walk rather than geodesic motion (and are essentially the possible "boundary values" of bounded harmonic functions). In some special cases, say a drunkard randomly walking on a 3-regular tree seems to actually be moving purposefully towards some point at infinity—just not at unit speed the way a geodesic would, but in general these theories diverge.

Less classical is the connection of large-scale ideas to rigidity in topology and in global analysis (which then feeds back, e.g., to differential geometry in a different way than classical rigidity theory). Interestingly, because as we have already explained, the universal cover of any manifold with fundamental group π is coarse quasi-isometric to π, the large-scale geometry of π can give information about many aspects of the topology and analysis of such manifolds.

So I will move very briefly to the topic of topological rigidity.[7] We will say that a closed manifold M is topologically rigid if any manifold homotopy equivalent to it is homeomorphic to it.[8] Usually, we will be interested in not only M being rigid, bus also in $M \times T$ (M cross a torus) for any dimensional torus, being rigid. This avoids some pathologies in the subject, but is perhaps hard to motivate. In any case, the sphere and the torus of arbitrary dimension are examples of (strongly) rigid manifolds.

Using L^2-cohomology, (a large-scale theory of the universal cover) for example, it is possible to show that no manifold M whose fundamental group has torsion can be very strongly topologically rigid (see [CW03]). However, in the opposite extreme, manifolds with contractible universal cover are expected to be topologically rigid—this is an important conjecture of Borel.[9]

Tacit in such a discussion is that global conditions restrict the possible tangent bundles (and other characteristic classes) of manifolds, just based on their fundamental groups. This idea is made precise in a general conjecture of Novikov, as extended to the setting of C^*-algebras by Baum and Connes and Kasparov. It asserts, roughly speaking, that the group cohomology of the fundamental group of a manifold gives rise to restrictions on the elliptic operators (e.g., the signature operator) on the manifold (and therefore to its homotopy type, in light of Hodge theory).[10] Almost all we know about this problem is obtained indirectly from the large-scale geometry of the fundamental group.

For example, we know the Novikov conjecture for groups with finite asymptotic dimension (a large-scale variant of Lebesgue's covering dimension for usual spaces) [Yu98], or for groups which have effectively proper Lipschitz maps into Hilbert (and some other Banach) space ([STY02], [Yu00], and others)—which then implies Novikov's conjecture, e.g., for amenable, hyperbolic and all linear groups. (For amenable and hyperbolic groups, the Baum–Connes conjecture is actually known; see [HK01] and [MY02], [Laf12], respectively. The result for linear groups now has two proofs [GHW05] and [GTY12], but only for the Novikov conjecture, not for Baum and Connes). I recommend Yu's ICM survey [Yu06] for some of this.[11]

Note though that we have connected here to the discussion we had earlier about the distortion of finite metric spaces! Gromov's paper [Gro03] shows hows how to embed expanders into group theory (to prevent effectively proper embedding into

[7] I must confess to be writing a book on related topics [We].

[8] More precisely, we wish that every homotopy equivalence is homotopic to a homeomorphism.

[9] See [B], [Lüc10] for a discussion of the current state of the art on this important problem. It is known, for example, for $K \; G / \Gamma$ for cocompact lattices (and with suitable interpretation in the non-cocompact case as well) by work of Farrell and Jones as well as aspherical manifolds (of dimension $>$ 4) whose fundamental groups are word hyperbolic by the work of Bartels and Luck.

[10] The papers [GL83], [Ros83], [SY79] are quite analogous developments for the rather different problem of existence of positive scalar curvature metrics.

[11] See also [FRR] for a somewhat older survey that includes topological methods as well.

Hilbert space), and this ultimately leads to counterexamples [HLS02] to a strong form of the Baum–Connes conjecture.

To continue in this vein would easily be possible, and even a sketchy overview of the area in which this book is embedded could easily be book length. Attractive as that possibility is, this section of the *Bulletin of the AMS* is not its place. (My favorite place to search is Gromov's web page [Gr0]; one book length overview of this area is Gromov's [Gro93].)

The book under discussion is a very good introduction to many of the themes mentioned above and some not mentioned—focusing on the large-scale geometric and analytic aspects, while leaving applications for more advanced treatments. Many of the topics they considered have proven themselves to be important in problems related to the Novikov and Baum–Connes conjectures—not surprising, given the interests of the authors. This includes discussions of asymptotic dimension, amenability, Property (T), including the construction of non-Lie theoretic groups with this property, expanders, and embeddings into Hilbert space. They go further in some directions, explain coarse homology theories, and also some information about group actions on other Banach spaces besides Hilbert space. Hyperbolic groups are introduced but not really studied in any detail; however, that is a subject which has, by now, several very good introductions (e.g., [Gro87], [CDP90], [GdlH]). On the other hand, an excellent more general introduction to metric geometry is [BBI01].

For students of index theory, high-dimensional topology, and noncommutative geometry, this text will be invaluable. Indeed, with this set alone, the authors will have surpassed their stated goal from the preface. However, I anticipate that this book will also find many adherents among students of other disciplines, such as workers on analysis on metric measure spaces (see, e.g., [Hei01], [CK10], [Nao10]), whose work has had lovely application to problems of metric embedding. The discussion of compression is a nice counterpoint to probabilistic methods; see [Aus11], [ANP09]. Geometric group theorists would do well to read this book as it is written from the point of view of people who seriously seek to apply their ideas, and surely the same can be said for those who study the geometric theory of Banach spaces. I plan to assign it to my students.

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