Around the orbit equivalence theory of the free groups, cost and ℓ^2 Betti numbers[*]

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Abstract

Abstract: The goal of this series of lectures is to present an overview of the theory of orbit equivalence, with a particular focus on the probability measure preserving actions of the free groups.

I will start by giving the basis of the theory of orbit equivalence and explain the theory of cost. In particular, prove such statements as the induction formula and the computation of the cost of free actions of some countable groups, including free groups. This will be related to the fundamental group of equivalence relations. I intend to present Abert-Nikolov theorem relating the cost of profinite actions to the rank gradient of the associated chain of subgroups. I will consider a recent result of F. Le Maître establishing a perfect connection between the cost of a probability measure preserving action with the number of topological generators of the associated full group. I shall also discuss the number of non orbit equivalent actions of countable groups.

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[*] Preparatory and personal notes for various masterclasses on Ergodic Theory, Orbit Equivalence, + later notes added. Trying to keep track of new developments in cost theory. Comments are welcome.

1 Standard Equivalence Relations

1.1 Standard Equivalence Relations

Let (X, μ) be a standard Borel space where μ is an atomless probability measure. Such a space is measurably isomorphic with the interval $[0, 1]$ equipped with the Lebesgue measure.

Let Γ be a countable group and α an action of Γ on (X, μ) by probability measure preserving (**p.m.p.**) Borel automorphisms.

In this measured context, null sets are neglected. Equality for instance is always understood almost everywhere.

The action α is ***(essentially) free*** if for μ-a.e. $x \in X$ one has $\gamma.x = x \implies \gamma = id$

The action is ***ergodic*** if the dynamics is indecomposable, i.e., whenever X admits a partition $X = A \sqcup \complement A$ into invariant Borel subsets, then one of them is trivial, i.e. $\mu(A)\mu(\complement A) = 0$.

1.1 Example

1. **Rotations**. *$\mathbb{Z}^n$ acts on the unit circle $\mathbb{S}^1$ (with normalized Lebesgue measure) by rationally independent rotations.*
2. **Linear actions on the tori**. *The standard action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$ on the n-torus $\mathbb{R}^n/\mathbb{Z}^n$ with the Lebesgue measure. The behavior is drastically different for $n \geq 3$ and for $n = 2$.*
   – The higher dimensional case was central in the super-rigidity results of Zimmer [Zim84] and Furman [Fur99a, Fur99b].
   – The 2-dimensional case $\mathrm{SL}(2, \mathbb{Z}) \curvearrowright \mathbb{T}^2$ played a particularly important role in the recent developments of the theory, mainly because of its relation with the semi-direct product $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$, in which $\mathbb{Z}^2$ has the so called **relative property (T)***, while $\mathrm{SL}(2, \mathbb{Z})$ is virtually a free group.*
3. **Actions on manifolds**. *Volume-preserving group actions on finite volume manifolds.*
4. **Lattices**. *Two lattices Γ, Λ in a Lie group G (or more generally in a locally compact second countable group). The actions by left multiplication (resp. right multiplication by the inverse) on G induce actions on the finite measure standard spaces $\Gamma \curvearrowright G/\Lambda$ and $\Lambda \curvearrowright \Gamma\backslash G$ preserving the measure induced by the Haar measure.*
5. **Compact actions**. *A compact group K, its Haar measure μ and the action of a countable subgroup Γ by left multiplication on K.*
6. **Bernoulli shift actions**. *Let (X_0, μ_0) be a standard probability measure space, possibly with atoms[1]. The* **standard Bernoulli shift action** *of Γ is the action on the space X_0^Γ of sequences $(x_\gamma)_{\gamma\in\Gamma}$ by shifting the indices $g.(x_\gamma)_{\gamma\in\Gamma} = (x_{g^{-1}\gamma})_{\gamma\in\Gamma}$, together with the Γ-invariant product probability measure $\otimes_\Gamma \mu_0$. In particular, every countable group admits at least one p.m.p. action. The action is free and ergodic iff Γ is infinite.*

   More generally, consider some action $\Gamma \curvearrowright \mathtt{V}$ of Γ on some countable set $\mathtt{V}$. The **generalized Bernoulli shift action** *of Γ is the action on the space $X^{\mathtt{V}}$ of sequences $(x_v)_{v\in\mathtt{V}}$ by shifting the indices $g.(x_v)_{v\in\mathtt{V}} = (x_{g^{-1}.v})_{v\in\mathtt{V}}$, with the invariant product probability measure.*
7. **Profinite actions**. **Profinite actions**. *Consider an action $\Gamma \curvearrowright (\mathtt{T}, v_0)$ of Γ on a locally finite rooted tree. The action preserves the equiprobability on the levels. The induced limit probability measure on the set of ends of the tree is Γ-invariant.*

   If Γ is residually finite and Γ_i is a decreasing chain of finite index subgroups with trivial intersection, consider the action of Γ by left multiplication on the profinite completion $\varprojlim \Gamma/\Gamma_i$. A rooted tree $(\mathtt{T}, (v_0 = \Gamma/\Gamma_0))$ is naturally built with vertex set (of level i) the cosets Γ/Γ_i and edges given by the reduction maps $\Gamma/\Gamma_{i+1} \to \Gamma/\Gamma_i$.

   The action is ergodic iff it is transitive on the levels.

A p.m.p. action α of a countable group Γ on a probability space (X, μ) produces the **orbit equivalence relation** :

$$\mathscr{R}_\alpha = \{(x, \gamma.x) : x \in X,\ \gamma \in \Gamma\} \tag{1}$$

This is an instance of a **p.m.p. countable standard equivalence relation**. As a subset of $X \times X$, the orbit equivalence relation $\mathscr{R} = \mathscr{R}_\alpha$ is just the union of the graphs of the $\gamma \in \Gamma$. It enjoys the following:

1.2 Proposition (Properties of the equivalence relation)

1. *The equivalence classes (or* **orbits***) of $\mathscr{R}$ are countable;*

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[1] for instance $X_0 = \{0, 1\}$ and $\mu_0(\{0\}) = 1 - p$, $\mu_0(\{1\}) = p$ for some $p \in (0, 1)$. The only degenerate situation one wishes to avoid is X_0 consisting of one single atom.

2. *$\mathscr{R}$ is a Borel subset of $X \times X$;*
3. *The measure is* **invariant** *under $\mathscr{R}$: every* **partial isomorphism**[2] *whose graph is contained*[3] *in $\mathscr{R}$ preserves the measure μ.*

1.3 Definition (p.m.p. countable standard equivalence relation)
An equivalence relation $\mathscr{R}$ on (X, μ) satisfying the above three properties of Proposition 1.2 is called a **measure preserving countable standard equivalence relation** *or shortly a* **p.m.p. equivalence relation**.

Comments on the need for such an axiomatization:
– Restrictions (see Subsection 1.5);
– Measured foliations.

1.4 Exercise
Prove item 3 of Proposition 1.2: Every partial isomorphism whose graph is contained in $\mathscr{R}$ preserves the measure μ.
[***hint*** *: For any partial isomorphism $\varphi : A \to B$, consider a partition of the domain A into pieces A_γ where $\gamma \in \Gamma$ coincide with φ.*]

1.5 Exercise
a) Two commuting actions of Γ on $\mathscr{R}_\alpha$: σ_l and σ_r on the first (resp. second) coordinate.
b) The identification $X \times \Gamma \simeq \mathscr{R}_\alpha$ via $(x, \gamma) \mapsto (x, \alpha(\gamma^{-1})(x))$ is equivariant for the diagonal Γ-action (α,left multiplication) and σ_l the action on the first coordinate.

1.6 Theorem (Feldman-Moore [FM77])
Any measure preserving countable standard equivalence relation $\mathscr{R}$ is the orbit equivalence relation $\mathscr{R}_\alpha$ for some action α of some countable group G.

The question of finding a freely acting G in Th. 1.6 remained open until A. Furman's work [Fur99b] exhibiting a lot of examples where this is impossible.

1.2 Orbit Equivalence

1.7 Definition (Orbit equivalence)
Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be p.m.p. equivalence relations on (X_i, μ_i) for $i = 1, 2$. We say that $\mathscr{R}_1$ is **orbit equivalent (OE)** *to $\mathscr{R}_2$ and we write*

$$\mathscr{R}_1 \overset{\mathrm{OE}}{\sim} \mathscr{R}_2 \tag{2}$$

if there exists a Borel bijection $f : X_1 \to X_2$ such that $f_*(\mu_1) = \mu_2$ and $\mathscr{R}_2(f(x)) = f(\mathscr{R}_1(x))$ for (almost) every $x \in X$.

Basic Questions
- Different groups giving OE actions? $\rightsquigarrow$ Examples
- One group giving many non-OE actions? $\rightsquigarrow$ Section 8.

Examples of different groups with OE actions are given by the following exercises.

1.8 Exercise (Odometer)
Show that the natural action α of the countable group $\Gamma = \oplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ (=restricted product) on $\{0, 1\}^{\mathbb{N}}$ (where the i-th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by flipping the i-th coordinate[4]*) has (almost) the same orbits as the* **odometer $\mathbb{Z}$-action** *(like the milometer of a car : "the" generator of $\mathbb{Z}$ acts by adding 1 to the first term of the sequence with carried digit e.g. $(1, 1, 1, 0, 0, 0, 1, \ldots) + 1 = (0, 0, 0, 1, 0, 0, 1, \ldots)$).*
[***hint*** *: Show that on a conull set two sequences are in the same class iff they coincide outside a finite window.*]

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[2] A ***partial isomorphism*** $\varphi : A \to B$ is a Borel isomorphism between two Borel subsets: $A = \mathrm{dom}(\varphi)$, called the ***domain*** of φ and $B = \mathrm{im}(\varphi)$ called its ***image*** of X.

[3] I.e., $\forall x \in \mathrm{dom}(\varphi),\ (x, \varphi(x)) \in \mathscr{R}$.

[4] This is the natural action by multiplication of the dense subgroup Γ of the compact group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ (unrestricted product). This is also the natural action of the countable discrete abelian group Γ on its Pontryagin dual $\hat{\Gamma}$.

1.9 Exercise
1) Show that if $\Gamma_j \overset{\alpha_j}{\curvearrowright} X_j$ is orbit equivalent with $\Lambda_j \overset{\beta_j}{\curvearrowright} X_j$, for $j = 1, \ldots n$, then the product actions (where Γ_j acts trivially on the k-th coordinate when $j \neq k$) are orbit equivalent

$$\left(\prod_{j=1}^{n} \Gamma_j \curvearrowright \prod_{j=1}^{n} X_j\right) \overset{\mathrm{OE}}{\sim} \left(\prod_{j=1}^{n} \Lambda_j \curvearrowright \prod_{j=1}^{n} X_j\right).$$

*[**hint** : Two points of the product are in the same (product-)orbit iff their are in the same orbit coordinate-wise.]*
2) Show that the odometer $\mathbb{Z}$-action is orbit equivalent with a free $\mathbb{Z}^n$-action.
*[**hint** : The $\mathbb{Z}$-action is OE with the $\Gamma = \oplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$-action on $\hat{\Gamma}$. Observe that $\oplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = \prod_{j=1}^{n} \oplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = \oplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.]*

1.10 Theorem (Dye [Dye59, Dye63])
Any two[5] ergodic probability measure preserving (p.m.p.) actions of $\mathbb{Z}$ are orbit equivalent.

1.11 Theorem (Ornstein-Weiss [OW80])
Any p.m.p. action of an amenable group is hyperfinite. Any free p.m.p. action of an infinite amenable group Γ_1 is orbit equivalent with some free p.m.p. action of any other infinite amenable group Γ_2. In particular (by Theorem 1.10) any two ergodic p.m.p. actions of any two infinite amenable groups are orbit equivalent.

A p.m.p. equivalence relation $\mathscr{R}$ is said to be ***hyperfinite*** if there exists an increasing sequence $(\mathscr{R}_n)_n$ of ***finite*** (i.e., for every x, the orbit $\mathscr{R}_n(x)$ is finite) standard subrelations that exhausts $\mathscr{R}$ (i.e. for every x, $\mathscr{R}(x) = \cup_n \nearrow \mathscr{R}_n(x)$). The point here, beside finiteness, is the standardness of the $\mathscr{R}_n$, i.e. that they appear as Borel subsets of $X \times X$. The real content of Dye's theorem is that any two p.m.p. ergodic hyperfinite relations are mutually OE, a fact which reflects the uniqueness of the hyperfinite II_1 factor.

1.12 Proposition
A p.m.p. equivalence relation on (X, μ) is hyperfinite if and only if it is OE to a $\mathbb{Z}$-action.

1.13 Corollary
Free p.m.p. actions of $\mathbf{F}_n$ ($n \geq 2$) are OE to free actions of uncountably many groups (free products of amenable ones).

1.3 Some Exercises

1.14 Exercise
The standard p.m.p. equivalence relation $\mathscr{R}$ admits a **fundamental domain**[6] *iff almost each class is finite.*
*[**hint** : Write $\mathscr{R} = \mathscr{R}_G$ for a countable group $G = \{g_i\}_{i \in \mathbb{N}}$ (by Feldman-Moore). Identifying X with $[0, 1]$, the map $J : x \mapsto \inf_i \{g_i(x)\}$ is Borel. Then $J(x) \in \mathscr{R}[x]$ whenever the classes are finite. Define D as $\{x : J(x) = x\}$.*
Conversely, let $D_\infty \subset D$ be the (Borel !) part of a fundamental domain D, corresponding to infinite classes. For $x \in D_\infty$, define $i_n(x) := \min\{i : \mathrm{card}\{g_1(x), g_2(x), \ldots, g_i(x)\} = n\}$ and $\psi_n(x) = g_{i_n(x)}(x)$. The Borel sets $\psi_n(D_\infty)$ are pairwise disjoint and all have the same measure (in a finite measure space !!).]

1.15 Exercise (Complete sections)
The following are equivalent:
(i) The classes of the standard p.m.p. equivalence relation $\mathscr{R}$ are almost all infinite
*(ii) there is a decreasing family $(E_n)_{n \in \mathbb{N}}$ of **complete sections**[7] with measures tending to 0*
(iii) $\forall \varepsilon > 0$, there is a complete section E with measure $\mu(E) \leq \varepsilon$
(iv) for every Borel subset F with $\mu(F) > 0$ the classes of the **restricted equivalence relation** *$\mathscr{R}\!\upharpoonright\! F$ are almost all infinite.*

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[5] Recall that the space (X, μ) is assumed to be atomless and that all such measured standard Borel space are isomorphic.

[6] A ***fundamental domain*** is a Borel subset D of X that meets almost (i.e. up to a union of classes of measure 0) each class exactly once.

[7] A ***complete section*** is a Borel subset meeting (almost every) equivalence class.

[***hint*** *: We take the same notations as in exercise 1.14.* $\mathscr{R}=\mathscr{R}_G$ *for a countable group* $G=\{g_i\}_{i\in\mathbb{N}}$, $X\simeq[0,1]$ *and the map* $J: x\mapsto \inf_i\{g_i(x)\}$ *is Borel.*

– Let $E_n:=\{x: 0\leq x-J(x)<2^{-n}\}$*. These subsets form a decreasing family of complete sections. Observe that* $\cap_n E_n=\{x: x=J(x)\}$ *so that the classes meeting the set* $D:=\cap_n E_n$ *meet it exactly once, i.e.* D *is a fundamental domain of its saturation. One deduce (exercise 1.14) that* $\mu(D)=0$ *when the* $\mathscr{R}$*-classes are infinite, so that* $\mu(E_n)\to 0$*.*

– If $(E_n)_n$ *is a decreasing sequence of complete sections whose intersection is a null-set, then for almost every* $x\in X$*, the intersection* $\mathscr{R}[x]\cap E_n$ *is a non-stationary decreasing sequence so that almost surely* $\#\mathscr{R}[x]=\infty$*.*

– Up to restricting to finite orbits in Y*, a fundamental domain for* $\mathscr{R}\restriction Y$ *would also be a fundamental domain for its saturation* $\mathscr{R}.Y$*.*]

1.16 Exercise

Assume $\mathscr{R}$ *is ergodic.*

1) Let $A,B\subset X$ *be Borel subsets s.t.* $\mu(A)=\mu(B)>0$*. There exists a partial isomorphism* $\varphi\in[[\mathscr{R}]]$ *in the* **full groupoid**[8] *with domain* A *and target* B*.*

2) Show that every partial isomorphism $\varphi: A\to B\in[[\mathscr{R}]]$ *extends to an element* $\psi\in[\mathscr{R}]$ *of the* **full group**[9]*, i.e. the restriction* $\psi\restriction A=\varphi$*.*

[***hint*** *:* $\mu(\complement A)=\mu(\complement B)$*.*]

3) Show that the full group $[\mathscr{R}]$ *is uncountable.*

[[***Solution*** *: An example of solution: Consider two disjoint non-null Borel subsets* A *and* B *and* $\varphi: A\to B\in[[\mathscr{R}]]$*. Identify* A *with an interval* $A\overset{h}{\simeq}[0,\mu(A)]$ *and set* $A_t=h^{-1}([0,t])$*. Now extend the restriction* $\varphi_t\restriction A_t$ *to an element* $\psi_t\in[\mathscr{R}]$ *by defining* ψ_t *to be* φ_t^{-1} *on the image* $B_t:=\varphi(A_t)$ *(so that* ψ_t *will be an involution) and the identity outside* $A_t\cup B_t$*. We thus get a one-parameter family of elements of* $[\mathscr{R}]$*. They are pairwise distinct since their* **support**[10] *has measure* $2.\mu(A_t)=2t$*.*

Another example: Choose a countable partition $X=\sqcup_{i\in\mathbb{N}}A_i$*, and subdivide each* A_i *into two parts of same measure* $A_i=A_i^+\sqcup A_i^-$*. For each* i*, choose a* $\varphi_i\in[[\mathscr{R}]]$ *such that* $\varphi_i(A_i^+)=A_i^-$*, such that* φ_i *is defined to be the inverse on* A_i^-*:* $\varphi_i\restriction A_i^-=(\varphi\restriction A_i^+)^{-1}$ *and* $\mathrm{dom}(\varphi_i)=A_i$*. For each sequence* $u=(u_i)\in\{0,1\}^{\mathbb{N}}$*, define the isomorphism* ψ_u *by its restrictions on the* A_i*'s, to be* φ_i *whenever* $u_i=1$ *and to be the identity whenever* $u_i=0$*.*]]

1.17 Exercise

(not so easy!) Prove Proposition 1.12.

1.18 Exercise

(not so easy!) Prove that an increasing union of p.m.p. hyperfinite equivalence relation is itself hyperfinite.

[[***Solution*** *: Let* $\mathscr{R}=\cup_n\nearrow\mathscr{R}_n$ *and* $\mathscr{R}_n=\cup_p\nearrow\mathscr{R}_n^p$ *be increasing union of equivalence relations where the* $\mathscr{R}_n^p$ *are finite. Each* $\mathscr{R}_i$ *is up to a* μ*-null set the orbit equivalence relation of some transformation* $T_i: X\to X$ *(Prop. 1.12). Let* $(\epsilon_n)_n$ *be a decreasing sequence of positive numbers tending to* 0 *(for instance* $\epsilon_n=\frac{1}{n}$*). For each* n*, there is a (smallest) integer* p_n *such that the approximation* $\mathscr{R}_n^{p_n}$ *to* $\mathscr{R}_n$ *satisfies, for each* $i=1,2,\cdots,n$*:* $\mu\{x\in X|(x,T_i(x))\notin\mathscr{R}_n^{p_n}\}\leq\frac{\epsilon_n}{2^n}$*. Set* $\mathscr{S}_k:=\bigcap_{n=k}^{\infty}\mathscr{R}_n^{p_n}$*. Observe that (1)* $\mathscr{S}_k$ *is finite (it is contained in* $\mathscr{R}_k^{p_k}$*); (2) The sequence* $(\mathscr{S}_k)_k$ *is increasing.*

We now show that: $\bigcup_{k=1}^{\infty}\nearrow\mathscr{S}_k=\mathscr{R}$ *up to a* μ*-null set. For each* k *and each* $i=1,2,\cdots,k$*, we have* $\{x\in X|(x,T_i(x))\notin\mathscr{S}_k\}=\bigcup_{n=k}^{\infty}\{x\in X|(x,T_i(x))\notin\mathscr{R}_n^{p_n}\}$*. Thus*

$$\mu\left(\{x\in X|(x,T_i(x))\notin\mathscr{S}_k\}\right)\leq\sum_{n=k}^{\infty}\mu\left(\{x\in X|(x,T_i(x))\notin\mathscr{R}_n^{p_n}\}\right)\leq\sum_{n=k}^{\infty}\frac{\epsilon_n}{2^n}\leq\epsilon_k.$$

It follows that $\mu\left(\{x\in X|(x,T_i(x))\notin\cup_{k=1}^{\infty}\mathscr{S}_k\}\right)=0$*. And thus* $\cup_{k=1}^{\infty}\mathscr{S}_k$ *contains all the* $\mathscr{R}_i$ *up to a* μ*-null set, i.e.* μ*-a.s.* $\cup_{k=1}^{\infty}\mathscr{S}_k=\mathscr{R}$*.* ■

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[8] The ***full groupoid*** of $\mathscr{R}$, denoted $[[\mathscr{R}]]$, is the set of all partial isomorphisms φ whose graph is contained in $\mathscr{R}$, i.e. $(x,\varphi(x))\in\mathscr{R}$ for all $x\in\mathrm{dom}(\varphi)$.

[9] The ***full group*** of $\mathscr{R}$, denoted $[\mathscr{R}]$, is the subgroup of $\mathrm{Aut}(X,\mu)$ consisting of all (global) isomorphisms $\psi: X\to X$ whose graph is contained in $\mathscr{R}$ (considered up to equality almost everywhere) (see Section 4 and particularly Definition 4.2).

[10] The ***support*** of an isomorphism of (X,μ) is the complement of its fixed-points set.

1.4 Some Orbit Equivalence Invariants

1. Amenability.
2. Kazhdan Property (T).
3. Cost.
4. ℓ^2 Betti numbers.
5. Euler Characteristic $\chi(\Gamma)$.
6. Haagerup[11] Property (a-T-amenability).

1.5 Restrictions, SOE and Fundamental Group

Let $Y \subset X$ be a non-null Borel subset and let $\mathscr{R}$ be a p.m.p. equivalence relation on X. We denote by $\mathscr{R}\restriction Y$ the ***restriction*** of $\mathscr{R}$ to Y, i.e. $\mathscr{R}\restriction Y := \mathscr{R} \cap Y \times Y$. We consider the ***normalized measure*** $\mu_Y := \frac{\mu\restriction Y}{\mu(Y)}$ on Y obtained by dividing out the restriction $\mu\restriction Y$ of the measure to Y by $\mu(Y)$. Then $\mathscr{R}_Y$ is a p.m.p. countable standard measure preserving equivalence relation on (Y, μ_Y).

1.19 Definition (Stable Orbit equivalence)
Let $\mathscr{R}_1$ *and* $\mathscr{R}_2$ *be p.m.p. equivalence relations on* (X_i, μ_i) *for* $i = 1, 2$. *We say that* $\mathscr{R}_1$ *is* **stably orbit equivalent (SOE)** *to* $\mathscr{R}_2$ *and we write*

$$\mathscr{R}_1 \overset{\text{SOE}}{\sim} \mathscr{R}_2 \tag{3}$$

if there exists **complete sections**[12] $Y_i \subset X_i$, $i = 1, 2$, *and a Borel bijection* $Y_1 \xrightarrow{f} Y_2$ *which preserves the restricted relations*[13] *and scales the measure*[14], *i.e* $f_*(\mu_1 \restriction Y_1) = \lambda(\mathscr{R}_1, \mathscr{R}_2)\, \mu_2 \restriction Y_2$. *The scalar* $\lambda(\mathscr{R}_1, \mathscr{R}_2) = \frac{\mu_1(Y_1)}{\mu_2(Y_2)}$ *is called the* **compression constant**.

1.20 Exercise
1) Let $\mathscr{R}$ *be a p.m.p. equivalence relation on* (X, μ) *and let* A_1, A_2 *be two complete sections. Show that the restrictions* $\mathscr{R}\restriction A_i$ *are SOE.*
[***hint*** *: Consider elements of the full group* $[\mathscr{R}]$ *sending part of* A_1 *in* A_2*.*]
2) Show that if $\mathscr{R}_1$ *and* $\mathscr{R}_2$ *are p.m.p. equivalence relations on* (X_i, μ_i) *for* $i = 1, 2$ *which are SOE, then there is a p.m.p. equivalence relation* $\mathscr{R}$ *on some* (X, μ) *and two complete sections* $A_1, A_2 \subset X$ *such that* $\mathscr{R}_i \simeq \mathscr{R}\restriction A_i$.
[***hint*** *: Consider the quotient space* $X = (X_1 \sqcup X_2)/(Y_1 \overset{f}{\sim} Y_2)$ *where* Y_1, Y_2 *are identified via* f, *equipped with the natural normalized measure and check that the equivalence relation* $\mathscr{R}$ *generated by the* $\mathscr{R}_i$ *on* X_i *is suitable.*]

Let $\mathscr{R}$ be a p.m.p. equivalence relation on (X, μ). The ***fundamental group*** of $\mathscr{R}$ denoted by $\mathcal{F}(\mathscr{R})$ is the multiplicative subgroup of $\mathbb{R}^+$

$$\mathcal{F}(\mathscr{R}) := \left\{ \frac{\mu(A)}{\mu(B)} : \mathscr{R}\restriction A \text{ is OE to } \mathscr{R}\restriction B \right\} \tag{4}$$

1.21 Exercise
Show that $\mathcal{F}(\mathscr{R})$ *is indeed a group.*
[***hint*** *: Clearly stable under taking inverses.*]

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[11] Uffe Haagerup was a Danish Mathematician. He sadly passed away in July 2015.
[12] For the definition, see footnote 7.
[13] i.e. $f(\mathscr{R}_1\restriction Y_1) = \mathscr{R}_2\restriction Y_2$.
[14] In other words, f preserves the normalized measures

2 Graphings, Cost

The ***cost*** of a p.m.p. equivalence relation $\mathscr{R}$ has been introduced by Levitt [Lev95]. It has been studied intensively in [Gab98, Gab00]. See also [KM04, Kec10, Fur09] and the popularization paper [Gab10]. When an equivalence relation is generated by a group action, the relations between the generators of the group introduce redundancy in the generation, and one can decrease this redundancy by using instead ***partial isomorphisms***.

2.1 Definitions

2.1 Definition (Graphing)
A **graphing** *is a countable family* $\Phi = (\varphi_i)_{i\in I}$ *of p.m.p.* **partial isomorphisms**[15] *of* (X,μ)

$$\varphi_i : \mathrm{dom}(\varphi_i) \to \mathrm{im}(\varphi_i)$$

A graphing ***generates*** *a p.m.p. standard equivalence relation* $\mathscr{R}_\Phi$*:*
the smallest equivalence relation such that $x \in \mathrm{dom}(\varphi_i) \Rightarrow x \sim \varphi_i(x)$*.*

2.2 Exercise
a) Show that $x\mathscr{R}_\Phi y \Leftrightarrow$ *there is a word* $w = \varphi_{i_n}^{\varepsilon_n}\cdots\varphi_{i_2}^{\varepsilon_2}\varphi_{i_1}^{\varepsilon_1}$ $(\varepsilon_j = \pm 1)$ *in* $\Phi\cup\Phi^{-1}$ *such that* $x \in \mathrm{dom}(w)$ *and* $w(x) = y$*.*
b) Show that $\mathscr{R}_\Phi$ *is p.m.p.*

2.3 Example
Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ *be a p.m.p. action of a countable group* Γ*. Then the graphing* $\Phi = (\alpha_\gamma)_{\gamma\in\Gamma}$*, generates the relation* $\mathscr{R}_\alpha$*, i.e.,* $R_\Phi = R_\alpha$*. If* S *is a generating subset of the group* Γ*, then the graphing* $\Psi = (\alpha_s)_{s\in S}$ *also generates the relation* $\mathscr{R}_\alpha$*.*

2.4 Definition
Let $\mathscr{R}$ *be an equivalence relation on* (X,μ)*. A countable graphing* Φ *on* X *is said to be a graphing of* $\mathscr{R}$ *if* $\mathscr{R} = \mathscr{R}_\Phi$*.*

2.5 Example
Let $\{1,\theta_1,\theta_2\}$ *be* $\mathbb{Q}$*-independent real numbers. Consider the* $\mathbb{Z}^2$ *action on the unit circle* $\mathbb{S}^1$ *given by the rotations* r_{θ_1} *and* r_{θ_2}*. Let* I_ϵ *be an arc of length* $\epsilon > 0$ *in* $\mathbb{S}^1$*. Consider the partial isomorphisms* $\Phi := (r_{\theta_1}, r_{\theta_2}|_{I_\epsilon})$*.*
Prove that Φ *is a graphing for* $R_{\mathbb{Z}^2}$*.*
[***hint*** *: Use the ergodicity of* r_{θ_1}*.*]

2.6 Definition (Graphing, Treeing)
Let Φ *be a graphing on* (X,μ) *that generates the p.m.p. equivalence relation* $\mathscr{R}$*. It equips each* $x \in X$ *with a pointed, directed* **graph** $\Phi[x]$*, thus explaining the terminology:*
– the vertices of $\Phi[x]$ *are the elements of* X *which are in the* $\mathscr{R}$*-class of* x*;*
– it is pointed at x*;*
– a directed edge u *to* v *whenever* $u \in \mathrm{dom}(\varphi)$ *and* $\varphi(u) = v$*. Such an edge gets moreover the* **label** φ*.*

Following S. Adams [Ada90], we say that Φ *is a* **treeing** *when almost all the* $\Phi[x]$ *are trees (i.e. the underlying unoriented graphs have no cycle).*

More globally, putting all these together, the **Cayley graph** $\mathrm{Cayl}(\mathscr{R},\Phi)$ of $\mathscr{R}$ with respect to Φ is the following oriented graph:
– the set of vertices is $\mathcal{V} = \mathscr{R}$
– the set of positively oriented edges $\mathcal{E}^+ = \{[(x,u),(x,\varphi(u))] : (x,u) \in \mathscr{R}, , \varphi \in \Phi, u \in \mathrm{dom}(\varphi)\}$. The edge $[(x,u),(x,\varphi(u))]$ is labelled φ.
For each $x \in X$, the pre-image of x in $\mathscr{R}$ under the first projection, i.e. the set $\{(x,u) : u \in \mathscr{R}[x]\}$ is equipped with the graph structure denoted by $\Phi[x]$. This is a ***measurable field of graphs***: $x \mapsto \Phi[x]$.
Observe that Φ equips each class with a graph structure when one forgets the pointing. It induces a ***distance*** d_Φ on the classes, that can be extended by the value $d_\Phi(x,y) = +\infty$ when x and y are not $\mathscr{R}_\Phi$-equivalent.

The valency of $x \in X$ in $\Phi[x]$ is the number of domains and images $\mathrm{dom}(\varphi), \mathrm{im}(\varphi)$ in which it is contained: $v_\Phi(x) = \sum_{i\in I}(\mathbf{1}_{A_i} + \mathbf{1}_{B_i})(x)$.

The ***cost*** of Φ is the number of generators weighted by the measure of their support:

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[15] i.e. ***partially defined isomorphisms*** $\varphi : \mathrm{dom}(\varphi) \overset{\sim}{\to} \mathrm{im}(\varphi)$ between Borel subsets of X.

2.7 Definition (Cost of a Graphing)

$$\mathscr{C}(\Phi)=\sum_{i\in I}\mu(\mathrm{dom}(\varphi_i))=\sum_{i\in I}\mu(\mathrm{im}(\varphi_i))=\frac{1}{2}\int_X v_\Phi d\mu. \tag{5}$$

The ***cost of*** $\mathscr{R}$ is the infimum over the costs of its generating graphings; it is by definition an OE-invariant:

2.8 Definition (Cost of an Equivalence Relation)

$$\mathscr{C}(\mathscr{R}) \quad = \quad \inf\{\mathscr{C}(\Phi):\mathscr{R}_\Phi=\mathscr{R}\}=\inf\{\nu(A):A\subset\mathscr{R}, A \text{ generates } \mathscr{R}\}. \tag{6}$$

Compare with the formula $n(\Gamma)=\min\{\text{card } A: A\subset\Gamma, A \text{ generates } \Gamma\}$ defining the minimal number of generators of a group.

2.9 Definition (Costs of a Group)
The **cost of a group**[16] *is the infimum of the costs over all its free p.m.p. actions.*
The **sup-cost of a group** *is the supremum of the costs over all its free p.m.p. actions.*

$$\mathscr{C}_*(\Gamma) \quad = \quad \inf\left\{\mathscr{C}(\mathscr{R}_{\Gamma\curvearrowright^{\alpha}X}):\alpha \text{ free p.m.p. action of } \Gamma\right\}. \tag{7}$$

$$\mathscr{C}^*(\Gamma) \quad = \quad \sup\left\{\mathscr{C}(\mathscr{R}_{\Gamma\curvearrowright^{\alpha}X}):\alpha \text{ free p.m.p. action of } \Gamma\right\}. \tag{8}$$

The group Γ *has* **fixed price** *if* $\mathscr{C}_*(\Gamma)=\mathscr{C}^*(\Gamma)$, *i.e. all its free p.m.p. actions have the same cost (no example of a non-fixed price group is known ; see Question 2.59).*

2.10 Exercise
Show the equivalence of the various definitions (eq. (5) and (6))
a) $\sum_{i\in I}\mu(\mathrm{dom}(\varphi_i))=\frac{1}{2}\int_X v_\Phi d\mu$.
b) $\inf\{\mathscr{C}(\Phi):\mathscr{R}_\Phi=\mathscr{R}\}=\inf\{\nu(A):A\subset\mathscr{R}, A \text{ generates } \mathscr{R}\}$.

[[***Solution*** *: For a)* $\int_X v_\Phi(x)d\mu(x)=\int_X\sum_{i\in I}(\mathbf{1}_{A_i}+\mathbf{1}_{B_i})(x)d\mu(x)=\sum_{i\in I}\int_X\mathbf{1}_{A_i}d\mu(x)+\sum_{i\in I}\int_X\mathbf{1}_{B_i}(x)d\mu(x)=\sum_{i\in I}\mu(A_i)+\sum_{i\in I}\mu(B_i)$.]]

2.11 Remark (Min and Max cost)
Both extrema (7 and 8) are indeed attained.
-For the infimum cost, $\mathscr{C}_*(\Gamma)$, *consider a diagonal product action over a sequence of actions with cost tending to the infimum [Gab00, VI.21].*
-As for the supremum cost, $\mathscr{C}^*(\Gamma)$, *it is realized by any standard Bernoulli action* $\Gamma\curvearrowright(X_0,\mu_0)^\Gamma$ *(Abért-Weiss [AW13]).*

2.12 Theorem (Factors and subgroups)
a) If $\Gamma\curvearrowright^{\beta}(Y,\nu)$ *is a factor of* $\Gamma\curvearrowright^{\alpha}(X,\mu)$ *(for free actions) then* $\mathscr{C}(\mathscr{R}_\alpha)\leq\mathscr{C}(\mathscr{R}_\beta)$.
b) If $\Lambda<\Gamma$ *is a subgroup then* Γ *admits a free action whose restriction to* Λ *realizes* $\mathscr{C}_*(\Lambda)$.

◇
*[***hint** : a) Pull-back any graphing of $\mathscr{R}_\beta$ to a graphing of $\mathscr{R}_\alpha$).
b) Use co-induction from a Λ-action realizing the cost of Λ.*]*

■ ↑

2.2 Finite Equivalence Relations

Recall that an equivalence relation is ***finite*** if for (almost) every x, the orbit $\mathscr{R}_n(x)$ is finite, and that it admits fundamental domains (and they all have the same measure).

2.13 Theorem (Levitt)
Let $\mathscr{R}$ *be a p.m.p. finite equivalence relation and* D *a fundamental domain. Then*

$$\mathscr{C}(\mathscr{R})=1-\mu(D) \tag{9}$$

Moreover, a graphing Φ *of* $\mathscr{R}$ *realizes the equality* $\mathscr{C}(\Phi)=\mathscr{C}(\mathscr{R})$ *iff* Φ *is a treeing.*

2.14 Corollary (Cost of Finite Groups)
Every free p.m.p. action of a finite group Γ *has cost* $\mathscr{C}(\mathscr{R})=1-\frac{1}{|\Gamma|}$.

---

[16] aka infimum cost, or minimal cost, see Remark 2.11.

$\diamondsuit$*Proof of Theorem 2.13.* Let Φ be a graphing of $\mathscr{R}$. Let's concentrate on the graphs $\Phi[x]$ for $x \in D$. We consider the following Borel functions $D \to \mathbb{N}$:

$$\begin{array}{rcl} x \overset{s}{\mapsto} s(x) & := & \text{number of vertices } \Phi[x] = |\mathscr{R}[x]| \\ x \overset{a}{\mapsto} a(x) & := & \text{number of edges } \Phi[x] = \dfrac{1}{2} \displaystyle\sum_{y \in \mathscr{R}[x]} v_{\Phi}(y) \end{array}$$

Like in every finite connected graph, in $\Phi[x]$ we have $s(x) - 1 \leq a(x)$, with equality iff the graph is a tree. By integrating on D, it comes:

$$1 - \mu(D) = \int_D s(x) - 1 \; d\mu(x) \leq \int_D a(x) \; d\mu(x) = \mathscr{C}(\Phi), \tag{10}$$

with equality iff almost every $\Phi[x]$ is a tree. ■ ↑

2.15 Proposition (See also Cor. 2.34 and [Lev95, Th. 2])
If the cost of the p.m.p. $\mathscr{R}$ is strictly smaller than the measure of the ambient space, (i.e. $\mathscr{C}(\mathscr{R}) < 1$) then $\mathscr{R}$ has a non-null set of finite classes.

$\diamondsuit$*Sketch of proof.* Let $\Phi = (\varphi_i : A_i \to B_i)$ is a graphing of $\mathscr{R}$ with *cost*$(\Phi) < \mu(X)$, define $f := \sum_{i \in I} \chi_{A_i} + \chi_{B_i}$ and $U_0 := \{x : f(x) = 0\}$, $U_1 := \{x : f(x) = 1\}$ and $U_+ := \{x : f(x) \geq 2\}$.

$$\begin{array}{rll} \mu(U_0)+ & \mu(U_1) + \mu(U_+) & = \mu(X) \\ & 1 \cdot \mu(U_1) + 2 \cdot \mu(U_+) & \leq 2 \cdot \mathscr{C}(\Phi) < 2.\mu(X) \end{array}$$

so that $2 \cdot \mu(U_0) + \mu(U_1) \geq 2 \cdot (\mu(X) - \mathscr{C}(\Phi)) = c > 0$.
In case $\mu(U_0) \neq 0$ we are done: the classes of points in U_0 are trivial.
Otherwise: $\mu(U_1) > c$ and we ***prune***: define $X^1 := X \setminus U_1$ and Φ^1 the graphing obtained by just removing the part of the generators that meet U_1– more precisely define φ_i^1 as the restriction of φ_i to $\mathrm{dom}(\varphi_i) \setminus ([U_1 \cap \mathrm{dom}(\varphi_i)] \cup \varphi_i^{-1}(U_1 \cap \mathrm{im}(\varphi_i)))$. We have $\mathscr{C}(\Phi^1) - \mu(X^1) = \mathscr{C}(\Phi) - \mu(X)$, and we continue inductively, by considering U_0^n, U_1^n, U_+^n, X^n and Φ^n such that Φ^n generates $\mathscr{R} \restriction X^n$ and $\mathscr{C}(\Phi^n) - \mu(X^n) = \mathscr{C}(\Phi) - \mu(X)$. At each step, one removes a part $\mu(U_1^n) \geq 2(\mu(X^n) - \mathscr{C}(\Phi^n)) = c > 0$ of the space. This cannot continue forever, so that at some stage $\mu(U_0^n) \neq 0$. And $\mathscr{R} \restriction U_0^n$ being trivial, the $\mathscr{R}$-classes of its saturation are finite. ■ ↑

2.3 Cost and Treeings

2.16 Definition (Treeing [Ada90])
A graphing Φ is said to be a **treeing** *if (almost) every $\Phi[x]$ is a tree.*

2.17 Example
a) Finite equivalence relations admit a treeing of cost $= 1 - \mu(D)$ (Th. 2.13).
b) Every hyperfinite equivalence relation admit a treeing of cost $= 1$ (Prop. 1.12).
c) Every free p.m.p. action of a free group $\mathbf{F}_n$ admits a treeing of cost $= n$. For $\{s_1, \ldots, s_n\}$ is a free generating set, $\Phi = (\alpha(s_1), \ldots, \alpha(s_n))$ is a treeing for $\mathscr{R}_\alpha$.

Recall Ornstein-Weiss' Th. 1.11:

2.18 Theorem (Ornstein-Weiss [OW80])
If Γ is an infinite amenable group, then for every p.m.p. action $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$, whose orbits are (almost all) infinite, the orbit equivalence relation $\mathscr{R}_\alpha$, is also generated by a $\mathbb{Z}$-action, and thus $\mathscr{R}_\alpha$ is treeable with a treeing of cost $\mathscr{C} = 1$.
In particular, infinite amenable groups have fixed price 1.

2.19 Remark
Consider a free p.m.p. action $\mathbf{F}_n \overset{\alpha}{\curvearrowright} (X, \mu)$, such that the free generators s_i act ergodically. By Dye and Ornstein-Weiss theorems (Th. 1.10 and 1.11), each $\alpha(\langle s_i \rangle)$ is orbit equivalent OE with an action of some (any) infinite amenable group Λ_i. Since the $\alpha(s_i)$'s act "freely and independently", the action $\mathbf{F}_n \overset{\alpha}{\curvearrowright} (X, \mu)$ is OE to a free action of the free product $\Lambda_1 * \cdots * \Lambda_n$.

At the opposite, Kazhdan property (T) are known to dislike the trees (their actions on trees have a global fixed point); they similarly dislike treeings.

2.20 Theorem ([AS90])
Infinite groups with **Kazhdan property (T)** *do not admit any treeable free action.*

2.21 Definition (Treeable, Strongly Treeable Groups)
A group is said to be

1. **treeable** *if it admits a treeable free p.m.p. action;*
2. **strongly treeable** *if all its free p.m.p. actions are treeable.*

2.22 Proposition
If Φ is a graphing of a p.m.p. equivalence relation $\mathscr{R}$ such that $\mathscr{C}(\Phi) = \mathscr{C}(\mathscr{R}) < \infty$, then Φ is a treeing.

$\diamondsuit$ If Φ is not a treeing, there exists a Φ-word $w \neq 1$ such that $U_w := \mu(\{x : w(x) = x\}) > 0$. Choose such a w of minimal length, say $w = \varphi_{i_n}^{\epsilon_n} \cdots \varphi_{i_1}^{\epsilon_1}$. By Lusin's theorem, there exists a non-null Borel subset $V \subset U_w$ whose images under the right subwords $\varphi_{i_j}^{\epsilon_j} \cdots \varphi_{i_1}^{\epsilon_1}$, $1 \leq j \leq n$, are pairwise disjoint.

We now define a sub-graphing Φ' by restricting φ_{i_1} to the complement $\mathrm{dom}(\varphi_{i_1}) \setminus V$ in case $\epsilon_1 = 1$ (resp. $\mathrm{dom}(\varphi_{i_1}) \setminus \varphi_1^{-1}(V)$ in case $\epsilon_1 = -1$). This sub-graphing still generates $\mathscr{R}$, since the "complementary" Φ'-word $\varphi_{i_n}^{\epsilon_n} \cdots \varphi_{i_2}^{\epsilon_2}$ connects $\varphi_{i_1}^{\epsilon_1}(x)$ to x for every $x \in V \subset U_w$. As a conclusion, if Φ is not a treeing, one can decrease it and continue to generate. If case it is finite, the cost decreases.
■ ↑

The above result (Prop. 2.22) states that when a (finite cost) graphing realizes the infimum in the definition (2.8) of the cost then it is a treeing. One central result in cost theory is the converse.

2.23 Theorem (Cost and treeings, Gaboriau [Gab00])
If Ψ is a treeing of a p.m.p. equivalence relation $\mathscr{R}$ then $\mathscr{C}(\Psi) = \mathscr{C}(\mathscr{R})$.

A proof of this theorem is given in section 3.

2.24 Corollary (Cost of some treeable groups, Gaboriau [Gab00])
The following groups are strongly treeable and have fixed price:

- $\mathscr{C}_*(\mathbf{F}_n) = \mathscr{C}^*(\mathbf{F}_n) = n$ *for the free group of rank n.*
- $\mathscr{C}_*(A *_C B) = \mathscr{C}^*(A *_C B) = 1 - (\frac{1}{|A|} + \frac{1}{|B|} - \frac{1}{|C|})$ *for any amalgamated free product of finite groups A, B, C.*
- *In particular,* $\mathscr{C}_*(\mathrm{SL}(2,\mathbb{Z})) = \mathscr{C}^*(\mathrm{SL}(2,\mathbb{Z})) = \frac{13}{12}$.

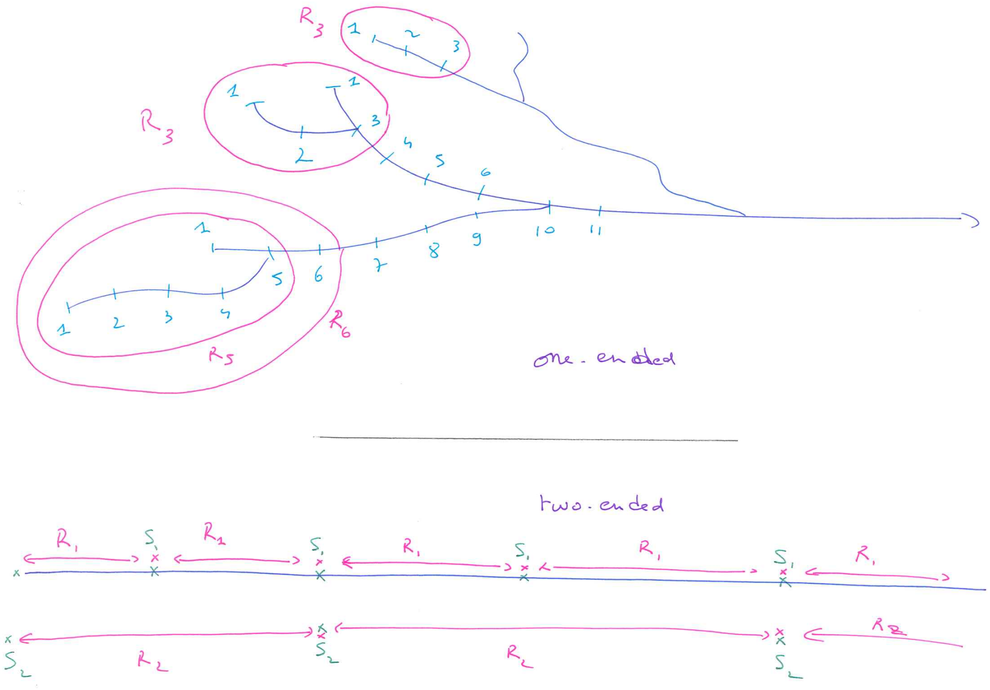
2.25 Corollary (Min-cost, treeability and anti-treeability)
If Γ admits a free p.m.p. **treeable** *action, then this action realizes the infimum $\mathscr{C}_*(\Gamma)$.*
In particular, if a non-amenable Γ admits a cost$= 1$ free p.m.p. action, then Γ is **non treeable**.

$\diamondsuit$ Consider the diagonal product of free p.m.p. actions α_n of Γ where α_0 is treeable and $\mathscr{C}(\mathscr{R}_{\alpha_n})$ tends to $\mathscr{C}_*(\Gamma)$. It is of minimal cost (by pulling-back efficient graphings for the α_n), it is treeable (by pulling-back a treeing of α_0) and both treeings (that of α_0 and the pulled-back one) have the same cost.

If Γ admits a free p.m.p. action with a cost $= 1$ treeing Ψ, then Γ is amenable:
$\mathscr{C}(\Psi) = \frac{1}{2}\int_X v_\Psi(x) d\mu(x) = 1$.
1) If $v_\Psi(x) < 2$ somewhere then prune the trees.
1)a) If it continues forever then (the trees have only one end and) $\mathscr{R}_\Psi$ is hyperfinite. Associate to each point x its stage of pruning $Pr(x)$. Now the $\mathscr{R}_n$-classes are the bushes above the points of level n (and singletons for the rest)(See Picture 1).
1)b) If this stops after finitely many steps $\rightsquigarrow$ see 2)
2) If $v_\Psi(x) \geq 2$ almost everywhere, then $v_\Psi(x) = 2$ a.e. ($\rightsquigarrow$ two-ended trees).
Choose a decreasing sequence of complete sections S_n ($\mu(S_n) \to 0$). The intersection of each tree with S_n cuts the tree into finite pieces (for otherwise, one could choose one or two points in the orbit equivalence class). The $\mathscr{R}_n$-classes are the pieces (See Picture 1).

In case, a mixed situation appears, split the relation into pieces, where the behavior is constant.
■ ↑

Figure 1: cost = 1 treeings, one or two ended



In [Gab00] the notion of ***free product decomposition*** $\mathscr{R} = \mathscr{R}_1 * \mathscr{R}_2$ (and more generally ***free product with amalgamation*** $\mathscr{R} = \mathscr{R}_1 *_{\mathscr{R}_3} \mathscr{R}_2$) of an equivalence relation over a subrelation is introduced (see also [Ghy95, Pau99]). Of course, when $\mathscr{R}$ is generated by a free action of a group Γ, any decomposition of $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$ induces the analogous decomposition of $\mathscr{R} = \mathscr{R}_{\Gamma_1} *_{\mathscr{R}_{\Gamma_3}} \mathscr{R}_{\Gamma_2}$.

2.26 Theorem (Free product with amalgamation over amenable [Gab00])
If $\mathscr{R} = \mathscr{R}_1 *_{\mathscr{R}_3} \mathscr{R}_2$ where $\mathscr{R}_3$ is hyperfinite (possibly trivial) (and where $\mathscr{R}_1, \mathscr{R}_2$ have finite cost[17]), then

$$\mathscr{C}(\mathscr{R}_1 *_{\mathscr{R}_3} \mathscr{R}_2) = \mathscr{C}(\mathscr{R}_1) + \mathscr{C}(\mathscr{R}_2) - \mathscr{C}(\mathscr{R}_3). \tag{11}$$

2.27 Corollary (Free product with amalgamation over amenable [Gab00])
If $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$ is an amalgamated free product of two countable groups (with finite cost[18]) over an amenable group Γ_3, then

$$\mathscr{C}_*(\Gamma) = \mathscr{C}_*(\Gamma_1) + \mathscr{C}_*(\Gamma_2) - \mathscr{C}_*(\Gamma_3), \qquad \text{where } \mathscr{C}_*(\Gamma_3) = 1 - \frac{1}{|\Gamma_3|}. \tag{12}$$

Moreover, if Γ_1 and Γ_2 have fixed price, then so has Γ. In particular, for free products

$$\mathscr{C}_*(\Gamma_1 * \Gamma_2) = \mathscr{C}_*(\Gamma_1) + \mathscr{C}_*(\Gamma_2) \tag{13}$$

Since cost of free actions is increasing under factors (and amenable groups have fixed price), it is easy to build a free p.m.p. of Γ that realizes both $\mathscr{C}_*(\Gamma_1)$ and $\mathscr{C}_*(\Gamma_2)$.

2.28 Remark
This Corollary 2.27 continues to hold without the finite cost assumptions: use exactly the same proof as in [Gab00, Theorem IV.15] with the cost replaced by the mean value of "degree minus 2" which is exactly twice the "cost minus the measure of the space" when the cost is finite. This

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[17] This assumption can be removed. See Remark 2.28.

[18] This assumption can be removed. See Remark 2.28.

allow the treatment of infinite costs. Another way adopted by A. Carderi in his master thesis (see [Car11]) consists in using groupoids, groupoid cost and the monotonicity of groupoid cost under factors together with a theorem in the Ph-D thesis of A. Alvarez [Alv08] on groupoids factoring onto free products.

2.29 Corollary (Surface groups [Gab00])
Surface groups[19] *have fixed price. More precisely, the fundamental group of an orientable surface of genus* g *has cost* $\mathscr{C}_*(\pi_1(S_g)) = \mathscr{C}^*(\pi_1(S_g)) = 2g - 1$.

2.30 Remark (Strong treeability)
Surface groups *are treeable, since they are lattices in* $\mathrm{SL}(2, \mathbb{R})$, *just as the free group* $\mathbf{F}_2$.
Bridson, Tweedale, Wilton proved that **elementarily free groups** *are treeable [BTW07].*
Recently, Conley-Gaboriau-Marks-Tucker–Drob [CGMTD20] proved that the **surface groups** *and* **elementarily free groups** *are* **strongly treeable**.

There is also A notion of HNN-extensions is also considered [Gab00, Définition IV.20] with a similar addition formula for the cost [Gab00, Corollaire IV.20] which translates for groups to the following:

2.31 Corollary (HNN-extensions over amenable [Gab00])
If $\Gamma = \Gamma_1 *_{\Gamma_3}$ *is an over an amenable group* Γ_3, *then*

$$\mathscr{C}_*(\Gamma) = \mathscr{C}_*(\Gamma_1) + 1 - \mathscr{C}_*(\Gamma_3), \qquad \text{where } \mathscr{C}_*(\Gamma_3) = 1 - \frac{1}{|\Gamma_3|}. \tag{14}$$

Moreover, if Γ_1 *and* Γ_2 *have fixed price, then so has* Γ.

2.4 Induction Formula

2.32 Proposition (Induction Formula [Gab00])
Let $Y \subset X$ *be a Borel subset which meets all the equivalence classes of the p.m.p. equivalence relation* $\mathscr{R}$. *The cost of* $\mathscr{R}$ *and the cost of the restriction* $\mathscr{R}\restriction Y$ *are related according to the following formula:*

$$\mathscr{C}(\mathscr{R}) - 1 = \mu(Y)\ \big(\mathscr{C}(\mathscr{R}\restriction Y) - 1\big). \tag{Induction Formula}$$

Of course, the cost of $\mathscr{R}\restriction Y$ is computed using the restricted normalized measure $\overline{\mu}_Y = \frac{\mu_{\restriction Y}}{\mu(Y)}$. The elements of the proof of the induction formula are given on page 15.

This formula smells a bit like the ***Schreier's Index formula*** [Sch27], and this is not a coincidence.
Recall: A subgroup Λ of a free group Γ is a free group (Nielsen [Nie21][20] finitely generated case, Schreier [Sch27] infinitely generated).
A finite index subgroup Λ of a free group Γ of rank $n < \infty$ has rank: $\mathrm{rk}(\Lambda) - 1 = [\Gamma : \Lambda](\mathrm{rk}(\Gamma) - 1)$

2.33 Exercise
Test the formula in the case of a profinite action associated with a chain of finite index normal subgroups (Γ_i), *when taking* Y *to be the shadow of a finite level vertex.*

2.34 Corollary (This is also Prop. 2.15)
If (almost) all the classes of the p.m.p. equivalence relation $\mathscr{R}$ *are infinite, then* $\mathscr{C}(\mathscr{R}) \geq 1$.

◇*Proof of the corollary.* Considering a sequence of complete section Y_n with measure tending to 0 (see Exercise 1.15), one gets $\mathscr{C}(\mathscr{R}) = 1 + \mu(Y_n)\ \big(\mathscr{C}(\mathscr{R}\restriction Y) - 1\big) \geq 1 - \mu(Y_n)$. ■ ↑

2.35 Corollary (SOE groups)
If Γ_1 *and* Γ_2 *admit SOE free p.m.p. actions* $\Gamma_i \curvearrowright (X_i, \mu_i)$ *then* $\mu_1(Y_1)(\mathscr{C}_*(\Gamma_1) - 1) = \mu_2(Y_2)(\mathscr{C}_*(\Gamma_2) - 1)$, *with the notations of Definition 1.19. In particular if* $\mathscr{C}_*(\Gamma_1) \neq 1, \infty$ *then the compression constant is constraint.*

The diagonal action of $\Gamma_i \curvearrowright (X_i, \mu_i)$ with a free p.m.p. actions realizing the infimum cost of Γ_i remains SOE with a free action of the other group Γ_j $(j \neq i)$ with the same compression constant. ■ ↑

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[19] ***Surface groups*** are the fundamental groups (in the algebraic topology sense of H. Poincaré) of closed surfaces.

[20] Jakob Nielsen (1890, Mjels, Als - 1959, Helsingør) was a Danish mathematician, professor of mathematics at the University of Copenhagen 1951-1955

2.36 Corollary (Fundamental Group)
If $1 < \mathscr{C}(\mathscr{R}) < \infty$, *then the* **fundamental group** $\mathcal{F}(\mathscr{R}) = \{1\}$.

$\diamondsuit$*Proof.* If $\mathscr{R}\restriction A \overset{\mathrm{OE}}{\sim} \mathscr{R}\restriction B$ then $\mathscr{C}_{\mu_A}(\mathscr{R}\restriction A) = \mathscr{C}_{\mu_B}(\mathscr{R}\restriction B)$. On the other hand, $\mathscr{C}(\mathscr{R}) - 1 = \mu(A)(\mathscr{C}_{\mu_A}(\mathscr{R}\restriction A) - 1) = \mu(B)(\mathscr{C}_{\mu_B}(\mathscr{R}\restriction B) - 1)$, so that if $\mathscr{C}(\mathscr{R}) - 1 \notin \{0, \infty\}$ then $\mu(A) = \mu(B)$. ■ ↑

2.37 Definition (Relative cost)
The **rel-cost** *of a pair* $(\mathscr{S} < \mathscr{R})$ *of a p.m.p. equivalence relation* $\mathscr{R}$ *and an equivalence sub-relation* $\mathscr{S}$ *is the infimum of the costs of the graphings* Φ *which together with* $\mathscr{S}$ *generate* $\mathscr{R}$*:*

$$\textit{rel-}\mathscr{C}(\mathscr{R};\mathscr{S}) := \inf\{\mathscr{C}(\Phi) : \mathscr{S} \vee \Phi = \mathscr{R}\}. \tag{15}$$

The **notation** $\mathscr{S} \vee \Phi$ means the equivalence relation generated by $\mathscr{S}$ and Φ, i.e. the smallest equivalence relation containing $\mathscr{S}$ and $\{(x, \varphi(x)) : \varphi \in \Phi, x \in \mathrm{dom}(\varphi)\}$.

2.38 Proposition (Relative-cost, cf. [Gab00, Lem. III.5])
If $\mathscr{S} < \mathscr{R}$ *and* $\mathscr{S}$ *is* **infinite**[21], *then*

$$\mathscr{C}(\mathscr{R}) - \mathscr{C}(\mathscr{S}) \leq \textit{rel-}\mathscr{C}(\mathscr{R};\mathscr{S}) \leq \mathscr{C}(\mathscr{R}) - 1. \tag{16}$$

In particular, if $\mathscr{C}(\mathscr{S}) = 1$, *then* $\textit{rel-}\mathscr{C}(\mathscr{R};\mathscr{S}) = \mathscr{C}(\mathscr{R}) - 1$.

$\diamondsuit$ If $\Phi_{\mathscr{S}}$ generates $\mathscr{S}$ and Φ is such that $\mathscr{S} \vee \Phi = \mathscr{R}$, then $\Phi_{\mathscr{S}} \vee \Phi$ generates $\mathscr{R}$; so that $\mathscr{C}(\mathscr{R}) - \mathscr{C}(\Phi_{\mathscr{S}}) \leq \mathscr{C}(\Phi)$. And the first inequality follows.

If Y be a complete section for $\mathscr{S}$, then $\mathscr{S} \vee \mathscr{R}\restriction Y = \mathscr{R}$, so that (measured with the ambiant measure μ)

$$\text{rel-}\mathscr{C}(\mathscr{R};\mathscr{S}) \leq \mathscr{C}_\mu(\mathscr{R}\restriction Y);$$

while by the induction formula (Proposition 2.32)

$$\mathscr{C}_\mu(\mathscr{R}\restriction Y) - \mu(Y) = \mu(Y)\big(\mathscr{C}_{\mu_Y}(\mathscr{R}\restriction Y) - 1\big) = \mathscr{C}_\mu(\mathscr{R}) - 1.$$

Since $\mu(Y)$ can be chosen arbitrarily small, the second inequality follows. ■ ↑

2.39 Corollary (Free product with amalgamation all F.P. 1)
If $\Gamma_1, \Gamma_2, \Gamma_3$ *are countable groups with fixed price* 1, *then*

$$\mathscr{C}^*(\Gamma_1 *_{\Gamma_3} \Gamma_2) = \mathscr{C}_*(\Gamma_1 *_{\Gamma_3} \Gamma_2) = 1 \tag{17}$$

Denoting $\mathscr{R}_i = \mathscr{R}_{\alpha(\Gamma_i)}$ for a free p.m.p. action α of $\Gamma = \Gamma_1 *_{\Gamma_3} \Gamma_2$, we have rel-$\mathscr{C}(\mathscr{R}_j;\mathscr{R}_3) = 0 = \inf\{\mathscr{C}(\Phi_j) : \mathscr{R}_3 \vee \Phi_j = \mathscr{R}_j\}$, $j = 1, 2$, and $\mathscr{R}_3 \vee \Phi_1 \vee \Phi_2$ generates $\mathscr{R}_{\alpha(\Gamma)}$.

2.40 Corollary (Subrelations with infinite intersection)
If $\mathscr{R}$ *is generated by a family of subrelations* $(\mathscr{R}_i)$ *such that almost every class of the intersections* $\mathscr{R}_i \cap \mathscr{R}_{i+1}$ *is infinite then* $\mathscr{C}(\mathscr{R}) \leq 1 + \sum_i(\mathscr{C}(\mathscr{R}_i) - 1)$.
In particular, if all the $\mathscr{R}_i$ *have cost* $= 1$, *then* $\mathscr{C}(\mathscr{R}) = 1$.

$\diamondsuit$ This follows directly from Proposition 2.38. First generate $\mathscr{R}_1$, then the amount of generators needed to get $\mathscr{R}_2$ from $\mathscr{R}_1 \cap \mathscr{R}_2$ is less than $\mathscr{C}(\mathscr{R}_2) - 1$. To generate $\mathscr{R}_n$ out of $(\mathscr{R}_1 \vee \mathscr{R}_2 \vee \cdots \vee \mathscr{R}_{n-1}) \cap \mathscr{R}_n$ requires an amount of cost less than $\mathscr{C}(\mathscr{R}_n) - 1$. ■ ↑

2.41 Corollary (Increasing cheap family)
If $\mathscr{R}_n$ *is an increasing sequence of infinite p.m.p. equivalence relations such that* $\mathscr{C}(\mathscr{R}_n) \to 1$, *then* $\mathscr{C}(\cup_n \mathscr{R}_n) = 1$.

Compare with Question 2.62.

$\diamondsuit$ Choose an infinite cost $= 1$ (for instance hyperfinite) p.m.p. sub-relation $\mathscr{S}$. For every $\epsilon > 0$, choose a sub-sequence $(\mathscr{R}_{n_k})$ such that $\sum_k(\mathscr{C}(\mathscr{R}_{n_k}) - 1) < \epsilon$. Then $\mathscr{R} = \cup_n \mathscr{R}_n = \cup_k \mathscr{R}_{n_k}$ and rel-$\mathscr{C}(\cup_k \mathscr{R}_{n_k};\mathscr{S}) \leq \sum_k$ rel-$\mathscr{C}(\mathscr{R}_{n_k};\mathscr{S}) \leq \sum_k(\mathscr{C}(\mathscr{R}_{n_k}) - 1) < \epsilon$. So that $\mathscr{C}(\mathscr{R}) \leq \mathscr{C}(\mathscr{S}) + \epsilon$. ■ ↑

---

[21] (Almost) all its classes are infinite.

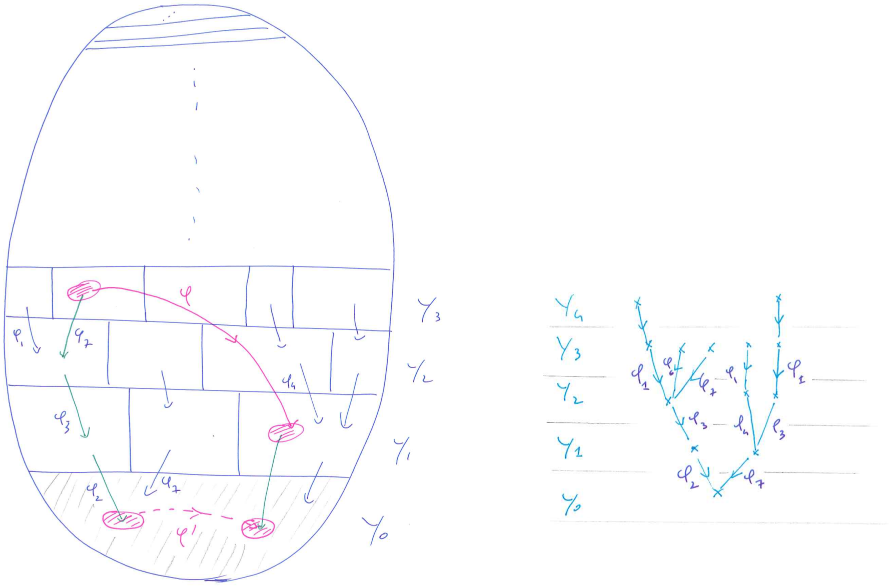
$\diamondsuit$*Proof of Proposition 2.32 (Induction formula).* Let $\Phi = (\varphi_i)_{i\in I}$ a graphing of $\mathscr{R}$. We shall produce a graphing Φ_Y of $\mathscr{R}\restriction Y$ satisfying $\mathscr{C}((\Phi) - 1 = \mu(Y)\ (\mathscr{C}(\Phi_Y) - 1)$.

Let $Y_0 := Y$ and $Y_i := \{x \in X : d_\Phi(x, Y_0) = i\}$, where d_Φ is the distance on the classes defined by the graph structure $\Phi[x]$. Since Y meets all the equivalence classes, this defines a partition $X = \sqcup_i Y_i$. Up to subdividing the generators in Φ by partitioning domains and images, and up to replacing certain generators by the inverse, one can assume that for each generator $\varphi \in \Phi$, domains and images are each entirely contained in some level and that moreover it "descends": $\mathrm{dom}(\varphi) \subset Y_i$ and $\mathrm{im}(\varphi) \subset Y_j$ with $i \geq j$. This doesn't change the cost. Choose a well-order on the family of generators Φ.

For every point $x \in Y_i$ for some $i > 0$, there is a generator $\varphi \in \Phi$ descending it to the next level: $x \in \mathrm{dom}(\varphi)$ and $\varphi(x) \in Y_{i-1}$. If φ is the smallest such generator, then we declare that $x \in Y^\varphi$. These Borel sets form a partition $X \setminus Y = \sqcup_{\varphi\in\Phi} Y^\varphi$.

Consider the sub-graphing $\Psi_v := (\varphi\restriction Y^\varphi)$ consisting in the restrictions of the generators $\varphi \in \Phi$ to the subsets Y^φ of their domain. This is a ***treeing*** with fundamental domain Y: each $x \in Y_i$, $i > 0$, has a unique representative in the next level Y_{i-1} (see Picture 2). The union of the domains is a partition $X \setminus Y = \sqcup_{\varphi\in\Phi} Y^\varphi$, so that $\mathscr{C}(\Psi_v) = 1 - \mu(Y)$.

Figure 2: Induction Formula, descending levels - vertical treeing



Let now Φ_h be the complementary graphing consisting in the restrictions of the generators $\varphi \in \Phi$ to the subsets $\mathrm{dom}(\varphi) \setminus Y^\varphi$. Its cost is $\mathscr{C}(\Phi_h) = \mathscr{C}(\Phi) - 1 + \mu(Y)$.

Let's consider now the finer partition defined according to the Φ_v-path to Y: $X = Y \cup \sqcup_w Y^w$ where the w range over the reduced Ψ_v-words, such that $x \in Y^w$ iff w is the (unique) Ψ_v-word such that $x \in \mathrm{dom}(w)$ and $w(x) \in Y$.

Up to subdividing the generators of Φ_h by partitioning domains and images, one can assume that domains and images are each entirely contained in some Y^w. This doesn't change its cost.

We now ***slide*** the generators of Φ_h along Ψ_v. For every $\varphi \in \Phi_h$ such that $\mathrm{dom}(\varphi) \subset Y^{w_1}$ and $\mathrm{im}(\varphi) \subset Y^{w_2}$ define $\varphi_Y := w_2\varphi w_1^{-1}$ and its domain and image are contained in Y (See φ' on Picture 2). We set $\Phi_Y := (\varphi_Y)_{\varphi\in\Phi_h}$ and observe that

$$\mathscr{C}(\Phi_Y) = \sum_{\varphi\in\Phi_h} \mathrm{dom}(\varphi_Y) = \sum_{\varphi\in\Phi_h} \mathrm{dom}(\varphi) = \mathscr{C}(\Phi_h) = \mathscr{C}(\Phi) - 1 + \mu(Y)$$

Claim:
– $\Phi_Y \vee \Psi_v$ *generates* $\mathscr{R}$.
– Φ_Y *generates the restriction* $\mathscr{R}\restriction Y$.

Each element of Φ_Y, Ψ_v or Φ_Y belongs to $[[\mathscr{R}]]$. Any Φ-word m defines uniquely a $\Psi_v \vee \Phi_h$-word and by sliding along Ψ_v a $\Phi_Y \vee \Psi_v$-word m'.

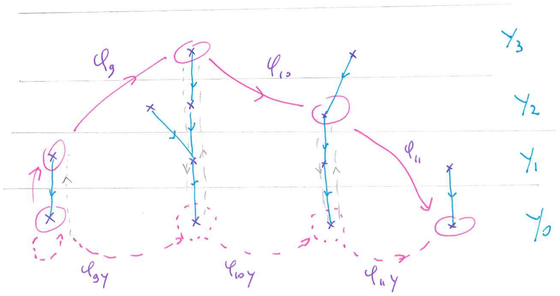
Observe that Ψ_v being a treeing with fundamental domain Y and the generators of Φ_Y having domain and image in Y, it follows that: if m connects two points $y, y' \in Y$, then writing m' as a product of sub-words alternatively taken from Φ_Y and Ψ_v, the associated reduced word $\mathrm{red}(m')$ consists in letters with domain and image in Y, i.e. consists in letters only taken from Φ_Y. It follows that Φ_Y generates $\mathscr{R}\restriction Y$ (See Picture 3). And, once the measure is normalized:

$$\mathscr{C}_{\mu_Y}(\mathscr{R}\restriction Y) \leq \frac{1}{\mu(Y)}\big(\mathscr{C}_\mu(\Phi) - 1 + \mu(Y)\big)$$
$$\mu(Y)\big(\mathscr{C}_{\mu_Y}(\mathscr{R}\restriction Y) - 1\big) \leq \mathscr{C}_\mu(\Phi) - 1$$

And since this is for every generating Φ,

$$\mu(Y)\big(\mathscr{C}_{\mu_Y}(\mathscr{R}\restriction Y) - 1\big) \leq \mathscr{C}_\mu(\mathscr{R}) - 1.$$

Figure 3: Induction Formula, sliding graphings



Conversely, if Φ_2 is a graphing of $\mathscr{R}\restriction Y$, then $\Phi_2 \vee \Psi_v$ generates $\mathscr{R}$ and taking the normalization of the measure into account, one sees that

$$\mathscr{C}(\mathscr{R}) \leq \mathscr{C}_\mu(\Phi_2 \vee \Psi_v) = \mu(Y)\mathscr{C}_{\mu_Y}(\Phi_2) + \mathscr{C}_\mu(\Psi_v)$$
$$\mathscr{C}(\mathscr{R}) \leq \mathscr{C}_\mu(\Phi_2 \vee \Psi_v) = \mu(Y)\mathscr{C}_{\mu_Y}(\Phi_2) + 1 - \mu(Y)$$

And since this is for every generating Φ_2,

$$\mathscr{C}(\mathscr{R}) - 1 \leq \mu(Y)\big(\mathscr{C}_{\mu_Y}(\mathscr{R}\restriction Y) - 1\big).$$

■ ↑

2.5 Commutations

The material of this section is essentially extracted form [Gab00].

By ***chain-commuting*** family in a group Γ, we mean a family F of elements for which the ***commutation graph*** (i.e. the graph with vertices F and an edge between two elements of F iff they commute) is connected. These groups are also known as called ***right angle groups***.

2.42 Theorem (Chain-commuting groups, [Gab00, Crit. VI.24])
If Γ is generated by a **chain-commuting** *family of infinite order elements, then $\mathscr{C}_*(\Gamma) = \mathscr{C}^*(\Gamma) = 1$.*

This is more generally true for a group Γ generated by a family of subgroups Γ_i of fixed price $= 1$ such that $\Gamma_i \cap \Gamma_{i+1}$ is infinite (Apply Corollary 2.40).

2.43 Corollary
The following groups admit chain-commuting infinite order generators and thus have fixed price $= 1$

- $\mathbf{F}_p \times \mathbf{F}_q$;
- $\mathbb{Z}^n$;

- $\mathrm{SL}(n,\mathbb{Z})$ *for* $n \geq 3$ *(special linear group);*
- $\mathrm{MCG}(\Sigma_g)$ *for* $g \geq 3$ *(mapping class group).*
- $\mathrm{Out}(\mathbf{F}_n)$, $n \geq 3$ *(Outer automorphisms of free group)*
- *Right angle Artin groups (***RAAG***) with connected associated graph.*

More generally,

2.44 Theorem
If Γ *is an increasing union of subgroups* $\Gamma = \cup_{n=0,\cdots} \nearrow \Gamma_n$ *such that* $\Gamma_{n+1} = \langle \Gamma_n, \gamma_{n+1} \rangle$ *is generated by* Γ_n *and some element* $\gamma_{n+1} \in \Gamma$*, satisfying* $|\gamma_{n+1}^{-1}\Gamma_n\gamma_{n+1} \cap \Gamma_n| = \infty$*, then for every free p.m.p. action* $\Gamma \curvearrowright^{\alpha} (X,\mu)$*, the rel-cost of* $\mathscr{R}_{\Gamma_0 \curvearrowright^{\alpha} X}$ *in* $\mathscr{R}_{\Gamma \curvearrowright^{\alpha} X}$ *is trivial:*

$$\text{rel-}\mathscr{C}(\mathscr{R}_{\alpha(\Gamma)}; \mathscr{R}_{\alpha(\Gamma_0)}) = 0.$$

In particular,

$$\mathscr{C}(\mathscr{R}_{\Gamma \curvearrowright^{\alpha} X}) \leq \mathscr{C}(\mathscr{R}_{\Gamma_0 \curvearrowright^{\alpha} X}).$$

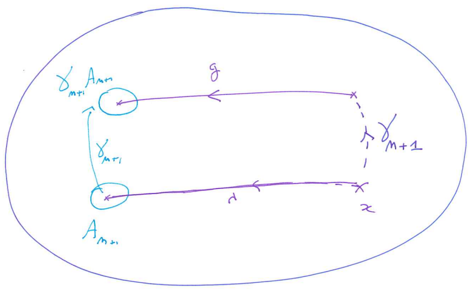
The same proof gives the same result if one replaces free actions by actions for which almost every $(\gamma_{n+1}^{-1}\Gamma_n\gamma_{n+1} \cap \Gamma_n)$-orbit is infinite. This is a direct application of [Gab00, Lemme V.3].

$\diamondsuit$*Proof of Th. 2.44.* We consider a free p.m.p. action α. The group $\Lambda_n := \gamma_{n+1}^{-1}\Gamma_n\gamma_{n+1} \cap \Gamma_n$ is infinite. For any $\epsilon_n > 0$, choose a Borel subset A_{n+1} that meets (almost) every Λ_n-orbit. Consider the partial isomorphism $\varphi_{n+1} := \alpha(\gamma_{n+1}) \upharpoonright A_{n+1}$ (defined as the restriction of $\alpha(\gamma_{n+1})$ to A_{n+1}).
Claim. The smallest equivalence relation $\mathscr{S}_n$ *generated by* $\mathscr{R}_n := \mathscr{R}_{\alpha(\Gamma_n)}$ *and* φ_{n+1} *is* $\mathscr{R}_{n+1}$ *itself.* For (almost) every $x \in X$, there is some element $\lambda \in \Lambda_n < \Gamma_n$ such that $\lambda.x \in A_n$. Since $g^{-1}\gamma_{n+1}\lambda = \gamma_{n+1}$ for some $g \in \Gamma_n$ (i.e. $\lambda = \gamma_{n+1}^{-1}g\gamma_{n+1}$), the following points are $\mathscr{S}_n$-equivalent:

$$x \overset{\mathscr{R}_n}{\sim} \lambda(x)_{\in A_n} \overset{\varphi_{n+1}}{\sim} \gamma_{n+1}\lambda(x) \overset{\mathscr{R}_n}{\sim} g^{-1}\gamma_{n+1}\lambda(x) = \gamma_{n+1}(x). \tag{18}$$

It follows that $\mathscr{R}_{\alpha(\Gamma)}$ is generated by a generating system for $\mathscr{R}_{\alpha(\Gamma_0)}$ together with $(\varphi_1, \varphi_2, \cdots)$ of cost $\sum_n \mu(A_n) = \sum_n \epsilon_n$. Considering ϵ_n of the form $\frac{1}{2^n}\epsilon$, this shows that $\mathscr{R}_{\alpha(\Gamma_0)}$ has rel-cost $= 0$ in $\mathscr{R}_{\alpha(\Gamma)}$. ■ ↑

Figure 4: commutations



2.45 Corollary (Infinite normal subgroup)
If $\Lambda \triangleleft \Gamma$ *is an infinite normal subgroup, then for every free p.m.p. action* $\Gamma \curvearrowright^{\alpha} (X,\mu)$*:*

$$\mathscr{C}(\mathscr{R}_{\Gamma \curvearrowright^{\alpha} X}) \leq \mathscr{C}(\mathscr{R}_{\Lambda \curvearrowright^{\alpha} X}).$$

■ ↑

More generally,

2.46 Corollary (Commensurated subgroups)
Assume N is a **commensurated**[22] *subgroup of an infinite countable group Γ. For every free p.m.p. action $\Gamma \curvearrowright^{\alpha} (X,\mu)$:*

$$\mathscr{C}(\mathscr{R}_{\Gamma\curvearrowright^{\alpha} X}) \leq \mathscr{C}(\mathscr{R}_{N\curvearrowright^{\alpha} X}).$$

2.47 Example
For many reasons, the Baumslag-Solitar groups $\mathrm{BS}(p,q) = \langle a, t : ta^pt^{-1} = a^q\rangle$ have fixed price 1. For instance the subgroup generated by a is commensurated. For another argument, $\mathrm{BS}(p,q)$ decomposes as an HNN-extension of $\mathbb{Z}$ over $\mathbb{Z}$ then use Corollary 2.31.

2.48 Example
$\mathrm{SL}(n,\mathbb{Z})$ is commensurated in $\Gamma = \mathrm{SL}(n, \mathbb{Z}[\frac{1}{p_1}, \frac{1}{p_2}, \cdots \frac{1}{p_r}])$ and in $\Gamma = \mathrm{SL}(n,\mathbb{Q})$. Since $\mathrm{SL}(n,\mathbb{Z})$ has fixed price 1 (Corollary 2.43) when $n \geq 3$, then the same holds for Γ.

2.49 Corollary (Direct products, commuting subgroups, infinite center)
The group Γ has fixed price $= 1$ in the following situations:

1. *If $\Gamma = \Lambda \times \Delta$ is a* **direct product** *of infinite groups such that Λ contains a fixed price $= 1$ subgroup Λ_0 (for instance an infinite amenable subgroup).*
2. *If Γ is generated by two* **commuting** *infinite subgroups Λ and Δ such that Λ contains a fixed price $= 1$ subgroup Λ_0.*
3. *If the* **center** *of Γ contains a fixed price $= 1$ subgroup (eg. an infinite order element).*

■ ↑

2.50 Proposition (Direct products, min-cost)
If $\Gamma = \Lambda \times \Delta$ is a direct product of infinite groups, then $\mathscr{C}_*(\Gamma) = 1$.

This admits of course generalizations similar to that of Corollary 2.49.

◊ Consider a product $\Lambda \times \Delta \curvearrowright^{\sigma\times\tau} (Y \times Z)$ of free p.m.p. actions $\Lambda \curvearrowright^{\sigma} Y$ and $\Delta \curvearrowright^{\tau} Z$. Choose an infinite order element t in the ***full group***[23] of $\Delta \curvearrowright^{\tau} Z$. Restrict a family (λ_i) of generators of Λ to rapidly decreasing Borel sets of the form $A_i \times Z$. Check that the graphing $\Phi = (t, \gamma_i \restriction A_i \times Z)$ generates a equivalence sub-relation $\mathscr{R}_\Phi$ which contains the Λ-orbits of the Λ-action on $Y \times Z$. Then use the usual trick to extend $\mathscr{R}_\Phi$ to $\mathscr{R}_{\Lambda\times\Delta}$ for a small cost. ■ ↑

2.51 Theorem
Let Γ be a lattice in a semi-simple connected Lie group with real rank ≥ 2. If Γ is non-cocompact or if Γ is reducible, then Γ has fixed price $= 1$.

◊ This is essentially done by using Th. 2.44 to a well chosen sequence of subgroups including a maximal parabolic subgroup (see [Gab00, VI.28]). ■ ↑

Let $L = (\mathtt{V}, \mathtt{E})$ be a finite graph[24] with edges (v,w) labelled by integers $m_{v,w} \in \{2, 3, \cdots\}$. The ***Artin group*** associated with L is the group with presentation given by the generators a_v indexed by the vertices $\mathtt{V}$ and relations indexed by the edges $\mathtt{E}$:

$$\langle (a_v)_{v\in\mathtt{V}} | \underbrace{a_v a_w a_v \cdots}_{m_{v,w} \text{ terms}} = \underbrace{a_w a_v \cdots}_{m_{v,w} \text{ terms}} \quad \text{for } (v,w) \in \mathtt{E} \rangle$$

For instance for $m_{v,w} = 3$, the relation associated with the edge (v,w) is $a_v a_w a_v = a_w a_v a_w$.

Right angle Artin groups correspond to the situation where all the labels of the edges are $m_{v,w} = 2$. Thus, two generators either commute or generate a free subgroup $\mathbf{F}_2$.

2.52 Theorem (Cost of Artin groups [KN14])
If A_L is an **Artin group** *with connected associated graph L. Then A_L has fixed price $= 1$.*
More generally, $\mathscr{C}_*(A_L) = \mathscr{C}^*(A_L)$ equals the number of connected components of L.

---

[22] ***Commensurated***: for every $g \in \Gamma$ the conjugate $g^{-1}Ng$ is commensurable with N i.e. $g^{-1}Ng \cap N$ has finite index in N

[23] See Def. 4.2.

[24] no loop, no double edges.

◇ If a, b form an edge in L, they generate a subgroup $A_{a,b} = \langle a, b | (ab)^m = (ba)^m \rangle$ of A_L, whose infinite cyclic subgroup generated by $(ab)^m$ is central. Thus $A_{a,b}$ has fixed price $= 1$ by Corollary 2.49. If (a, b) and (b, c) are two edges with a common vertex b, then the subgroup $A_{a,b} \cap A_{b,c}$ contains the infinite order element b. The result for connected L follows as an application of corollary 2.40. The Artin groups associated with the connected components $L_1, L_2, \cdots, L_r$ of L assemble to form a free product decomposition $A_L = A_{L_1} * A_{L_2} * \cdots * A_{L_r}$. The general result then follow from Corollary 2.27. ■ ↑

A generalization of a theorem of Schreier (for the free groups) [Sch27]:

2.53 Theorem (Finite cost normal subgroup [Gab02b, Th. 3.4])
If $1 \to \Lambda \to \Gamma \to Q \to 1$ is an exact sequence of infinite groups, and $\mathscr{C}_*(\Lambda) < \infty$, then $\mathscr{C}_*(\Gamma) = 1$.
If Γ is moreover non-amenable, then Γ is non treeable.

◇
*[**hint** : Get some inspiration from the proof of Proposition 2.50 starting with a Γ-action satisfying Theorem 2.12 (b). For the moreover part, see Cor 2.25.]*

■ ↑

2.54 Proposition (Bound for a finite cost increasing union)
Consider an increasing sequence of p.m.p. equivalence relations $\mathscr{R}_n$ such that $\mathscr{R} := \cup_n \nearrow \mathscr{R}_n$ has finite cost, then

$$\mathscr{C}(\cup_n \nearrow \mathscr{R}_n) \leq \liminf \mathscr{C}(\mathscr{R}_n).$$

◇ Since the result is trivial when $\liminf \mathscr{C}(\mathscr{R}_n) = +\infty$, WLOG one can assume that the sequence $(\mathscr{C}(\mathscr{R}_n))_n$ converges to $c := \liminf \mathscr{C}(\mathscr{R}_n) < \infty$. Let $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_k, \cdots)$ be a finite cost graphing of $\mathscr{R}$. Up to subdividing the domains of the φ_j, one can assume that each φ_j belongs to some full groupoid $[[\mathscr{R}_{n_j}]]$ and up to extraction of a subsequence of $(\mathscr{R}_n)_n$ one can assume that $\varphi_n \in [[\mathscr{R}_n]]$ (and indeed since the sequence is increasing $\varphi_j \in [[\mathscr{R}_n]]$ for all $j \leq n$). If follows that $\mathscr{R}$ can be generated by a graphing of $\mathscr{R}_n$ together with the remaining generators $(\varphi_{n+1}, \varphi_{n+2}, \cdots)$ from Φ. In particular $\mathscr{C}(\mathscr{R}) \leq \underbrace{\mathscr{C}(\mathscr{R}_n) + 1/2^n}_{\to c} + \underbrace{\mathscr{C}(\varphi_{n+1}, \varphi_{n+2}, \cdots)}_{\to 0}$ and the result follows. ■ ↑

2.55 Corollary (of Corollary 2.41 and Proposition 2.54)
Let $\Gamma = \cup_n \nearrow \Gamma_n$ be an increasing union of groups Γ_n.

1. *If $\mathscr{C}_*(\Gamma_n) = 1$ then $\mathscr{C}_*(\Gamma) = 1$.*
2. *If $\mathscr{C}^*(\Gamma_n) = 1$ then $\mathscr{C}^*(\Gamma) = 1$.*
3. *If $\mathscr{C}_*(\Gamma) < \infty$ then $\mathscr{C}_*(\Gamma) \leq \liminf \mathscr{C}_*(\Gamma_n)$.*
4. *If $\mathscr{C}^*(\Gamma) < \infty$ then $\mathscr{C}^*(\Gamma) \leq \liminf \mathscr{C}^*(\Gamma_n)$.*

Observe that there is a free p.m.p. action of Γ which realizes at the same time $\mathscr{C}_*(\Gamma)$ and all the $\mathscr{C}_*(\Gamma_n)$ (use co-induction from Γ_n to Γ and a direct product action). Similarly the Bernoulli shift actions of Γ restricts to Bernoulli shift actions of Γ_n thus all realizing the $\mathscr{C}^*$ (see Remark 2.11).

2.56 Corollary (Cost of $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_d}])$ and $\mathrm{SL}(2, \mathbb{Q})$)
$\mathscr{C}_*(\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_d}])) = \mathscr{C}_*(\mathrm{SL}(2, \mathbb{Q})) = 1$.
$\mathscr{C}^*(\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_d}]))$ *and* $\mathscr{C}^*(\mathrm{SL}(2, \mathbb{Q}))$ *are* $\leq \mathscr{C}^*(\mathrm{SL}(2, \mathbb{Z})) = 1 + \frac{1}{12}$.

◇ Since $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_d}])$ is a lattice in $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Q}_{p_1}) \times \mathrm{SL}(2, \mathbb{Q}_{p_2}) \times \cdots \times \mathrm{SL}(2, \mathbb{Q}_{p_d})$, it is thus ME with a $d + 1$-fold direct product of $\mathbf{F}_2$. It infimum cost follows from Corollary 2.35. As for $\Gamma = \mathrm{SL}(2, \mathbb{Q})$, it an increasing union of $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_d}])$. Thus $\mathscr{C}_*(\Gamma) = 1$ by Corollary 2.55. All these groups contain $\mathrm{SL}(2, \mathbb{Z})$ as a commensurated subgroup. Thus Corollary 2.46 gives the upper bound $\mathscr{C}^*(\mathrm{SL}(2, \mathbb{Z})) = 1 + \frac{1}{12}$ (Corollary 2.24). ■ ↑

2.57 Exercise
1) Exhibit examples of increasing sequences of fixed price groups Γ_n such that $\mathscr{C}_*(\Gamma_n) \to \infty$ while $\mathscr{C}^*(\cup \nearrow \Gamma_n) = 1$.
*[**hint** : For instance groups of the form $\mathbf{F}_p \times \oplus_{i=1}^{q} \mathbb{Z}/2\mathbb{Z}$ for well chosen sequences of p and q.]*
2) Exhibit examples of decreasing sequences of fixed price groups Γ_n such that $\mathscr{C}^*(\Gamma_n) = 1$ while $\mathscr{C}_*(\cap \searrow \Gamma_n) = 7$.

[***hint*** : *Ex:* $\Gamma_n := \mathbf{F}_7 \times (\oplus_{i\geq n}\mathbb{Z}/2\mathbb{Z})$. *Since* $\Gamma_\infty = \cap_n \Gamma_n = \mathbf{F}_7$, *we have* $\mathscr{C}_*(\Gamma_\infty) = \mathscr{C}^*(\Gamma_\infty) = 7$ *while* $\mathscr{C}^*(\Gamma_n) = \mathscr{C}_*(\Gamma_n) = 1$.]

2.6 Some Open Problems

2.58 Question (Strong Treeability Problem)
Can you find treeable but non-strongly treeable groups?

2.59 Question (Fixed Price Problem)
Can you find a group Γ admitting free p.m.p. actions of different costs?
i.e. can you find non-fixed price groups?

2.60 Question (Fixed Price Problem for direct products)
If $\Gamma = \Lambda \times \Delta$ is a direct product of infinite groups, does it have fixed price $= 1$?

REM: In case Λ contains a fixed price $= 1$ subgroup, this is Cor. 2.49. It can be done when Λ contains arbitrarily large finite subgroups. In general, one knows that these groups admit at least one cost $= 1$ free action by Prop. 2.50.

2.61 Question (Cost vs First ℓ^2 Betti Number Problem, [Gab02a, p. 129])
Can you find groups such that $\mathscr{C}_*(\Gamma) - 1 > \beta_1(\Gamma) - \beta_0(\Gamma)$?
Can you find groups admitting free p.m.p. actions such that $\mathscr{C}(\mathscr{R}_{\Gamma\curvearrowright^\alpha X}) - 1 > \beta_1(\Gamma) - \beta_0(\Gamma)$?
Can you find p.m.p. equivalence relations $\mathscr{R}$ such that $\mathscr{C}(\mathscr{R}) - 1 > \beta_1(\mathscr{R}) - \beta_0(\mathscr{R})$?

<u>REM</u> : In any case, the inequality $\mathscr{C}(\cdot) - 1 \geq \beta_1(\cdot) - \beta_0(\cdot)$ is proved [Gab02a].

2.62 Question (Semi-Continuity of the Cost ?)
Is there an example of an increasing sequence of p.m.p. equivalence relations $\mathscr{R}_n$ such that:
$\mathscr{C}(\cup_n \nearrow \mathscr{R}_n) > \liminf \mathscr{C}(\mathscr{R}_n)$*?*

REM: Observe that such an example would deliver a counter-example to Question 2.61, since $\mathscr{C}(\mathscr{R}_n) - 1 \geq \beta_1(\mathscr{R}_n) - \beta_0(\mathscr{R}_n)$ and $\beta_1(\cup_n \nearrow \mathscr{R}_n) \leq \liminf \beta_1(\mathscr{R}_n)$.
REM: Observe this cannot happen when $\liminf \mathscr{C}(\mathscr{R}_n) = 1$ (see Corollary 2.41) or when $\mathscr{C}(\cup_n \nearrow \mathscr{R}_n) < \infty$ (see Proposition 2.54).

2.63 Question (Cost vs Kazhdan Property (T) Problem)
Is it true that the cost of infinite Kazhdan property (T) groups is 1*?*
Observe that $\beta_1(\Gamma) = 0$. *See Th.* 2.67 *for some information.*
Added: This question has been answered by T. Hutchcroft and G. Pete [HP18] using percolation methods: infinite Kazhdan Property (T) groups have cost 1*. Their fixed price problem remains open.*

2.64 Question
If Φ is a graphing of $\mathscr{R}$, can one always approximate the cost of $\mathscr{R}$ by a sequence of **subgraphings**[25]*?*
I.e. can one find a sequence $(\Phi_n)_n$ such that each Φ_n is a subgraphing of Φ, generate $\mathscr{R}$ and $\mathscr{C}(\Phi_n) \to \mathscr{C}(\mathscr{R})$?

2.7 Additional Results

The following result of G. Hjorth is very powerful to relate treeability with actions of free groups.

2.65 Theorem (Hjorth [Hjo06])
If a p.m.p. ergodic equivalence relation $\mathscr{R}$ admits a treeing of cost $= n$, then there is a free action of the free group $\mathbf{F}_n$ producing $\mathscr{R}$.

2.66 Theorem (Gaboriau -Lyons [GL09])
For every non-amenable group Γ, there is a free p.m.p. action $\Gamma \curvearrowright (X,\mu)$ and a free ergodic $\mathbf{F}_2$-action $\mathbf{F}_2 \curvearrowright (X,\mu)$ such that the $\mathbf{F}_2$-orbits are contained in the Γ-orbits.

Concerning this result, see also section 8.5 “Comments on von Neumann’s problem”. Concretely in our proof, the Γ-action is the Bernoulli shift action $\Gamma \curvearrowright^\alpha ([0,1]^\Gamma, \mathrm{Leb}^\Gamma)$.

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[25] A ***subgraphing*** of a graphing $\Phi = (\varphi_i)_{i\in I}$ is a graphing whose partial isomorphisms are restrictions of the φ_i to Borel subsets.

2.67 Theorem (Ioana-Kechris-Tsankov [IKT09])
If Γ is a Kazhdan property (T) group and (Q,ϵ) is a Kazhdan pair, then:

$$\mathscr{C}_*(\Gamma) \leq |Q|\left(1-\frac{\epsilon^2}{2}\right)+\frac{|Q|-1}{2|Q|-1}. \tag{19}$$

If moreover Q contains an element of infinite order, then

$$\mathscr{C}_*(\Gamma) \leq |Q|-\frac{\epsilon^2}{2}. \tag{20}$$

A ***chain*** of subgroups of Γ is a decreasing sequence of finite index subgroups $\Gamma=\Gamma_0 \geq \Gamma_1, \geq \cdots \geq \Gamma_i \geq \cdots$ and the ***rank*** $\mathrm{rk}(\Gamma)$ of a group Γ is the smallest cardinal of generating subset of Γ.

2.68 Theorem (Abert-Nikolov [AN12])
The cost of a free **profinite action** $\Gamma \curvearrowright (X,\mu)=\varprojlim(\mathtt{V}_i,\mu_i)$ *of a finitely generated group Γ is related to the rank gradient of the associated chain of finite index subgroups (Γ_i) by the formula*

$$\mathscr{C}\left(\Gamma \curvearrowright \varprojlim(\mathtt{V}_i,\mu_i)\right)-1=\lim_{i\to\infty}\frac{\mathrm{rk}(\Gamma_i)-1}{[\Gamma:\Gamma_i]} \tag{21}$$

The quantity

$$\mathrm{gradrk}(\Gamma,(\Gamma_i)) := \lim_{i\to\infty}\frac{\mathrm{rk}(\Gamma_i)-1}{[\Gamma:\Gamma_i]} \tag{22}$$

introduced by Lackenby [Lac05] is called the ***rank gradient*** of the chain $(\Gamma_i)_i$.

2.69 Corollary (Rank gradient of the amenable groups)
Let Γ be a finitely generated **amenable** *group and $\Gamma=\Gamma_0 \geq \Gamma_1, \geq \cdots \geq \Gamma_i \geq \cdots$ a chain of finite index normal subgroups with trivial intersection. Then* $\mathrm{gradrk}(\Gamma,\Gamma_i)=0$.

2.70 Exercise
Consider a free ergodic **profinite action** $\Gamma \curvearrowright \lim(\mathtt{V}_i,\mu_i)$. *To a choice of a path from the root to an end corresponds the* **chain** $(\Gamma_i)_i$ *of stabilizers of the vertices encountered. This is a decreasing sequence of finite index subgroups of Γ.*
1) Observe that replacing the path by another one, simply replaces each Γ_i by a conjugate subgroup (thus does not modify the sequence of ranks).
2) Observe that the ergodicity assumption means that the action is level-transitive.
3) Show that the freeness assumption can be translated into the **Farber's condition** *:*

$$\forall \gamma \in \Gamma \setminus \{1\}, \quad \lim_{i\to\infty}\frac{\textit{number of conjugates of } \Gamma_i \textit{ in } \Gamma \textit{ that contain } \gamma}{\textit{number of conjugate of } \Gamma_i \textit{ in } \Gamma}=0. \tag{23}$$

Show that this condition (23) implies that Γ is **residually finite** *and that it is automatically satisfied when the chain is made of normal subgroups with trivial intersection.*

2.71 Theorem (Carderi-Gaboriau -de la Salle [CGd18, Cor. 4.11], see also [AT17])
If Γ is finitely generated, has fixed price $\mathscr{C}_*(\Gamma)$ and $(\Gamma_i)_i$ is any (non necessarily nested) Farber sequence[26]*, then we have:*

$$\lim_{n\to\infty}\frac{\mathrm{rk}(\Gamma_i)-1}{[\Gamma:\Gamma_i]}+1=c\mathscr{C}(\mathrm{Sch}(\Gamma/\Gamma_i,S))=\mathscr{C}_*(\Gamma),$$

where $c\mathscr{C}(\mathrm{Sch}(\Gamma/\Gamma_i,S))$ is the combinatorial cost of the associated sequence of Schreier graphs (see [CGd18]).

A group Γ is ***boundedly generated*** if there exist some integer m and $g_1, g_2, \cdots, g_m \in \Gamma$ such that the following equality of sets holds: $\Gamma=\langle g_1\rangle\langle g_2\rangle\cdots\langle g_m\rangle$.

2.72 Theorem (Shusterman, [Shu16])
Residually finite boundedly generated groups have cost $=1$.

---

[26] A sequence of subgroups Γ_i of Γ is ***Farber sequence*** if it satisfies the condition (23).

Indeed, the rank of subgroups in boundedly generated groups grows sublinearly with the index. It follows that their profinite actions all have cost 1 (Th. 2.68). More precisely Shusterman proved ([Shu16, Th. 1.2]): Let m be a positive integer, let Γ be an m-boundedly generated group and let $\Lambda \triangleright \Gamma$ be a subgroup of finite index. Then $\mathrm{rk}(\Lambda) \leq m \log_2([\Gamma : \Lambda]) + m$.

2.73 Question
Do they have fixed price?

A group Γ is ***inner amenable*** if the action of Γ on itself by conjugation admits an atomless invariant mean.

2.74 Theorem (Inner amenable groups, Tucker-Drob [Tuc14])
If Γ is an infinite inner amenable group then Γ has fixed price $= 1$.

The ***Tarski number*** $\mathcal{T}(\Gamma)$ is the minimum number of pieces in a paradoxical decomposition of Γ.

2.75 Theorem (Ershov-Golan-Sapir [EGS15])
Let Γ be a group generated by 3 elements and such that $\mathscr{C}_*(G) \geq 5/2$, then $\mathcal{T}(\Gamma) \leq 6$.

2.76 Question ([EGS15, Problem. 5.7])
Let Γ be a finitely generated group with $\mathscr{C}_*(G) > 1$.

(a) *Is it true that $\mathcal{T}(\Gamma) \leq 6$?*

(b) *If Γ is not a torsion group, is it true that $\mathcal{T}(G) \leq 5$?*

2.8 A "mercuriale", list of costs

Group Γ	$\mathscr{C}_*(\Gamma)$	Fixed price	ref
Γ finite	$1 - \frac{1}{\lvert\Gamma\rvert}$	Yes	
Γ generated by g elements	$\mathscr{C}^*(\Gamma) \leq g$		
Γ infinite amenable	$\mathscr{C}_*(\Gamma) = 1$	Yes	(1)
$\mathbf{F}_n$	n	Yes	
$\pi_1(S_g)$	$2g - 1$	Yes	(3)
Lattice in $\mathrm{SO}(2,1)$	$\mathscr{C}_*(\Gamma) = 1 - \chi$	Yes	(2)
$\Gamma = \Gamma_1 * \Gamma_2$	$\mathscr{C}_*(\Gamma_1) + \mathscr{C}_*(\Gamma_2)$	see (3)	(3)
$\mathrm{SL}(2,\mathbb{Z})$	$1 + \frac{1}{12}$	Yes	(3)
$(\mathbf{F}_m \times \mathbf{F}_n) * \mathbf{F}_k$	$k + 1$	Yes	(3)
$\Gamma_1 *_{\Gamma_3} \Gamma_2$, all Γ_i fixed price 1	1	Yes	(4)
$\mathrm{SL}(n,\mathbb{Z})$, $n \geq 3$	1	Yes	(14)
Lattice in $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$	1	?	(5)
direct products $\Gamma \times \Lambda$ of infinite groups	1	?	(6)
$\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_l}$	1	Yes	(6)
$(\bigoplus_{n\in\mathbb{N}} \mathbf{F}_2) \times \mathbb{Z}$	1	Yes	(6)
$\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$	1	Yes	(7)
Baumslag-Solitar $\mathrm{BS}(p,q)$	1	Yes	(8)
$\Gamma = \cup_n \nearrow \mathrm{PSL}(n,\mathbb{Z})$	1	Yes	(9)
$\mathrm{SL}(n,\mathbb{Z}[\frac{1}{p_1},\cdots,\frac{1}{p_d}])$ and $\mathrm{SL}(n,\mathbb{Q}))$, $n \geq 3$	1	Yes	(11)
$\mathrm{SL}(2,\mathbb{Z}[\frac{1}{p_1},\cdots,\frac{1}{p_d}])$ and $\mathrm{SL}(2,\mathbb{Q})$	1	?	(10)
infinite-conjugacy-class inner amenable groups	1	Yes	(12)
groups with a normal fixed price 1 subgroup	1	Yes	
Thompson's group F	1	Yes	
non-cocompact lattices in connected semi-simple Lie groups of $\mathbb{R}$-rank ≥ 2	1	Yes	(13)
cocompact lattices in connected semi-simple Lie groups of $\mathbb{R}$-rank ≥ 2	1	?	(5)
Γ right-angled group	1	Yes	(14)
$\mathrm{MCG}(\Sigma_g)$, $g \geq 3$,	1	Yes	(14)
$\mathrm{Out}(\mathbf{F}_n)$ for $n \geq 3$,	1	Yes	(14)
RAAG with connected graphs	1	Yes	(14)
A_L ***Artin group*** with defining graph L	# conn. comp. A_L	Yes	(15)
infinite Kazhdan (T) groups	1	?	(16)

1. see Theorem 2.18.
2. Γ is strongly treeable [CGMTD20].
3. Fixed price when Γ_1 and Γ_2 have fixed price [Gab00] . Moreover, for amalgamated free products over Γ_3 amenable $\mathscr{C}_*(\Gamma_1 *_{\Gamma_3} \Gamma_2) = \mathscr{C}_*(\Gamma_1) + \mathscr{C}_*(\Gamma_2) - \left(1 - \frac{1}{|\Gamma_3|}\right)$. See Corollary 2.27.
4. Corollary 2.39.
5. Corollary 2.35.
6. See Proposition 2.50. Fixed price when one of the factors contains a fixed price 1 subgroup, see Corollary 2.49.
7. See Theorem 2.53.
8. See Corollary 2.47.
9. See Corollary 2.55.
10. See Corollary 2.56
11. $\mathrm{SL}(n, \mathbb{Z})$ is commensurated in both. See Corollary 2.46 and Example 2.48.
12. See Theorem 2.74.
13. See [Gab00, VI.28.(a)].
14. Groups generated by a chain-commuting family of infinite order elements. Cf. Th. 2.42 and Corollary 2.43
15. See Theorem 2.52.
16. T. Hutchcroft and G. Pete [HP18]. See Question 2.63

3 A Proof: Treeings realize the cost

The goal of this section is to prove the following theorem.

3.1 Theorem (Gaboriau [Gab00], also Th. 2.23 in these notes)
If Ψ is a treeing of a p.m.p. equivalence relation $\mathscr{R}$ then $\mathscr{C}(\Psi)=\mathscr{C}(\mathscr{R})$.

We'll start proving Th. 2.23 in the case where Ψ is a treeing with finite cost. We'll then extend it to the case where Ψ has infinite cost (section 3.4). For mnemonic reason, we'll try to use letters without loops for graphings thought of as treeings $(\Psi,\cdots)$ and letters with loops $(\Theta,\Phi,\cdots)$ for non-treeings. We will follow the strategy of [Gab98]:
We have a treeing Ψ of $\mathscr{R}$ on the one hand. We will start on the other hand with a graphing Θ whose cost is close to $\mathscr{C}(\mathscr{R})$ and will modify it finitely many times in a way that does not increase its cost (or more precisely $\mathscr{C}(\Theta)$ minus the measure of the space) so as to eventually get a graphing Θ_n which is a subgraphing of the treeing Ψ. But a subgraphing of a treeing does not generate if it is a strict subgraphing. Thus the final graphing is indeed the treeing itself, showing that $\mathscr{C}(\Psi)\leq\mathscr{C}(\Theta)$. In order to make sure that our process involves only finitely many steps, we start (section 3.1) by choosing our Θ to be nicely related to Ψ.

3.1 Adapted Graphing

3.2 Lemma (Adapted Graphing [Gab00, Prop. IV.35])
Let $\Psi=(\psi_j)_{j\in J}$ be a graphing made of finitely many partial isomorphisms (J finite and thus Φ and also $\mathscr{R}$ have finite cost) of $\mathscr{R}$ and $\epsilon>0$. Then there exists an ϵ-efficient generating graphing Θ of $\mathscr{R}$ and a constant $L\geq 1$ such that:

1. *$\mathscr{C}(\Theta)\leq\mathscr{C}(\mathscr{R})+\epsilon$ (ϵ-efficiency);*
2. *each $\theta\in\Theta$ coincides on its whole domain $\mathrm{dom}(\theta)$ with **one** Ψ-word of length $\leq L$*
$$m_\theta=\psi_{t(\theta,l(\theta))}^{\varepsilon(\theta,l(\theta))}\ \cdots\ \psi_{t(\theta,j)}^{\varepsilon(\theta,j)}\ \cdots\ \psi_{t(\theta,2)}^{\varepsilon(\theta,2)}\ \psi_{t(\theta,1)}^{\varepsilon(\theta,1)} \tag{24}$$
of length $l(\theta)\leq L$, with $t(\theta,j)\in J$ et $\varepsilon(\theta,j)=\pm1$,
3. *each domain $\mathrm{dom}(\psi_j)$ decomposes into finitely many pieces on which ψ_j coincides with **one**[27] Θ-word of length $\leq L$*

In particular, for (almost) every $x\in X$ the graphs $\Psi[x]$ and $\Theta[x]$ are uniformly bi-Lipschitz, i.e. for every $\mathscr{R}$-equivalent points $(x_1,x_2)\in\mathscr{R}$:

$$\frac{1}{L}.d_\Psi(x_1,x_2)\overset{\text{by (2.)}}{\leq}d_\Theta(x_1,x_2)\overset{\text{by (3.)}}{\leq}L.d_\Psi(x_1,x_2). \tag{25}$$

◇*Proof of Lemma 3.2.* Let Φ be an auxiliary $\frac{\epsilon}{3}$-efficient graphing of $\mathscr{R}$, i.e. $\mathscr{C}(\Phi)\leq\mathscr{C}(\mathscr{R})+\frac{\epsilon}{3}$. Up to subdividing the domains, one can assume (without changing the cost) that each $\varphi\in\Phi$ coincides on its whole domain with one Ψ-word[28].

Choosing an enumeration of the countable family of Φ-words, define for each $j\in J$, the Borel set W_n^j where ψ_j does not ***coincide***[29] with one of the n first Φ-words. They satisfy $\lim_{n\to\infty}\mu(W_n^j)=0$. There is an n_0 such that $\sum_{j\in J}\mu(W_{n_0}^j)\leq\frac{\varepsilon}{3}$. Let Φ_0 the finite family of Φ-generators appearing as letters in the writing of the n_0 first Φ-words. Define Θ as the union of Φ_0, of the restrictions of each ψ_j to $W_{n_0}^j$:
$$\Theta:=\Phi_0\cup(\psi_j\restriction W_{n_0}^j)_{j\in J} \tag{26}$$
One has $\mathscr{C}(\Theta)=\underbrace{\mathscr{C}(\Phi_0)}_{\leq\mathscr{C}(\Phi)}+\underbrace{\mathscr{C}((\psi_j\restriction W_{n_0}^j)_{j\in J})}_{\leq\frac{\varepsilon}{3}}\leq\mathscr{C}(\mathscr{R})+\varepsilon$. This gives item (1).
This new graphing Θ satisfies:
– Θ *generates* $\mathscr{R}$: Ψ generates and the "missing-in-Θ" part of the generators ψ_j for $j\in J$ are replaced by a Φ_0-word on the missing part $\mathrm{dom}(\psi_j)\setminus W_{n_0}^j$.
– The generators $\theta\in\Phi_0$ (finite number) coincide (just as those of Φ) each on its domain with **one** Ψ-word m_θ, of length bounded by say L_1; while each $\theta\in\Theta\setminus\Phi_0$ being of the form $\psi_j\restriction W_{n_0}^j$ coincides

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[27] There is no choice when Ψ is a treeing. Otherwise, a choice is made in the proof.
[28] and such a word is chosen, for instance after an enumeration of the Ψ-words.
[29] The set where two partial isomorphisms u and v coincide is $\{x\in X:u(x)=v(x)\}$.

on its domain with **one** Ψ-letter. This shows item (2).
– The domain of each "missing" generator ψ_j for $j \in J$ decomposes into a number of pieces on which it coincides with one of the n_0 first Φ-words: finitely many words, thus finitely many pieces. Equally well it coincides on each piece with one Φ_0-word, of length bounded by say L_2, the maximum of the lengths of the n_0 first Φ-words. Taking $L = \max\{L_1, L_2\}$, this shows item (3). Lemma 3.2 is proved. ■

↑

3.3 Remark
The above Lemma 3.2 also holds true if one just assume Ψ to have finite cost. In the case the set J of indices is not finite, one starts by choosing $J_0 \subset J$ to be a finite subset such that $\sum_{j\in J\setminus J_0} \mathscr{C}(\{\psi_j\}) \leq \frac{\varepsilon}{3}$. Then one chooses n_0 as above, but only for the $j \in J_0$. Then one sets

$$\Theta := \Phi_0 \cup (\psi_j \restriction W^j_{n_0})_{j\in J_0} \cup (\psi_j)_{j\in J\setminus J_0}. \tag{27}$$

And one has $\mathscr{C}(\Theta) = \underbrace{\mathscr{C}(\Phi_0)}_{\leq \mathscr{C}(\Phi)} + \underbrace{\mathscr{C}((\psi_j \restriction W^j_{n_0})_{j\in J_0})}_{\leq \frac{\varepsilon}{3}} + \underbrace{\mathscr{C}((\psi_j)_{j\in J\setminus J_0})}_{\leq \frac{\varepsilon}{3}} \leq \mathscr{C}(\mathscr{R}) + \varepsilon$. *The "missing-in-$\Theta$" generators of Ψ are just the same as above.*

3.2 Expanded Graphing

Now we shall expand Θ in accordance with Ψ.
By eq. (24), each $\theta \in \Theta$ coincides on its whole domain $\mathrm{dom}(\theta)$ with **one** Ψ-word

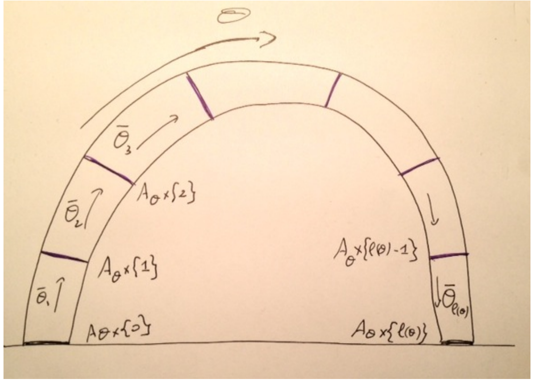
$$m_\theta = \psi^{\varepsilon(\theta,l(\theta))}_{t(\theta,l(\theta))} \cdots \psi^{\varepsilon(\theta,j)}_{t(\theta,j)} \cdots \psi^{\varepsilon(\theta,2)}_{t(\theta,2)} \psi^{\varepsilon(\theta,1)}_{t(\theta,1)} \tag{28}$$

of length $l(\theta) \leq L$, with $t(\theta, j) \in J$ et $\varepsilon(\theta, j) = \pm 1$,

We consider the spaces $\overline{A}_{\theta,j} = A_\theta \times \{j\}$ with $A_\theta \simeq \mathrm{dom}(\theta)$ for $j \in \{0, 1, \cdots, l(\theta)\}$, equipped with the restricted measure, together with the isomorphisms for $j = 1, \cdots, l(\theta)$:

$$\overline{\theta}_j : \begin{pmatrix} \overline{A}_{\theta,j-1} & \to & \overline{A}_{\theta,j} \\ (x, j-1) & \mapsto & (x, j) \end{pmatrix}$$

Figure 5: Expansion of θ.



We define the measure[30] space $(\overline{X}, \overline{\mu})$ as the disjoint union of $X_0 = X$ and of all the $A_\theta \times \{0\}$, $A_\theta \times \{1\}, \cdots, A_\theta \times \{l(\theta)\}$ after the (measure preserving) identifications, for the various $\theta \in \Theta$:

$$\begin{array}{ccc} \overline{A}_{\theta,0} & \overset{\sim}{\to} & \mathrm{dom}(\theta) \subset X_0 \\ (x,0) & \mapsto & x \end{array} \qquad\qquad \begin{array}{ccc} \overline{A}_{\theta,l(\theta)} & \overset{\sim}{\to} & \mathrm{im}(\theta) \subset X_0 \\ (x,0) & \mapsto & \theta(x) \end{array}$$

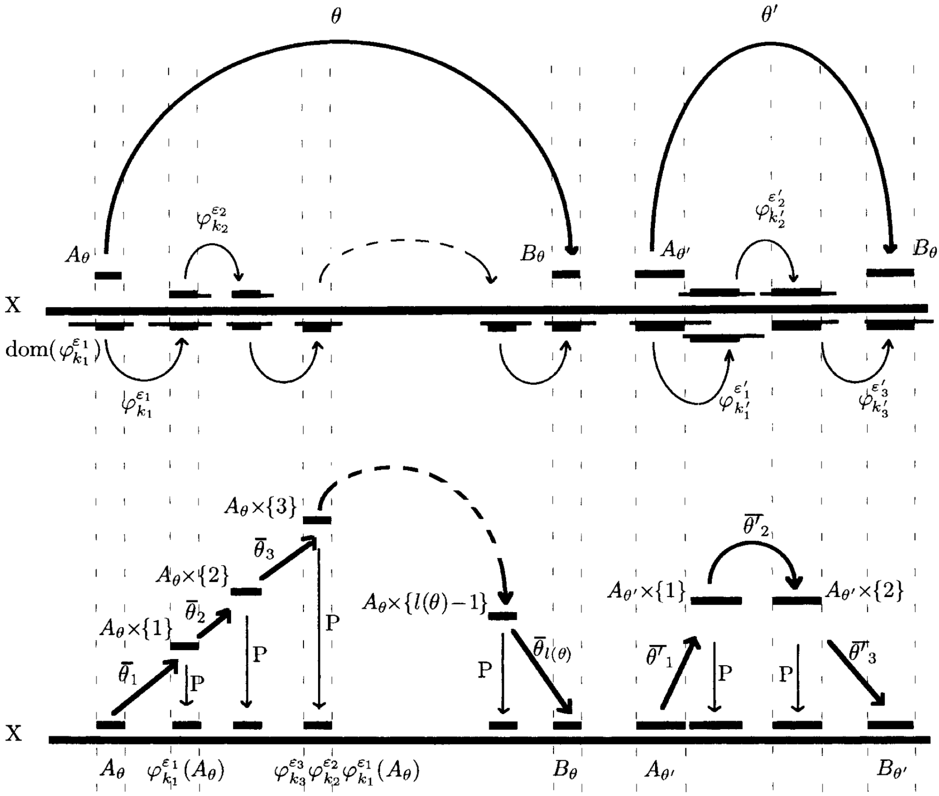
---

[30] Its total measure is indeed

$$\overline{\mu}(\overline{X}) = \mu(X) + \sum_{\theta\in\Theta} (l(\theta) - 1)\, \mu(\mathrm{dom}(\theta)) \tag{29}$$

.

Figure 6: Expanded graphing (picture borrowed from [Gab00]).



The map $\overline{\Theta} \to (\Psi \cup \Psi^{-1})$ sending the generator $\overline{\theta}_j$ to the j-th letter $\psi_{t(\theta,j)}^{\varepsilon(\theta,j)}$ of the word m_θ (eq. 28) extends to a ***word-morphism***[31] from the $\overline{\Theta}$-words to the Ψ-words:

$$(\overline{\Theta} \cup \overline{\Theta}^{-1})^* \xrightarrow{\mathfrak{P}_*} (\Psi \cup \Psi^{-1})^* \tag{30}$$

We also define a map

$$\left(\begin{array}{rcl} \overline{X} & \xrightarrow{\mathfrak{P}} & X \\ \bar{x} \in X_0 = X & \mapsto & \mathfrak{P}(\bar{x}) = \bar{x} \in X \\ \bar{x} = (x,j) \in A_\theta \times \{j\} & \mapsto & \mathfrak{P}(\bar{x}) = \mathfrak{P}_* \left(\psi_{t(\theta,j)}^{\varepsilon(\theta,j)} \ \cdots \ \psi_{t(\theta,2)}^{\varepsilon(\theta,2)} \ \psi_{t(\theta,1)}^{\varepsilon(\theta,1)}\right)(x) \end{array}\right)$$

It has finite fibers, it is measure preserving where it is injective, it is injective on the domains $\mathrm{dom}(\overline{\theta}_j)$ and if $\bar{x} \in \mathrm{dom}(\overline{\theta}_j)$, then $\mathfrak{P}(\bar{x}) \in \mathrm{dom}(\mathfrak{P}_*(\overline{\theta}_j))$ and the following diagrams commute:

$$\begin{tikzcd} \mathrm{dom}(\overline{\theta}_j) \arrow[r, "\overline{\theta}_j"] \arrow[d, "\mathfrak{P}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{im}(\overline{\theta}_j) \arrow[d, "\mathfrak{P}"] \\ \mathrm{dom}(\mathfrak{P}_*(\overline{\theta}_j)) \arrow[r, "\mathfrak{P}_*(\overline{\theta}_j)=\psi_{t(\theta,j)}^{\varepsilon(\theta,j)}"] & \mathfrak{P}_*(\mathrm{im}(\overline{\theta}_j)) \end{tikzcd} \tag{31}$$

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[31] compatible with concatenations and reductions.

$$\begin{tikzcd}
\operatorname{dom}(\theta) = \overline{A}_{\theta,0} \arrow[r, "\overline{\theta}_1"] \arrow[d, "="'] \arrow[dr, phantom, "\circlearrowright"] & \overline{A}_{\theta,1} \arrow[r, "\overline{\theta}_2"] \arrow[d, "\mathfrak{P}"'] \arrow[dr, phantom, "\circlearrowright"] & \dots \arrow[r, "\overline{\theta}_{l(\theta)-1}"] & \overline{A}_{\theta,l(\theta)-1} \arrow[r, "\overline{\theta}_{l(\theta)}"] \arrow[d, "\mathfrak{P}"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{im}(\theta) = \overline{A}_{\theta,l(\theta)} \arrow[d, "="] \\
\operatorname{dom}(\theta) = A_{\theta,0} \arrow[r, "\psi_{t(\theta,1)}^{\varepsilon(\theta,1)}"] & A_{\theta,1} \arrow[r, "\psi_{t(\theta,2)}^{\varepsilon(\theta,2)}"] & \dots \arrow[r, "\psi_{t(\theta,l(\theta)-1)}^{\varepsilon(\theta,l(\theta)-1)}"] & A_{\theta,l(\theta)-1} \arrow[r, "\psi_{t(\theta,l(\theta))}^{\varepsilon(\theta,l(\theta))}"] & \operatorname{im}(\theta) = A_{\theta,l(\theta)}
\end{tikzcd} \tag{32}$$

where $A_{\theta,j}$ is the image of $\operatorname{dom}(\theta)$ under the j-th subword $A_{\theta,j} := \psi_{t(\theta,j)}^{\varepsilon(\theta,j)} \ \cdots \ \psi_{t(\theta,1)}^{\varepsilon(\theta,1)}(\operatorname{dom}(\theta))$.

The collection $\overline{\Theta}$ of $\overline{\theta}_j$, for $\theta \in \Theta, j \in \{1, \cdots, l(\theta)\}$ forms a graphing between Borel subsets of $(\overline{X}, \overline{\mu})$. Its cost (for the non-normalized measure $\overline{\mu}$) is $\mathscr{C}_{\overline{\mu}}(\overline{\Theta}) = \sum_{\theta\in\Theta} l(\theta)\, \mu(\operatorname{dom}(\theta))$, so that (with eq. (29))

$$\mathscr{C}_{\mu}(\Theta) - \mu(X) = \mathscr{C}_{\overline{\mu}}(\overline{\Theta}) - \overline{\mu}(\overline{X}) \tag{33}$$

Claim: Two points $\bar{x}$ and $\bar{y}$ are in the same $\overline{\Theta}$-orbit if and only if $\mathfrak{P}(\bar{x})$ and $\mathfrak{P}(\bar{y})$ are in the same Ψ-orbit. And indeed, the graphings are uniformly quasi-isometric:

$$d_{\Psi}(\mathfrak{P}(\bar{x}), \mathfrak{P}(\bar{y})) \leq d_{\overline{\Theta}}(\bar{x}, \bar{y}) \leq C_1 d_{\Psi}(\mathfrak{P}(\bar{x}), \mathfrak{P}(\bar{y})) + C_2. \tag{34}$$

Since $\mathfrak{P}_*$ sends letters to letters, $\mathfrak{P}$ decreases the lengths. This gives the first inequality.
Consider now a Ψ-word m sending $\mathfrak{P}(\bar{x})$ to $\mathfrak{P}(\bar{y})$. Each Ψ-letter gives a Θ-word of length $\leq L$. Each Θ-letter delivers an $\overline{\Theta}$-word of length $\leq L$ (so that each Ψ-letter gives an $\overline{\Theta}$-word of length $\leq L^2$). The $\overline{\Theta}$-distance between $\mathfrak{P}(\bar{x})$ and $\mathfrak{P}(\bar{y})$ is thus $\leq |m|L^2$ and $C_1 = L^2$. Now the $\overline{\Theta}$-distance between $\bar{x}$ and $\mathfrak{P}(\bar{x})$ is bounded by $L + L^3$. Indeed $\bar{x} = (x, j)$ in some $\overline{A}_{\theta,j}$ (and for some $x \in X$) so that $\bar{x} = \overline{\theta}_j \overline{\theta}_{j-1} \cdots \overline{\theta}_1(x)$, while $\mathfrak{P}(\bar{x}) = \mathfrak{P}_*(\overline{\theta}_j \overline{\theta}_{j-1} \cdots \overline{\theta}_1)(x) = \mathfrak{P}_*(\overline{\theta}_j)\mathfrak{P}_*(\overline{\theta}_{j-1}) \cdots \mathfrak{P}_*(\overline{\theta}_1)(x)$ gives a Ψ-word of length j, leading in turn to a $\overline{\Theta}$-word of length $\leq jL^2$. And $C_2 = 2(L + L^3)$.

3.3 Foldings

Let's rename $(\overline{\sigma_i})_i$ the generators of the graphing $\overline{\Theta}$. Choose an enumeration of the finite number K^0 of pairs $(\overline{\sigma_i}^{\epsilon_i}, \overline{\sigma_j}^{\epsilon_j})$ with $\sigma_i \neq \sigma_j$, with $\epsilon_i, \epsilon_j \in \{\pm 1\}$ and such that $\mathfrak{P}_*(\overline{\sigma_i}^{\epsilon_i}) = \mathfrak{P}_*(\overline{\sigma_j}^{\epsilon_j})$. For the first one, say $(\overline{\sigma_i}^{\epsilon_i}, \overline{\sigma_j}^{\epsilon_j})$, let's consider the set

$$\overline{U}^1 := \{\bar{x} \in \overline{X} : \bar{x} \in \operatorname{dom}(\overline{\sigma_i}^{\epsilon_i}) \cap \operatorname{dom}(\overline{\sigma_j}^{\epsilon_j}),\ \overline{\sigma_i}^{\epsilon_i}(\bar{x}) \neq \overline{\sigma_j}^{\epsilon_j}(\bar{x})\} \tag{35}$$

Observe that $\overline{\sigma_i}^{\epsilon_i}(\bar{x})$ and $\overline{\sigma_j}^{\epsilon_j}(\bar{x})$ are in the same $\mathfrak{P}$-fiber[32].

Let Π^1 the quotient map to the quotient space $\overline{X}^1 := \overline{X}/[\overline{\sigma_i}^{\epsilon_i}(\bar{x}) \sim \overline{\sigma_j}^{\epsilon_j}(\bar{x})$ for $\bar{x} \in \overline{U}^1]$, with the natural measure $\bar{\mu}^1$. Define the quotient maps $\mathfrak{P}^1$ and $\mathfrak{P}^1_*$ such that $\mathfrak{P} = \mathfrak{P}^1 \circ \Pi^1$ and the "quotient" partial isomorphims $\overline{\sigma_j}^1$ defined by:

$$\begin{tikzcd}
\overline{X} \supset \operatorname{dom}(\overline{\sigma_j}) \arrow[r, "\overline{\sigma_j}"] \arrow[d, "\Pi^1"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{im}(\overline{\sigma_j}) \subset \overline{X} \arrow[d, "\Pi^1"] \\
\overline{X}^1 \supset \Pi^1(\operatorname{dom}(\overline{\sigma_j})) = \operatorname{dom}(\overline{\sigma_j}^1) \arrow[r, "\overline{\sigma_j}^1"] \arrow[d, "\mathfrak{P}^1"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{im}(\overline{\sigma_j}^1) = \Pi^1(\operatorname{im}(\overline{\sigma_j})) \subset \overline{X}^1 \arrow[d, "\mathfrak{P}^1"] \\
X \arrow[r, "\mathfrak{P}_*(\overline{\sigma_j}) = \mathfrak{P}^1_*(\overline{\sigma_j}^1)"] & X
\end{tikzcd}$$

and $\mathfrak{P}^1_*(\overline{\sigma_j}^1) = \mathfrak{P}_*(\overline{\sigma_j})$.

Now, the partial isomorphisms $(\overline{\sigma_i}^1)^{\epsilon_i}$ and $(\overline{\sigma_j}^1)^{\epsilon_j}$ coincide on the Borel set $\Pi(\overline{U}^1)$. Let's remove this part from the domain of, say $(\overline{\sigma_i}^1)^{\epsilon_i}$ and let's continue to call $(\overline{\sigma_i}^1)^{\epsilon_i}$ the restriction to the rest. The new graphing thus constructed on $\overline{X}^1$ is called Θ^1. Since $\bar{\mu}(\overline{U})$ is precisely the decrease both in the measure of the space and in the cost of the graphing, we have

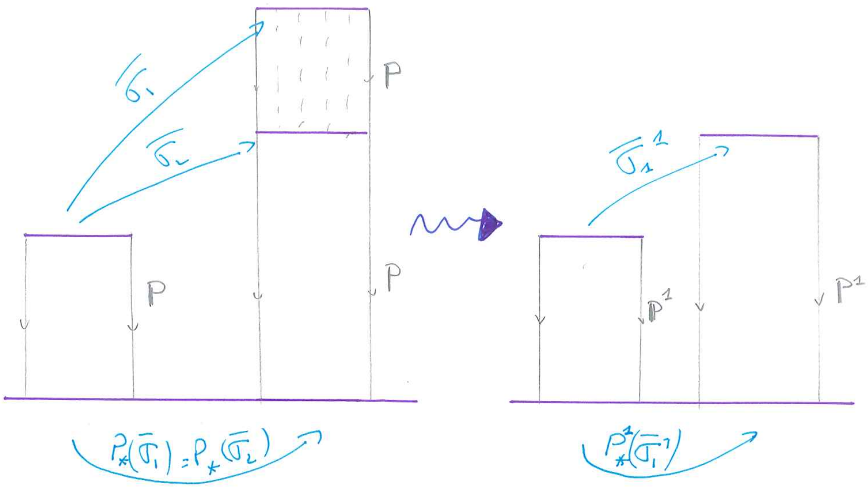
$$\mathscr{C}_{\mu}(\Theta) - \mu(X) = \mathscr{C}_{\overline{\mu}}(\overline{\Theta}) - \overline{\mu}(\overline{X}) = \mathscr{C}_{\overline{\mu}^1}(\overline{\Theta}^1) - \overline{\mu}^1(\overline{X}^1) \tag{36}$$

The finite number K^1 of pairs $((\overline{\sigma_i}^1)^{\epsilon_i}, (\overline{\sigma_j}^1)^{\epsilon_j})$ with $\sigma_i \neq \sigma_j$, with $\epsilon_i, \epsilon_j \in \{\pm 1\}$ and such that $\mathfrak{P}^1_*((\overline{\sigma_i}^1)^{\epsilon_i}) = \mathfrak{P}^1_*((\overline{\sigma_j}^1)^{\epsilon_j})$ naturally injects in K^0. So that one can proceed the same way as we did

---

[32] $\mathfrak{P}(\overline{\sigma_i}^{\epsilon_i}(\bar{x})) = \mathfrak{P}_*(\overline{\sigma_i}^{\epsilon_i})\mathfrak{P}(x) = \mathfrak{P}_*(\overline{\sigma_j}^{\epsilon_j})\mathfrak{P}(x) = \mathfrak{P}(\overline{\sigma_j}^{\epsilon_j}(\bar{x}))$.

Figure 7: Foldings



for the next pair, and so on until we reach the last pair where we produce $\overline{X}^M, \bar{\mu}^M$ together with a graphing $\overline{\Theta}^M = (\overline{\sigma}^M)$ such that

$$\mathscr{C}_\mu(\Theta) - \mu(X) = \mathscr{C}_{\overline{\mu}}(\overline{\Theta}) - \overline{\mu}(\overline{X}) = \mathscr{C}_{\overline{\mu}^M}(\overline{\Theta}^M) - \overline{\mu}^M(\overline{X}^M) \tag{37}$$

and maps $\mathfrak{P}^M$, $\mathfrak{P}^M_*$.

Two points $\bar{x}$ and $\bar{y}$ in the same $\mathfrak{P}$-fiber have distance $d_{\overline{\Theta}}(\bar{x}, \bar{y}) \leq C_2$ (eq. 34). The image by $\mathfrak{P}_*$ of a $\overline{\Theta}$-word $\bar{\omega}$ between them is a non-reduced Ψ-word since it gives, in the tree $\Psi[\mathfrak{P}(u)]$, a path from $\mathfrak{P}(u)$ to $\mathfrak{P}(v) = \mathfrak{P}(u)$. So that this word $\bar{\omega}$ had to cross a foldable pair of edges.

If $\mathfrak{P}$ was not injective, then for any pair of points $\bar{x}$ and $\bar{y}$ in the same $\mathfrak{P}$-fiber, their $\overline{\Theta}$-distance would have decreased along the process: $d_{\overline{\Theta}^M}(\Pi^M(\bar{x}), \Pi^M(\bar{y})) < d_{\overline{\Theta}}(\bar{x}, \bar{y})$.

Applying several (finitely many) times this whole folding process decreases the distance in the fibers so as to reach a stage where $\mathfrak{P}^N : \overline{X}^N \to X$ is injective (indeed an isomorphism). Then each $\overline{\sigma}^N \in \overline{\Theta}^M$ coincides on its whole domain with the Ψ-letter $\mathfrak{P}^N_*(\overline{\sigma}^N)$ of the **treeing** Ψ. Moreover, $\overline{\Theta}^M$ has the same classes as Ψ (thus, for any $\psi \in \Psi$ and any $x \in \mathrm{dom}(\psi)$, there is a $\overline{\sigma}^N \in \overline{\Theta}^M$ such that $\psi(x) = \overline{\sigma}^N(x)$) so that:

$$\mathscr{C}_\mu(\Psi) \leq \mathscr{C}_\mu(\overline{\Theta}^M). \tag{38}$$

On the other hand $\overline{\Theta}^M$, satisfy the equality similar to (37), so that:

$$\mathscr{C}_\mu(\Psi) - \mu(X) \leq \mathscr{C}_\mu(\overline{\Theta}^M) - \mu(X) = \mathscr{C}_\mu(\Theta) - \mu(X). \tag{39}$$

■ ↑

3.4 Infinite cost

Let's extend Th. 2.23 when the treeing Ψ contains infinitely many elements (for instance if $\mathscr{C}(\Psi) = \infty$). Let Θ be another graphing of $\mathscr{R}_\Psi$, and let's show that $\mathscr{C}(\Theta) \geq \mathscr{C}(\Psi)$. Call $\Psi_q = (\psi_1, \psi_2, \ldots, \psi_q)$ the treeing consisting in the first q generators.

Up to subdivisions, one can assume that each generator in Θ can be expressed as a single Ψ-word (without change of cost).

For a fixed q, "the generators $\psi_i \in \Psi_q$ can be expressed using a finite part of Θ, up to a small error": There is a finite subgraphing $\Theta_{r_q} = (\theta_1, \theta_2, \ldots, \theta_{r_q})$ of Θ and Borel sets $D_1 \subset \mathrm{dom}(\psi_1), D_2 \subset \mathrm{dom}(\psi_2), \ldots, D_q \subset \mathrm{dom}(\psi_q)$, of measure $< 1/2^q$ such that the points x and $\psi_i(x)$ are Θ_{r_q}-equivalent, for every $i = 1, \ldots, q$ and for all $x \in \mathrm{dom}(\psi_i) \setminus D_i$.

On the other hand, there is a big enough $p \geq q$ such that the generators of Θ_{r_q} can be expressed as Ψ_p-words. The graphing Φ made of the following three parts: Θ_{r_q}, the restrictions of ψ_i to D_i (for $i = 1, \ldots, q$) and $\Psi_p \setminus \Psi_q$ generates $\mathscr{R}_{\Psi_p}$.

$$\Phi = \Theta_{r_q} \vee (\psi_i \restriction D_i)_{i=1,\cdots,q} \vee \Psi_p \setminus \Psi_q$$

It follows that :

$$\mathscr{C}(\Theta_{r_q}) + q/2^q + \mathscr{C}(\Psi_p \setminus \Psi_q) = \mathscr{C}(\Phi) \geq \mathscr{C}(\mathscr{R}_{\Psi_p}) \overset{(*)}{=} \mathscr{C}(\Psi_p) = \mathscr{C}(\Psi_p \setminus \Psi_q) + \mathscr{C}(\Psi_q),$$

where the first part of the proof (treeing with finitely many generators) shows the equality (*).
We deduce $\mathscr{C}(\Theta) \geq \mathscr{C}(\Theta_{r_q}) \geq \mathscr{C}(\Psi_q) - q/2^q$. This last quantity goes to $\mathscr{C}(\Psi)$ when q goes to ∞. ■

↑

4 Full Group

The ***uniform topology*** on $\mathrm{Aut}(X,\mu)$ is induced by the bi-invariant and complete ***uniform metric***:

$$d_u(S,T)=\mu\{x : S(x)\neq T(x)\} \tag{40}$$

4.1 Exercise
Show that the uniform metric d_u *on* $\mathrm{Aut}(X,\mu)$ *is bi-invariant and complete. Show that it is not separable.*
[***hint*** *: Consider rotations on* $\mathbb{S}^1$*.*]

4.2 Definition (Full Group)
The **full group** *of* $\mathscr{R}$ *denoted by* $[\mathscr{R}]$ *is defined as the subgroup of* $\mathrm{Aut}(X,\mu)$ *whose elements have their graph contained in* $\mathscr{R}$*:*

$$[\mathscr{R}]:=\{T\in\mathrm{Aut}(X,\mu) : (x,T(x))\in\mathscr{R} \text{ for a.a. } x\in X\}. \tag{41}$$

Two such isomorphisms agreeing almost everywhere are thus considered equal.

It was introduced and studied by Dye [Dye59], and it is clearly an OE-invariant. But conversely, its algebraic structure is rich enough to remember the equivalence relation:

4.3 Theorem (Dye's reconstruction theorem [Dye63])
Two ergodic p.m.p. equivalence relations $\mathscr{R}_1$ *and* $\mathscr{R}_2$ *are OE iff their full groups are algebraically isomorphic; moreover the isomorphism is then implemented by an orbit equivalence.*

The full group has very nice properties.

With the uniform topology, given by the bi-invariant metric $d_u(T,S)=\mu\{x : T(x)\neq S(x)\}$ the full group of a standard (countable classes) p.m.p. equivalence relation is Polish[33]. In general, it is not locally compact[34].

4.4 Exercise
Show that $[\mathscr{R}]$ *does not contain any 1-parameter group, i.e. every continuous group homomorphism* $\mathbb{R}\to[\mathscr{R}]$ *is trivial.*
[***hint*** *: Consider the* **supports**[35] *of the elements in the group generated by an element close to the identity.*]

4.5 Theorem (Bezuglyi-Golodets [BG80, Kec10])
The full group is a **simple group** *iff* $\mathscr{R}$ *is ergodic.*

On the other hand (when taking its topology into account), [Dye63, prop. 5.1] the closed normal subgroups of $[\mathscr{R}]$ are in a natural bijection with $\mathscr{R}$-invariant subsets of X.
It satisfies this very remarkable, ***automatic continuity*** property:

4.6 Theorem (Kittrell-Tsankov [KT10])
If $\mathscr{R}$ *is ergodic, then every group homomorphism* $f:[\mathscr{R}]\to G$ *with values in a separable topological group is indeed continuous.*

Hyperfiniteness translates into an abstract topological group property:

4.7 Theorem (Giordano-Pestov [GP07])
Assuming $\mathscr{R}$ *ergodic,* $\mathscr{R}$ *is hyperfinite iff* $[\mathscr{R}]$ *is extremely amenable.*

Recall that a topological group G is ***extremely amenable*** if every continuous action of G on a (Hausdorff) compact space has a fixed point.

Closely related to the full group, is the ***automorphism group***

$$\mathrm{Aut}(\mathscr{R}):=\{T\in\mathrm{Aut}(X,\mu) : (x,y)\in\mathscr{R}\Rightarrow (T(x),T(y))\in\mathscr{R}\} \tag{42}$$

It contains the full group as a normal subgroup. The quotient is the ***outer automorphism group***

$$\mathrm{Out}(\mathscr{R})=\mathrm{Aut}(\mathscr{R})/[\mathscr{R}]. \tag{43}$$

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[33] homeomorphic to a complete metric space that has a countable dense subset.
[34] In fact, it is homeomorphic with the separable Hilbert space ℓ^2 [KT10].
[35] The complement of the fixed-point set.

In his very rich monograph [Kec10], Kechris studied the continuity properties of the cost function on the space of actions and proved that the condition $\mathscr{C}(\mathscr{R}) > 1$, for an ergodic $\mathscr{R}$, forces its outer automorphism group to be Polish[36].

Kechris [Kec10] also introduced the topological OE-invariant $t([\mathscr{R}])$ and initiated the study of its relations with the cost.

4.8 Definition (Number of Topological Generators [Kec10])
The **number of topological generators** $t([\mathscr{R}])$ *is the minimum number of generators of a dense subgroup of the full group* $[\mathscr{R}]$.

When $\mathscr{R}$ is generated by a free ergodic action of $\mathbf{F}_n$, Miller obtained the following lower bound: $n+1 \leq t([\mathscr{R}])$, and [KT10] proved that $t([\mathscr{R}_{\mathrm{hyp}}]) \leq 3$ and that $t([\mathscr{R}]) \leq 3(n+1)$. Quite recently, Matui [Mat11] proved that for an infinite hyperfinite equivalence relation, one has $t([\mathscr{R}_0]) = 2$. A series of results by Matui [Mat06, Mat11] and Kittrell-Tsankov [KT10] led to the following estimate between the floor of the cost and the number of topological generators: $\lfloor \mathscr{C}(\mathscr{R}) \rfloor + 1 \leq t([\mathscr{R}]) \leq 2(\lfloor \mathscr{C}(\mathscr{R}) \rfloor + 1)$. Recently, Le Maître obtained the optimal value:

4.9 Theorem (Le Maître [LM13])
If $\mathscr{R}$ is a p.m.p. ergodic equivalence relation, then

$$\lfloor \mathscr{C}(\mathscr{R}) \rfloor + 1 = t([\mathscr{R}]).$$

Moreover, for every $\epsilon > 0$, there is $t([\mathscr{R}])$-tuple of topological generators of $[\mathscr{R}]$ such that the sum of the measures of the **supports** *is smaller than $\mathscr{C}(\mathscr{R}) + \epsilon$.*

4.10 Theorem (Le Maître [LM14])
For every free p.m.p. action $\mathbf{F}_p \curvearrowright^{\alpha} (X, \mu)$ of the free group, $t([\mathscr{R}_{\mathbf{F}_p \curvearrowright^{\alpha} (X,\mu)}]) = p + 1$.
For every free p.m.p. action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright^{\alpha} (X, \mu)$ of $\mathrm{SL}(n, \mathbb{Z})$ $(n \geq 2)$, one has $t([\mathscr{R}_{\mathrm{SL}(n,\mathbb{Z}) \curvearrowright^{\alpha} (X,\mu)}]) = 2$.

Observe that the ergodic case is a corollary of Th. 4.9.

Recall that: $\mathscr{C}(\mathrm{SL}(n, \mathbb{Z}) \curvearrowright^{\alpha} (X, \mu)) = \begin{cases} \frac{13}{12} & \text{for } n = 2; \\ 1 & \text{for } n > 2. \end{cases}$

---

[36] At this point, the relevant topology comes from the ***weak topology*** on $\mathrm{Aut}(X, \mu)$, i.e. the topology induced by the metric:

$$\delta_w(S, T) = \sum_n \frac{1}{2^n} \mu(\{S(A_n) \triangle T(A_n)\}) \tag{44}$$

where $\{A_n\}$ is a dense family of Borel sets in the measure algebra of (X, μ), and the associated complete metric:

$$\overline{\delta}_w(S, T) = \delta_w(S, T) + \delta_w(S^{-1}, T^{-1}). \tag{45}$$

5 ℓ^2-Betti Numbers

Usually ***Betti numbers*** are defined as dimension or rank of a vector space, a module or a group appearing as homology or cohomology of some objects. And ℓ^2 refers to the framework of Hilbert spaces.

Reference to add and comment: [Ati76, Con79, CG86, Eck00, Lüc02].

Simplicial Complexes:

A ***simplicial complex*** L is the combinatorial data of
– a finite or countable set $V^{(0)}$, the ***vertices***
– a collection of finite subsets of $V^{(0)}$ whose elements are called ***simplices***
such that
– each singleton $\{v\}$ is a simplex;
– each part of a simplex is itself a simplex.

A simplex form with $n+1$ vertices is called an ***n-simplex***; its dimension is n. The collection of the n-simplices is denoted $L^{(n)}$.

The ***space of n-chains*** of L will be the space with the family of n-simplices as a basis, where the term basis has to be understood in the sense of $\mathbb{Z}$-modules, resp. $\mathbb{K}$-vector spaces ($\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$), resp. Hilbert spaces according to whether we consider chains with coefficients in $\mathbb{Z}$, $\mathbb{K}$ or ℓ^2-chains.

$$C_n(L, \mathbb{Z}) := \left\{ \sum_{\text{finite}} a_i \sigma_i : a_i \in \mathbb{Z} \right\}$$

$$C_n(L, \mathbb{K}) := \left\{ \sum_{\text{finite}} a_i \sigma_i : a_i \in \mathbb{K} \right\}$$

$$C_n^{(2)}(L) := \left\{ \sum_{\text{infinite}} a_i \sigma_i : a_i \in \mathbb{K}, \sum |a_i|^2 < \infty \right\}$$

A simplex $\sigma = \{v_0, \cdots, v_n\}$ of dimension $n \geq 2$ has two orientations (corresponding to the orbits of the alternating group $\mathfrak{A}_{n+1}$ on its possible total orders $[v_1, v_5, \cdots, v_n, v_2]$). These two orientations are considered to correspond to opposite simplices in the sequel.

For an oriented n-simplex $\sigma = [v_0, \cdots, v_n]$, its boundary $\partial_n \, \sigma$ is given by

$$\partial_n \, [v_0, \, \ldots, \, v_n] = \sum_{i=0}^{n} (-1)^i \, [v_0, \, \ldots, \, \widehat{v_i}, \, \ldots, \, v_n].$$

This extends linearly to a map $\partial_n : C_n(L, K) \to C_{n-1}(L, K)$, for all $n \geq 1$, and these maps satisfy $\partial_n \, \partial_{n+1} = 0$. We thus have a ***chain complex***

$$0 \leftarrow C_0(L) \overset{\partial_1}{\leftarrow} C_1(L) \leftarrow \cdots \overset{\partial_n}{\leftarrow} C_n(L) \leftarrow \cdots .$$

5.1 Definition (Simplicial Homology)

The **simplicial homology** *of a simplicial complex L is defined as the sequence of quotients*

$$H_n(L, \mathbb{K}) = \ker \, \partial_n / \mathrm{Im} \;\, \partial_{n+1}, \qquad n \geq 0, \qquad \mathbb{K} \in \{\mathbb{Z}, \, \mathbb{Q}, \, \mathbb{R}, \, \mathbb{C}\}.$$

5.2 Theorem

A simplicial complex L is ***n*-connected** *iff it is connected, it has $\pi_1(L) = 0$ and $H_i(L, \mathbb{Z}) = 0$, $\forall \, 1 \leq i \leq n$.*

Let's take this as a definition.

5.3 Exercise

Check that the boundary of the simplex with the opposite orientation is the opposite of the boundary $\partial_n \sigma^{opp} = -\partial_n \sigma$.

5.1 ℓ^2-Homology and ℓ^2-Cohomology

In general, the boundary operators ∂_n, $n \geq 1$ do not extend to bounded operators on the spaces of ℓ^2-chains. We thus impose the extra condition that the simplicial complex L is ***uniformly locally bounded***, i.e., each n-simplex belongs to at most $N_n < \infty$ simplices of dimension $n+1$.

5.4 Exercise
If a simplicial complex L is uniformly locally bounded, then the boundary map ∂_n extends to a bounded linear operator $\partial_n : C_n^{(2)}(L) \to C_{n-1}^{(2)}(L)$ and $\partial_n \, \partial_{n+1} = 0$.

We thus have a ***chain complex*** of Hilbert spaces

$$0 \leftarrow C_0^{(2)}(L) \overset{\partial_1}{\leftarrow} C_1^{(2)}(L) \leftarrow \cdots \overset{\partial_n}{\leftarrow} C_n^{(2)}(L) \leftarrow \cdots .$$

The ℓ^2***-homology*** of a uniformly locally bounded simplicial complex L is defined as the quotients:

$$H_n^{(2)}(L) = \ker\, \partial_n / \mathrm{Im}\ \partial_{n+1},\ n \geq 0.$$

Since in general Im ∂_{n+1} doesn't need to be a closed subspace of ker ∂_n, and thus the quotient space may not be a nice topological vector space, we define:

5.5 Definition (Reduced ℓ^2-Homology)
The **reduced ℓ^2-homology** *of a uniformly locally bounded simplicial complex L is defined as:*

$$\overline{H}_n^{(2)}(L) = \ker\, \partial_n / \overline{\mathrm{Im}\ \partial_{n+1}},\ n \geq 0.$$

Since the space of ℓ^2-chains are Hilbert spaces, taking the adjoint of the boundary operators, we obtain a ***co-chain complex***

$$0 \to C_0^{(2)}(L) \overset{\partial_1^*}{\to} C_1^{(2)}(L) \overset{\partial_2^*}{\to} \cdots \to C_n^{(2)}(L) \overset{\partial_{n+1}^*}{\to} \cdots .$$

5.6 Definition (Reduced ℓ^2-Cohomology)
The **reduced ℓ^2-cohomology** *of a uniformly locally bounded simplicial complex L is defined as the quotients:*

$$\overline{H}^n_{(2)}(L) = \ker\, \partial_{n+1}^* / \overline{\mathrm{Im}\ \partial_n^*}, \qquad n \geq 0.$$

Decomposing as an orthogonal sum $\ker\, \partial_n = \overline{\mathrm{Im}\ \partial_{n+1}} \oplus \mathcal{H}_n^{(2)}$, we define $\mathcal{H}_n^{(2)}$ and get

$$\overline{H}_n^{(2)}(L) \cong \mathcal{H}_n^{(2)},\ \forall\ n \geq 0. \tag{46}$$

Further, since $(\ker\, \partial_n)^{\perp} = \overline{\mathrm{Im}\ \partial_n^*}$, and $\overline{\mathrm{Im}\ \partial_{n+1}} = (\ker\, \partial_{n+1}^*)^{\perp}$, we obtain the overlapping decompositions

$$\begin{array}{lcll}
C_n^{(2)}(L) & = & \qquad\qquad \mathrm{Ker}\ \partial_n \qquad\qquad\quad \overset{\perp}{\oplus}\ \ \mathrm{Ker}\ \partial_n^{\perp} & \\
C_n^{(2)}(L) & = & \overbrace{\overline{\mathrm{Im}}\ \partial_{n+1}\ \overset{\perp}{\oplus}\ \underbrace{\mathcal{H}_n^{(2)}(L)}}\underbrace{\ \overset{\perp}{\oplus}\ \ \mathrm{Ker}\ \partial_n^{\perp}} & \Rightarrow \mathcal{H}_n^{(2)}(L) \simeq \mathrm{Ker}\ \ \partial_n / \overline{\mathrm{Im}}\ \ \partial_{n+1} = \bar{H}_n^{(2)}(L) \\
C_n^{(2)}(L) & = & \overline{\mathrm{Im}}\ \partial_{n+1}\ \overset{\perp}{\oplus} \qquad\qquad \mathrm{Ker}\ \ \partial_{n+1}^* & \Leftarrow \overline{\mathrm{Im}\ \ \partial_{n+1}}^{\perp} = \mathrm{Ker}\ \ \partial_{n+1}^*
\end{array}$$

Thus we observe that

$$\overline{H}^n_{(2)}(L) \cong \mathcal{H}_n^{(2)},\ \forall n \geq 0.$$

5.7 Proposition
$\mathcal{H}_n^{(2)} = \ker\, \Delta_n$*, where Δ is the Laplace operator defined by* $\Delta_n = \partial_n^* \, \partial_n + \partial_{n+1} \, \partial_{n+1}^*,\ n \geq 0$.

5.8 Exercise
Show that Δ_n is a positive operator. Prove the proposition.

5.9 Definition (Harmonic ℓ^2-n-Chains)
$\mathcal{H}_n^{(2)}$ *is called the space of harmonic ℓ^2-n-chains.*

5.2 Some Computations

(1) If L is the straight line simplicial complex

$$\ldots\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\bullet\text{——}\ldots\ ,$$

then $\overline{H}_*^{(2)}(L) = 0 = \overline{H}^*_{(2)}(L)$.

(2) If L is the Cayley graph of the free group $\mathbf{F}_2$ with respect to a generating set with 2 elements, then $H_1(L, \mathbb{Z}) = 0$, but $H_1^{(2)}(L) \neq 0$. The other homologies are all zero.

5.10 Exercise
Prove these facts.

5.3 Group Actions on Simplicial Complexes

All the simplicial complexes, unless otherwise mentioned, will be uniformly locally bounded.

Let a group Γ act freely[37] on a simplicial complex L.

Choose one oriented n-simplex for each Γ-orbit of n-simplices and call this family $\{\sigma_i\}$. Then we have

$$L^{(n)} = \sqcup_i\ \Gamma \cdot \sigma_i.$$

This gives an orthogonal decomposition

$$\begin{aligned} C_n^{(2)}(L) &= \left\{ \sum_n a_n \sigma_n : \sum_n |a_n|^2 < \infty \right\} \\ &= \left\{ \sum_i \sum_{\gamma \in \Gamma} a_{\gamma,\, i} \gamma \cdot \sigma_i : \sum_{\gamma,\, i} |a_{\gamma,\, i}|^2 < \infty \right\} \\ &= \bigoplus i \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \cdot \sigma_i : \sum_\gamma |a_\gamma|^2 < \infty \right\} \\ &\cong \bigoplus_i \ell^2(\Gamma). \end{aligned}$$

In fact, the above identification is Γ-equivariant (for left regular representation of Γ), and

$$\overline{H}_n^{(2)}(L) \cong \mathcal{H}_n^{(2)} \overset{\Gamma}{\hookrightarrow} C_n^{(2)} \cong \bigoplus_i \ell^2(\Gamma).$$

Thus $\overline{H}_n^{(2)}(L)$ is an $L\Gamma$-Hilbert module, and the ***ℓ^2-Betti numbers*** for the Γ-action on L are defined as

$$\beta_n^{(2)}(L, \Gamma) := \dim_{L\Gamma} \overline{H}_n^{(2)}(L),\ n \geq 0. \tag{47}$$

Here, the dimension on the right is the von Neumann dimension of Γ-Hilbert space.

5.11 Theorem
If L is an **acyclic** *(i.e. n-connected for all n) simplicial complex and it is co-compact with respect to a free action of a group Γ, then the Betti numbers $\beta_n^{(2)}(L, \Gamma)$, $n \geq 0$ do not depend upon L.*

5.4 ℓ^2-Betti Numbers of Groups

Let L and L' be two uniformly locally bounded simplicial complexes and suppose a group Γ acts freely on them. L and L' are said to be Γ-equivariantly ***homotopy equivalent*** if their respective chain complexes $C_\bullet(L, \mathbb{Z})$ and $C_\bullet(L', \mathbb{Z})$ are homotopy equivalent by a Γ-equivariant homotopy.

5.12 Theorem
If two simplicial complexes L and L' are Γ-cocompact, and Γ-equivariantly homotopy equivalent, then

$$\overline{H}_*^{(2)}(L) \cong \overline{H}_*^{(2)}(L').$$

In particular, $\beta_*^{(2)}(L, \Gamma) = \beta_*^{(2)}(L', \Gamma)$.

---

[37] The action is orientation preserving and no simplex is fixed.

5.13 Definition

If L is Γ-cocompact and acyclic (n-connected $\forall\, n \geq 0$), then define the ℓ^2**-Betti numbers** *of the group Γ for all $0 \leq i$:*

$$\beta_i^{(2)}(\Gamma) := \beta_i^{(2)}(L, \Gamma). \tag{48}$$

In general, a countable group does not admit such a space to act freely upon.

5.14 Theorem

If L is Γ-cocompact and n-connected, then the following does not depend on the particular L, and define the ℓ^2**-Betti numbers** *of the group Γ for $0 \leq i \leq n$:*

$$\beta_i^{(2)}(\Gamma) := \beta_i^{(2)}(L, \Gamma). \tag{49}$$

If Γ acts freely on L, and L/Γ is compact, then we consider the Euler characteristic

$$\begin{aligned}\chi(L/\Gamma) &:= \sum_n (-1)^n \dim_{L\Gamma} C_n^{(2)}(L) \\ &= \sum_n (-1)^n \, \# \{n\text{-simplices in } L/\Gamma\} \\ &= \sum_n (-1)^n \dim_{\mathbb{C}} H_i(L/\Gamma).\end{aligned}$$

The equalities above are not hard to prove. Furthermore, we have:

5.15 Proposition

$\chi(L/\Gamma) = \chi^{(2)}(L, \Gamma) := \sum_n (-1)^n \beta_n^{(2)}(L, \Gamma)$.

5.16 Exercise

Prove this proposition.

[***hint*** *: Use the orthogonal decomposition $C_i^{(2)}(L) = (\ker \partial)^\perp \oplus \mathcal{H}_i \oplus (\ker \partial^*)^\perp$, and Rank Nullity Theorem.*]

5.17 Proposition (Reciprocity Formula)

If Λ is a finite index subgroup of Γ and Γ acts freely cocompactly on L, then L is also Λ-cocompact and

$$\beta_*^{(2)}(L, \Lambda) = [\Gamma : \Lambda]\, \beta_*^{(2)}(L, \Gamma).$$

5.18 Exercise

Prove the reciprocity formula.

[***hint*** *: Γ splits into finitely many Λ-cosets.*]

5.19 Theorem (Lück [Lüc94])

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be a decreasing sequence of normal, finite index subgroups of Γ such that $\cap_i \Gamma_i = \{1\}$. Suppose that $\Gamma_i \curvearrowright L$ is a cocompact action for all $i \in \mathbb{N}$. Then

$$\frac{b_n(L/\Gamma_i)}{[\Gamma : \Gamma_i]} \to \beta_n^{(2)}(L, \Gamma),\ \forall\, n \geq 0, \tag{50}$$

where $b_n(L/\Gamma_i)$ denotes the usual n-Betti number of the compact space L/Γ_i.

Note: Farber [Far98] has shown that this theorem remains valid if instead of normality and trivial intersection one assumes the Farber's condition (23) form exercise 2.70. Gaboriau-Bergeron [BG04] have further extended this result by removing the Farber condition: The sequence (50) still converges but the limit is different in general and is interpreted as some foliation Betti numbers (see [BG04]).

5.20 Remark

Cheeger and Gromov [CG86] defined the ℓ^2 Betti numbers for general countable groups by considering their actions on topological spaces.

5.5 Some Properties

Some properties of ℓ^2-Betti numbers.

1. For free products, we have

$$\beta_1^{(2)}(\Gamma_1 * \Gamma_2) = \beta_1^{(2)}(\Gamma_1) + \beta_1^{(2)}(\Gamma_2) + 1 - [\beta_0^{(2)}(\Gamma_1) + \beta_0^{(2)}(\Gamma_2)],$$

   where the sum in the square bracket vanishes if the groups Γ_i, $i = 1, 2$ are infinite; and

$$\beta_n^{(2)}(\Gamma_1 * \Gamma_2) = \beta_n^{(2)}(\Gamma_1) + \beta_n^{(2)}(\Gamma_2), \forall n \geq 2.$$

2. (Mayer-Vietoris + Cheeger-Gromov) For free products with amalgamation over an infinite amenable subgroup, we have

$$\beta_n^{(2)}(\Gamma_1 *_{\Gamma_3} \Gamma_2) = \beta_n^{(2)}(\Gamma_1) + \beta_n^{(2)}(\Gamma_2), \forall n \geq 0.$$

3. (Künneth) For direct products, we have

$$\beta_n^{(2)}(\Gamma_1 \times \Gamma_2) = \sum_{i+j=n} \beta_i^{(2)}(\Gamma_1)\, \beta_j^{(2)}(\Gamma_2).$$

4. (Poincaré Duality) For the fundamental group Γ of a closed aspherical manifold of dimension p, we have

$$\beta_n^{(2)}(\Gamma) = \beta_{n-p}^{(2)}(\Gamma).$$

5.6 A list of ℓ^2 Betti Numbers.

We give some ℓ^2 Betti numbers:

Group Γ	$\beta_*^{(2)}(\Gamma)$
Γ finite	$(\frac{1}{\lvert\Gamma\rvert}, 0, 0, \ldots)$
Γ generated by g elements	$\beta_1^{(2)}(\Gamma) \leq g - 1$
Γ infinite amenable	$(0, 0, 0, \ldots)$
$\mathbf{F}_n$	$(0, n-1, 0, \ldots)$
$\pi_1(S_g)$	$(0, 2g-2, 0, \ldots)$
Lattice in $\mathrm{SO}(p, q)$	$\beta_d^{(2)}(\Gamma) = \begin{cases} \chi^{(2)}(\Gamma) & \text{if } d = pq/2 \\ 0 & \text{otherwise} \end{cases}$
Lattice in $\mathrm{SL}(n, \mathbb{R})$, $n > 2$	$(0, 0, 0, \ldots)$
$\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_l}$	$\beta_d^{(2)}(\Gamma) = \begin{cases} \prod_{j=1}^{l}(p_j - 1) & \text{if } d = l \\ 0 & \text{otherwise} \end{cases}$
$(\mathbf{F}_m \times \mathbf{F}_n) * \mathbf{F}_k$	$(0, k, (m-1)(n-1), 0, \ldots)$
$(\bigoplus_{n\in\mathbb{N}} \mathbf{F}_2) \times \mathbb{Z}$	$(0, 0, 0, \ldots)$
one-relator group $\Gamma = \langle g_1, \cdots, g_k \lvert r\rangle$ $r = w^m$ with max m	$\beta_d^{(2)}(\Gamma) = \begin{cases} k - 1 - \frac{1}{m} & \text{if } d = 1 \\ 0 & \text{otherwise} \end{cases}$
$\Gamma = \mathrm{MCG}(S_g)$ Mapping class group and B_j Bernoulli number	$\beta_d^{(2)}(\Gamma) = \begin{cases} \frac{\lvert B_{2g}\rvert}{4g(g-1)} & \text{if } d = 3g - 3 \\ 0 & \text{otherwise} \end{cases}$

For one-relator groups, see [DL07].

6 L^2-Betti Numbers for p.m.p. Equivalence Relations and Proportionality Principle

In a somewhat similar manner, there is a well defined notion of L^2-Betti numbers $\beta_i^{(2)}(\mathscr{R})$ for equivalence relations, this is the central result of [Gab02a]. An important feature is that they coincide with group-ℓ^2-Betti numbers in case of free p.m.p. actions.

6.1 Theorem (Gaboriau [Gab02a])
If $\Gamma \curvearrowright X$ freely, then $\beta_i^{(2)}(\mathscr{R}_\Gamma) = \beta_i^{(2)}(\Gamma)$, $\forall i \geq 0$.

6.2 Corollary (Gaboriau [Gab02a, Th. 3.2])
If Γ_1 and Γ_2 have free OE actions, then $\beta_*^{(2)}(\Gamma_1) = \beta_*^{(2)}(\Gamma_2)$.

6.3 Corollary (Gaboriau [Gab02a])
If Γ_1 and Γ_2 have free SOE actions, with associated complete sections A_1 and A_2 respectively, then $\frac{\beta_*^{(2)}(\Gamma_1)}{\mu_1(A_1)} = \frac{\beta_*^{(2)}(\Gamma_2)}{\mu_2(A_2)}$.

In particular, one obtains the following general proportionality principle. It was previously known for lattice in various Lie groups (quite easy for cocompact lattices, a bit harder for non-cocompact ones: see [Gro93, §8] and further references to articles of Cheeger and Gromov there).

6.4 Theorem (Proportionality principle, Gaboriau [Gab02a, Cor. 0.2])
If Γ and Λ are lattices in a locally compact second countable group G, then for every $i \geq 0$ we have

$$\frac{\beta_i^{(2)}(\Gamma)}{\mathrm{Haar}(G/\Gamma)} = \frac{\beta_i^{(2)}(\Lambda)}{\mathrm{Haar}(G/\Lambda)}. \tag{51}$$

This common quantity is by definition the ***i-th ℓ^2-Betti number of*** G and is denoted $\beta_i^{(2)}(G)$. It is well defined for every locally compact second countable group G that admits a lattice, and once the Haar measure is prescribed. This indirect definition has been made direct and extended to locally compact second countable unimodular groups G that admit no lattice (see [Pet13]). The equivalence of the two definitions has been proved in [KPV15].

⇝ Notes to be developed...

7 An ℓ^2-Proof of “Treeings realize the cost” (Th. 2.23)

We give an ℓ^2-proof of the central cost theorem. This proof has already been presented during the “Borel Seminar” (Berne, 2002) and “Instructional Workshop on Operator Algebras/Non-commutative Geometry” (Chennai, 2008).

7.1 Theorem
If Ψ is a treeing of $\mathscr{R}$, then it realizes the cost of $\mathscr{R}$:

$$\mathscr{C}(\mathscr{R}) = \mathscr{C}(\Psi)$$

Let Φ be another graphing of $\mathscr{R}$. We have to show that $\mathscr{C}(\Phi) \geq \mathscr{C}(\Psi)$.

We get fields of graphs, equipped with a discrete action of $\mathscr{R}$ together with an $\mathscr{R}$-equivariant isomorphism between their vertex sets.

$$\begin{tikzcd}
\Sigma_\Psi & \Sigma_\phi & \Sigma_\Psi \\
X \arrow[u] & X \arrow[u] & X \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{(0)}_\Psi \arrow[r, phantom, "="] & \Sigma^{(0)}_\phi \arrow[r, phantom, "="] & \Sigma^{(0)}_\Psi \\
X \arrow[u] \arrow[r, phantom, "="] & X \arrow[u] \arrow[r, phantom, "="] & X \arrow[u]
\end{tikzcd}$$

Up to subdividing their domains each $\varphi \in \Phi$ could be expressed by **replacing Ψ-words** w_φ, and each $\psi \in \Psi$ as replacing Φ-words ω_ψ (we do not effectively subdivide the domains!).

This induces measurable fibred maps:

$$\begin{tikzcd}
C_1(\Psi[x], \mathbb{Z}) \arrow[r, "g^x"] \arrow[d, "\partial_1^x"] & C_1(\Phi[x], \mathbb{Z}) \arrow[r, "f^x"] \arrow[d, "\partial_1^x"] & C_1(\Psi[x], \mathbb{Z}) \arrow[d, "\partial_1^x"] \\
C_0(\Psi[x], \mathbb{Z}) \arrow[r, phantom, "="] & C_0(\Phi[x], \mathbb{Z}) \arrow[r, phantom, "="] & C_0(\Psi[x], \mathbb{Z})
\end{tikzcd}$$

s.t. $\partial_1^x \circ g^x = \partial_1^x$ and $\partial_1^x \circ f^x = \partial_1^x$.

It follows that $f^x \circ g^x = id$ since each equation $\partial_1^x(c) = v_1 - v_0$ has a single solution c for $v_1, v_2 \in \Psi^0[x]$, the set of vertices of the tree $\Psi[x]$.

- These maps extend to the ℓ^2 setting as soon as they give bounded operators, in particular as soon as $\|w_\varphi\|_\Psi$ and $\|\omega_\psi\|_\Phi$ are uniformly bounded. If it is the case, they integrate as $M_{\mathscr{R}}$-equivariant bounded operators between the Hilbert $M_{\mathscr{R}}$-modules:

$$\begin{tikzcd}
\int_X^{\oplus} C_1^{(2)}(\Psi[x])\, d\mu(x) \arrow[r, "G"] \arrow[d, "\partial_1"] & \int_X^{\oplus} C_1^{(2)}(\Phi[x])\, d\mu(x) \arrow[r, "F"] \arrow[d, "\partial_1"] & \int_X^{\oplus} C_1^{(2)}(\Psi[x])\, d\mu(x) \arrow[d, "\partial_1"] \\
\int_X^{\oplus} C_0^{(2)}(\Psi[x])\, d\mu(x) \arrow[r, "="] & \int_X^{\oplus} C_0^{(2)}(\Phi[x])\, d\mu(x) \arrow[r, "="] & \int_X^{\oplus} C_0^{(2)}(\Psi[x])\, d\mu(x)
\end{tikzcd}$$

s.t. $F \circ G = Id$. This leads, by the rank-nullity theorem, to

$$\begin{aligned}
\dim_{M_{\mathscr{R}}} \int_X^{\oplus} C_1^{(2)}(\Phi[x])\, d\mu(x) &\geq \dim_{M_{\mathscr{R}}} \int_X^{\oplus} C_1^{(2)}(\Psi[x])\, d\mu(x) && (52) \\
\mathscr{C}(\Phi) &\geq \mathscr{C}(\Psi) && (53)
\end{aligned}$$

- If the length of the words $\|w_\varphi\|_\Psi$ and $\|\omega_\psi\|_\Phi$ are NOT uniformly bounded.

Let $\Phi_n = (\varphi_n)$ be obtained by removing from the domain of the φ's the locus where the replacing Ψ-words have length $\geq n$.

For Ψ_n, we'll have two conditions:
Let $\Psi_n = (\psi_n)$ obtained by removing from the domain of the ψ's the locus where the replacing Φ-words use pieces from $\varphi \setminus \varphi_n$ and also the locus where the replacing Φ-words have length $\geq n$.

We have $\mathscr{C}(\Psi_n) \to_n \mathscr{C}(\Psi)$ and $\mathscr{C}(\Phi_n) \to_n \mathscr{C}(\Phi)$.

Now, the fields g^x, f^x induce bounded operators for the restricted fields of graphs

$$\begin{tikzcd}
\int_X^{\oplus} C_1^{(2)}(\Psi_n[x])\, d\mu(x) \arrow[r, "G"] \arrow[d, phantom, "\|"] & \int_X^{\oplus} C_1^{(2)}(\Phi_n[x])\, d\mu(x) \arrow[r, "F"] \arrow[d, phantom, "\|"] & \int_X^{\oplus} C_1^{(2)}(\Psi[x])\, d\mu(x) \arrow[d, phantom, "\|"] \\
C_1^{(2)}(\Psi_n) \arrow[r, "G"] & C_1^{(2)}(\Phi_n) \arrow[r, "F"] & C_1^{(2)}(\Psi)
\end{tikzcd},$$

where $F \circ G = Id$ in restriction to $C_1^{(2)}(\Psi_n)$. The rank nullity theorem then gives

$$\begin{array}{rcl}
\dim_{M_{\mathscr{R}}} \int_X^{\oplus} C_1^{(2)}(\Phi_n[x])\, d\mu(x) & \geq & \dim_{M_{\mathscr{R}}} \int_X^{\oplus} C_1^{(2)}(\Psi_n[x])\, d\mu(x) \\
\mathscr{C}(\Phi_n) & \geq & \mathscr{C}(\Psi_n) \\
\downarrow & n \to \infty & \downarrow \\
\mathscr{C}(\Phi) & & \mathscr{C}(\Psi)
\end{array}$$

■ ↑

8 Uncountably Many Actions up to OE

8.1 Review of results

8.1 Theorem (Dye [Dye59])
Any two ergodic p.m.p. free actions of $\Gamma_1 \simeq \mathbb{Z}$ and $\Gamma_2 \simeq \mathbb{Z}$ are orbit equivalent.

8.2 Theorem (Ornstein-Weiss [OW80])
Any two ergodic p.m.p. free actions of any two infinite amenable groups are orbit equivalent.

8.3 Theorem (Connes-Weiss [CW80])
Any countable group that is ***neither amenable nor Kazhdan property (T)*** *admits at least two non OE p.m.p. free ergodic actions.*

$\rightsquigarrow$ strong ergodicity (Schmidt, [Sch80]) + Gaussian actions.

8.4 Theorem (Bezuglyĭ-Golodets [BG81])
The first examples of groups with ***uncountably many non OE*** *p.m.p. free ergodic actions, for a somewhat circumstantial family of groups, introduced by [McDuff 1969].*

8.5 Theorem (Gefter-Golodets [GG88])
Non uniform ***lattices in higher rank simple Lie groups*** *with finite center produce uncountably many non OE p.m.p. free ergodic actions.*

$\rightsquigarrow$ Relies on Zimmer's super rigidity for cocycles.
Ex. $\mathrm{SL}(n,\mathbb{Z})$ $n \geq 3$.

8.6 Theorem (Hjorth 2002, [Hjo05])
Each infinite group with ***Kazhdan property (T)*** *produces uncountably many non OE p.m.p. free ergodic actions.*

8.7 Theorem (Monod-Shalom [MS06])
There exists a ***continuum*** *of finitely generated groups, each admitting a* ***continuum*** *of p.m.p. free actions, such that no two actions in this whole collection are orbit equivalent.* [38]

The family of groups is: $\Gamma = \Gamma_1 \times \Gamma_2$ where $\Gamma_i = A * B$ range over all free products of any two torsion-free infinite countable groups.
$\rightsquigarrow$ Relies on bounded cohomology.
Ex. non trivial ($l \geq 2$) products of free groups $\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_l}$, $p_i \geq 2$.

On the other hand, the situation for the free groups themselves or $\mathrm{SL}(2,\mathbb{Z})$ remained unclear and arouse the interest of producing more non OE free ergodic actions of $\mathbf{F}_n$, i.e. in producing ways to distinguish them from the OE point of view.

8.2 What about the free group itself ?

The number of non OE p.m.p. free ergodic actions of the free group $\mathbf{F}_p$ ($2 \leq p \leq \infty$) was shown to be ≥ 2 (Connes-Weiss [CW80]); ≥ 3 (Popa [Pop06a])[39]; ≥ 4 (Hjorth, [Hjo05])[40].

8.8 Theorem (Gaboriau -Popa [GP05])
For each $2 \leq n \leq \infty$ there exists an uncountable family of non stably orbit equivalent (non-SOE) free ergodic p.m.p. actions α_t of $\mathbf{F}_n$.
Moreover the corresponding equivalence relations $\mathscr{R}_{\alpha_t,\mathbf{F}_n}$ have at most countable fundamental group (trivial in the case $n < \infty$) and at most countable outer automorphism group. [41]

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[38] first version 2002.
[39] first version 2001.
[40] first version 2002.
[41] first version 2003.

The proof of this result Th. 8.8, as well as Th. 8.11, Th. 8.18 and the refined versions sect. 8.7, uses deeply the theory of ***rigid action*** of S. Popa ([Pop06a]).

The result Th.8.8 is an existence result. Ioana [Ioa09] gave an explicit 1-parameter family of actions of $\mathbf{F}_n$. Consider the action of $\mathbf{F}_n$ on the torus $\mathbb{T}^2$ via a fixed embedding in $\mathrm{SL}(2,\mathbb{Z})$. Consider a fixed epimorphism $\pi : \mathbf{F}_n \to \mathbb{Z}$ and for $t \in (0, \frac{1}{2}]$ the Bernoulli shift action of $\mathbf{F}_n$ through π on the 2-points probability space $\mathbf{F}_n \twoheadrightarrow \mathbb{Z} \curvearrowright (\{0,1\}, t\delta_0 + (1-t)\delta_1)^{\mathbb{Z}}$ with masses t and $1-t$.

The diagonal action of $\mathbf{F}_n$ on the product $\mathbb{T}^2 \times (\{0,1\}, t\delta_0 + (1-t)\delta_1)^{\mathbb{Z}}$ provides a family of pairwise non-equivalent, free, ergodic actions.

8.3 More groups

8.9 Theorem (Ioana [Ioa07])
Given any group G of the form $G = H \times K$ with H non amenable and K infinite amenable, there exist a sequence σ_n of free ergodic, non-strongly ergodic p.m.p. non SOE actions. [42]

$\rightsquigarrow$ introduced an invariant for p.m.p. actions $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ denoted $\chi_0(\sigma; G)$ and defined as the "intersection" of the 1-cohomology group $H^1(\sigma, G)$ with Connes' invariant $\chi(M)$ of the crossed-product von Neumann algebra $L^\infty X \rtimes_\alpha \Gamma$

8.10 Theorem (S. Popa [Pop06b])
If Γ contains an infinite normal subgroup with the relative Kazhdan property (T), then Γ admits a continuum of non OE actions.

Relies on explicit computation of 1-cohomology groups of actions. For each infinite abelian group H, Popa constructs an action α_H of Γ with $H^l(\sigma_H, \Gamma) = \mathrm{char}(\Gamma) \times H$.
$\rightsquigarrow$ First explicit use of the 1-cohomology groups (considered for instance by Feldman-Moore in [FM77]) to distinguish non OE actions.

8.4 Almost all non-amenable groups

8.11 Theorem (Ioana [Ioa11])
[43] *Let Γ be a countable group which contains a copy of the free group $\mathbf{F}_2$. Then*

1. *Γ has uncountably many non-OE actions.*
2. *Any Λ ME to Γ has uncountably many non-OE actions.*

In fact the above actions may be taken not only Orbit inequivalent, but also von Neumann inequivalent.

8.5 Comments on von Neumann's problem

Amenability of groups is a concept introduced by J. von Neumann in his seminal article [vN29] to explain the so-called Banach-Tarski paradox.

In particular, if Γ contains $\mathbf{F}_2$, then Γ is not amenable.

8.12 Question (von Neumann's Problem)
If Γ is non-amenable, does it contain $\mathbf{F}_2$?

A. Ol′šanskiĭ [1982] gave a negative answer. The examples he constructed of groups with all proper subgroups cyclic (1980) in both cases torsion-free and torsion (the so-called "Tarski monsters") are non-amenable.

Still, this characterization could become true after relaxing the notion of "containing a subgroup"...

K. Whyte (1999) gave a very satisfactory **geometric group-theoretic** solution:

8.13 Theorem (Whyte [Why99])
A finitely generated group Γ is non-amenable iff it admits a partition with uniformly Lipschitz-embedded copies of the regular 4-valent tree.

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[42] First version 2004.
[43] first version 2006.

There is also a reasonable solution in the measure theoretic context.

8.14 Theorem (Gaboriau -Lyons 2007 [GL09])
For any countable discrete non-amenable group Γ, *there is a measurable ergodic essentially free action* σ *of* $\mathbf{F}_2$ *on* $([0,1]^\Gamma, \mu)$ *such that a.e.* Γ*-orbit of the Bernoulli shift decomposes into* $\mathbf{F}_2$*-orbits.*

In other words, the orbit equivalence relation of the $\mathbf{F}_2$-action is contained in that of the Γ-action: $\mathscr{R}_{\sigma(\mathbf{F}_2)} \subset \mathscr{R}_\Gamma$.

The key point (we don't touch here) in proving that an ergodic equivalence relation $\mathscr{R}$ contains the orbits of a free $\mathbf{F}_2$-action (as in Theorem 2.66 or 8.14) is to find an ergodic sub-equivalence relation $\mathscr{S} < \mathscr{R}$ of cost > 1. The use of Theorem 8.15 proved independently by Kechris-Miller and Pichot allows then to conclude.

When $\mathscr{R}$ is ergodic, it contains an ergodic hyperfinite sub-equivalence relation $\mathscr{S}_0$. One can extend the graphing $\mathscr{G}$ in Theorem 8.15 so as to contain an ergodic treeing $\mathscr{T}_0$ of $\mathscr{S}_0$, with the result that $\mathscr{T}$ contains an ergodic cost 1 subtreeing $\mathscr{T}_0$. This $\mathscr{T}_0$ makes easy the realization of $\mathscr{T}$ by a free action of $\mathbf{F}_2$ when $\mathscr{C}(\mathscr{T}) = 2$ (resp. of a certain ergodic subrelation $\mathscr{T}_1 \subset \mathscr{T}$ of cost 2: use a restriction $\mathscr{T} \upharpoonright B$ to some Borel subset B of measure $1/p$ with p an integer so that the cost becomes $\mathscr{C}(\mathscr{T} \upharpoonright B) \geq 3$ (by the induction formula – Proposition 2.32) then restrict the induced treeing to a cost 2 subrelation containing $\mathscr{T}_0 \upharpoonright B$ and then pick a finite index subrelation of well chosen index (using the Schreier formula) so that it induces $\mathscr{T}_1$ of cost 2 on X).

8.15 Theorem (Kechris-Miller [KM04], Pichot [Pic05])
Let $\mathscr{R}$ *be a p.m.p. equivalence relation on* (X, μ) *and* $\mathscr{G}$ *be a generating (oriented) graphing. Then* $\mathscr{G}$ *admits a subtreeing* $\mathscr{T}$ *of cost* $\geq \mathscr{C}(\mathscr{R})$. *Moreover,* $\mathscr{T}$ *can be assumed to contain any prescribed subtreeing* $\mathscr{T}_0 \subset \mathscr{G}$.

We propose here a shorter proof, in the spirit of the hint of Exercise 1.15.
$\Diamond$*Proof of Theorem 8.15*: The subset $\mathscr{G}$ of $(\mathscr{R}, \nu)$ is equipped with the restriction of the measure ν and has total mass $c := \nu(\mathscr{G}) = \mathscr{C}(\mathscr{G})$. The base space being assumed atomless, one can pick a measure preserving isomorphism

$$\eta : ([0,c], \lambda) \to (\mathscr{G}, \nu \upharpoonright \mathscr{G}),$$

where λ is the Lebesgue measure. WLOG one can assume that $\mathscr{T}_0$ is the η-image of the initial segment $[0, \mathscr{C}(\mathscr{T}_0)]$.

Any measurable subset $A \subset [0,c]$ defines a subgraphing $\mathscr{G}_A = \eta(A) \subset \mathscr{G}$ of cost $\lambda(A)$. If A, B are measurable subsets of $[0,c]$, we say that $\mathscr{G}_B$ is realized in $\mathscr{G}_A$ (denote $\mathscr{G}_B \prec \mathscr{G}_A$) if $\mathscr{G}_B$ is contained in the equivalence relation generated by $\mathscr{G}_A$. Define $f(\tau) := \inf \left\{ t \colon \mathscr{G}_{\{\tau\}} \prec \mathscr{G}_{[0,t]} \right\}$ i.e. the infimum of the $t \in [0,c]$ such that the end-points of the edge $\eta(\tau)$ are already connected by a path of edges in $\eta([0,t])$. Define $A_n := \{\tau \colon \tau \in [f(\tau), f(\tau) + 2^{-n}]\}$ and $A_\infty := \{\tau \colon \tau = f(\tau)\}$. We have two lemmas:

8.16 Lemma
$\mathscr{T} := \mathscr{G}_{A_\infty}$ *is a subtreeing of* $\mathscr{G}$ *(containing* $\mathscr{T}_0$*).*

$\mathscr{G}_{A_\infty}$ is indeed the minimal spanning forest associated with η^{-1}. It clearly contains $\mathscr{T}_0$. ■ ↑

8.17 Lemma
For all $n \geq 1$, *the subgraphing* $\mathscr{G}_{A_n} \subset \mathscr{G}$ *generates* $\mathscr{R}$.

We show that $\mathscr{G}_{A_n}$ is generating by induction on p by showing that all the $\mathscr{G}_{[0,p2^{-n}] \cap [0,c]}$ (and thus $\mathscr{G}_{[0,c]}$) are realized in $\mathscr{G}_{A_n}$: First of all, $\mathscr{G}_{[0,2^{-n}] \cap [0,c]}$ is realized in $\mathscr{G}_{A_n}$. Assume now that $\mathscr{G}_{[0,p2^{-n}] \cap [0,c]}$ is realized in $\mathscr{G}_{A_n}$. Let $\tau \in [p2^{-n}, (p+1)2^{-n}] \cap [0,c]$. If $\tau \in A_n$ then $\mathscr{G}_{\{\tau\}} \in \mathscr{G}_{A_n}$ and we are done. Otherwise, $f(\tau) < \tau - 2^{-n}$. Thus $\mathscr{G}_{\{\tau\}} \prec \mathscr{G}_{[0,p2^{-n}] \cap [0,c]} \prec \mathscr{G}_{A_n}$ and we are done by transitivity of $\prec$.
■ ↑

Since $\mathscr{G}_{A_n}$ is generating, $\lambda(A_n) = \mathscr{C}(\mathscr{G}_{A_n}) \geq \mathscr{C}(\mathscr{R})$ for all $n \geq 1$ and since $A_\infty = \cup_n \searrow A_n$, we get $\lambda(A_\infty) = \mathscr{C}(\mathscr{G}_{A_\infty}) \geq \lim \searrow \lambda(A_n) \geq \mathscr{C}(\mathscr{R})$. This finishes the proof of Theorem 8.15 ■ ↑

8.6 Conclusion

Taking advantage of Theorem 8.14, I. Epstein generalized Ioana's theorem:

8.18 Theorem (Epstein, 2007 [Eps08])
If Γ *is non-amenable, then* Γ *admits continuum many orbit inequivalent free p.m.p. ergodic actions.*

Thus leading to the complete solution of this long standing problem.

8.7 Refined versions

Recall: To free p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$, Murray and von Neumann associated a von Neumann algebra: the ***crossed-product*** or ***group-measure-space construction*** denoted $A \rtimes_{\alpha} \Gamma$ where $A = L^{\infty}(X,\mu)$.

Free p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$ and $\Lambda \curvearrowright^{\sigma} (X,\mu)$ are **orbit equivalent** iff the crossed-products are isomorphic via an isomorphism which sends A to A: $(A \subset A \rtimes_{\alpha} \Gamma) \simeq (A \subset A \rtimes_{\sigma} \Lambda)$; and **stably orbit equivalent** if (say for ergodic actions) there is some $t > 0$ such that $(A \subset A \rtimes_{\alpha} \Gamma)^{t} \simeq (A \subset A \rtimes_{\sigma} \Lambda)$.

8.19 Definition
The actions are ***von Neumann equivalent*** *if* $A \rtimes_{\alpha} \Gamma \simeq A \rtimes_{\sigma} \Lambda$.

(this is weaker than OE).

REM. Connes-Jones [CJ82] example of a factor $M \simeq M \otimes R$ with two non OE Cartan subalgebras provides precisely an instance of von Neumann equivalent actions that are not OE.

In fact the above actions in Theorem 8.8 may be taken not only Orbit inequivalent, but also von Neumann inequivalent.

8.20 Theorem (Ioana [Ioa11])
If Γ is non-amenable, then Γ admits continuum many von Neumann inequivalent free p.m.p. ergodic actions.

8.21 Theorem (Törnquist [Tör06])
Consider the orbit equivalence relation on measure preserving free ergodic actions of the free group $\mathbf{F}_2$.
– The equivalence relation E_0 can be Borel reduced to it.
– It cannot be classified by countable structures.

8.22 Theorem (Epstein-Ioana-Kechris-Tsankov [IKT09])
Let Γ be a non-amenable group. Consider the orbit equivalence relation on measure preserving, free ergodic[44] *actions of Γ.*
– The equivalence relation E_0 can be Borel reduced to it.
– It cannot be classified by countable structures.

If Γ is not amenable, orbit equivalence of such actions is unclassifiable in various strong senses.

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[44] In fact even mixing actions.

9 A Proof: The Free Group $\mathbf{F}_\infty$ has Uncountably Many non OE Actions

We will prove the following:

9.1 Theorem (Gaboriau -Popa [GP05])
The free group $\mathbf{F}_n$, $n = 3, 4, \cdots, \infty$ admits uncountably many free p.m.p. ergodic actions that are pair-wise Orbit inequivalent.

Let's prove it for $\mathbf{F}_\infty$ first. Why is it "easier" ? The group $\mathbf{F}_\infty$ contains uncountably many distinct subgroups isomorphic with $\mathbf{F}_\infty$.

The following arguments relies on the notion of rigid action of S. Popa, and the presentation below benefited inspiration from A. Ioana [Ioa09].

Point 1 – For any p.m.p. equivalence relation $\mathscr{R}$ on (X, μ), the full group $[\mathscr{R}]$ and the group

$$\mathbb{U} := \mathcal{U}L^\infty(X, \mu) = \{f \in L^\infty(X, \mu) : f(x) \in \mathbb{S}^1, \text{ a.e. } x\} \tag{54}$$

of functions on X with values in the group of modulus $= 1$ complex numbers are in semi-direct product:

$$\mathbb{U} \rtimes [\mathscr{R}] \tag{55}$$

where $\psi f \psi^{-1}(x) = f(\psi^{-1}x)$, for $f \in \mathbb{U}$ and $\psi \in [\mathscr{R}]$.

With the natural measure $(\mathscr{R}, \tilde{\mu})$ we have the Hilbert space $\mathcal{H} = L^2(\mathscr{R}, \tilde{\mu})$ and two commuting representations of $\mathbb{U} \rtimes [\mathscr{R}]$:
For $f \in \mathbb{U}$ and $\psi \in [\mathscr{R}]$ and for $\xi \in L^2(\mathscr{R}, \tilde{\mu})$,

$$\pi^l(f\psi)\xi(x, y) := f(x) \quad \xi(\psi^{-1}.x, y) \tag{56}$$

$$\pi^r(f\psi)\xi(x, y) := \overline{f}(y) \quad \xi(x, \psi^{-1}.y) \tag{57}$$

Point 2 – Recall the definition of ***relative property (T)*** for $H < G$ (countable discrete groups): $\forall\delta > 0$, $\exists K$ finite subset of G and $\exists\epsilon > 0$ (K is called a **critical set** and (K, ϵ) a **critical pair**) s.t. if a representation $(\pi, \mathcal{H})$ admits a (K, ϵ)-invariant unit vector ξ, then π admits an H-invariant unit vector ξ_0 near ξ: $\|\xi_0 - \xi\| < \delta$.
REM: The equivalence with the usual definition is quite easy when the subgroup is normal. It is more involved in general, but true (Jolissaint [Jol05]).

Point 3 – Recall Burger theorem: For every non-amenable subgroup $\Gamma < \mathrm{SL}(2, \mathbb{Z})$, the induced pair $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma$ has relative property (T).

Choose some $\Gamma_0 \simeq \mathbf{F}_2$. By Fourier: $\Gamma_0 \curvearrowright \widehat{\mathbb{Z}^2} = \mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$.

Where is hidden the relative property (T) on the Fourier side ? $\rightsquigarrow$ in the **intimate relations between the group and the space**: "The action has property (T) relative to the space $\mathbb{T}^2$".

In fact $\widehat{\mathbb{T}^2} = \widehat{\widehat{\mathbb{Z}^2}} = \mathbb{Z}^2$. Thus $\mathbb{Z}^2$ is hidden in the space as a family of functions, the characters

$$\mathbb{T}^2 \to \mathbb{S}^1 \tag{58}$$

$$\chi_{n_1,n_2} : \quad (z_1, z_2) \mapsto z_1^{n_1} z_2^{n_2} \tag{59}$$

$$\begin{aligned} \Gamma_0 \curvearrowright \mathbb{T}^r \quad \rightsquigarrow \quad & \Gamma_0 \curvearrowright L^\infty(\mathbb{T}^2) \\ & \Gamma_0 \curvearrowright \{\chi_{n_1,n_2} : (n_1, n_2) \in \mathbb{Z}^2\} \simeq \mathbb{Z}^2 \end{aligned}$$

leading back to the semi-direct product $\mathbb{Z}^2 \rtimes \Gamma_0$.

Point 4 – Consider some $\Gamma \simeq \mathbf{F}_\infty < \mathrm{SL}(2, \mathbb{Z})$ and its natural action $\Gamma \curvearrowright \mathbb{T}^2$ on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

This $\mathbf{F}_\infty$ admits a free generating set made of elements that act ergodically (individually).

(Fourier expansion: $\gamma \in \mathrm{SL}_2$ acts ergodically iff it does not have a root of the unit as an eigenvalue (contrarily to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$). Then by basic move, one may transform any free generating system into one whose elements are hyperbolic (see [GP05, lem. 8]).

Let $s_1, s_2,\ a_1, a_2, \cdots, a_n, \cdots$ be this free generating set for $\mathbf{F}_\infty$

$$\mathbf{F}_\infty = \mathbf{F}\langle s_1, s_2\rangle * \mathbf{F}\langle a_n, n \in \mathbb{N}\rangle \tag{60}$$

$$\mathbf{F}_\infty = \underbrace{\mathbf{F}\langle s_1, s_2\rangle}_{\curvearrowright \mathbb{T}^2 \text{ with rel. (T)}} * \mathbf{F}\langle a_n, n \in \mathbb{N}\rangle$$

Point 5 – For each subset $I \subset \mathbb{N}$ define

$$\begin{aligned} \Gamma_I &:= \mathbf{F}\langle s_1, s_2\rangle * \mathbf{F}\langle a_n, n \in I\rangle \qquad < \mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2, \text{ free action} && (61)\\ \mathscr{R}_I &:= \mathscr{R}_{\Gamma_I \curvearrowright \mathbb{T}^2} && (62) \end{aligned}$$

REM:

$$\begin{aligned} &\Gamma_\emptyset = \mathbf{F}\langle s_1, s_2\rangle\\ &\mathbf{F}_\mathbb{N} = \mathbf{F}\langle s_1, s_2\rangle * \mathbf{F}\langle a_n, n \in \mathbb{N}\rangle, \text{ the original } \mathbf{F}_\infty\\ &\Gamma_\emptyset < \Gamma_I < \mathbf{F}_\mathbb{N}\\ &\text{if } |I| = \infty, \text{ then } \Gamma_I \simeq \mathbf{F}_\infty \end{aligned}$$

$\rightsquigarrow$ we get a continuum of
– group actions $\Gamma_I \curvearrowright \mathbb{T}^2$, **each having the original** $\Gamma_\emptyset = \mathbf{F}\langle s_1, s_2\rangle \curvearrowright \mathbb{T}^2$ **subaction**
– equivalence relations $\mathscr{R}_I$ with $\mathscr{R}_\emptyset \subset \mathscr{R}_I \subset \mathscr{R}_\mathbb{N}$

Point 6 – Observe: $\mathscr{R}_I \upharpoonright A = \mathscr{R}_J \upharpoonright A$ for some Borel subset $A \subset \mathbb{T}^2$ with $\mu(A) > 0$ iff $I = J$.
Indeed, by the freeness of the action of $\Gamma_\mathbb{N}$ and the freeness of the generating set $\mathscr{R}_I = \mathscr{R}_J$ iff $I = J$ ($n \in I \setminus J$: no way to get a_n back from Γ_J).
But also (by ergodicity of the common sub-relation $\mathscr{R}_\emptyset$, any x, y have $\Gamma_\emptyset$-representatives in A) $\mathscr{R}_I \upharpoonright A = \mathscr{R}_J \upharpoonright A$ iff $\mathscr{R}_I = \mathscr{R}_J$.

Point 7 – Our goal: We will show that, for every $I_0 \in \mathcal{P}_\infty(\mathbb{N})$, the set

$$\{I \in \mathcal{P}_\infty(\mathbb{N}) : \mathscr{R}_I \overset{\mathrm{OE}}{\sim} \mathscr{R}_{I_0}\} \tag{63}$$

is at most countable. i.e. We will show that the relation on $\mathcal{P}_\infty(\mathbb{N})$

$$I \sim J \iff \mathscr{R}_I \overset{\mathrm{OE}}{\sim} \mathscr{R}_J \tag{64}$$

has at most countable classes.
$\rightsquigarrow$ if you pack together the I's giving OE actions, you get uncountably many packs.

Point 8 – Fix some $I_0 \in \mathcal{P}_\infty(\mathbb{N})$. For our **goal** we will show that if one picks **uncountably** many times some I's in $\{I \in \mathcal{P}_\infty(\mathbb{N}) : \mathscr{R}_I \overset{\mathrm{OE}}{\sim} \mathscr{R}_{I_0}\}$ then at least two of them are equal (!).

Point 9 – Let's concentrate on the common $\Gamma_\emptyset = \mathbf{F}_2$-action

$$\begin{aligned} \mathbf{F}_2 \curvearrowright \mathbb{T}^2 \quad \rightsquigarrow \quad & \mathbf{F}_2 \curvearrowright L^\infty(\mathbb{T}^2) && (65)\\ & \mathbf{F}_2 \curvearrowright \mathcal{U}L^\infty(\mathbb{T}^2) \text{ the unitaries of } L^\infty, \text{ i.e. functions taking values in } \mathbb{S}^1 && (66)\\ & \mathbf{F}_2 \curvearrowright \{\chi_{n_1,n_2} : (z_1, z_2) \mapsto z_1^{n_1} z_2^{n_2}, (n_1, n_2) \in \mathbb{Z}^2\} \ \simeq \ \mathbb{Z}^2 && (67)\\ & \text{leading to the standard matrix multiplication action } \rightsquigarrow \mathbb{Z}^2 \rtimes \mathbf{F}_2 && (68) \end{aligned}$$

for which, the subgroup $\mathbb{Z}^2 < \mathbb{Z}^2 \rtimes \Gamma_\emptyset$ has relative property (T).
$\rightsquigarrow$ Choose a critical pair for $\delta < \sqrt{2}$: $\exists (K, \epsilon)$, $K \subset \Gamma_\emptyset$ finite, $\epsilon > 0$ such that...

Point 10 – Let's now concentrate on the maximal $\mathscr{R}_\mathbb{N}$ and the "maximal universe" $\mathcal{H} = L^2(\mathscr{R}_\mathbb{N}, \tilde{\mu})$ in which all our situation lives. We have two commuting representations of $\mathcal{U}L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\mathbb{N}]$ on $L^2(\mathscr{R}_\mathbb{N}, \tilde{\mu})$.
For $f \in L^\infty(\mathbb{T}^2)$ and $\psi \in [\mathscr{R}_\mathbb{N}]$ and for $\xi \in L^2(\mathscr{R}_\mathbb{N}, \tilde{\mu})$

$$\begin{aligned} \pi^l(f\psi)\xi(x,y) &:= f(x) \ \ \xi(\psi^{-1}.x, y) && (69)\\ \pi^r(f\psi)\xi(x,y) &:= \overline{f}(y) \ \ \xi(x, \psi^{-1}.y) && (70) \end{aligned}$$

Claim: Observe that the particular vector $\mathbf{1}_\Delta \in L^2(\mathscr{R}_\mathbb{N}, \tilde{\mu})$, the characteristic function of the diagonal

$$\Delta := \{(x,x) : x \in \mathbb{T}^2\} \in \mathscr{R}_\mathbb{N} \subset \mathbb{T}^2 \times \mathbb{T}^2$$

satisfies: for $f \in \mathcal{U}L^\infty(\mathbb{T}^2)$ and $\psi \in [\mathscr{R}_\mathbb{N}]$

$$\pi^r(f\psi)\mathbf{1}_\Delta = \pi^l((f\psi)^{-1})\mathbf{1}_\Delta. \tag{71}$$

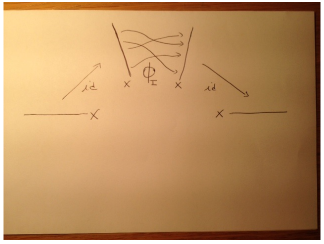
$$\pi^r(f\psi)\mathbf{1}_\Delta(x,y) = \overline{f}(y)\mathbf{1}_\Delta(x, \psi^{-1}y) \quad = \quad \begin{cases} 0 & \text{when } x \neq \psi^{-1}y \\ \overline{f}(y) & \text{when } x = \psi^{-1}y. \end{cases}$$

Since, $\underbrace{(f\psi)^{-1} = \psi^{-1}f^{-1}}_{\in \mathcal{U}L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\mathbb{N}]} = (\underbrace{\psi^{-1} \cdot f^{-1}}_{\in \mathcal{U}L^\infty(\mathbb{T}^2)})\psi^{-1}$ where $(\psi^{-1} \cdot f^{-1})(x) = \bar{f}(\psi x)$, we get:

$$\begin{aligned} \pi^l((f\psi)^{-1})\mathbf{1}_\Delta(x,y) = \bar{f}(\psi x)\mathbf{1}_\Delta(\psi x, y) \quad &= \quad \begin{cases} 0 & \text{when } \psi x \neq y \\ \overline{f}(\psi x) & \text{when } \psi x = y \end{cases} \\ &= \quad \pi^r(f\psi)\mathbf{1}_\Delta(x,y). \end{aligned}$$

Point 11 – Consider an Orbit Equivalence $\mathscr{R}_{I_0} \underset{\Phi_I}{\overset{\mathrm{OE}}{\sim}} \mathscr{R}_I$ for some I

Figure 8: An OE ϕ_I.



$$\begin{tikzcd} & \mathscr{R}_{I_0} \arrow[r, "\phi_I", "\mathrm{OE}"'] & \mathscr{R}_I \arrow[dr, phantom, "\subset" sloped] & \\ \mathscr{R}_0 \arrow[ur, phantom, "\subset" sloped] & & & \mathscr{R}_\mathbb{N} \end{tikzcd} \tag{72}$$

ϕ_I induces a natural group homomorphism $\mathcal{U}L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\emptyset] \overset{\widetilde{\phi_I}}{\to} \mathcal{U}L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\mathbb{N}]$

Through ϕ_I, the inclusions (for $\mathscr{R}_0$) $\mathbb{Z}^2 < L^\infty(\mathbb{T}^2)$ and $\mathbf{F}_2 = \Gamma_0 < [\mathscr{R}_0]$ get “twisted” (for $\mathscr{R}_\mathbb{N}$) :

$$\widetilde{\phi_I}(\mathbf{F}_2) < [\mathscr{R}_\mathbb{N}] \qquad (x \overset{\gamma}{\mapsto} \phi_I \circ \gamma \circ \phi_I^{-1}(x)) \tag{73}$$

$$\widetilde{\phi_I}(\mathbb{Z}^2) < L^\infty(\mathbb{T}^2) \quad (x \mapsto \chi_{n_1,n_2} \circ \phi_I^{-1}(x)) \tag{74}$$

And thus, we get a ϕ_I-twisted embedding

$$\widetilde{\phi_I}(\mathbb{Z}^2 \rtimes \mathbf{F}_2) < \mathcal{U}L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\mathbb{N}]. \tag{75}$$

We thus get two commuting “twisted” representations of $\mathbb{Z}^2 \rtimes \mathbf{F}_2$ on $L^2(\mathscr{R}_\mathbb{N}, \tilde{\mu})$:

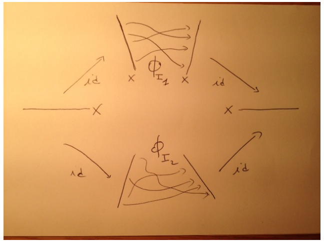
$$\pi_I^l(\chi_{n_1,n_2}\ \gamma)\xi(x,y) \quad := \quad \chi_{n_1,n_2}({\phi_I}^{-1}(x)) \ \ \xi\Big((\phi_I\ \gamma^{-1}\phi_I^{-1})(x), y\Big) \tag{76}$$

$$\pi_I^r(\chi_{n_1,n_2}\ \gamma)\xi(x,y) \quad := \quad \overline{\chi_{n_1,n_2}}({\phi_I}^{-1}(y)) \ \ \xi\Big(x, (\phi_I\ \gamma^{-1}\phi_I^{-1})(y)\Big) \tag{77}$$

Point 12 – Given two Orbit Equivalences $\mathscr{R}_{I_0} \underset{\Phi_{I_1}}{\overset{\mathrm{OE}}{\sim}} \mathscr{R}_{I_1}$ and $\mathscr{R}_{I_0} \underset{\Phi_{I_2}}{\overset{\mathrm{OE}}{\sim}} \mathscr{R}_{I_2}$ for some $I_1, I_2 \in \mathcal{P}_\infty(\mathbb{N})$

$$\begin{tikzcd} & \mathscr{R}_{I_0} \arrow[r, "\phi_{I_1}", "\mathrm{OE}"'] & \mathscr{R}_{I_1} \arrow[dr, phantom, "\subset" sloped] & \\ \mathscr{R}_0 \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & & \mathscr{R}_\mathbb{N} \\ & \mathscr{R}_{I_0} \arrow[r, "\phi_{I_2}", "\mathrm{OE}"'] & \mathscr{R}_{I_2} \arrow[ur, phantom, "\subset" sloped] & \end{tikzcd}$$

Figure 9: Two OE ϕ_1 and ϕ_{I_2}.



We get two embeddings of $\mathbb{Z}^2 \rtimes \mathbf{F}_2)$ in $L^\infty(\mathbb{T}^2) \rtimes [\mathscr{R}_\mathbb{N}]$ and thus two commuting twisted representations and the diagonal representation

$$\pi^l_{I_1}\pi^r_{I_2}(\chi\gamma)\xi(x,y) := \chi(\phi_{I_1}^{-1}(x)) \;\; \overline{\chi}(\phi_{I_2}^{-1}(y)) \;\; \xi((\phi_{I_1}\gamma\, \phi_{I_1}^{-1})^{-1}(x), (\phi_{I_2}\gamma\, \phi_{I_2}^{-1})^{-1}(y)) \tag{78}$$

Point 13 – Let's see whether $\xi := \mathbf{1}_\Delta$, the characteristic function of the diagonal is almost invariant for some (I_1, I_2), for $\delta > 0$, for the elements in the critical set (K, ϵ). For $k \in K$, the representations being unitary and commuting:

$$\begin{aligned} \|\pi^l_{I_1}\pi^r_{I_2}(k)\mathbf{1}_\Delta - \mathbf{1}_\Delta\| \quad &= \quad \|\pi^l_{I_1}(k)\mathbf{1}_\Delta - \underbrace{\pi^r_{I_2}(k^{-1})\mathbf{1}_\Delta}_{\substack{/\!/ \\ \pi^l_{I_2}(k)\mathbf{1}_\Delta}}\| && (79) \\ \text{from eq. (71)} & && (80) \end{aligned}$$

Point 14 – In the **separable** Hilbert space $\bigoplus_{K=\{k_1,k_2,\cdots,k_d\}} L^r(\mathscr{R}_\mathbb{N}, \tilde{\mu})$, the uncountable family of

$$(\pi^l_I(k_1)\mathbf{1}_\Delta, \pi^l_I(k_2)\mathbf{1}_\Delta, \cdots, \pi^l_I(k_d)\mathbf{1}_\Delta)$$

(associated with uncountably many I's taken from $\{I : \mathscr{R}_I \overset{\text{OE}}{\sim} \mathscr{R}_{I_0}\}$ has necessarily at least two elements closer than ϵ:

$$\exists J_1, J_2 \in \{I : \mathscr{R}_I \overset{\text{OE}}{\sim} \mathscr{R}_{I_0}\} : \quad \|\pi^l_{J_1}\pi^r_{J_2}(k)\mathbf{1}_\Delta - \mathbf{1}_\Delta\| < \epsilon \tag{81}$$

Point 15 – By relative property (T), there is a unit vector $\xi_0 \in L^2(\mathbb{T}^2, \tilde{\mu})$ δ-close to $\mathbf{1}_\Delta$ and $\mathbb{Z}^2$-invariant:

$$\begin{aligned} & \|\xi_0 - \mathbf{1}_\Delta\| < \delta && (82) \\ \forall \chi \in \mathbb{Z}^2, \quad & \pi^l_{J_1}\pi^r_{J_2}(\chi)\xi_0 = \xi_0 && (83) \\ \forall \chi \in \mathbb{Z}^2, \quad & \chi(\phi_{J_1}^{-1}(x))\; \overline{\chi}(\phi_{J_2}^{-1}(y))\; \xi_0(x,y) = \xi_0(x,y) && (84) \\ \forall \chi \in \mathbb{Z}^2, \quad & \chi(\phi_{J_1}^{-1}(x))\; \xi_0(x,y) = \chi(\phi_{J_2}^{-1}(y))\; \xi_0(x,y) && (85) \end{aligned}$$

Point 16 – By (eq (82)) and δ small enough ($\delta < \sqrt{2}$ so that Pythagoras...), the set $A := \{x \in X : \xi_0(x,x) \neq 0\}$ is **non-negligible**. Eq (85) then gives on A:

$$\begin{aligned} \forall \chi \in \mathbb{Z}^2, \quad & \chi(\phi_{J_1}^{-1}(x))\; \xi_0(x,x) = \chi(\phi_{J_2}^{-1}(x))\; \xi_0(x,x) && (86) \\ \forall x \in A \quad & \chi(\phi_{J_1}^{-1}(x)) = \chi(\phi_{J_2}^{-1}(x)) && (87) \end{aligned}$$

When applied to $\chi = \chi_{1,0} : (z_1, z_2) \mapsto z_1$ (resp. $\chi = \chi_{0,1} : (z_1, z_2) \mapsto z_2$), this shows that the first (resp. second) coordinate of $\phi_{J_1}^{-1}(x)$ and $\phi_{J_2}^{-1}(x)$ coincide on A, i.e.

$$\begin{aligned} \phi_{J_1}^{-1} = \phi_{J_2}^{-1} \quad & \text{on } A && (88) \\ \phi_{J_2} \circ \phi_{J_1}^{-1} = id_A \quad & \text{on } A && (89) \end{aligned}$$

But $\phi_{J_2} \circ \phi_{J_1}^{-1}$ defines an OE between $\mathscr{R}_{J_1}$ and $\mathscr{R}_{J_2}$, and on A, it is the identity: $\mathscr{R}_{J_1} \restriction A = \mathscr{R}_{J_2} \restriction A$. The Observation from point 6 implies that we reached our goal (from point 7):

$$J_1 = J_2.$$

Point 17 – Indeed, eventually, one can treat $\mathbf{F}_n$, $n \geq 3$, exactly the same way. One chooses a free subgroup

$$\mathbf{F}_n \quad = \quad \mathbf{F}\langle s_1, s_2\rangle * \mathbf{F}\langle a_1\rangle * \mathbf{F}_{n-3} < \mathrm{SL}(2,\mathbb{Z}) \tag{90}$$

such that $\mathbf{F}\langle s_1, s_2\rangle$ (and why not also a_1) acts ergodically. By Dye's theorem [Dye59] there is a free action[45] of $\Lambda_{\mathbb{N}} := \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ with the same orbits as $\langle a_1\rangle \curvearrowright \mathbb{T}^2$.

Now repeat the above argument with

$$\Lambda_I \quad := \quad \bigoplus_{i\in I} \mathbb{Z}/2\mathbb{Z} \tag{91}$$

$$\Gamma_I \quad := \quad \mathbf{F}\langle s_1, s_2\rangle * \Lambda_I * \mathbf{F}_{n-3}. \tag{92}$$

Let's concentrate on the infinite subsets $I \subset \mathbb{N}$. We get a family of subgroups Γ_I of $\Gamma_{\mathbb{N}}$ (the Γ_I are indeed pairwise isomorphic, but we don't need this fact) and the sub-families of them leading to pairwise orbit equivalent actions are at most countable. Each Λ_I being an infinite countable locally finite group it's action on $\mathbb{T}^2$ has the same orbits as some free $\mathbb{Z}$-action. So that eventually each $\Gamma_I \curvearrowright \mathbb{T}^2$ is orbit equivalent with a free action $\mathbf{F}_n \overset{\alpha_I}{\curvearrowright} \mathbb{T}^2$.

■ ↑

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[45] Let's say we take the standard Bernoulli shift action of $\Lambda_{\mathbb{N}}$ and pull-it back using an orbit equivalence with $\langle a_1\rangle \curvearrowright \mathbb{T}^2$. The (small) advantage is that every infinite subgroup of $\Lambda_{\mathbb{N}}$ still acts ergodically.

Index

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