INTEGRAL AND RATIONAL MAPPING CLASSES

FEDOR MANIN and SHMUEL WEINBERGER

Abstract

Let X and Y be finite complexes. When Y is a nilpotent space, it has a rationalization $Y \to Y_{(0)}$ which is well understood. Early on it was found that the induced map $[X,Y] \to [X,Y_{(0)}]$ on sets of mapping classes is finite-to-one. The sizes of the preimages need not be bounded; we show, however, that, as the complexity (in a suitable sense) of a rational mapping class increases, these sizes are at most polynomial. This "torsion" information about $[X,Y]$ is in some sense orthogonal to rational homotopy theory but is nevertheless an invariant of the rational homotopy type of Y in at least some cases. The notion of complexity is geometric, and we also prove a conjecture of Gromov regarding the number of mapping classes that have Lipschitz constant at most L.

1. Introduction

One of the great successes of homotopy theory is the complete algebraicization of rational homotopy theory by Quillen in [14] and Sullivan in [16]. In particular, while the main objects of study are best understood as infinite complexes, the finiteness theorem of Sullivan and Hilton, Mislin, and Roitberg, quoted below, and related results allow the ideas to be applied to the homotopy theory of based maps between finite complexes $X \to Y$, where Y is a nilpotent space.

To each nilpotent complex Y of finite type[1] is associated a (functorially constructed) rationalization $Y_{(0)}$, characterized by the condition that the map $Y \to Y_{(0)}$ induces an isomorphism on $\pi_i(-) \otimes \mathbb{Q}$. In this context, the finiteness theorem says the following.

THEOREM ([16, Theorem 10.2(i)] or [10, Corollary II.5.4])
If X is a finite complex, then the map between (based or unbased) sets of homotopy classes $[X,Y] \to [X,Y_{(0)}]$ is finite-to-one.

DUKE MATHEMATICAL JOURNAL
Vol. 169, No. 10, © 2020 DOI 10.1215/00127094-2020-0012
Received 1 March 2018. Revision received 11 February 2020.
First published online 9 June 2020.
2010 *Mathematics Subject Classification*. Primary 55P62; Secondary 53C23.

[1] A CW complex is of finite type if it is homotopy equivalent to one with finitely many cells in every dimension.

This paper is devoted to understanding this more quantitatively. If X is the sphere, say, then the number of preimages of a rational map is independent of the map—when it is nonzero—since it is the cardinality of the kernel of the map $\pi_i(Y) \to \pi_i(Y) \otimes \mathbb{Q}$. However, even for as simple a target as S^4, there are examples where the cardinalities of these fibers can be unbounded. If $X = S^3 \times S^4$, then the sizes depend on the degree of the map restricted to the S^4 factor.

The precise statement that we will make is that the size of these preimages grows like a polynomial in the complexity of the homotopy class. But this forces the question of defining complexity.

There are two solutions to this new problem, both suggested by Gromov's well-known paper [8].

(1) Replace X and Y by manifolds with boundary, and consider the norms that the maps induce on sufficiently large finite-dimensional algebras of differential forms. Unfortunately, unlike the minimal model, which is unique up to isomorphism (see Section 2 for relevant concepts in the homotopy theory of commutative differential graded algebras [DGAs]), the algebras of differential forms are not. We will see that, in consequence, although the notion of polynomially bounded is well defined, the degree of the polynomial is not.

(2) Fix (possibly piecewise or cellwise) Riemannian metrics on X and Y, and view the Lipschitz constant of $f : X \to Y$ as the complexity of the map. Minimizing this over representatives gives us a measure of complexity for homotopy classes. For example, the complexity of a degree d map $S^n \to S^n$ is roughly $d^{1/n}$.

The notion in (1) does not require the homomorphism to be integral. One just asks for a map of DGAs with certain properties and considers the impacts on norms. For integral classes, this quantity can be estimated from the map itself and bounded in terms of its Lipschitz constant.

On the other hand, the notion in (2) is only defined for integral mapping classes and, moreover, depends a priori on the metric. However, any homotopy equivalence between finite metric complexes is homotopic to a Lipschitz homotopy equivalence in the obvious sense. Hence, asymptotics with respect to it (up to a multiplicative constant) are actually homotopy invariants.

The precise interconnection between (1) and (2) is complex. Gromov noted in [8] and J. Maher showed in more detail in his unpublished thesis [11] that the rational invariants of Lipschitz maps to a nilpotent complex (and therefore the notion in (1)) are bounded by a polynomial in the Lipschitz constant. In fact, it turns out that the minimal Lipschitz constant is likewise polynomially bounded by the notion in (1); indeed, in [12] the first author shows this by way of a purely rational notion of dilatation which is equivalent to (2) up to a multiplicative constant. This points to

(2) as the "correct" notion of complexity for rational mapping classes even from an algebraic-topological point of view.

We will show the following.

THEOREM 1.1
The size of the preimages of maps in $[X,Y] \to [X,Y_{(0)}]$ *is bounded by a polynomial in the complexity (in either sense).*

There is another phenomenon that is dual to this that also needs to be considered, namely, a statement about the density of the image of $[X,Y]$ in $[X,Y_{(0)}]$. The image is always discrete, but the density, that is, the number of image points in a "ball of radius 1," can grow as one moves farther from the zero map. However, this density turns out to be polynomial as well.

These conclusions can be summarized by the following theorem, proved by Gromov when X is a sphere in [8] and conjectured by him in [9, Chapter 7].

THEOREM 1.2
The number of homotopy classes of maps $[X,Y]$ *that have a representative with Lipschitz constant at most* L *is bounded by a polynomial in* L.

However, in Section 3, we will see that (contrary to the speculation in Gromov's book; see [9, p. 358]) the number is not necessarily asymptotic to a polynomial. We give an example that has an extra $\log(L)$ factor.

In all cases, we work with based maps, but the corresponding results for unbased maps follow easily. Moreover, our examples are all in the realm of simply connected spaces, where these notions are equivalent.

1.1. Rational invariance

The techniques in the proofs of Theorems 1.1 and 1.2 are variations of the work of Sullivan. It is natural to ask whether the growth rates in these problems, which we call *torsion growth*[2] and *growth*, respectively, are actually invariants of the rational homotopy type of $Y_{(0)}$ and not just the integral homotopy type of Y. It is unclear whether this is true in general, but we prove the following partial result.

THEOREM 1.3
Let X *and* Y *be finite metric complexes with* Y *simply connected. If* Y *(resp.,* X*) is a*

[2] We hope not confusingly, since there is only growth in situations where the mapping set does *not* have a group structure.

space with positive weights*, then the asymptotic behavior of the growth* $g_{[X,Y]}$ *and the torsion growth* $\mathrm{tg}_{[X,Y]}$ *depends only on the rational homotopy type of* Y *(resp.,* X*).*

Having positive weights is a technical condition on the rational homotopy type of a simply connected space first introduced by Morgan and Sullivan.[3] The main property of such spaces is that they have a large family of "telescoping" automorphisms. Many naturally occurring simply connected spaces have positive weights; in particular, all of our examples do. Therefore, for example, the $L^8 \log L$ growth we show for $[(S^3 \times S^4)^{\#2}, S^4]$ is a rational homotopy invariant. On the other hand, this theorem does not give any information about non-simply-connected nilpotent spaces.

For more general spaces, we can only say something much weaker.

PROPOSITION 1.4
Given a rational homotopy equivalence $Y \to Z$ *between finite nilpotent complexes, for any finite complex* X*, the induced map* $[X, Y] \to [X, Z]$ *is uniformly finite-to-one; that is, preimages of classes have bounded size.*

This is insufficient even to prove rational invariance of torsion growth because, for example, there may be classes in $[X, Z_{(0)}]$ that have quickly growing preimages in $[X, Z]$ but no preimages at all in $[X, Y]$. The difficulties may be number-theoretic: in general, the integral classes of homomorphisms $X \to Y$ are integer points of an arbitrarily complicated algebraic variety cut out by the differentials of X and Y. It is unclear what rationally invariant estimates can be found in general.

1.2. Structure of the paper

Section 2 provides the background about DGAs and their connection to rational homotopy theory. We recommend the (impatient) reader skip partway through to Section 3, which gives examples of the various phenomena that this paper grapples with, and then go back to complete Section 2. In Section 4, we explain methods (1) and (2) for defining sizes of maps, and, in Section 5, we prove our main theorem by combining the ideas of Section 4 with Sullivan's inductive method. Finally, the last section addresses the rational invariance problem.

2. Homotopy theory of DGAs

In this section, we sketch out the homotopy theory of DGAs, following the treatment in [7, Chapters IX and X]. Their relatively explicit formulation helps us obtain quantitative bounds. This justifies a thorough exposition, as the formalism may differ from more abstract modern treatments.

[3] According to [3], although this class of spaces was studied earlier by Mimura and Toda in [13].

A *(commutative) DGA* will always denote a cochain complex of $\mathbb{Q}$- or $\mathbb{R}$-vector spaces equipped with a graded commutative multiplication which satisfies the (graded) Leibniz rule.[4] The prototypical example of an $\mathbb{R}$-DGA is the algebra of smooth forms on a manifold or piecewise smooth forms on a simplicial complex. On a simplicial complex X, one can also define a $\mathbb{Q}$-DGA A^*X of *polynomial forms* (see [7, Chapter VIII] for a detailed exposition). In the rest of the section, we will denote $\mathbb{Q}$ or $\mathbb{R}$ by $\mathbb{F}$.

The cohomology of a DGA is the cohomology of the underlying cochain complex. The relative cohomology $H^n(\varphi)$ of a DGA homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is defined to be the cohomology of the cochain complex

$$C^n(\varphi) = \mathcal{A}^n \oplus \mathcal{B}^{n-1}$$

with the differential given by $d(a,b) = (da, \varphi(a) - db)$. This cohomology fits, as expected, into the obvious exact sequence involving $H^*(\mathcal{A})$ and $H^*(\mathcal{B})$.

Given a finite-dimensional vector space V, we write $H^*(\mathcal{A}; V^*)$, where V^* is the dual of V, for the cohomology of the cochain complex $\mathrm{Hom}(V, \mathcal{A})$. By the universal coefficient theorem, this is naturally isomorphic to $\mathrm{Hom}(V, H^*(\mathcal{A}))$, but we will refer to individual cochains in the former format.

A *weak equivalence* between DGAs $\mathcal{A}$ and $\mathcal{B}$ is a homomorphism $\mathcal{A} \to \mathcal{B}$ which induces an isomorphism on cohomology.

An algebra $\mathcal{A}$ is *simply connected* if $\tilde{H}^0(\mathcal{A}) = H^1(\mathcal{A}) = 0$. If $\mathcal{A}$ is simply connected and of *finite type* (i.e., it has finite-dimensional cohomology in every degree), then it has a *minimal model*: a weak equivalence $m_{\mathcal{A}} : \mathcal{M}_{\mathcal{A}} \to \mathcal{A}$, where $\mathcal{M}_{\mathcal{A}}$ is freely generated as an algebra by finite-dimensional vector spaces V_n in degree n, written

$$\mathcal{M}_{\mathcal{A}} = \bigwedge_{n=2}^{\infty} V_n,$$

and the differential satisfies

$$dV_n \subseteq \bigwedge_{k=2}^{n-1} V_k.$$

In other words, $\mathcal{M}_{\mathcal{A}}$ can be built up via a sequence of *elementary extensions*

$$\mathcal{M}_{\mathcal{A}}(n+1) = \mathcal{M}_{\mathcal{A}}(n) \otimes \wedge V_{n+1}$$

with a differential extending that on $\mathcal{M}_{\mathcal{A}}(n)$, starting with $\mathcal{M}_{\mathcal{A}}(1) = \mathbb{Q}$ or $\mathbb{R}$. We refer to elements of the V_n's as *indecomposables*. We will often define finitely generated

[4]We use the abbreviation "DGA" for "differential graded algebra," following [7] and [6]. In other areas this abbreviation may be reserved for augmented algebras. Minimal algebras have a natural augmentation which sends indecomposables to zero, but cochain algebras generally do not.

free DGAs by indicating the degree of generators as superscripts in parentheses: $a^{(3)}$ means that a is an indecomposable generator in degree 3.

In particular, if Y is a manifold or simplicial complex which is simply connected and of finite cohomological type, then the algebras of forms $\mathcal{A} = A^*Y$ or Ω^*Y each have minimal models, both of which we will call $m_Y : \mathcal{M}_Y^* \to \mathcal{A}$. (This notational confusion will not cause us any problems.) This models the Postnikov tower of Y: each $V_n \cong \operatorname{Hom}(\pi_n(Y), \mathbb{F})$ and the differential on V_n is dual to the k-invariant of the fibration $Y_{(n)} \to Y_{(n-1)}$. This is shown inductively via the obstruction theory discussed below.

More generally, suppose that Y is a *nilpotent space*; that is, its fundamental group is nilpotent and acts nilpotently on the higher homotopy groups. Then it still has a minimal model $m_Y : \mathcal{M}_Y^* \to \mathcal{A}$, in the sense that it is built as a limit of extensions $\mathcal{M}_Y^*(n+1) = \mathcal{M}_Y^*(n) \otimes \wedge V_{n+1}$, but now n can no longer denote the degree of the extension as the degrees

$$1 \le \deg V_1 \le \cdots \le V_n \le V_{n+1} \le \cdots$$

need not be strictly increasing. In other words, the kth Postnikov stage yields a finite sequence of elementary extensions which correspond to a decomposition of the Postnikov stage $K(\pi_k(Y), k) \to Y_{(k)} \to Y_{(k-1)}$ into a sequence of principal fibrations. Even more generally, we say $\mathcal{A}$ is *geometric for* a space Y if there is a weak equivalence $\mathcal{A} \to A^*Y$ or Ω^*Y.

2.1. *Obstruction theory*

Given a principal fibration $K(\pi, n) \to E \to B$, obstruction theory gives an exact sequence of based sets

$$H^n(X;\pi) \to [X, E] \to [X, B] \to H^{n+1}(X;\pi)$$

of sets of based homotopy classes (see, e.g., [7, Proposition 14.3]). Moreover, over a given map $f : X \to B$, there is an exact sequence of groups

$$\cdots \to \pi_1(E^X, \tilde{f}) \to \pi_1(B^X, f) \to H^n(X;\pi) \to \{\text{lifts of } f\} \to 0,$$

where the set of lifts is not a group but has a transitive action by $H^n(X;\pi)$.

We now give DGA versions of these statements. First define homotopy of DGA homomorphisms as follows: $f, g : \mathcal{A} \to \mathcal{B}$ are homotopic if there is a homomorphism

$$H : \mathcal{A} \to \mathcal{B} \otimes \wedge(t^{(0)}, dt^{(1)})$$

such that $H|_{\substack{t=0\\dt=0}} = f$ and $H|_{\substack{t=1\\dt=0}} = g$. We think of $\wedge(t, dt)$ as an algebraic model for the unit interval and this notion as an abstraction of the map induced by an ordinary smooth or simplicial homotopy. In particular, it defines an equivalence relation (see [7, Corollary 10.7]).

We also introduce some notation which is useful for constructing homotopies between DGA homomorphisms. For any DGA $\mathcal{A}$, define an operation $\int_0^t : \mathcal{A} \otimes \wedge(t, dt) \to \mathcal{A} \otimes \wedge(t, dt)$ of degree -1 by

$$\int_0^t a \otimes t^i = 0, \qquad \int_0^t a \otimes t^i \, dt = (-1)^{\deg a} a \otimes \frac{t^{i+1}}{i+1}$$

and an operation $\int_0^1 : \mathcal{A} \otimes \wedge(t, dt) \to \mathcal{A}$ of degree -1 by

$$\int_0^1 a \otimes t^i = 0, \qquad \int_0^1 a \otimes t^i \, dt = (-1)^{\deg a} \frac{a}{i+1}.$$

These provide a formal analogue of fiberwise integration; in particular, they satisfy the identities

$$d\Big(\int_0^t u\Big) + \int_0^t du = u - u|_{\substack{t=0 \\ dt=0}} \otimes 1, \tag{2.1}$$

$$d\Big(\int_0^1 u\Big) + \int_0^1 du = u|_{\substack{t=1 \\ dt=0}} - u|_{\substack{t=0 \\ dt=0}}. \tag{2.2}$$

Now we state the main lemma of obstruction theory, which gives the conditions under which a map can be extended over an elementary extension.

PROPOSITION 2.3 ([7, Proposition 10.4])
Let $\mathcal{A} \otimes \wedge V$ be a degree n elementary extension of a DGA $\mathcal{A}$. Suppose that we have a diagram of DGAs

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \big\downarrow & & \big\downarrow h \\ \mathcal{A} \otimes \wedge V & \xrightarrow{g} & \mathcal{C} \end{array}$$

with $g|_{\mathcal{A}} \simeq hf$ by a homotopy $H : \mathcal{A} \to \mathcal{C} \otimes \wedge(t, dt)$. Then the map $O : V \to \mathcal{B}^{n+1} \oplus \mathcal{C}^n$ given by

$$O(v) = \Big(f(dv), g(v) + \int_0^1 H(dv)\Big)$$

defines an obstruction class $[O] \in H^{n+1}(h : \mathcal{B} \to \mathcal{C}; V^*)$ to producing an extension $\tilde{f} : \mathcal{A} \otimes \wedge V \to \mathcal{B}$ of f with $h \circ \tilde{f} \simeq g$ via a homotopy $\tilde{H}$ extending H.

When the obstruction vanishes, there are maps $(b,c): V \to (\mathcal{B}^n, \mathcal{C}^{n-1})$ such that $d(b,c) = O$; that is,

$$db(v) = f(dv),$$
$$dc(v) = h \circ b(v) - g(v) - \int_0^1 H(dv).$$

Then for $v \in V$ we can set $\tilde{f}(v) = b(v)$ and

$$\tilde{H}(v) = h \circ \tilde{f}(v) + \int_0^t H(dv) + d\big(c(v) \otimes t\big).$$

This gives a specific formula for the extension.

There is also a relative version of this proposition, as in [7, Lemma 10.5]. This can be used to prove the following.

PROPOSITION 2.4
Let $\mathcal{A} \otimes \wedge V$ be a degree n elementary extension of a DGA $\mathcal{A}$. Let $h : \mathcal{B} \to \mathcal{C}$ be a surjection of DGAs, and let $\mathcal{A} \otimes \wedge V \xrightarrow{\varphi} \mathcal{C}$ be a map. Then there is an exact sequence of based sets

$$H^n(h : \mathcal{B} \to \mathcal{C}; V^*) \to [\mathcal{A} \otimes \wedge V, \mathcal{B}]_\varphi \to [\mathcal{A}, \mathcal{B}]_{\varphi|_\mathcal{A}} \to H^{n+1}(h : \mathcal{B} \to \mathcal{C}; V^*)$$

of homotopy classes of lifts of φ. Moreover, for every lift $\psi : \mathcal{A} \to \mathcal{B}$ of $\varphi|_\mathcal{A}$, there are an exact sequence of groups and a set

$$[\mathcal{A}, \mathcal{B} \otimes \wedge e^{(1)}]_{\psi \otimes 1} \xrightarrow{\mathcal{O}} H^n(h : \mathcal{B} \to \mathcal{C}; V^*) \to \left\{ \begin{array}{c} \textit{extensions of } \psi \\ \textit{in } [\mathcal{A} \otimes \wedge V, \mathcal{B}]_\varphi \end{array} \right\} \to 0.$$

Here in the first term, we are looking at lifts of $\psi \otimes 1$ as a map

$$\mathcal{A} \to (\mathcal{B} \otimes \wedge e) / \ker h \otimes (e),$$

that is, self-homotopies of ψ which project to $\varphi|_\mathcal{A} \otimes 1$, and the obstruction $\mathcal{O}$ sends

$$\psi + \eta \otimes e \mapsto (\eta d|_V, 0).$$

This is a mild extension of [7, Proposition 14.4] and is proved in essentially the same way.

In the case of spaces, a principal fibration $K(\pi,n) \to E \to B$ induces a fibration $K(\pi,n)^X \to E^X \to B^X$ of spaces of based maps, for any CW-complex X. The homotopy exact sequence of this fibration is

$$\cdots \to H^{n-k}(X;\pi) \to \pi_k(E^X, \tilde{f}) \to \pi_k(B^X, f) \xrightarrow{\iota_k} H^{n-k+1}(X;\pi) \to \cdots$$
$$\to \pi_1(B^X, f) \xrightarrow{\iota_1} H^n(X;\pi) \to \left\{ \begin{matrix} \text{homotopy classes} \\ \text{of lifts of } f \end{matrix} \right\} \to 0, \tag{2.5}$$

where $\tilde{f} : X \to E$ is an arbitrary lift of the map $f : X \to B$.

The analogous long exact sequence for DGAs can be proved by an application of Proposition 2.4. Let $\mathcal{A} \otimes \wedge V$ be an n-dimensional elementary extension of a minimal DGA $\mathcal{A}$, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then there is an exact sequence of groups

$$\cdots \to H^{n-k}(\mathcal{B}; V^*) \to \left[\mathcal{A} \otimes \wedge V, \mathcal{B} \otimes \mathbb{F}\langle e^{(k)}\rangle\right]_{\tilde{\varphi}}$$
$$\to \left[\mathcal{A}, \mathcal{B} \otimes \mathbb{F}\langle e^{(k)}\rangle\right]_{\varphi} \xrightarrow{\iota_k} H^{n-k+1}(\mathcal{B}; V^*) \to \cdots$$
$$\to [\mathcal{A}, \mathcal{B} \otimes \wedge e^{(1)}]_{\varphi} \xrightarrow{\iota_1} H^n(\mathcal{B}; V^*) \to \left\{ \begin{matrix} \text{homotopy classes} \\ \text{of extensions of } \varphi \end{matrix} \right\} \to 0. \tag{2.6}$$

Here again $\tilde{\varphi} : \mathcal{A} \otimes \wedge V \to \mathcal{B}$ is an arbitrary extension of φ, and $e^{(k)}$ represents a k-dimensional generator with $de = 0$ and $e^2 = 0$. (Note that $\mathbb{F}\langle e^{(k)}\rangle$ is not minimal when k is even.)

When B is nilpotent, $\mathcal{A}$ is a minimal model for B, $V = \pi \otimes \mathbb{Q}$, and $\mathcal{B}$ is geometric for X, there is a homomorphism between these two sequences; in fact, by induction on elementary extensions, when B is a rational space, this homomorphism is an isomorphism. Therefore, for any nilpotent space this homomorphism is the tensor product with $\mathbb{Q}$, as shown, for example, by Sullivan as part of the proof of [16, Theorem 10.2(i)].

The group operation on $[\mathcal{A}, \mathcal{B} \otimes \mathbb{F}\langle e^{(k)}\rangle]_{\varphi}$ is given as follows. We can represent any element as $F = \varphi + \eta \otimes e$, where $\eta : \mathcal{A}^* \to \mathcal{B}^{*-k}$ satisfies the identities $d\eta = \eta d$ and

$$\eta(uv) = (-1)^{\deg v}\eta(u)\varphi(v) + \varphi(u)\eta(v). \tag{2.7}$$

Then we define the operation $\boxplus$ on such elements by the formula

$$(\varphi + \eta \otimes e) \boxplus (\varphi + \zeta \otimes e) = \varphi + (\eta + \zeta) \otimes e.$$

When we view e as the volume element on S^k, this operation is homotopic to the image of the usual operation in π_k by an Eckmann–Hilton argument. We can then identify ι_k with

$$\varphi + \eta \otimes e \mapsto \eta \circ d|_V : V \to \mathcal{B}^{n-k+1}.$$

3. Torsion and density growth

The finite-to-one-ness statement mentioned in the Introduction was proved by Sullivan as part of the following more general result (see [16, Theorem 10.2(i)] and its proof).

THEOREM

Let X be a finite complex, and let Y be a nilpotent space of finite type (over the integers). Then

(1) *the localization map* $\mathrm{loc} : [X, Y] \to [X, Y_{(0)}]$ *is finite-to-one; and*

(2) *for all $i > 0$ and $f : X \to Y$, the map*

$$\pi_i(Y^X, f) \otimes \mathbb{Q} \to \pi_i\big((Y_{(0)})^X, f_{(0)}\big)$$

induced by localization is an isomorphism.

One might hope that the finiteness in (1) is *uniform*, that is, that cardinalities of preimages of points are bounded by some constant $N(X, Y)$. Indeed, this is obviously true when $X = S^n$, since in that case the correspondence is a group homomorphism and each such preimage is a coset of the kernel. In general, however, the size of this preimage may grow without bound depending on the rational homotopy class; we then say that $[X, Y]$ exhibits *torsion growth*. Instead, the quantitative version of Sullivan's theorem is provided by Theorem 4.6, which implies Theorems 1.1 and 1.2.

In this section, we provide three examples of torsion growth and related phenomena that motivate the rest of the discussion.

Example 3.1

By obstruction theory, the homotopy class of a map $f : S^3 \times S^4 \to S^4$ is determined by the degree on the S^4 factor and a Hopf invariant on the top-dimensional cell. However, some combinations determine homotopic maps. To see this, we fix two maps $f_1, f_2 : S^3 \times S^4 \to S^4$ with degree d and Hopf invariants h_1, h_2 which factor through a map $S^3 \times S^4 \to S^7 \vee S^4$ given by pinching off a disk in the top cell and projecting the rest onto the S^4 factor, as in Figure 1.

Suppose that $H : S^3 \times S^4 \times I \to S^4$ is a homotopy between two such maps. The original maps factor through $S^7 \vee S^4$, so such a homotopy factors as

$$S^3 \times S^4 \times I \xrightarrow{H_1} U \xrightarrow{H_2} S^4,$$

where U is given by collapsing each end of the cylinder to a copy of $S^7 \vee S^4$. This U is homotopy equivalent to $S^4 \times S^4$ minus two open disks; here the first S^4 factor is $S^3 \times I$ modulo the ends. Then H_2 sends the boundaries of the two disks to S^4 via maps of Hopf invariant h_1 and h_2 and the second S^4 factor via a map of degree d. It

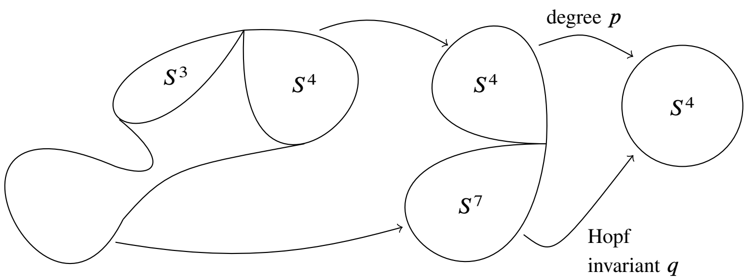

Figure 1. Construct maps $S^3 \times S^4 \to S^4$ by "budding off" a small ball and then projecting the rest onto the S^4 factor.

remains to determine the degree on the first S^4 factor, which we call a. In order for the map to be defined on the top cell, the Hopf invariant on its boundary must be 0; that is,

$$2ad = h_1 - h_2.$$

Thus, a is determined by d, h_1, and h_2, and such a homotopy can be constructed if and only if $h_1 - h_2$ is an integer multiple of $2d$. In other words, maps $S^3 \times S^4 \to S^4$ which have degree d on the S^4 have a nontrivial Hopf invariant modulo $2d$. For $d \neq 0$, this gives $2d$ elements of $[X, Y]$ above a single element of $[X, Y_{(0)}]$.

Repeating this example with maps $S^3 \times \mathbb{HP}^n \to \mathbb{HP}^n$ gives $(n+1)d^n$ different maps of degree d on the second factor. Thus, the torsion growth of $[X, Y]$ may be an arbitrarily large polynomial in the rational homotopy invariants. The second part of Theorem 4.6 shows that this is the worst that can happen.

Note that torsion growth occurs when the obstruction theory is affected by what happens in lower dimensions. A similar situation may lead to another kind of growth which is also limited by Theorem 4.6. Namely, the "density" of rational homotopy classes which come from genuine, integral homotopy classes in $[X, Y]$ (*integral classes* for short) may grow as we look at larger balls in $\mathrm{Hom}(\mathcal{M}_Y^*, \mathcal{M}_X^*)$.

Example 3.2
Consider now the space $X = S^3 \times (S^4 \vee S^4)$. Similarly to the previous example, elements of $[X, S^4]$ are determined by degrees α_1 and α_2 on the two copies of S^4 and Hopf invariants β_1 and β_2 on the two 7-cells. We now translate this into rational homotopy theory. The two spaces have minimal models $\mathcal{M}_{S^4}^* = \langle a^{(4)}, b^{(7)} \mid da = 0, db = a^2 \rangle$ and

$$\mathcal{M}_X^* = \langle x^{(3)}, y_1^{(4)}, y_2^{(4)}, z_{11}^{(7)}, z_{12}^{(7)}, z_{22}^{(7)}, \ldots \mid dz_{ij} = y_i y_j, \ldots \rangle$$

(omitting higher-degree terms), and homomorphisms are given by

$$a \mapsto \alpha_1 y_1 + \alpha_2 y_2,$$
$$b \mapsto \alpha_1^2 z_{11} + 2\alpha_1\alpha_2 z_{12} + \alpha_2^2 z_{22} + \beta_1 x y_1 + \beta_2 x y_2$$

for any $\alpha_i, \beta_i \in \mathbb{Q}$. However, when $\alpha_1\beta_2 = \alpha_2\beta_1$, a homotopy to the homomorphism with $\beta_1 = \beta_2 = 0$ is given by

$$a \mapsto \alpha_1 y_1 + \alpha_2 y_2 + \frac{\beta_1}{\alpha_1} x \otimes dt,$$
$$b \mapsto \alpha_1^2 z_{11} + 2\alpha_1\alpha_2 z_{12} + \alpha_2^2 z_{12} + (\beta_1 y_1 + \beta_2 y_2) x \otimes t.$$

Extending by linearity, we see that the representatives of a given rational homotopy class form a line of slope α_2/α_1 in the (β_1, β_2)-plane. (Thus, the space of homotopy classes in this plane is 1-dimensional.) The lines which pass through lattice points are integral. Thus, when α_1 and α_2 are relatively prime, a ball of radius R in this plane contains $2\max\{\alpha_1, \alpha_2\}R$ integral classes.[5] As we allow α_1 and α_2 to increase, this density grows, and the total number of integral classes in an R-ball in $\mathrm{Hom}(\mathcal{M}_Y^*, \mathcal{M}_X^*)$ is $\sim R^4$, rather than $\sim R^3$, as one may expect purely by looking at the dimension of the space of rational homotopy classes.[6]

Thus, Theorem 4.6 may be rephrased as saying that both the torsion growth and density growth of $[X, Y]$ are always at worst polynomial.

Finally, we compute an example in which only looking at the volume growth of the space of DGA homomorphisms actually yields the wrong overall bound on the growth of $[X, Y]$. This is in contrast with the previous two examples, where the number of distinct maps of degree zero with at most a given Lipschitz constant, which is determined by the Hopf invariant, swamps the "extra" elements coming from the torsion and density growth. Indeed, in this example, the correct bound is not even polynomial!

Example 3.3
Let $X = (S^3 \times S^4)\#(S^3 \times S^4)$. Using the same method as in Example 3.1, we see the following:

- The homotopy class of a map $X \to S^4$ is determined by degrees d_1 and d_2 on the two S^4 factors and a Hopf invariant h on the 7-cell.

[5] By repeating the analysis in the previous example, one can see that the size of the preimage of this ball in $[X, Y]$, without identifying classes which are the same rationally, is $2\max\{\alpha_1, \alpha_2\}R \pm \gcd\{\alpha_1, \alpha_2\}$, regardless of what the gcd is.
[6] This relies on the fact that a positive fraction (namely, $6/\pi^2$) of all pairs of numbers are relatively prime.

- The invariant h is well defined modulo $2\gcd(d_1, d_2)$.

We now estimate the number of homotopy classes which have representatives with Lipschitz constant at most L. By a minimal model analysis, such a homotopy class must have $d_i = O(L^4)$ and (if $d_1 = d_2 = 0$) $h = O(L^8)$. Conversely, any homotopy class with $d_i \leq L^4$ and $h \leq L^8$ can be realized with Lipschitz constant $O(L)$ by the construction in Example 3.1. The number of such homotopy classes is

$$2L^8 + 4 \sum_{0<d<L^4} 2d + \sum_{0<|d_1|,|d_2|<L^4} 2\gcd(d_1, d_2).$$

Clearly, the last term is asymptotically at least as large as the other two. Now, if $N \geq k$, then the proportion of pairs $0 < a, b \leq N$ with $\gcd(a,b) = k$ is

- at most that of pairs for which k divides both a and b (i.e., $1/k^2$); and
- at least

$$\frac{1}{4}\Big(\frac{1}{k^2} - \sum_{\ell=2}^{\infty} \frac{1}{(k\ell)^2}\Big) = \frac{1}{4k^2}\Big(2 - \frac{\pi^2}{6}\Big).$$

  Here the factor of $1/4$ comes from accommodating the possibility that k does not evenly divide N, and the summation comes from an overcount of all the pairs which have gcd divisible by and strictly larger than k.

Therefore, to within a multiplicative constant, the number of integral homotopy classes with these bounds is

$$\sum_{0<|d_1|,|d_2|<L^4} 2\gcd(d_1, d_2) \sim \sum_{k=1}^{L^4} \frac{L^8}{k^2} \cdot 2k \sim L^8 \log L.$$

This is, therefore, the growth function of $[X, S^4]$.

4. Polynomially bounded functionals on $[X, Y]$

We now introduce some vocabulary to talk about the quantitative properties of the fundamental correspondences in rational homotopy theory. Let X be a finite simplicial complex, and let Y be a nilpotent space of finite $\mathbb{Q}$-homological type. We will study the set of homotopy classes from X to Y by studying their images in the homotopy classes of DGA homomorphisms $\mathcal{M}_Y^* \to A^*X$ from the minimal model of Y to the simplexwise polynomial forms on X. Although the domain and range are both potentially infinite-dimensional as vector spaces, there is a finite-dimensional vector subspace of $\mathrm{Hom}_{\mathbb{Q}\text{-v.s.}}(\mathcal{M}_Y^*, A^*X)$ which contains representatives of every homotopy class, as a consequence of the following lemma.

LEMMA 4.1
There is a finite-dimensional space $W \subset A^*X$ such that every homotopy class of

maps $\mathcal{M}_Y^* \to A^*X$ *has a representative where the images of all the indecomposables are in* W.

Proof
We show this by induction on elementary extensions. In the base case, there is one DGA map $\mathbb{Q} \to A^*X$, whose image is in $\mathbb{Q}$. Now suppose that every homotopy class of maps $\mathcal{M}_Y^*(k) \to A^*X$ has a representative such that the indecomposables land in a finite-dimensional subspace W_k. Suppose that $\mathcal{M}_Y^*(k+1) = \mathcal{M}_Y^*(k) \otimes \wedge V$, where V is of degree n_{k+1}. Then for any such representative, dV lands in the finite-dimensional subspace $D_{k+1} \subset \mathbb{Q}[W_k]$ consisting of $(n_{k+1}-1)$-coboundaries. Let S_{k+1} be a finite-dimensional subspace of A^*X such that $d|_{S_{k+1}}$ is an isomorphism to D_{k+1}. Finally, let $\mathcal{H}^{n_{k+1}}(X;\mathbb{Q})$ be a subspace of A^*X containing a representative for each element of $H^{n_{k+1}}(X;\mathbb{Q})$. By obstruction theory, every homotopy class of DGA maps $\mathcal{M}_Y^*(k+1) \to A^*X$ has a representative in

$$W_{k+1} := W_k + S_{k+1} + \mathcal{H}^{n_{k+1}}(X;\mathbb{Q}).$$

Then we can set $W = W_r$, where r is maximal such that $n_r \leq \dim X$. □

Note that, given a minimal model $m_X : \mathcal{M}_X^* \to A^*X$ for X, we can always choose $W \subset A^*X$ as in the proof to be a subset of the image of m_X. This is true even if X is not nilpotent: in this case a minimal model can still be chosen for A^*X, although it may be infinite-dimensional in each degree and may not be a good homotopical model for X. We fix the notation $W(m_X, Y)$ for the vector space constructed in this way, as it will be useful later. We also write $\mathbb{Q}[W]$ for the subalgebra of A^*X generated by W. Since W, as constructed in the proof, has no elements of degree zero, this is still finite-dimensional in each degree and zero in degrees greater than $\dim X$, allowing us to use $\operatorname{Hom}(\mathcal{M}_Y^*, \mathbb{Q}[W])$ as a larger but still finite-dimensional substitute for the set of homomorphisms which send indecomposables of $\mathcal{M}_Y^*$ to W.

Now let $F : [X,Y] \to \mathbb{R}$ be a functional.[7] We say that F is *polynomially bounded with respect to* a W as above if, for some (equivalently, any) choice of norms on W and $\mathcal{M}_Y^{\leq \dim X}$, there is some p such that, for every $\alpha \in [X,Y]$,

$$\bigl|F(\alpha)\bigr| = O\Bigl(\bigl(\min\bigl\{\|\varphi\|_{\mathrm{op}} : \varphi \in \operatorname{Hom}\bigl(\mathcal{M}_Y^*, \mathbb{Q}[W]\bigr) \text{ with } [\varphi] = \alpha_{(0)}\bigr\}\bigr)^p\Bigr). \tag{4.2}$$

Here $\|\cdot\|_{\mathrm{op}}$ represents the operator norm and $\alpha_{(0)}$ is the image of α in $[\mathcal{M}_Y^*, A^*(X)]$. We say that F is *polynomially bounded* if it is polynomially bounded with respect to all choices of W. Likewise, if for some (not necessarily every!) W the reverse inequality holds; that is, for some $p > 0$,

[7] In the old-fashioned sense of a mapping assigning numerical values to elements of a function space.

$$F(\alpha) = \Omega\Big(\big(\min\big\{\|\varphi\|_{\mathrm{op}} : \varphi \in \operatorname{Hom}\big(\mathcal{M}_Y^*, \mathbb{Q}[W]\big) \text{ with } [\varphi] = \alpha_{(0)}\big\}\big)^p\Big), \tag{4.3}$$

then we say that the functional F is *polynomially bounded below*.

The degree p in (4.2) may certainly depend on the choice of W. For example, one may take $X = S^3$, $Y = S^2$, and F to simply be the absolute value of the Hopf invariant. Here we have $\mathcal{M}_X^* = \langle x^{(3)} | dx = 0\rangle$ and

$$\mathcal{M}_Y^* = \langle a^{(2)}, b^{(3)} | da = 0, db = a^2\rangle.$$

The obstruction-theoretic choice of W as in Lemma 4.1 is a purely 3-dimensional one generated by a volume form on S^3; the map $a \mapsto 0, b \mapsto kd\,\mathrm{vol}_{S^3}$ is homotopic to the pullback of a map of Hopf invariant k, so the bound for this choice is linear. But we could also, for example, choose $W = \langle f^*d\,\mathrm{vol}_{S^2}, d\,\mathrm{vol}_{S^3}\rangle$, where $f : S^3 \to S^2$ is the Hopf fibration. Then a map of Hopf invariant k^2 can be represented by $a \mapsto kf^*d\,\mathrm{vol}_{S^2}, b \mapsto 0$,[8] and so the polynomial bound on F is only quadratic.

It is probably the case, although we do not know of specific examples, that functionals may be polynomially bounded with respect to some W without being polynomially bounded.[9] On the other hand, it is always enough to test the specific W that we have already constructed.

LEMMA 4.4
Fix a minimal model $m_X : \mathcal{M}_X^* \to A^*X$. Whenever a functional F on $[X, Y]$ is polynomially bounded with respect to $W(m_X, Y)$, it is polynomially bounded. Similarly, F is polynomially bounded below if and only if (4.3) holds for $W(m_X, Y)$.

When X is also nilpotent, we may also choose $\mathbb{Q}[W] = m_X(\mathcal{M}_X^{\leq \dim X})$. In this case, it is possible to define the degree of polynomiality of a functional F as its degree with respect to this W. This may be different from the degree with respect to the Lipschitz norm given by the minimal Lipschitz constant of a representative; in this paper we will take the latter as more natural, as explained in the Introduction.

Proof of Lemma 4.4
Write $W = W(m_X, Y)$. For every k, fix subspaces S_k and $\mathcal{H}^{n_k}(X; \mathbb{Q})$ of W as constructed in the proof of Lemma 4.1.

Let W' be another finite-dimensional subset of A^*X such that maps to W' contain representatives for all of $[X, Y]$. It is enough to show that there is a polynomial

[8]Note that we still need to include $d\,\mathrm{vol}_{S^3}$ in W in order to represent maps with Hopf invariant not a perfect square.
[9]This is because there are ways to define W so that the images of indecomposables must be related via essentially arbitrary systems of rational Diophantine equations, and very little is known about how the minimal size of solutions to these depends on parameters.

$P(t)$ such that, for every $\varphi' : \mathcal{M}_Y^* \to \mathbb{Q}[W']$, there is a $\varphi : \mathcal{M}_Y^* \to \mathbb{Q}[W]$ which is homotopic to φ' as a map to A^*X such that $\|\varphi\|_{\mathrm{op}} \leq P(\|\varphi'\|_{\mathrm{op}})$. If F is bounded with respect to W by a polynomial P_0, then it is bounded with respect to W' by $P \circ P_0$.

We show this by induction on elementary extensions. The point is to move $\varphi'|_{\mathcal{M}_Y^*(k)}$ to a $\varphi_k : \mathcal{M}_Y^*(k) \to \mathbb{Q}[W]$ which is sufficiently nearby, in the sense that there is a polynomial-size homotopy from one to the other. This allows us to lift φ_k to a φ_{k+1} which is still not too far away.

Formally, we keep track of the operator norm of φ_k (which sends indecomposables to $W(m_X, Y)$) and of the homotopy H_k between $\varphi'|_{\mathcal{M}_Y^*(k)}$ and φ_k (which sends indecomposables to $U_k \otimes \langle t^{\leq k}, dt\rangle$ for some finite-dimensional U_k which depends only on W and W'). At the $(k+1)$st step, lifting the homotopy increases these operator norms by at most a polynomial, again depending only on W and W'.

Specifically, we define φ_{k+1} and H_{k+1} as follows. Write $\mathcal{M}_Y^*(k+1) = \mathcal{M}_Y^*(k) \otimes \wedge V$; choose a finite subspace $S' \subset A^*X$ such that $d|_{S'}$ is an isomorphism to the space of coboundaries in the subspace

$$S_{k+1} + \mathcal{H}^{n_{k+1}}(X;\mathbb{Q}) + (U_k + W')^{n_{k+1}} \subset A^*X.$$

By the discussion after Proposition 2.3, to extend φ_k and H_k it is enough to choose (b, c) satisfying

$$db(v) = \varphi_k(dv),$$
$$dc(v) = b(v) - \varphi'(v) - \int_0^1 H_k(dv)$$

for $v \in V$. Then we can set $\varphi_{k+1}(v) = b(v)$ and

$$H_{k+1}(v) = \varphi_{k+1}(v) + \int_0^t H_k(dv) + d\big(c(v) \otimes t\big).$$

To choose b and c in a polynomially bounded way, first let $\tilde{b}(v) = (d|_{S_{k+1}})^{-1} \times (\varphi_k(dv)) \in S_{k+1}$. Then

$$\tilde{b}(v) - \varphi'(v) - \int_0^1 H_k(dv)$$

is a cycle in $S_{k+1} + U_k + W'$; let $a(v)$ be the representative of its homology class in $\mathcal{H}^{n_{k+1}}(X;\mathbb{Q})$. Then we choose $b(v) = \tilde{b}(v) - a(v)$ and $c(v)$ to be the antiderivative in S' of

$$\tilde{b}(v) - a(v) - \varphi'(v) - \int_0^1 H_k(dv).$$

All four of these terms are polynomially bounded in terms of φ_k and H_k, with the polynomial depending only on the differential on $\mathcal{M}_Y^*$ and the structure of W and W'. □

All the above results, starting with Lemma 4.1, also hold for the de Rham algebra $\Omega^* X$; in this case one should talk of $\mathbb{R}[W]$ rather than $\mathbb{Q}[W]$.

We now give the two main examples of polynomially bounded functionals which motivate the definitions.

THEOREM 4.5
Suppose that X and Y are finite complexes with piecewise Riemannian metrics and with Y nilpotent. Then the functional $\operatorname{Lip} : [X, Y] \to \mathbb{R}^+$ *given by*

$$\operatorname{Lip} \alpha = \inf\{\operatorname{Lip} f : f \text{ is a Lipschitz representative of } \alpha\}$$

is polynomially bounded below.

THEOREM 4.6
For finite complexes X and Y with Y nilpotent and for any $W \subset A^* X$ as in the statement of Lemma 4.1, the number of homotopy classes in $[X, Y]$ whose image in $[\mathcal{M}_Y, A^* X]$ has a representative in the R-ball in $\operatorname{Hom}(\mathcal{M}_Y^*, W)$ *is bounded by a polynomial in R. In particular, the functional $\# : [X, Y_{(0)}] \to \mathbb{R}^+$, which measures the size of the preimage in $[X, Y]$ of each class under composition with the rationalization map, is polynomially bounded.*

These two results combine to immediately yield Gromov's conjecture.

COROLLARY 4.7
If X and Y are finite complexes and Y is nilpotent, then the number of homotopy classes of maps $X \to Y$ which have representatives with Lipschitz constant at most L is bounded by a polynomial in L.

The proof of Theorem 4.6 is deferred to Section 5.

Before we begin the proof of Theorem 4.5, we remark that, since we have not used the equivalence between the rational homotopy categories of spaces and DGAs, everything discussed thus far in this section is true for real DGAs as well as rational ones. In other words, if X is a smooth manifold, perhaps with boundary, then we can replace $A^* X$ with $\Omega^* X$ without changing any of the arguments.

Proof of Theorem 4.5
We first note that this property is invariant under Lipschitz homotopy equivalence.

Therefore, we can assume that X and Y are compact Riemannian manifolds with boundary by embedding them in some high-dimensional Euclidean space and thickening.

From here, the proof follows the same outline as that of Lemma 4.4. Let $f : X \to Y$ be a map with Lipschitz constant L. Fix real minimal models $m_X : \mathcal{M}_X^* \to \Omega^* X$ and $m_Y : \mathcal{M}_Y^* \to \Omega^* Y$. (Here again, the fact that $\mathcal{M}_X^*$ may not be a good homotopical model is immaterial.) We would like to show that, for some polynomial P depending only on m_Y and m_X, $f^* m_Y : \mathcal{M}_Y^* \to \Omega^* X$ is homotopic to a map $\varphi : \mathcal{M}_Y^* \to W(m_X, Y)$ such that $\|\varphi\| \le P(L)$.

This is proved by induction on the minimal model of Y. We let the ∞-norm be our chosen norm on $\Omega^* X$. This gives us an operator norm on maps $\mathcal{M}_Y^* \to \Omega^* X$ which is well defined up to a constant. Note that the Lipschitz condition implies that, for $\omega \in \Omega^n(Y)$,

$$\|f^*\omega\|_\infty \le L^n \|\omega\|_\infty.$$

In other words, $\|f^* m_Y\|_{\text{op}} \le C(\dim X, m_Y) L^{\dim X}$.

At each stage of the induction we produce maps $\varphi_k : \mathcal{M}_Y^*(k) \to W(m_X, Y)$ and $H_k : \mathcal{M}_Y^*(k) \to \Omega^* X \otimes \wedge(t, dt)$ such that

- $\|\varphi_k\| \le P_k(L)$; and
- if we write

$$H_k(y) = \sum_{i=0}^{r} I_{k,i}(y) \otimes t^i + \sum_{j=0}^{s} J_{k,j}(y) \otimes t^j \, dt,$$

  then r and s depend only on X and Y, $\|I_{k,i}\|_{\text{op}} \le P_k(L)$, and $\|J_{k,j}\|_{\text{op}} \le P_k(L)$.

The induction step proceeds exactly as in Lemma 4.4, except that, since $f^* m_Y$ is not guaranteed to land in a finite subspace, neither can we guarantee this about H_k. Therefore, we also cannot choose antiderivatives from a finite subspace. Instead we use the following lemma, which dates back to [8] and is proved carefully for simplicial complexes as [12, Lemma 2.2].

LEMMA 4.8 (Coisoperimetric inequality)
Given a compact Riemannian manifold X, there is a constant $I(X, n-1)$ such that any exact form $\beta \in \Omega^n(X)$ has an antidifferential $\alpha \in \Omega^{n-1}(X)$ with $\|\alpha\|_\infty \le I(X, n-1)\|\beta\|_\infty$.

We then produce φ_{k+1} and H_{k+1} as before. Write $\mathcal{M}_Y^*(k+1) = \mathcal{M}_Y^*(k) \otimes \wedge V_{k+1}$. First we produce $\tilde{b}(v)$ for each element v of a basis for V_{k+1}. This gives a cycle

$$\tilde{b}(v) - f^* m_Y(v) - \int_0^1 H_k(dv)$$

whose ∞-norm is polynomially bounded in L; this also gives a bound on its cohomology class, obtained by integrating it against cycles generating $H_{n_{k+1}}(X;\mathbb{R})$. Thus, we can choose $a(v)$ and $b(v)$ as before. Applying the coisoperimetric inequality to

$$\tilde{b}(v) - a(v) - f^* m_Y(v) - \int_0^1 H_k(dv),$$

we get a polynomially bounded $c(v)$ and finally obtain φ_{k+1} and H_{k+1}, which also have polynomial estimates. □

5. Quantitative finiteness

We now demonstrate Theorem 4.6. Sullivan's result is proved by using obstruction-theoretic exact sequences and the five lemma; for the quantitative version, we will develop some quantitative homological algebra.

Definition

Let $h : A \to V$ be a homomorphism from a finitely generated group to a normed $\mathbb{Q}$-vector space. We say that h is

- *C-injective* if, for every 1-ball B in V, $\#h^{-1}(B) \le C$;
- *C-surjective* if every point of V is within C of $h(A)$.

LEMMA 5.1 (Quantitative four lemmas)

Suppose that

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 \\
\downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_4} \\
V_1 & \xrightarrow{m_1} & V_2 & \xrightarrow{m_2} & V_3 & \xrightarrow{m_3} & V_4
\end{array}$$

are exact sequences with the A_i's finitely generated groups and the V_i's finite-dimensional normed $\mathbb{Q}$- or $\mathbb{R}$-vector spaces such that m_1 and m_3 have operator norm at most 1. *Let τ be a constant such that m_2 satisfies*

$$\min\{\|u\| : u \in m_2^{-1}(v)\} \le \tau\|v\| \quad \textit{for every } v \in m_2(V_2).$$

(1) *If φ_2 is C_2-injective, φ_4 is C_4-injective, and φ_1 is C_1-surjective, then φ_3 is $(C_1+\tau)^{\operatorname{rk} m_1} \tau^{\operatorname{rk} m_2} C_2 C_4$-injective.*

(2) *If φ_1 is C_1-surjective, φ_3 is C_3-surjective, and φ_4 is C_4-injective, then φ_2 is $(C_1 + 3\tau C_3^{\operatorname{rk} m_3 + 1} C_4)$-surjective.*

We remark that the groups A_i are not necessarily abelian, although the φ_i's of course factor through the abelianization map.

Proof

We use a quantitative version of the usual diagram-chasing arguments for proving the four lemmas.

For the injectivity four lemma, we would like to show that, for every 1-ball B in V_3,

$$\#\varphi_3^{-1}(B) \le (C_1 + \tau)^{\operatorname{rk} m_1} \tau^{\operatorname{rk} m_2} C_2 C_4.$$

First note that $\# f_3(\varphi_3^{-1}(B)) \le C_4$. Thus, it is enough to show that, for any $a \in A_4$,

$$\#\big(f_3^{-1}(a) \cap \varphi_3^{-1}(B)\big) \le (C_1 + \tau)^{\operatorname{rk} m_1} \tau^{\operatorname{rk} m_2} C_2.$$

By shifting the center of B by $-\varphi_3(\tilde{a})$, where $\tilde{a}$ is an arbitrary preimage of a, we see that it is enough to show this for $a = \mathrm{id}$. To do that, we will show that every element in $\varphi_3^{-1}(B) \cap \ker f_3$ has a preimage in A_2 which lands within distance τ of a C_1-ball $\tilde{B}$ in a $\operatorname{rk} m_1$-dimensional affine subspace $v_2 + m_1(V_1)$.

Choose the center of $\tilde{B}$ to be an arbitrary preimage v_2 of the center of B. Given $b \in \varphi_3^{-1}(B) \cap \ker f_3$, choose a preimage $\tilde{b} \in A_2$: we know that $\varphi_2(\tilde{b})$ is at most distance τ from $v_2 + m_1(V_1)$. Then we can choose $s \in A_1$ such that $m_1 \circ \varphi_1(s)$ is at most distance C_1 from $\varphi_2(\tilde{b}) - v_2$, and therefore $\tilde{b} - f_1(s)$ is the preimage we are looking for. This completes the proof of the injectivity lemma.

For the surjectivity lemma, choose $v \in V_2$; we would like to show that there is an $a \in A_2$ such that $\varphi_2(a)$ is contained in a $(C_1 + 3\tau C_3^{\operatorname{rk} m_3 + 1} C_4)$-ball around v. For this, we will show that there is an element $b \in A_2$ such that

$$\big\| m_2\big(v - \varphi_2(b)\big)\big\| \le 3 C_3^{\operatorname{rk} m_3 + 1} C_4.$$

We can find a point $v' \in \ker m_2$ whose distance from $v - \varphi_2(b)$ is at most $3\tau \times C_3^{\operatorname{rk} m_3 + 1} C_4$. Then there is an $\tilde{a} \in A_1$ such that $m_1 \circ f_1(\tilde{a})$ is within C_1 of v', and we can use $a = b + f_1(\tilde{a})$.

It remains to find b. If $\|m_2(v)\| \le 3 C_3^{\operatorname{rk} m_3 + 1} C_4$, then we can use $b = 0$. Otherwise, we show by induction that we can reduce to this case. Let $N = C_3^{\operatorname{rk} m_3} C_4$, and consider the $N + 1$ disjoint C_3-balls B_i around $\frac{i}{N} m_2(v)$, $i = 0, 1, \dots, N$. Each of these has a preimage point $c_i \in A_3$; moreover, $\varphi_4 \circ f_3$ sends each of the c_i's to the C_3-ball around zero in V_4, which means that they have at most $C_3^{\operatorname{rk} m_3} C_4$ distinct images under f_3. By the pigeonhole principle, there are $i < j$ such that some $c_i \in \varphi_3^{-1}(B_i)$ and $c_j \in \varphi_3^{-1}(B_j)$ have $f_3(c_i) = f_3(c_j)$. Then $c_j - c_i = f_2(b')$ for some $b' \in A_2$. Moreover,

$$\left\| m_2\big(v - \varphi_2(b')\big) \right\| \leq \left\| m_2(v) \right\| - C_3.$$

Now we repeat this process with $m_2(v - \varphi_2(b'))$; after a finite number of steps, we get an element of length at most $3C_3^{\mathrm{rk}\, m_3+1} C_4$ and can set b to be the sum of all the b''s used along the way. □

We are now ready to prove the theorem along with the following extra statements.

LEMMA 5.2

Let X and Y be finite complexes with Y nilpotent.

(i) *For every k, there is a polynomial P such that the rationalization map*

$$\pi_1\big((Y_k)^X, f\big) \to \big[\mathcal{M}_Y^*(k), A^*X \otimes \wedge e^{(1)}\big]_{f^* m_Y}$$

is $P(\mathrm{Lip}\, f)$-surjective, where the norm on the latter is given by the operator norm on the indecomposables,

$$\|\gamma\|_l = \inf\big\{\max_{i \leq k} \|\eta|_{V_i}\|_{\mathrm{op}} | \eta : \mathcal{M}_Y^*(k) \to A^*X \text{ s.t. } [f^* m_Y + \eta \otimes e] = \gamma\big\}.$$

(ii) *For every k, there is a polynomial P_k' such that the map*

$$\left\{\begin{matrix} \text{homotopy classes of} \\ \text{lifts of } f_{(k-1)} \text{ to } Y_k \end{matrix}\right\} \to \left\{\begin{matrix} \text{homotopy classes of} \\ \text{extensions of } f^* m_Y(k-1) \text{ to } \mathcal{M}_Y^*(k) \end{matrix}\right\}$$

is $P_k'(\mathrm{Lip}\, f)$-injective, where the norm on the set of extensions is induced by the obstruction in $H^k(X; \pi_k(Y) \otimes \mathbb{Q})$ to homotoping to some fixed extension.

Remark 5.3

A similar proof, applied to a different portion of the long exact sequence, simultaneously proves that

$$\pi_i(Y^X, f) \to \pi_i\big((Y_{(0)})^X, f_{(0)}\big)$$

is $P(\mathrm{Lip}\, f)$-injective and

$$\pi_{i+1}(Y^X, f) \to \pi_{i+1}\big((Y_{(0)})^X, f_{(0)}\big)$$

is $P(\mathrm{Lip}\, f)$-surjective, for norms similar to those in Lemma 5.2(i). Thus, we recover quantitative versions of the entirety of Sullivan's result.

Proof

Write Y as an inverse limit of a tower of spaces

$$\cdots \to Y_k \to Y_{k-1} \to \cdots \to Y_0 = *,$$

where each $Y_k \to Y_{k-1}$ is a principal $K(A_k, n_k)$-fibration, $n_k \geq n_{k-1}$. Fix W as in the previous section; let $\varphi : \mathcal{M}_Y^* \to A^*X$ be a homomorphism which sends indecomposables to W and is homotopic to f^*m_Y.

Our goal here is to understand the behavior of $[X, Y]$, that is, π_0 of the mapping space Y^X. To do this, we need to also consider the behavior of $\pi_1((Y_k)^X, f)$ at various stages k and with various basepoints f and its rationalization homomorphism to

$$\Pi(k, \varphi) := \left[\mathcal{M}_Y^*(k), A^*X \otimes \wedge e^{(1)}\right]_\varphi.$$

By induction on k, we will construct polynomials P_k such that, for the norm in the statement of the lemma, the homomorphisms

$$\pi_1\big((Y_k)^X, f\big) \to \Pi(k, \varphi)$$

are $P_k(\|\varphi\|)$-surjective. In turn, we will use this to construct polynomials P'_k such that the homomorphisms

$$\begin{Bmatrix} \text{homotopy classes of} \\ \text{lifts of } f \text{ to } Y_k \end{Bmatrix} \to \begin{Bmatrix} \text{homotopy classes of} \\ \text{extensions of } \varphi \text{ to } \mathcal{M}_Y^*(k) \end{Bmatrix}$$

are $P'_k(\|\varphi\|)$-injective, where the set of extensions is given a group structure by fixing a basepoint. Then the number of classes in $[X, Y]$ which map to the R-ball of $\operatorname{Hom}(\mathcal{M}_Y^*, \mathbb{Q}[W])$ is at most

$$R^{\sum_{k=1}^r \dim H^{n_k}(X; A_k \otimes \mathbb{Q})} \prod_{k=1}^r P'_k(R), \tag{5.4}$$

where $r = \max\{i : n_i \leq \dim X\}$. This is the estimate we are looking for (although it may often be a drastic overcount). The Lipschitz estimates then follow from Theorem 4.5.

We now produce the polynomials P_k. Of course, we can take $P_0 = 0$, since both groups are trivial. For general k, we inductively apply the surjectivity four lemma to the subsequence

$$H^{n_k-1}(X; A_k) \to \pi_1\big((Y_k)^X, \tilde{f}\big) \to \pi_1\big((Y_{k-1})^X, f\big) \to H^{n_k}(X; A_k) \tag{5.5}$$

of the exact sequence (2.5) and the corresponding subsequence

$$H^{n_k-1}(A^*X; A_k \otimes \mathbb{Q}) \to \Pi(k, \varphi) \xrightarrow{\rho} \Pi(k-1, \varphi) \xrightarrow{\iota_1} H^{n_k}(A^*X; A_k \otimes \mathbb{Q}) \tag{5.6}$$

of (2.6). To do this, we must put norms on the vector spaces in (5.6) that satisfy the relevant compatibility conditions. Note that the groups in (5.5) are finitely generated as noted by Sullivan and perhaps already Serre.

First, let $\hat{W} \supseteq W$ be a finite-dimensional subspace such that every homotopy class of homomorphism $\mathcal{M}_Y^* \to A^*X \otimes \wedge e^{(1)}$ has a representative which lands in $\mathbb{Q}[\hat{W}] \otimes \wedge e$. Such a subspace can be found by the method of Lemma 4.1. We put norms on each of the groups $\operatorname{Hom}(A_k, \mathbb{Q})$ (V_k for short) and on the degree at most $\dim X$ vectors in $\mathbb{Q}[\hat{W}]$. This gives a well-defined operator norm on maps $V_k \to \mathbb{Q}[\hat{W}]$ with fixed-degree image—for example, on cochains that land in $\mathbb{Q}[\hat{W}]$.

Now for $i = n_k$ and $n_k - 1$ we define norms on $H^i(A^*X; V_k^*)$ by minimizing over cochain representatives that land in $\mathbb{Q}[\hat{W}]$:

$$\|\alpha\|_H = \inf\bigl\{\|\psi\|_{\mathrm{op}} \,|\, \psi : V_k \to \mathbb{Q}[\hat{W}] \text{ s.t. } [\psi] = \alpha\bigr\}.$$

Similarly, for $\gamma \in \Pi(k, \varphi)$ we take the minimum over representatives of γ of the operator norm on indecomposables, which we call the left norm:

$$\|\gamma\|_l = \inf\bigl\{\max_{i \le k} \|\eta|_{V_i}\|_{\mathrm{op}} \,|\, \eta : \mathcal{M}_Y^*(k) \to \mathbb{Q}[\hat{W}] \text{ s.t. } [\varphi + \eta \otimes e] = \gamma\bigr\}.$$

Finally, for $\gamma \in \Pi(k-1, \varphi)$ we need to use a right norm which combines the left norm (for $\mathcal{M}_Y^*(k-1)$) and the homology norm on the image under ι_1:

$$\|\gamma\|_r = \inf\left\{\max\bigl\{\max_{i \le k-1} \|\eta|_{V_i}\|_{\mathrm{op}}, \bigl\|(\eta \circ d)|_{V_k}\bigr\|_{\mathrm{op}}\bigr\} \,\middle|\, \begin{array}{l} \eta : \mathcal{M}_Y^*(k-1) \to \mathbb{Q}[\hat{W}] \\ \text{s.t. } [\varphi + \eta \otimes e] = \gamma \end{array}\right\}.$$

It is easy to see that, under these norms, the outer two maps of (5.6) are norm-nonincreasing. Moreover, for the restriction map

$$\rho : \bigl(\Pi(k, \varphi), \|\cdot\|_l\bigr) \to \bigl(\Pi(k-1, \varphi), \|\cdot\|_r\bigr)$$

there is a constant τ_k, which is determined by the differentials on V_k and $\mathbb{Q}[\hat{W}]$ and is therefore independent of φ, such that

$$\min\bigl\{\|\tilde{\gamma}\|_l : \tilde{\gamma} \in \rho^{-1}(\gamma)\bigr\} \le \tau \|\gamma\|_r \text{ for every } \gamma \in \rho\bigl(\Pi(\varphi, k)\bigr).$$

Finally, in order to induct, we need to compare the left and right norms on $\Pi(k-1, \varphi)$. Indeed, there is a polynomial Q_k such that, for $\gamma \in \Pi(k-1, \varphi)$,

$$\|\gamma\|_r \le Q_k\bigl(\|\varphi\|\bigr) \cdot \|\gamma\|_l.$$

This is because, for any $u = dv$, $v \in V_k$, $\eta(u)$ decomposes by repeated applications of (2.7) as

$$\eta(u) = \sum_i \varphi(u_i)\eta(y_i),$$

where the y_i's are indecomposable. This bounds $\eta|_{d(V_k)}$ in terms of η applied to indecomposables.

Therefore, by the surjectivity four lemma,

$$P_k(\|\varphi\|) \le C_{k,-1} + 3\tau_k C_{k,0}\big(Q_k(\|\varphi\|)P_{k-1}(\|\varphi\|)\big)^{\operatorname{rk}\iota_1+1},$$

where $C_{k,-1}$ and $C_{k,0}$ depend only on X and Y. Now we apply the injectivity four lemma to the sequence

$$\pi_1\big((Y_{k-1})^X, f\big) \to H^{n_k}(X; A_k) \to \{\text{lifts of } f\} \to 0$$

and the corresponding sequence of vector spaces, letting the norm on the set of lifts be induced by that on $H^{n_k}(X; V_k^*)$. We get that the rationalization on the set of lifts is $P'_k(\|\varphi\|)$-injective, where

$$P'_k(\|\varphi\|) = C_{k,0}\big(Q_k(\|\varphi\|)P_{k-1}(\|\varphi\|) + 1\big)^{\operatorname{rk}\iota_1},$$

where $C_{k,0} = C_{k,0}(X, Y)$ is the same as above. Now, distances under this norm are a lower bound for distances under the operator norm on $\operatorname{Hom}(\mathcal{M}_Y^*, \mathbb{Q}[W])$; this proves the bound (5.4) and the theorem. □

6. Rational invariance

In this section we prove the statements about rational invariance given in Section 1.1. We first restate Theorem 1.3.

THEOREM

Let X and Y be finite metric complexes with Y simply connected. If Y (resp., X) is a space with positive weights, *then the asymptotic behavior of $g_{[X,Y]}$ and $\operatorname{tg}_{[X,Y]}$ depends only on the rational homotopy type of Y (resp., X).*

A simply connected space Y has ($\mathbb{Q}$-) *positive weights* (see [3] or [5]) if the indecomposables of its minimal DGA split as $U_1 \oplus U_2 \oplus \cdots \oplus U_r$ so that for every $t \in \mathbb{Q}$ there is an automorphism φ_t sending $v \mapsto t^i v$, $v \in U_i$. Examples include formal spaces in [15], coformal spaces in [5], as well as homogeneous spaces and other spaces whose indecomposables split as $V_0 \oplus V_1$, where $dV_0 = 0$ and $dV_1 \subset \bigwedge V_0$. In particular, the spaces in Section 3 all have positive weights. The lowest-dimensional nonexample, as far as we know, is a complex given in [13], which is constructed by attaching a 12-cell to $S^3 \vee \mathbb{C}\mathbf{P}^2$; other, much higher-dimensional nonexamples are given in [2] and [1].

Proof
Suppose that Y and Y' are rationally equivalent simply connected finite complexes with positive weights. This implies (see [3]) that these spaces are 0-*universal*; in particular, there are maps

$$Y \xrightarrow{\varphi} Y' \xrightarrow{\psi} Y$$

inducing rational equivalences. We can assume that these maps are Lipschitz; moreover, by Proposition 1.4, there are constants $C(\varphi, X)$ and $C'(\psi, X)$ such that the maps $[X, Y] \to [X, Y'] \to [X, Y]$ induced by φ and ψ are, respectively, C-to-one and C'-to-one. Then we immediately see that, for any X,

$$g_{[X,Y]}(L) \leq C g_{[X,Y']}\big(\mathrm{Lip}(\varphi) \cdot L\big) \leq C'C g_{[X,Y]}\big(\mathrm{Lip}(\psi)\,\mathrm{Lip}(\varphi) \cdot L\big),$$
$$\mathrm{tg}_{[X,Y]}(L) \leq C\,\mathrm{tg}_{[X,Y']}\big(\mathrm{Lip}(\varphi) \cdot L\big) \leq C'C\,\mathrm{tg}_{[X,Y]}\big(\mathrm{Lip}(\psi)\,\mathrm{Lip}(\varphi) \cdot L\big).$$

Since all these functions are polynomial, this means that they are within a multiplicative constant of each other. A similar argument works for rationally equivalent X and X' with positive weights. □

It remains to prove Proposition 1.4, which we again restate.

PROPOSITION
Given a rational homotopy equivalence $\varphi : Y \to Z$ between finite nilpotent complexes, for any finite complex X, the induced map $[X, Y] \to [X, Z]$ is uniformly finite-to-one; that is, preimages of classes have size bounded by some $C(\varphi, X)$.

For the second part of Theorem 1.3, which concerns the domain, we will also need the following dual statement.

PROPOSITION
Given a map $\varphi : X \to X'$ between finite complexes where the relative homology groups $H^*(X', X)$ are finite, for any simply connected finite complex Y, the induced map $[X', Y] \to [X, Y]$ is uniformly finite-to-one.

Proof of both propositions
To bound the size of the preimage of a homotopy class, we use obstruction theory on the relative Postnikov tower

$$\begin{array}{ccl} & & P_n \\ & \overset{\varphi_n}{\nearrow} & \big\downarrow p_n \\ & & \vdots \\ & & \big\downarrow p_3 \\ & \overset{\varphi_2}{\nearrow} & P_2 \\ & & \big\downarrow p_2 \\ Y & \xrightarrow{\varphi_0 = \varphi} & P_1 = P_0 = Z \end{array}$$

of the map $\varphi : Y \to Z$. Here, P_k is a space such that $\pi_i(P_k, Y) = 0$ for $i \leq k$ and $\pi_i(Z, P_k) = 0$ for $i > k$. The map p_k therefore only has one nonzero (and finite) relative homotopy group, $\pi_k(Z, Y)$. This means that the obstruction to homotoping two lifts of a map $X \to P_k$ to P_{k+1} lies in $H^k(X; \pi_k(Z, Y))$, which is again finite. Thus, there are at most

$$\prod_{k=1}^{\dim X} \left| H^k\big(X; \pi_k(Z, Y)\big) \right|$$

homotopy classes of maps $X \to Y$ going to any homotopy class of maps $Y \to Z$.

For the dual proposition, we can use the dual argument to show that the size of the preimage is bounded by

$$\prod_{k=1}^{\dim X} \left| H^k\big(X', X; \pi_k(Y)\big) \right|,$$

which is also finite. □

Acknowledgments. The first author would like to thank the Mathematical Sciences Research Institute (MSRI) for its hospitality during September–October 2017, and both authors would like to thank the Israel Institute for Advanced Studies at Hebrew University for its hospitality during (the rest of) fall 2017. We would like to thank two anonymous referees for a number of useful comments drawing our attention to errors and disfluencies in the presentation.

References

[1] M. AMANN, *Degrees of self-maps of simply connected manifolds*, Int. Math. Res. Not. IMRN **2015**, no. 18, 8545–8589. MR 3417685. DOI 10.1093/imrn/rnu201. *(1966)*

[2] M. ARKOWITZ and G. LUPTON, *Rational obstruction theory and rational homotopy sets*, Math. Z. **235** (2000), no. 3, 525–539. MR 1800210. DOI 10.1007/s002090000144. *(1966)*

[3] R. Body, M. Mimura, H. Shiga, and D. Sullivan, *p-universal spaces and rational homotopy types*, Comment. Math. Helv. **73** (1998), no. 3, 427–442. MR 1633367. DOI 10.1007/s000140050063. *(1946, 1966, 1967)*

[4] A. K. Bousfield and V. K. Gugenheim, *On PL De Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179. MR 0425956. DOI 10.1090/memo/0179. *()*

[5] R. Douglas, *Positive weight homotopy types*, Illinois J. Math. **27** (1983), no. 4, 597–606. MR 0720095. *(1966)*

[6] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Math. **205**, Springer, New York, 2001. MR 1802847. DOI 10.1007/978-1-4613-0105-9. *(1947)*

[7] P. A. Griffiths and J. W. Morgan, *Rational Homotopy Theory and Differential Forms*, Progr. Math. **16**, Birkhäuser, Boston, 1981. MR 0641551. *(1946, 1947, 1948, 1949, 1950)*

[8] M. Gromov, *Homotopical effects of dilatation*, J. Differential Geometry **13** (1978), no. 3, 303–310. MR 0551562. *(1944, 1945, 1960)*

[9] ———, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progr. Math. **152**, Birkhäuser, Boston, 1999. MR 1699320. *(1945)*

[10] P. Hilton, G. Mislin, and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Math. Stud. **15**, Elsevier, New York, 1975. MR 0478146. *(1943)*

[11] J. T. Maher, *The geometry of dilatation and distortion*, Ph.D. dissertation, University of Chicago, 2004. MR 2717156. *(1944)*

[12] F. Manin, *Plato's cave and differential forms*, Geom. Topol. **23** (2019), no. 6, 3141–3202. MR 4039187. DOI 10.2140/gt.2019.23.3141. *(1944, 1960)*

[13] M. Mimura and H. Toda, *On p-equivalences and p-universal spaces*, Comment. Math. Helv. **46** (1971), no. 1, 87–97. MR 0285007. DOI 10.1007/BF02566829. *(1946, 1966)*

[14] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR 0258031. DOI 10.2307/1970725. *(1943)*

[15] H. Shiga, *Rational homotopy type and self maps*, J. Math. Soc. Japan **31** (1979), no. 3, 427–434. MR 0535089. DOI 10.2969/jmsj/03130427. *(1966)*

[16] D. Sullivan, *Infinitesimal computations in topology*, Publ. Math. Inst. Hautes Études Sci. **47** (1977), no. 1, 269–331. MR 0646078. *(1943, 1951, 1952)*

Manin

Department of Mathematics, University of California, Santa Barbara, Santa Barbara, California, USA; manin@math.ucsb.edu

Weinberger

Department of Mathematics, University of Chicago, Chicago, Illinois, USA; shmuel@math.uchicago.edu