GROUPS AND POLYTOPES

STEFAN FRIEDL, WOLFGANG LÜCK, AND STEPHAN TILLMANN

Abstract. In a series of papers the authors associated to an L^2-acyclic group Γ an invariant $\mathcal{P}(\Gamma)$ that is a formal difference of polytopes in the vector space $H_1(\Gamma;\mathbb{R})$. This invariant is in particular defined for most 3-manifold groups, for most 2-generator 1-relator groups and for all free-by-cyclic groups. In most of the above cases the invariant can be viewed as an actual polytope.

In this survey paper we will recall the definition of the polytope invariant $\mathcal{P}(\Gamma)$ and we state some of the main properties. We conclude with a list of open problems.

1. Introduction

1.1. **The Grothendieck group of polytopes.** A *polytope in a finite dimensional real vector space* V is defined as the convex hull of a finite non-empty subset of V. Given a polytope $\mathcal{P}$ we denote by

$$\overline{\mathcal{P}} \;:=\; \{-x \,|\, x \in \mathcal{P}\}$$

the mirror image of $\mathcal{P}$ in the origin.[1] We say that two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are *translation-equivalent* if there exists a vector $v \in V$ with $v + \mathcal{P} = \mathcal{Q}$. We denote by $\mathfrak{P}(V)$ the set of all translation-equivalence classes of polytopes in V.

The *Minkowski sum* of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ in V is defined as the polytope

$$\mathcal{P} + \mathcal{Q} \;:=\; \{p + q \,|\, p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}.$$

This turns $\mathfrak{P}(V)$ into an abelian monoid, where the identity element 0 is represented by any polytope consisting of a single point.

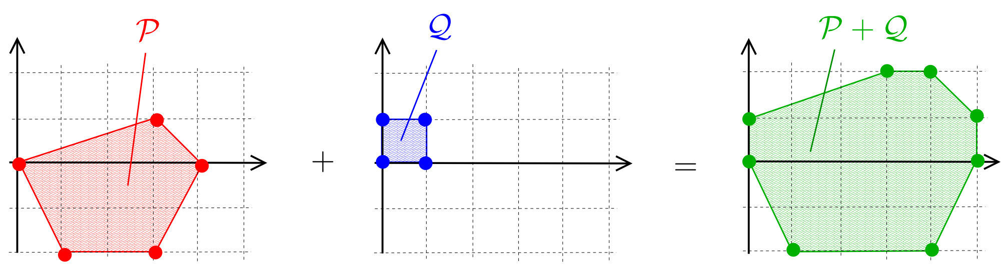

Figure 1.

It is straightforward to show, see e.g. [Sc93, Lemma 3.1.8], that $\mathfrak{P}(V)$ has the cancellation property, i.e. for any $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathfrak{P}(V)$ with $\mathcal{P} + \mathcal{Q} = \mathcal{P} + \mathcal{R}$ we have $\mathcal{Q} = \mathcal{R}$. We denote

Date: November 8, 2016.

2000 *Mathematics Subject Classification.* Primary 20J05; Secondary 20F65, 22E40, 57R19 .

Key words and phrases. Finitely presented group, L^2-acyclic groups, polytopes, BNS invariant.

[1]In the literature the mirror image of $\mathcal{P}$ in the origin is often denoted by $-\mathcal{P}$. In our paper $-\mathcal{P}$ will have a very different meaning.

by $\mathfrak{G}(V)$ the set of all equivalence classes of pairs $(\mathcal{P}, \mathcal{Q}) \in \mathfrak{P}(V)^2$, where we say that $(\mathcal{P}, \mathcal{Q}) \sim (\mathcal{P}', \mathcal{Q}')$ if $\mathcal{P} + \mathcal{Q}' = \mathcal{P}' + \mathcal{Q}$. Note that $\mathfrak{G}(V)$ is an abelian group, and since $\mathfrak{P}(V)$ has the cancellation property it follows that the map $\mathfrak{P}(V) \to \mathfrak{G}(V)$ given by $\mathcal{P} \mapsto (\mathcal{P}, 0)$ is a monomorphism. We will use this monomorphism to identify $\mathfrak{P}(V)$ with its image in $\mathfrak{G}(V)$. Given $\mathcal{P}$ and $\mathcal{Q} \in \mathfrak{P}(V)$ we may write $\mathcal{P} - \mathcal{Q} = (\mathcal{P}, \mathcal{Q})$. We refer to $\mathfrak{G}(V)$ as the *Grothendieck group of polytopes*.

Now let Γ be a group. We write $\mathfrak{P}(\Gamma) = \mathfrak{P}(H_1(\Gamma; \mathbb{R}))$ and we say a polytope in $H_1(\Gamma; \mathbb{R})$ is *integral* if all the vertices lie in the image of $H_1(\Gamma; \mathbb{Z}) \to H_1(\Gamma; \mathbb{R})$. Similarly we write $\mathfrak{G}(\Gamma) = \mathfrak{G}(H_1(\Gamma; \mathbb{R}))$ and we say an element in $\mathfrak{G}(\Gamma)$ is *integral* if it can be represented by the difference of two integral polytopes.[2] A group homomorphism $\varphi \colon \Gamma \to \Pi$ induces natural homomorphisms $\varphi_* \colon \mathfrak{P}(\Gamma) \to \mathfrak{P}(\Pi)$ and $\varphi_* \colon \mathfrak{G}(\Gamma) \to \mathfrak{G}(\Pi)$.

1.2. **L^2-acyclic groups and the Atiyah Conjecture.** We say that a group Γ is of *type F* if it admits a finite model for $K(\Gamma, 1)$. (A group of type F is well-known to be torsion-free, see e.g. [Br82, Corollary VIII.2.5] for a proof.) We say Γ is *L^2-acyclic* if all its L^2-Betti numbers $b_n^{(2)}(\Gamma)$ vanish. The following gives examples of L^2-acyclic groups of type F.

(1) Fundamental groups of admissible 3-manifolds (here a 3-manifold is *admissible* if it is connected, orientable, irreducible, its boundary is empty or a disjoint union of tori, and its fundamental group is infinite).
(2) Free-by-cyclic groups, i.e. groups that can be written as a semidirect product $F \rtimes \mathbb{Z}$ with F a free group.[3]
(3) Torsion-free groups with two generators and one non-trivial relator.

We refer to [LoL95, Theorem 0.1], [Lü02, Section 4.2] and [DL07] for proofs that these groups are indeed L^2-acyclic of type F.

In this paper we are mostly interested in groups for which the Whitehead group is trivial. It is conjectured that the Whitehead group is trivial for any torsion-free group. This conjecture has now been proven for large classes of groups. For example the Whitehead group is known to be trivial for all torsion-free hyperbolic groups [BLR08], all virtually solvable [We15] and the aforementioned three types of groups, see e.g. [AFW15, (C.36)] and [Wa78, p. 249 and p. 250] for details. We refer to [KLR16, Theorem 2] for a detailed list of all groups for which it is currently known that the Whitehead group is trivial.

Furthermore, for the most part we want to restrict ourselves to groups that satisfy the Atiyah Conjecture. A torsion-free group Γ satisfies the Atiyah Conjecture if given any $(m \times n)$-matrix A over $\mathbb{Z}[\Gamma]$ the L^2-dimension of the kernel of the map $r_A \colon l^2(\Gamma)^m \to l^2(\Gamma)^n$ defined by $v \mapsto v \cdot A$ is a natural number. The class of torsion-free groups that are known to satisfy the Atiyah Conjecture is considerably smaller than the class of torsion-free groups for which it is known that the Whitehead group is trivial. We refer to [LiL16, Theorem 2.3] for a comprehensive summary of what is known about the Atiyah Conjecture. For us it is of interest that the Atiyah Conjecture is known for fundamental groups of most admissible 3-manifolds and for free-by-cyclic groups, see e.g. [AFW15, (H.21)] and [Li93, Theorem 1.5]. In the following we refer to a group with trivial Whitehead group and which satisfies the

[2] The invariants of a group π we will define later on will lie in $\mathfrak{G}(H_1(\pi; \mathbb{R}))$. In fact they will lie in a subgroup given by polytopes with integral vertices. Therefore it makes sense to study the polytope group of integral polytopes, which has also been studied in its own sake by Funke [Fu16].

[3] Strictly speaking these groups should be called "free-by-infinite cyclic." But we follow the common convention to just say free-by-cyclic.

Atiyah Conjecture as a Wh-AC-*group*. It is an open question whether all torsion-free groups are Wh-AC-groups.

1.3. **The polytope invariant of L^2-acyclic groups of type F.** In Section 2 we will use Reidemeister torsion over an appropriate skewfield to associate to an L^2-acyclic Wh-AC-group Γ of type F an integral element $\mathcal{P}(\Gamma)$ of $\mathfrak{G}(\Gamma)$. We refer to $\mathcal{P}(\Gamma)$ as the *polytope invariant of* Γ. In Section 3.1 we will see that there exists an L^2-acyclic Wh-AC-group Γ of type F such that neither $\mathcal{P}(\Gamma)$ nor $-\mathcal{P}(\Gamma)$ is represented by an actual polytope.

The following theorem summarizes some of the structural properties of this invariant. We refer to Section 2.3 for details.

Theorem 1.1. (1) *Suppose A, B and C are L^2-acyclic* Wh-AC-*groups of type F. Let $\varphi\colon C \to A$ and $\phi\colon C \to B$ be monomorphisms. We consider the corresponding amalgamated product $A *_C B$ and denote by a, b and c the monomorphisms from A, B and C into $A *_C B$. Then $A *_C B$ is also L^2-acyclic of type F. Furthermore, if $A *_C B$ is a* Wh-AC-*group, then*

$$\mathcal{P}(A *_C B) \;=\; a_*(\mathcal{P}(A)) + b_*(\mathcal{P}(B)) - c_*(\mathcal{P}(C)).$$

(2) *Let*

$$1 \to K \xrightarrow{i_*} G \to B \to 1$$

be an exact sequence of groups of type F. Suppose G is a Wh-AC-*group and K is L^2-acyclic. Then*

$$\mathcal{P}(G) \;=\; i_*(\mathcal{P}(K)) \cdot \chi(B).$$

(3) *If Γ is the fundamental group of an aspherical n-dimensional manifold M and if Γ is an L^2-acyclic*[4] Wh-AC-*group of type F, then*

$$\mathcal{P}(\Gamma) \;=\; (-1)^{n+1} \cdot \overline{\mathcal{P}(\Gamma)} \in \mathfrak{G}(\Gamma).$$

The polytope invariant is in general rather difficult to calculate. The following theorem summarizes what is known for special classes of groups. In the interest of readability we keep the language somewhat informal. We refer to Section 3 for more carefully formulated statements.

Theorem 1.2. (1) *If $N \not\cong S^1 \times D^2$ is an admissible 3-manifold that is not a closed graph manifold, then $\mathcal{P}(N)$ equals the dual of the Thurston norm ball. In particular $\mathcal{P}(N)$ is an integral polytope and it determines and is determined by the Thurston norm.*

(2) *If Γ is a group that admits a presentation $\pi = \langle x, y | r\rangle$ such that r is non-empty, reduced and cyclically reduced, then $\mathcal{P}(\Gamma)$ can be easily read off the word r.*

(3) *If N is a closed admissible 3-manifold that is not a graph manifold and if N admits a CW-structure with one 0-cell, two 1-cells, two 2-cells and one 3-cell, then $\mathcal{P}(N)$ can be determined immediately from the corresponding chain complex of the universal cover.*

It follows in particular from Theorem 1.2 that for 3-manifolds to which (2) or (3) applies one can easily obtain the Thurston norm from the fundamental group or the chain complex.

[4] The Singer Conjecture [Sin77][Lü02, p. 421] predicts that M has zero L^2-Betti numbers if and only if $\chi(M) = 0$.

Next we turn to the question of what information the polytope invariant contains. The following theorem gives a partial answer to that question. It is again formulated in a slightly informal way, the precise statements will be given later in Sections 4 and 5.

Theorem 1.3. (1) *If $N \neq S^1 \times D^2$ is an admissible 3-manifold that is not a closed graph manifold, then we can mark some of the vertices of the polytope $\mathcal{P}(\pi_1(N))$ such that a class $\phi \in H^1(\pi_1(N);\mathbb{R})$ pairs maximally with a marked vertex if and only if it represents an element in the BNS-invariant* [BNS87] $\Sigma(\pi_1(N))$.

(2) *Let Γ be a group with $b_1(\Gamma) = 2$ that admits a presentation $\pi = \langle x, y | r\rangle$ such that r is non-empty, reduced and cyclically reduced. Then the following hold:*
   (a) *The "thickness" of $\mathcal{P}(\Gamma)$ is determined and determines the minimal complexity of "HNN-splittings" of Γ along groups.*
   (b) *We can mark some of the vertices of the polytope $\mathcal{P}(\Gamma)$ such that a class $\phi \in H^1(\Gamma;\mathbb{R})$ pairs maximally with a marked vertex if and only if it represents an element in the BNS-invariant $\Sigma(\Gamma)$.*

(3) [Funke–Kielak] *If Γ is an descending HNN-extension of a free group on two generators, then the geometry of $\mathcal{P}(\Gamma)$ is closely related to $\Sigma(\Gamma)$.*

Most of the results in this paper are already explicit or at least implicit in our previous papers [FT15, FST15, FL16a, FL16b]. Only Theorem 1.2 (3) is a completely new statement.

The paper is organized as follows. In Section 2 we will use Reidemeister torsion to introduce the polytope invariant $\mathcal{P}(\Gamma)$ for any L^2-acyclic Wh-AC-group Γ of type F. Furthermore we prove several statements regarding polytopes and Reidemeister torsions that will in particular imply Theorem 1.1. In Section 3 we give a more detailed discussion of the various statements of Theorem 1.2. In Sections 4 and 5 we will explain in more detail the statement and the references for Theorem 1.3. We conclude this paper with a long list of questions in Section 6.

Acknowledgments. We wish to thank the many people who shared their ideas and insight with us as we developed the subject. We are especially grateful to Nathan Dunfield, Florian Funke, Dawid Kielak and Kevin Schreve for their contributions.

The first author is supported by the SFB 1085 "Higher invariants" at the University of Regensburg, funded by the Deutsche Forschungsgemeinschaft (DFG). The second author is financially supported by the Leibniz-Preis granted by the DFG and the ERC Advanced Grant "KL2MG-interactions" (no. 662400) granted by the European Research Council. The third author is partially supported under the Australian Research Council's Discovery funding scheme (project number DP140100158).

2. Definition of the polytope invariant of groups

2.1. **Review of division and rational closure.** Let R be a subring of a ring S. The *division closure* $\mathcal{D}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is division closed, i.e., any element $x \in \mathcal{D}(R \subseteq S)$ which is invertible in S is already invertible in $\mathcal{D}(R \subseteq S)$. The *rational closure* $\mathcal{R}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is rationally closed, i.e., for any natural number n and matrix $A \in M_{n,n}(\mathcal{R}(R \subseteq S))$, if A is invertible over S, then A is already invertible over $\mathcal{R}(R \subseteq S)$. The division closure and the rational closure always exist. Obviously $R \subseteq \mathcal{D}(R \subseteq S) \subseteq \mathcal{R}(R \subseteq S) \subseteq S$.

Consider a group Γ. Let $\mathcal{N}(\Gamma)$ be the group von Neumann algebra which can be identified with the algebra $\mathcal{B}(L^2(\Gamma), L^2(\Gamma))^\Gamma$ of bounded left Γ-equivariant operators $L^2(\Gamma) \to L^2(\Gamma)$. Denote by $\mathcal{U}(\Gamma)$ the algebra of operators that are affiliated to the group von Neumann algebra, see [Lü02, Section 8] for details. This is the same as the Ore localization of $\mathcal{N}(\Gamma)$ with respect to the multiplicatively closed subset of non-zero divisors in $\mathcal{N}(\Gamma)$, see [Lü02, Theorem 8.22 (1)]. By the right regular representation we can embed $\mathbb{C}\Gamma$ and hence also $\mathbb{Z}\Gamma$ as a subring in $\mathcal{N}(\Gamma)$. We will denote by $\mathcal{R}(\Gamma)$ and $\mathcal{D}(\Gamma)$ the division and the rational closure of $\mathbb{Z}\Gamma$ in $\mathcal{U}(\Gamma)$. Summarizing we get a commutative diagram of inclusions of rings

$$\begin{tikzcd}
\mathbb{Z}\Gamma \arrow[r] \arrow[d] & \mathcal{N}(\Gamma) \arrow[dd] \\
\mathcal{D}(\Gamma) \arrow[d] & \\
\mathcal{R}(\Gamma) \arrow[r] & \mathcal{U}(\Gamma).
\end{tikzcd}$$

We will use these inclusions to identify each ring with its monomorphic images.

The following lemma is well-known to the experts, full references are given in [FL16b, Lemma 1.21].

Lemma 2.1. *Let C_* be a finite chain complex of free left-$\mathbb{Z}\Gamma$-modules. Then the following assertions are equivalent:*

1. *The L^2-Betti numbers of C_* are all zero.*
2. *The $\mathcal{R}(\Gamma)$-chain complex $\mathcal{R}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*$ is contractible.*

Lemma 2.2. *Let Γ be a group. If Γ is torsion-free and if Γ satisfies the Atiyah Conjecture, then the rational closure $\mathcal{R}(\Gamma)$ agrees with the division closure $\mathcal{D}(\Gamma)$ of $\mathbb{Z}\Gamma \subseteq \mathcal{U}(\Gamma)$. Furthermore, $\mathcal{D}(\Gamma)$ is a skew field.*

This lemma is proved in [Lü02, Lemma 10.39] for the Atiyah Conjecture over $\mathbb{C}$ with the division closure of $\mathbb{C}[\Gamma]$ instead of the division closure of $\mathbb{Z}[\Gamma]$, but the proof verbatim implies Lemma 2.2.

2.2. **The polytope homomorphism.** Throughout this section let Γ be a torsion-free group that satisfies the Atiyah Conjecture. By Lemma 2.2 $\mathcal{D}(\Gamma)$ is a skew field. Our goal is to associate to any element in the multiplicative group of units $\mathcal{D}(\Gamma)^\times = \mathcal{D}(\Gamma) \setminus \{0\}$ an integral element of $\mathfrak{G}(\Gamma)$. We outline the main steps in the construction and refer to [FL16a], where this map was first introduced, for more details.

Write $H := H_1(\Gamma; \mathbb{Z})/\text{torsion}$ and view H as a multiplicative group. Let $\mathrm{pr}\colon \Gamma \to H$ be the canonical projection and K be the kernel of pr. Choose a map of sets $s\colon H \to \Gamma$ with $\mathrm{pr} \circ s = \mathrm{id}_H$. We denote by $\mathbb{Z}[K] *_s H$ the ring, often referred to as the crossed product of $\mathbb{Z}[K]$ and H, whose underlying abelian group is given by finite formal sums $\sum_{h \in H} a_h h$ with each $a_h \in \mathbb{Z}[K]$ and for which multiplication is extended from the multiplication on $\mathbb{Z}[K]$ by the rule $g \cdot h = s(g) \cdot s(h) \cdot s((gh)^{-1}) \cdot gh$ for $g, h \in H$ and by the rule $h \cdot k = (hks(h)^{-1}) \cdot h$ for $h \in H$ and $k \in K$. It is straightforward to see that $\mathbb{Z}[K] *_s H$ is a ring and that

$$\begin{array}{ccc}
\mathbb{Z}[\Gamma] & \to & \mathbb{Z}[K] *_s H \\
\sum\limits_{g \in \Gamma} a_g g & \mapsto & \sum\limits_{g \in \Gamma} a_g g s(\mathrm{pr}(g))^{-1} \cdot \mathrm{pr}(g)
\end{array}$$

is a ring isomorphism, which we will sometimes use to identify these two rings. Given a non-zero element $\sum_{h\in H} a_h h \in \mathbb{Z}[K] *_s H$ we consider the integral polytope

$$\mathcal{P}\Big(\sum_{h\in H} a_h h\Big) \;=\; (\text{convex hull of } h \text{ with } a_h \neq 0) \;\in\; \mathcal{P}(H) \;=\; \mathcal{P}(\Gamma).$$

Defining $\mathcal{D}(K) *_s H$ analogously, given any non-zero element $f \in \mathcal{D}(K) *_s H$ we obtain the corresponding polytope $\mathcal{P}(f)$. Since $\mathcal{D}(K)$ is a skew field, in particular a domain, we obtain from elementary arguments that for any non-zero $f, g \in \mathcal{D}(K) *_s H$ we have

$$\mathcal{P}(f \cdot g) \;=\; \mathcal{P}(f) + \mathcal{P}(g). \tag{1}$$

We denote by $T = (\mathcal{D}(K) *_s H) \setminus \{0\}$ the set of all non-zero elements of the domain $\mathcal{D}(K) *_s H$. We can form the Ore localization $T^{-1}(\mathcal{D}(K) *_s H)$—a proof of this fact is for example given in [DLMSY03, Theorem 6.4] or [Lü02, Example 8.16]. There exists a canonical isomorphism

$$T^{-1}(\mathcal{D}(K) *_s H) \;\xrightarrow{\cong}\; \mathcal{D}(\Gamma),$$

which we will use to identify these rings, see e.g. [Lü02, Lemma 10.69]. (Again the lemma in [Lü02] is stated for the division closure of $\mathbb{C}[G]$ but it also holds with the same proof for the division closure of $\mathbb{Z}[G]$.) Now let $h = fg^{-1} \in \mathcal{D}(\Gamma) = T^{-1}(\mathcal{D}(K) *_s H)$ be non-zero. We define

$$\mathcal{P}(h) \;:=\; \mathcal{P}(f) - \mathcal{P}(g) \;\in\; \mathfrak{G}(\Gamma).$$

It follows from (1) that this is well-defined and that this map defines a group homomorphism $\mathcal{P}\colon \mathcal{D}(\Gamma)^\times \to \mathfrak{G}(\Gamma)$. Since the target is abelian this descends to a group homomorphism

$$\mathcal{P}\colon \mathcal{D}(\Gamma)^\times/[\mathcal{D}(\Gamma)^\times, \mathcal{D}(\Gamma)^\times] \;\to\; \mathfrak{G}(\Gamma).$$

One easily checks that this homomorphism is independent of the choice of s.

2.3. **The polytope invariant of an L^2-acyclic group of type F.** Given a ring R we denote by $K_1(R)$ the usual K_1-group, as defined in [Sil81, Ro94]. If $\mathbb{K}$ is a skew field, then the Dieudonné determinant, see [Sil81, Corollary 4.3 on page 133] and [Ro94], gives rise to an isomorphism

$$\det\colon K_1(\mathbb{K}) \;\xrightarrow{\cong}\; \mathbb{K}^\times_{\mathrm{ab}} := \mathbb{K}^\times/[\mathbb{K}^\times, \mathbb{K}^\times].$$

In the following, given a torsion-free group Γ that satisfies the Atiyah Conjecture we will use this canonical isomorphism to identify $K_1(\mathcal{D}(\Gamma))$ with $\mathcal{D}(\Gamma)^\times_{\mathrm{ab}}$.

Definition. (1) An *L^2-acyclic pair* (X, φ) consists of a finite connected CW-complex and a homomorphism $\varphi\colon \pi_1(X) \to \Gamma$ such that $b_i^{(2)}(X, \varphi) = 0$ for all i. Here $b_i^{(2)}(X, \varphi)$ denotes the L^2-Betti numbers of the covering space of X corresponding to φ, viewed as a Γ-CW-complex.

(2) Suppose that $(X, \varphi\colon \pi_1(X) \to \Gamma)$ *is an L^2-acyclic pair* and suppose that Γ is torsion-free and that it satisfies the Atiyah Conjecture. We denote by $\widetilde{X}$ the universal cover of X. By picking orientations of the cells of X and by picking lifts of the cells of X to $\widetilde{X}$ we can view $C_*(\widetilde{X})$ as a chain complex of based free $\mathbb{Z}[\pi_1(X)]$-left modules. Similarly we can view $\mathcal{D}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{X})$ as a chain complex of based free $\mathcal{D}(\Gamma)$-left modules. We had assumed that (X, φ) is L^2-acyclic. Together with Lemmas 2.1 and 2.2 this implies that the chain complex $\mathcal{D}(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} C_*(\widetilde{X})$ is acyclic. We denote

by $\tau(X,\varphi) \in K_1(\mathcal{D}(\Gamma))/\pm\Gamma = \mathcal{D}(\Gamma)^{\times}_{\mathrm{ab}}/\pm\Gamma$ the corresponding torsion as defined in [Mi66]. Furthermore we define the *polytope invariant of* (X,φ) as

$$\mathcal{P}(X,\varphi) \;:=\; -\mathcal{P}(\tau(X,\varphi)) \;\in\; \mathfrak{G}(\Gamma).$$

The minus sign in the definition of $\mathcal{P}(X,\varphi)$ might initially be a little surprising. This choice of sign ensures that in many situations of interest, the polytope invariant $\mathcal{P}(X,\varphi)$ lies in $\mathfrak{P}(\Gamma)$, i.e. it indeed can be represented by a polytope.

The following proposition summarizes some of the key properties of the polytope invariant of an L^2-acyclic pair. The proof of the proposition follows from standard properties of Reidemeister torsion and the fact that the map $\mathcal{P}\colon K_1(\mathcal{D}(\Gamma)) \to \mathfrak{G}(\Gamma)$ is a homomorphism. We refer to [FL16b, Theorem 2.5] for details.

Proposition 2.3. *Let* $(X,\varphi\colon \pi_1(X) \to \Gamma)$ *be an* L^2*-acyclic pair and suppose that* Γ *is a torsion-free group that satisfies the Atiyah Conjecture.*

(1) (Induction) *If* $\delta\colon \Gamma \to \Pi$ *is a monomorphism to a torsion-free group* Π *that satisfies the Atiyah Conjecture, then* $\mathcal{P}(X,\delta\circ\varphi) = \delta_*(\mathcal{P}(X,\varphi))$*, where* $\delta_*\colon \mathfrak{G}(\Gamma) \to \mathfrak{G}(\Pi)$ *is the induced map.*

(2) (Simple homotopy invariance) *If* $f\colon W \to X$ *is a simple homotopy equivalence of CW-complexes, i.e. if* $\mathrm{Wh}(f) = 0 \in \mathrm{Wh}(\pi_1(X))$*, then* $\mathcal{P}(W,\varphi\circ f_*) = \mathcal{P}(X,\varphi)$*.*

(3) (Homeomorphism invariance) *If* $f\colon W \to X$ *is a homeomorphism, then we have* $\mathcal{P}(W,\varphi\circ f_*) = \mathcal{P}(X,\varphi)$*.*

Now we can make the following two definitions.

(1) We say that a group Γ is L^2*-acyclic of type* F if it admits a finite $K(\Gamma,1)$ and if all its L^2-Betti numbers vanish. If Γ is a Wh-AC-group, then we define

$$\mathcal{P}(\Gamma) \;:=\; \mathcal{P}(X,\mathrm{id})$$

where X is any finite $K(\Gamma,1)$. It follows from Proposition 2.3 (2) that this definition does not depend on the choice of X. We refer to $\mathcal{P}(\Gamma)$ as the *polytope invariant* of Γ.

(2) Let M be a compact manifold and let $\varphi\colon \pi_1(M) \to \Gamma$ be a homomorphism to a group Γ that satisfies the Atiyah Conjecture. As above we say that (M,φ) is L^2-acyclic if $b_n^{(2)}(M,\varphi) = 0$ for all $n \in \mathbb{N}_0$. Now suppose that (M,φ) is L^2-acyclic. We pick a CW-structure X for M and we define $\mathcal{P}(M,\varphi) := \mathcal{P}(X,\varphi)$. It follows from Proposition 2.3 (3) that this definition does not depend on the choice of X. Sometimes we write $\mathcal{P}(M) := \mathcal{P}(M,\mathrm{id})$.

The following proposition collects a few more structural properties of the polytope invariant. These are again a consequence of [FL16b, Theorem 2.5].

Proposition 2.4. (1) *Let* $X = A \cup_C B$ *be a decomposition of a finite CW-complex* X *into two connected CW-complexes* A *and* B *such that* $C := A \cap B$ *is also connected. Let* $\varphi\colon \pi_1(X) \to \Gamma$ *be a homomorphism to a torsion-free group* Γ *that satisfies the Atiyah Conjecture. We denote by* $a\colon A \to X$*,* $b\colon B \to X$ *and* $c\colon C \to X$ *the inclusion maps. If* $(A,\varphi\circ a_*)$*,* $(B,\varphi\circ b_*)$ *and* $(C,\varphi\circ c_*)$ *are* L^2*-acyclic, then* (X,φ) *is also* L^2*-acyclic and*

$$\mathcal{P}(X,\varphi) \;=\; a_*(\mathcal{P}(A,\varphi\circ a_*)) + b_*(\mathcal{P}(B,\varphi\circ b_*)) - c_*(\mathcal{P}(C,\varphi\circ c_*)).$$

(2) *Let*

$$1 \to F \xrightarrow{i} E \to B \to 1$$

be a fibration of finite CW-complexes. Let $\varphi\colon \pi_1(E) \to \Gamma$ *be a homomorphism to a torsion-free group* Γ *that satisfies the Atiyah Conjecture. If* $(F, \varphi \circ i_*)$ *is* L^2*-acyclic, then*

$$\mathcal{P}(E, \varphi) \;=\; \chi(B) \cdot i_*(\mathcal{P}(F, \varphi \circ i_*)).$$

(3) *Let* M *be an* n*-dimensional closed orientable manifold and let* $\varphi\colon \pi_1(M) \to \Gamma$ *be a homomorphism to a group* Γ *that satisfies the Atiyah Conjecture. If* (M, φ) *is* L^2*-acyclic, then*

$$\mathcal{P}(M, \varphi) \;=\; (-1)^{n+1} \cdot \overline{\mathcal{P}(M, \varphi)} \;\in\; \mathfrak{G}(\Gamma).$$

It is clear that Theorem 1.1 is an immediate consequence of Propositions 2.3 and 2.4 and the definitions.

3. Examples

3.1. **Elementary examples.** We first consider the infinite cyclic group $\langle t \rangle$. In this case we take $X = K(\langle t \rangle, 1) = S^1$. The cellular chain complex of the universal cover of $\mathbb{R}$ is then isomorphic to

$$0 \;\to\; \mathbb{Z}[t^{\pm 1}] \xrightarrow{\cdot(1-t)} \mathbb{Z}[t^{\pm 1}] \;\to\; 0.$$

It follows that $\tau(X, \mathrm{id}) = (1-t)^{-1}$ and therefore $\mathcal{P}(\langle t \rangle) = -[0,1] \in \mathfrak{G}(\langle t \rangle) = \mathfrak{G}(\mathbb{R})$. In this case $\mathcal{P}(\langle t \rangle)$ is therefore not represented by a polytope.

Now consider $A = \langle s \rangle \times F_3$, where F_3 denotes, as usual, the free group on three generators. It follows from the above calculation and from Theorem 1.1 (2) that $\mathcal{P}(A)$ is represented by an interval of length $-\chi(F_3) = 2$. We also consider $B = \Gamma = \langle t \rangle \times (F_3 \times F_5)$. (Strictly speaking we do not know of a proof that B satisfies the Atiyah Conjecture, the following discussion implicitly assumes that this is the case so that we can define $\mathcal{P}(B)$.) Similar to the above we see that $\mathcal{P}(B)$ is represented by minus an interval of length $\chi(F_3) \cdot \chi(F_5) = 8$. Now let $C = \mathbb{Z}$ and pick a monomorphism $\varphi\colon C \to A$ that factors through a monomorphism $C \to F_3$ and similarly pick a monomorphism $\psi\colon C \to B$ that factors through a monomorphism $C \to F_3 \times F_5$. We denote by Γ the corresponding amalgamated product. It is a consequence of Theorem 1.1 (2) that $\mathcal{P}(\Gamma)$ is a difference of two intervals and it is straightforward to see that $\mathcal{P}(\Gamma)$ is neither represented by a polytope nor by minus a polytope.

3.2. **Groups with two generators and one relator.** In general it is very hard to compute the polytope invariant, the main difficulty lies in determining the Dieudonné determinant of a square matrix. There is only one situation in which the calculation of the Dieudonné determinant is straightforward, namely when the matrix is a 1×1-matrix. As we will see, this observation makes it straightforward to determine the polytope invariant for groups with two generators and one relator.

We say that a presentation $\pi = \langle x, y | r \rangle$ is *nice* if it satisfies the following conditions:

(1) r is a non-empty, reduced and cyclically reduced word, and
(2) $b_1(\Gamma_\pi) = 2$, where Γ_π denotes the group defined by the presentation π.

Following [FT15] we will associate to a nice presentation $\pi = \langle x, y | r\rangle$ a polytope $\mathcal{S}(\pi)$. The definition is illustrated in Figure 2. First we identify $H_1(\Gamma_\pi;\mathbb{Z})$ with $\mathbb{Z}^2$ such that x corresponds to $(1,0)$ and y corresponds to $(0,1)$. Then the relator r determines a discrete walk on the integer lattice in $H_1(\Gamma_\pi;\mathbb{R}) = \mathbb{Z}^2$ and the polytope $\mathcal{S}(\pi)$ is obtained from the convex hull of the trace of this walk in the following way:

(1) Start at the origin and walk across $\mathbb{Z}^2$ reading the word r from the left.
(2) Take the convex hull $\mathcal{C}$ of the set of all lattice points reached by the walk.
(3) An elementary argument, using the fact that r is reduced and cyclically reduced, shows that one can take the Minkowski difference with the square $\mathcal{Q}$ of length one, i.e. there exists a polytope $\mathcal{S}(\pi)$ with $\mathcal{S}(\pi) + \mathcal{Q} = \mathcal{C}$.

Figure 2 illustrates the construction of the polytope for the nice presentation

$$\pi = \langle x, y \,|\, yx^4yx^{-1}y^{-1}x^2y^{-1}x^{-2}y^2xy^{-1}xy^{-1}x^{-1}y^{-2}x^{-3}y^2x^{-1}\rangle.$$

For the final Minkowski difference see also Figure 1.

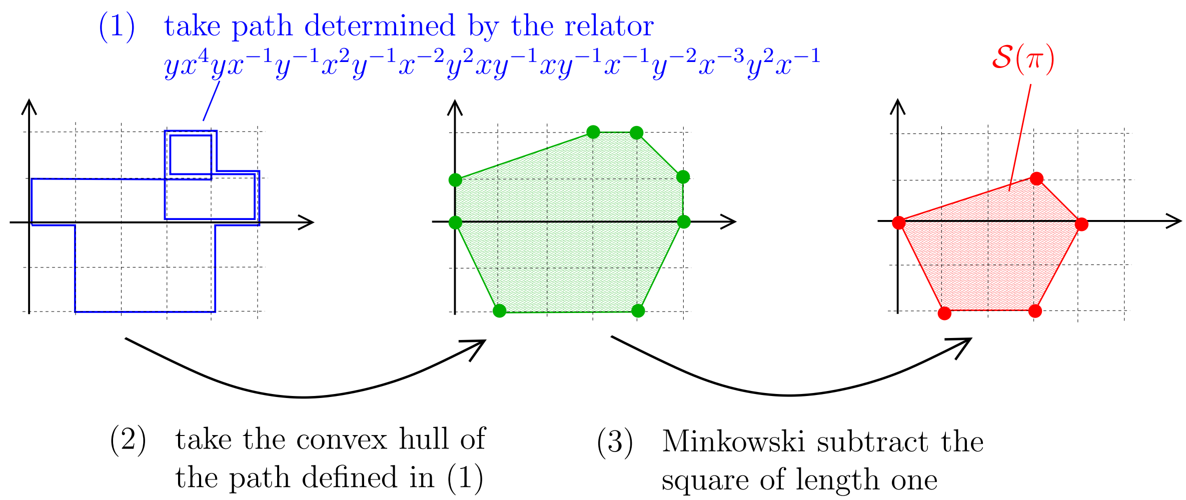

FIGURE 2. The polytope $\mathcal{S}(\pi)$ for a presentation π

We now view $\mathcal{S}(\pi)$ as an element in $\mathfrak{P}(\mathbb{Z}^2) = \mathfrak{P}(\Gamma_\pi)$, where we identify $\mathbb{Z}^2$ with $H_1(\Gamma_\pi;\mathbb{Z})$ by sending $(0,1)$ to x and $(0,1)$ to y. A priori $\mathcal{S}(\pi) \in \mathfrak{P}(\Gamma_\pi)$ is an invariant of the presentation and not of the underlying group Γ_π.

The following theorem gives another strong indication that the polytope is an invariant of the group.

Theorem 3.1. *Let $\pi = \langle x, y \,|\, r\rangle$ be a nice presentation. If Γ_π is torsion-free*[5] *and if Γ_π is a* Wh-AC*-group, then Γ_π is L^2-acyclic of type F and*

$$\mathcal{S}(\pi) \;=\; \mathcal{P}(\Gamma_\pi) \in \mathfrak{P}(\Gamma_\pi).$$

Remark. In [FT15, Theorem 1.3] it was shown $\mathcal{S}(\pi)$ is an invariant of the underlying group if Γ_π is residually a group that is elementary-amenable and torsion-free (without the assumption that Γ_π satisfies the Atiyah Conjecture). In [Wi14, Conjecture 1.9] Wise (see also [BK15,

[5]The group Γ_π is torsion-free if and only if the word r is not a proper power of another word, see e.g. [LS77, Proposition II.5.17] for details.

p. 2]) conjectured that most hyperbolic groups $\langle x,y|r\rangle$ act properly and cocompactly on a CAT(0) cube complex. By Agol's Theorem [Ag13] a proof of this conjecture would imply that such groups are virtually special. By [FST15, Theorem 3.8] together with [AFW15, (H.26)] this would imply that any word hyperbolic torsion-free group with a nice presentation $\langle x,y|r\rangle$ is residually a group that is elementary-amenable and torsion-free. Summarizing, a proof of Wise's Conjecture would show that $\mathcal{S}(\pi)$ is an invariant of the underlying group for torsion-free hyperbolic groups with a nice $(2,1)$-presentation.

Proof of Theorem 3.1. By [LS77, Proposition III.11.1] the 2-complex X corresponding to the given presentation is aspherical. We denote the universal of X by $\widetilde{X}$ and we write $\Gamma=\Gamma_\pi$. As usual we view $\mathbb{Z}[\Gamma]$ as a subring of $\mathcal{D}(\Gamma)$. The chain complex $\mathcal{D}(\Gamma)\otimes_{\mathbb{Z}[\Gamma]}C_*(\widetilde{X})$ is isomorphic to

$$0\,\to\,\mathcal{D}(\Gamma)\xrightarrow{\begin{pmatrix} r_x & r_y\end{pmatrix}}\mathcal{D}(\Gamma)^2\xrightarrow{\begin{pmatrix} x-1\\ y-1\end{pmatrix}}\mathcal{D}(\Gamma)\,\to\,0$$

where $r_x:=\frac{\partial r}{\partial x}$ and $r_x:=\frac{\partial r}{\partial x}$ denote the Fox derivatives [Fo53] of the word r with respect to the generators x and y. By [DL07] and Lemmas 2.1 and 2.2 this chain complex is acyclic. Since $y-1\neq 0\in\mathcal{D}(\Gamma)$ it follows from standard arguments, see e.g. [Tu01, Theorem 2.2] or [DFL16, Lemma 3.1] that the torsion of the chain complex equals $r_x\cdot(y-1)^{-1}\in\mathcal{D}(\Gamma)$. So we obtain

$$\mathcal{P}(\Gamma)\;=\;\mathcal{P}\big(\tau\big(\mathcal{D}(\Gamma)\otimes_{\mathbb{Z}[\Gamma]}C_*(\widetilde{X})\big)\big)\;=\;\mathcal{P}(r_x\cdot(y-1)^{-1})\;=\;\mathcal{P}(r_x)-\mathcal{P}(y-1).$$

But by [FT15, Proposition 3.5] the polytope $\mathcal{S}(\pi)$ agrees with $\mathcal{P}(r_x)-\mathcal{P}(y-1)$ in $\mathfrak{P}(\Gamma_\pi)$. □

In [FT15, Section 8] we also deal with presentations of the form $\langle x,y|r\rangle$, where r is non-empty, reduced and cyclically reduced, but where $b_1(\Gamma_\pi)=1$. In this case we can also 'naively' define a polytope (i.e. an interval) $\mathcal{S}(\pi)$ in $H_1(\Gamma_\pi;\mathbb{R})=\mathbb{R}$ and the obvious analogue of Theorem 3.1 holds.

3.3. **3-manifolds.** Let N be a 3–manifold. For each $\phi\in H^1(N;\mathbb{Z})$ there is a properly embedded oriented surface Σ, such that $[\Sigma]\in H_2(N,\partial N;\mathbb{Z})$ is the Poincaré dual to ϕ. Letting $\chi_-(\Sigma)=\sum_{i=1}^k\max\{-\chi(\Sigma_i),0\}$, where $\Sigma_1,\dots,\Sigma_k$ are the connected components of Σ, the *Thurston norm* of $\phi\in H^1(N;\mathbb{Z})$ is defined as

$$x_N(\phi)\;:=\;\min\big\{\chi_-(\Sigma)\,|\,\Sigma\text{ is a properly embedded surface with }\mathrm{PD}([\Sigma])=\phi\big\}.$$

Thurston [Th86] showed that x_N is a seminorm on $H^1(N;\mathbb{Z})$ and elementary arguments show that x_N extends to a seminorm x_N on $H^1(N;\mathbb{R})$. We denote by

$$\mathcal{T}(N)\;:=\;\{v\in H_1(N;\mathbb{R})\,|\,\phi(v)\leq 1\text{ for all }\phi\in H^1(N;\mathbb{R})\text{ with }x_N(\phi)\leq 1\}$$

the dual of the unit norm ball of x_N. Thurston [Th86] showed that $\mathcal{T}(N)$ is a polytope with integral vertices.

The following theorem is [FL16b, Theorem 3.35]. The proof relies on the recent work of Agol [Ag08, Ag13], Przytycki-Wise [PW12] and Wise [Wi09, Wi12a, Wi12b].

Theorem 3.2. *For any admissible 3-manifold $N\neq S^1\times D^2$ that is not a closed graph manifold we have*

$$\mathcal{T}(N)\;=\;2\cdot\mathcal{P}(\pi_1(N))\;\in\;\mathfrak{P}(\pi_1(N)).$$

The combination of Theorems 3.1 and 3.2 gives us the following corollary. This corollary was first proved in [FST15] using a different approach.

Corollary 3.3. *Let N be an admissible 3-manifold such that $\pi_1(N)$ admits a nice presentation $\pi = \langle x, y | r\rangle$. Then*
$$\mathcal{T}(N) \;=\; 2 \cdot \mathcal{S}(\pi).$$

3.4. **Closed 3-manifolds with a two-generator fundamental group.** The following theorem says that for "small" closed 3-manifolds one can easily obtain the Thurston norm from the chain complex of the universal cover. This theorem can be viewed as a version of Corollary 3.3 for closed 3-manifolds. It applies, in particular, to all closed admissible 3–manifolds of Heegaard genus equal to two. It is known through work of Kobayashi [Ko88] and Hempel [He01] in combination with Perelman's solution of the geometrisation conjecture that if the splitting distance of a genus two Heegaard splitting of a 3-manifold is larger than two, then the 3-manifold is hyperbolic (whence admissible). Maher [Ma10] used this fact to show that *most* 3–manifolds of Heegaard genus two are hyperbolic.

Theorem 3.4. *Let N be a closed admissible 3-manifold that is not a graph manifold. We write $\pi = \pi_1(N)$. Suppose N admits a CW-structure with one 0-cell, two 1-cells, two 2-cells and one 3-cell. The corresponding cellular chain complex of the universal cover $\widetilde{N}$ is of the form*
$$0 \to \mathbb{Z}[\pi] \xrightarrow{\begin{pmatrix} c_1 & c_2 \end{pmatrix}} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$
Then there exist $i, j \in \{1, 2\}$ with $c_i \neq 0$ and $a_j \neq 0$, and
$$\mathcal{T}(N) \;=\; \mathcal{P}(b_{3-i,3-j}) - \mathcal{P}(c_i) - \mathcal{P}(a_j).$$

Proof. As we pointed out earlier, the chain complex $\mathcal{D}(\pi) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{N})$ is acyclic. It thus follows that there exist $i, j \in \{1, 2\}$ with $c_i \neq 0$ and $a_j \neq 0$. Since $\mathcal{D}(\pi)$ is a skew field one can show, similar to [Tu01, Theorem 2.2], that
$$\tau(N) \;=\; c_i^{-1} \cdot b_{3-i,3-j} \cdot a_j^{-1}.$$
This implies that $\mathcal{P}(\pi_1(N)) = \mathcal{P}(b_{3-i,3-j}) - \mathcal{P}(c_i) - \mathcal{P}(a_j)$. The theorem now follows from Theorem 3.2. $\square$

4. Marked polytopes

A *marked polytope* is a pair $\mathcal{M} = (\mathcal{P}, \mathcal{V})$, where $\mathcal{P}$ is a polytope and $\mathcal{V}$ is a (possibly empty) subset of the set of vertices of $\mathcal{P}$. We refer to the vertices in $\mathcal{V}$ as the *marked vertices*. If $\mathcal{M}$ and $\mathcal{N}$ are two marked polytopes, then we define the (*marked*) *Minkowski sum* $\mathcal{M} + \mathcal{N}$ *of* $\mathcal{M}$ *and* $\mathcal{N}$ as the Minkowski sum of the corresponding polytopes, where the marked vertices of the Minkowski sum are precisely those that are the sum of a marked vertex of $\mathcal{M}$ and a marked vertex of $\mathcal{N}$. An example is given in Figure 3, where the marked vertices are indicated by a dot.

Marked polytopes appear naturally in many contexts:

(1) Let N be a 3-manifold. A class $\phi \in H^1(N; \mathbb{R})$ is called *fibered* if it can be represented by a non-degenerate closed 1-form. By [Ti70] an integral class $\phi \in H^1(N; \mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$ is fibered if and only if there exists a locally trivial fiber bundle

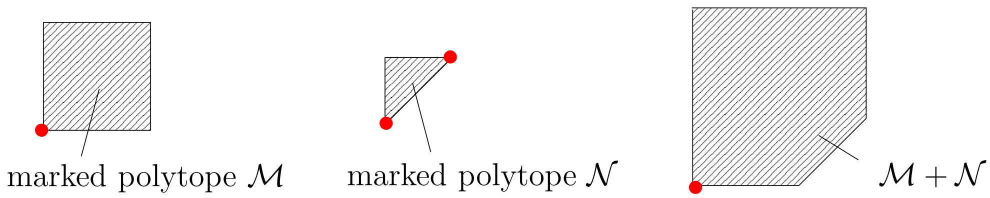

FIGURE 3. Example of the Minkowski sum of two marked polytopes.

$p\colon N \to S^1$ such that $p_* = \phi\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$. Thurston [Th86] showed that we can turn $\mathcal{T}(N)$ into a marked polytope $\mathcal{M}(N)$ which has the property that $\phi \in H^1(N;\mathbb{R}) = \operatorname{Hom}(H_1(N;\mathbb{R}),\mathbb{R})$ is fibered if and only if it pairs maximally with a marked vertex. This means that there exists a marked vertex v of $\mathcal{M}(N)$, such that $\phi(v) > \phi(w)$ for any vertex $w \neq v$ in the underlying polytope $\mathcal{T}(N)$.

(2) Let Γ be a torsion-free group. We will now see that we can associate a marked polytope to a non-zero element in $\mathbb{Z}[\Gamma]$. (In our subsequent discussion of this assignment we will use the notation introduced in Section 2.2.) We recall that we have an identification $\mathbb{Z}[\Gamma] = \mathbb{Z}[K] *_s H$, where $H = H_1(\Gamma;\mathbb{Z})/\text{torsion}$ and $K = \operatorname{Ker}(\Gamma \to H)$. Given a non-zero element $f = \sum_{h\in H} a_h h$ we consider the marked polytope $\mathcal{M}(f)$ which is given by the polytope

$$\mathcal{P}\Big(\sum_{h\in H} a_h h\Big) = (\text{convex hull of } h \text{ with } a_h \neq 0)$$

where we mark a vertex h of $\mathcal{P}(f)$ if $a_h = \pm k$ for some $k \in K$. If $\mathbb{Z}[\Gamma]$ is a domain, then for any non-zero $f, g \in \mathbb{Z}[\Gamma]$ we have $\mathcal{M}(f\cdot g) = \mathcal{M}(f) + \mathcal{M}(g)$.

(3) Let $\pi = \langle x, y | r\rangle$ be a nice presentation such that Γ_π is torsion-free. We showed in [FT15, Proposition 3.5] that there exists a unique marked polytope $\mathcal{M}(\pi)$ with $\mathcal{M}(\pi) + \mathcal{M}(y-1) = \mathcal{M}(\frac{\partial r}{\partial x})$.

The set of marked polytopes in a vector space forms a monoid under Minkowski sum. As illustrated in Figure 4 this monoid does *not* have the cancellation property. This implies that the monoid of marked polytopes does *not* inject into the corresponding Grothendieck group. Therefore we can not associate a meaningful notion of a difference of marked polytopes to an L^2-acyclic group π of type F at this point of time.

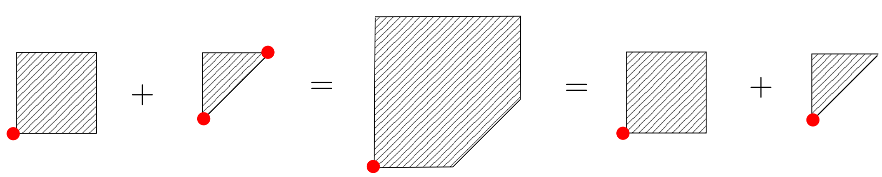

FIGURE 4. The Minkowski sum of marked polytopes does not have the cancellation property.

5. The polytope invariant and intrinsic properties of the group

In Section 2.3 we associated to an L^2-acyclic Wh-AC-group Γ of type F its polytope invariant $\mathcal{P}(\Gamma) \in \mathfrak{G}(\Gamma)$. In this section we want to determine information encoded in this invariant.

5.1. **The thickness of the polytope invariant and splittings of the group.** Given a polytope $\mathcal{P}$ in a real vector space V and given $\phi \in \operatorname{Hom}(V, \mathbb{R})$ we refer to

$$\operatorname{th}_\phi(\mathcal{P}) \; = \; \max\{\phi(x) - \phi(y) \,|\, x, y \in \mathcal{P}\} \; \in \; \mathbb{R}_{\geq 0}$$

as the *thickness of* $\mathcal{P}$ *in the* ϕ*-direction*. Since thickness is translation invariant we can also define $\operatorname{th}_\phi(\mathcal{P}) \in \mathbb{R}_{\geq 0}$ for any $\mathcal{P} \in \mathfrak{P}(V)$ and since thickness is additive under Minkowski sum we can also define $\operatorname{th}_\phi(\mathcal{P}) \in \mathbb{R}$ for any $\mathcal{P} \in \mathfrak{G}(V)$.

Given a polytope $\mathcal{P}$ in V we refer to

$$\mathcal{P}^{\mathrm{sym}} := \{\tfrac{1}{2}(p - q) \,|\, p, q \in \mathcal{P}\}$$

as the *symmetrization of* $\mathcal{P}$. This definition extends to a symmetrization map on $\mathfrak{P}(V)$ and $\mathfrak{G}(V)$. It is clear that the thickness of $\mathcal{P}$ depends only on the symmetrization of $\mathcal{P}$.

The following theorem is now a straightforward consequence of Theorem 3.2.

Theorem 5.1. *For any admissible 3-manifold $N \neq S^1 \times D^2$ that is not a closed graph manifold and any $\phi \in H^1(N; \mathbb{R})$ we have*

$$\operatorname{th}_\phi(\mathcal{P}(\pi_1(N))) \; = \; x_N(\phi).$$

Ideally one would like to generalize the statement of Theorem 5.1 to larger classes of groups. The first problem that arises is that there is no satisfactory purely group theoretic definition of the Thurston norm. We refer to [FSW15, FSW16] for several ideas and approaches.

In an attempt to generalize Theorem 5.1 we will work with the notion of a splitting of a group. Let Γ be a finitely presented group and let $\phi \colon \Gamma \to \mathbb{Z}$ be an epimorphism. Let B be a finitely generated group. A *splitting of* (Γ, ϕ) *over* B is an isomorphism

$$f \colon \Gamma \; \xrightarrow{\cong} \; \langle A, t \,|\, \mu(B) = tBt^{-1} \rangle$$

such that the following hold:

1. A is finitely generated,
2. B is a subgroup of A and $\mu \colon B \to A$ is a monomorphism,
3. $(\phi \circ f^{-1})(a) = 0$ for $a \in A$ and $(\phi \circ f^{-1})(t) = 1$.

It is well-known, see e.g. [BS78] or [St84, Theorem B*], that any such pair (Γ, ϕ) admits a splitting over a finitely generated group. We define the *splitting complexity of* (Γ, ϕ) as

$$c(\Gamma, \phi) \; := \; \min\{\operatorname{rank}(B) \,|\, (\Gamma, \phi) \text{ splits over } B\},$$

where $\operatorname{rank}(B)$ is defined as the minimal number of generators of B.

Theorem 5.2. *Let Γ be a torsion-free group that admits a nice presentation $\langle x, y | r\rangle$. Furthermore suppose that Γ satisfies the Atiyah Conjecture. Then for any epimorphism $\phi \colon \Gamma \to \mathbb{Z}$ we have*

$$c(\Gamma, \phi) - 1 \; = \; \operatorname{th}_\phi(\mathcal{P}(\Gamma)).$$

Remark. (1) In [FSW15] it was shown that the statement of the theorem also holds if π is the fundamental group of a knot complement. But the equality of Theorem 5.2 does not hold for the fundamental groups of all 3-manifolds. For example, if N is the 3-torus, then $\pi_1(N) = \mathbb{Z}^3$ and it follows from Theorem 1.1 and the discussion in Section 3.1 that $\mathcal{P}(\pi_1(N))$ is a point. In particular the thickness for each ϕ is zero. On the other hand it is straightforward to see that for any epimorphism $\phi \colon \pi_1(N) \to \mathbb{Z}$ we have $c(\pi_1(N), \phi) = 2$.

(2) The inequality $c(\Gamma, \phi) - 1 \geq \mathrm{th}_\phi(\mathcal{P}(\Gamma))$ was shown in [FT15, Proposition 7.6] if Γ is residually a group that is elementary-amenable and torsion-free (without the assumption that Γ satisfies the Atiyah Conjecture). As we remarked after Theorem 3.1, potentially most torsion-free groups with a nice presentation satisfy this condition.

Proof. The inequality $c(\Gamma, \phi) - 1 \leq \mathrm{th}_\phi(\mathcal{P}(\Gamma))$ was shown in [FT15, Proposition 7.3]. As was suggested to us by Nathan Dunfield, the proof of this inequality follows from a careful reading of the first step in the proof of the Freiheitssatz.

The proof of the reverse inequality follows along the same lines as the proof [FT15, Proposition 7.6] but one needs to replace the Ore localizations of the elementary-amenable and torsion-free groups by $\mathcal{D}(\Gamma)$. More precisely, it follows easily from the combination of Theorem 3.6(4) and Lemmas 4.3, 6.12 and 6.13 of [FL16a] together with a slight variation on Theorem 4.1 of [FL16a]. We leave the details to the reader. □

5.2. **The polytope invariant and its relation to the BNS-invariant.** Let Γ be a finitely generated group. The Bieri–Neumann–Strebel [BNS87] invariant $\Sigma(\Gamma)$ of Γ is by definition a subset of $S(\Gamma) := (\mathrm{Hom}(\Gamma, \mathbb{R}) \setminus \{0\})/\mathbb{R}_{>0}$. We refer to [BNS87] for the precise definition, but in order to give a flavour of the invariant we recall three properties:

(1) An epimorphism $\phi \in \mathrm{Hom}(\Gamma, \mathbb{Z})$ represents an element in $\Sigma(\Gamma)$ if and only if it corresponds to an *ascending HNN-extension.* To be precise: if and only if there exists an isomorphism
$$f \colon \Gamma \;\to\; \langle A, t \,|\, A = t^{-1}\varphi(A)t\rangle$$
where A is a finitely generated group and $\varphi \colon A \to A$ is a monomorphism, such that ϕ corresponds under f to the epimorphism given by $t \mapsto 1$ and $a \mapsto 0$ for all $a \in A$.[6]
(2) A non-trivial homomorphism $\phi \in \mathrm{Hom}(\Gamma, \mathbb{Z})$ has the property that ϕ and $-\phi$ represent elements in $\Sigma(\Gamma)$ if and only if $\mathrm{Ker}(\phi)$ is finitely generated.
(3) $\Sigma(\Gamma)$ is an open subset of $S(\Gamma)$.

The first two properties follow from [BNS87, Proposition 4.3] (see also [Br87, Corollary 3.2]) and the third one is [BNS87, Theorem A].

It is shown in [BNS87, Theorem E] that given a 3-manifold N the fibered classes of $H^1(N;\mathbb{R}) = \mathrm{Hom}(\pi_1(N), \mathbb{R})$ correspond precisely to the classes that lie in $\Sigma(\pi_1(N))$. The following is now a reformulation of the statement of the first example in Section 4.

Theorem 5.3. *Let $N \neq S^1 \times D^2$ be an admissible 3-manifold that is not a closed graph manifold. Then the polytope $\mathcal{P}(N)$ admits a marking with the property that for any non-trivial $\phi \in \mathrm{Hom}(\pi_1(N), \mathbb{R})$ we have*
$$[\phi] \in \Sigma(\pi_1(N)) \quad \Longleftrightarrow \quad \phi \text{ pairs maximally with a marked vertex of } \mathcal{P}(N).$$

For groups with a nice presentation $\langle x, y | r\rangle$ we obtain a similar theorem.

Theorem 5.4. *Let $\pi = \langle x, y \,|\, r\rangle$ be a nice presentation. A non-trivial class $\phi \in H^1(\Gamma_\pi; \mathbb{R})$ represents an element in $\Sigma(\Gamma_\pi)$ if and only if ϕ pairs maximally with a marked vertex of $\mathcal{M}(\pi)$.*

[6]Some authors, e.g. [FK16], call this a *descending* HNN-extension. We follow the convention used in [BS78, BNS87].

The theorem is stated as [FT15, Theorem 1.1]. In that paper the theorem is proved using the generalised Novikov rings of Sikorav [Sik87]. It can also be viewed as a reformulation of Brown's algorithm [Br87, Theorem 4.3].

Finally a relationship between the polytope invariant and the BNS-invariant was also found for a different class of groups by Funke–Kielak [FK16]. In order to state their theorem we need to introduce a few more definitions.

(1) Let $\varphi\colon \pi \to \pi$ be a monomorphism of a group π. We denote by
$$\pi *_\varphi \;:=\; \langle \pi, t \,|\, t^{-1}gt = \varphi(g), g \in \pi\rangle$$
the corresponding descending HNN-extension.[7] We refer to the epimorphism $\pi *_\varphi \to \mathbb{Z}$ given by $g \mapsto 0$ for all $g \in \pi$ and $t \to 1$ as the *canonical epimorphism*.

(2) Let $\mathcal{P}$ be a polytope in a vector space V and let $\phi \in \operatorname{Hom}(V, \mathbb{R})$. Following [FK16] we refer to
$$F_\phi(\mathcal{P}) \;=\; \{v \in \mathcal{P} \,|\, \phi(v) \leq \phi(w) \text{ for all } w \in \mathcal{P}\}$$
as the *ϕ-minimal face of $\mathcal{P}$*.

(3) Let V be a vector space and let $\mathcal{S} = [\mathcal{P}] - [\mathcal{Q}] \in \mathfrak{G}(V)$. Following [FK16] we say that $\phi, \psi \in \operatorname{Hom}(V, \mathbb{R})$ are *$\mathcal{S}$-equivalent* if $F_\phi(\mathcal{P}) = F_\psi(\mathcal{P})$ and $F_\phi(\mathcal{Q}) = F_\psi(\mathcal{Q})$. Note that this definition is independent of the choice of $\mathcal{P}$ and $\mathcal{Q}$.

The following is the main result of [FK16].

Theorem 5.5. *Let $\varphi\colon F_2 \to F_2$ be a monomorphism of the free group on two generators. Let $\phi\colon \Gamma := F_2 *_\varphi \to \mathbb{R}$ be a homomorphism such that the class $[\phi] \in S(\Gamma)$ is not represented by the canonical epimorphism. Then there exists an open neighborhood U of $[\phi]$ in $S(\Gamma)$ such that for every non-trivial $\psi\colon \Gamma \to \mathbb{R}$ with $[\psi] \in U$ that is $\mathcal{P}(\Gamma)$-equivalent to ϕ we have*
$$[-\phi] \in \Sigma(\Gamma) \quad \Longleftrightarrow \quad [-\psi] \in \Sigma(\Gamma).$$

6. Questions

We conclude this survey with a long list of questions and conjectures.

Conjecture 6.1. *If Γ is a free-by-cyclic group, then $\mathcal{P}(\Gamma) \in \mathfrak{G}(\Gamma)$ can be represented by a polytope.*

Conjecture 6.2. *Let $\Gamma \neq \mathbb{Z}$ be an L^2-acyclic* Wh-AC-*group of type F that admits a 2-dimensional $K(\Gamma, 1)$. Can $\mathcal{P}(\Gamma) \in \mathfrak{G}(\Gamma)$ be represented by a polytope?*

A proof of Conjecture 6.2 also proves Conjecture 6.1.

Before we state our next question we recall that the Thurston norm of an admissible hyperbolic 3-manifold N is a norm. (This is a direct consequence of the fact that hyperbolic admissible 3-manifolds are atoroidal, i.e. any embedded torus is boundary parallel, see e.g. [BP92, Proposition D.3.2.8].) This implies that if N is an admissible hyperbolic 3-manifold with $b_1(N) \geq 1$, then $\mathcal{P}(N) = \mathcal{P}(\pi_1(N))$ is not a point.

Question 6.3. *Let $\Gamma \neq \mathbb{Z}$ be an L^2-acyclic* Wh-AC-*group of type F that admits a 2-dimensional $K(\Gamma, 1)$. Furthermore suppose that Γ is a non-elementary hyperbolic group and that $b_1(\Gamma) \geq 1$. Does it follow that $\mathcal{P}(\Gamma) \neq 0 \in \mathfrak{G}(\Gamma)$?*

[7] In [FK16] we authors refer to such an HNN-extension as an "ascending HNN-extension", whereas here we stick with the convention eastablished above and that follows [BS78, BNS87].

Conjecture 6.4. *Let Γ be an L^2-acyclic* Wh-AC-*group of type F. If Γ is amenable and if Γ is not virtually $\mathbb{Z}$, then $\mathcal{P}(\Gamma) = 0 \in \mathfrak{G}(\Gamma)$.*

Question 6.5. *Let Γ be an L^2-acyclic* Wh-AC-*group of type F. Is the BNS-invariant related to $\mathcal{P}(\Gamma)$? Does an analogue of Theorem 5.5 hold?*

In general it might be too optimistic to expect a positive answer. It seems more likely that the question can be answered in the affirmative if Γ has a 2-dimensional $K(\Gamma, 1)$.

It is known that the BNS invariants of metabelian and 3–manifold groups are polyhedral. An affirmative (even partial) answer to Question 6.5 may establish polyhedrality of BNS invariants for new families of groups, as well as whether the polyhedra are rational or not. We can therefore also ask the following question.

Conjecture 6.6. *Let Γ be a hyperbolic L^2-acyclic group of type F with a 2-dimensional $K(\Gamma, 1)$. Then there exists an integral marked polytope $\mathcal{M} \subset H_1(\Gamma; \mathbb{R})$ such that a non-trivial $\phi \in H^1(\Gamma; \mathbb{R})$ represents an element in $\Sigma(\Gamma)$ if and only if ϕ pairs maximally with a marked vertex of $\mathcal{M}$.*

The combination of Theorem 3.6(4) and Lemmas 4.3, 6.12 and 6.13 of [FL16a] shows that if Γ is an L^2-acyclic Wh-AC-group of type F, then for any epimorphism $\phi\colon \Gamma \to \mathbb{Z}$ we have

$$\sum_{n \in \mathbb{N}_0} (-1)^{n+1} \cdot b_n^{(2)}(\operatorname{Ker}(\phi\colon \Gamma \to \mathbb{Z})) \;=\; \operatorname{th}_\phi(\mathcal{P}(\Gamma)).$$

Conjecture 6.7. *Let Γ be an L^2-acyclic* Wh-AC-*group of type F and let $n \in \mathbb{N}_0$. Then there exists a polytope $\mathcal{P}$ such that*

$$b_n^{(2)}(\operatorname{Ker}(\phi\colon \Gamma \to \mathbb{Z})) \;=\; \operatorname{th}_\phi(\mathcal{P})$$

for any epimorphism $\phi\colon \Gamma \to \mathbb{Z}$.

In Section 5.1 we saw that for 3-manifold groups and many two-generator one-relator groups the thickness of the polytope invariant is related to the complexity of splittings of the underlying group.

Question 6.8. *Let Γ be an L^2-acyclic* Wh-AC-*group of type F. What information does the thickness of the polytope invariant contain?*

As we mentioned before, the thickness of a polytope is an invariant of the symmetrized polytope.

Question 6.9. *What information does the polytope invariant contain, that cannot be obtained from the symmetrized polytope invariant?*

One partial answer is given by the discussion of the BNS-invariant. This invariant is in general not symmetric and if the polytope invariant is related to the BNS-invariant it cannot be symmetric in general. Nonetheless, there are many groups with empty BNS-invariant and non-symmetric polytope invariant.

Finally we want to discuss which elements of $\mathfrak{G}(\mathbb{Z}^n)$ can be realized as the polytope invariant of a manifold. Here we say that $\mathcal{P} \in \mathfrak{G}(\mathbb{Z}^n)$ *can be realized by a d-dimensional manifold* if there exists a pair (N, φ) where N is a closed L^2-acyclic d-dimensional manifold N such that $\pi_1(N)$ is a Wh-AC-group and where $\varphi\colon \mathbb{Z}^n \to H_1(N; \mathbb{Z})/\text{torsion}$ is an isomorphism, such that $\varphi_*(\mathcal{P}) = \mathcal{P}(N) \in \mathfrak{G}(\pi_1(N))$.

From our above results we know that not all $\mathcal{P} \in \mathfrak{G}(\mathbb{Z}^n)$ can be realized by manifolds. More precisely, in Proposition 2.4 we showed that polytopes realized by closed orientable manifolds have a symmetry. Furthermore in Theorem 3.2 we showed that the polytope invariant of an admissible 3-manifold $N \neq S^1 \times D^2$ that is not a closed graph manifold can be represented by a polytope. So we can only hope to realize elements of $\mathfrak{P}(\mathbb{Z}^n) \subset \mathfrak{G}(\mathbb{Z}^n)$ by 3-dimensional manifolds.

We have the following realization result.

Lemma 6.10. *Let $n \in \mathbb{N}$ and let $\mathcal{P} \in \mathfrak{G}(\mathbb{Z}^n)$. For any $d \geq 7$ we can realize $\mathcal{P} + (-1)^{d+1}\overline{\mathcal{P}}$ by a d-dimensional manifold.*

Proof. Let F be a free group on $n-1$ generators. We set $\Gamma = F \times \mathbb{Z}$. Then $K(\Gamma, 1)$ is 2-dimensional and it is L^2-acyclic by [Lü02, Theorem 1.35 (4)]. Furthermore Γ is a Wh-AC-group by [KLR16, Theorem 2] and [LiL16, Theorem 2.3]. We pick an isomorphism $\varphi\colon \mathbb{Z}^n \to H_1(\Gamma; \mathbb{Z})$ and we pick a set-theoretic section $s\colon H_1(\Gamma; \mathbb{Z}) \to \Gamma$ of the projection map $\Gamma \to H_1(\Gamma; \mathbb{Z})$.

Now let $\mathcal{P} \in \mathfrak{G}(\mathbb{Z}^n)$. We can find $a, b \in \mathbb{Z}[\mathbb{Z}^n]$ with $\mathcal{P} = \mathcal{P}(a) - \mathcal{P}(b)$. We write

$$\omega \;:=\; s(\varphi(a)) \cdot s(\varphi(b))^{-1} \;\in\; K_1(\mathcal{D}(\Gamma)) = \mathcal{D}(\Gamma)^{\times}_{ab}.$$

It follows immediately from the definitions that $\mathcal{P}(\omega) = \varphi(\mathcal{P}) \in \mathfrak{G}(\Gamma)$. Now let $d \geq 7$. It follows easily from a slight generalization of [FL16b, Lemma 2.9] that there exists a closed orientable d-dimensional manifold N with $\pi_1(N) = \Gamma$ such that

$$\tau(N, \mathrm{id}_{\pi_1 N}) \;=\; \omega \cdot \overline{\omega}^{(-1)^{d+1}} \;\in\; K_1(\mathcal{D}(\Gamma)) = \mathcal{D}(\Gamma)^{\times}_{ab}/\pm\Gamma.$$

It follows immediately that $\mathcal{P}(N) = \varphi(\mathcal{P}) + (-1)^{d+1}\overline{\varphi(\mathcal{P})} \in \mathfrak{G}(\Gamma)$. $\square$

Question 6.11. (1) *Let $\mathcal{P} \in \mathfrak{P}(\mathbb{Z}^n)$. Does there exist a closed orientable admissible 3-manifold that realizes $\mathcal{P} + \overline{\mathcal{P}}$?*

(2) *Let $\mathcal{P} \in \mathfrak{G}(\mathbb{Z}^n)$ and let $d \in \{4, 5, 6\}$. Does there exist an L^2-acyclic d-dimensional closed orientable manifold that realizes $\mathcal{P} + (-1)^{d+1}\overline{\mathcal{P}}$?*

The first question seems to be very hard since by Theorem 3.2 it is a reformulation of the question of which Thurston norm balls are realized by aspherical 3-manifolds. This question has been open since the 1970's. The second question might be more accessible, especially in dimensions 5 and 6. It is conceivable that in dimension 4 the answer depends on whether one studies topological or smooth manifolds.

References

[Ag08] I. Agol, *Criteria for virtual fibering*, J. Topol. 1 (2008), no. 2, 269–284.

[Ag13] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Documenta Math. 18 (2013), 1045–1087.

[AFW15] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, EMS Series of Lectures in Mathematics (2015)

[BLR08] A. Bartels, W. Lück and H. Reich, *Equivariant covers for hyperbolic groups*, Geom. Topol. 12 (2008), 1799–1882.

[BP92] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer, Berlin, 1992.

[BNS87] R. Bieri, W. D. Neumann and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. 90 (1987), 451-477.

[BS78] R. Bieri and R. Strebel, *Almost finitely presented soluble groups*, Comment. Math. Helv. 53 (1978), 258–278.

[Br82] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer, 1982.

[Br87] K. S. Brown, *Trees, valuations, and the Bieri-Neumann-Strebel invariant*, Invent. Math. 90 (1987), 479-504.

[BK15] J. Button and R. Kropholler, *Non Hyperbolic Free-By-Cyclic and One-Relator Groups*, preprint (2015)

[DL07] W. Dicks and P. Linnell, *L^2-Betti numbers of one-relator groups*, Math. Ann. 337, No. 4 (2007), 855–874.

[DLMSY03] J. Dodziuk, P. Linnell, V. Mathai, T. Schick and S. Yates, *Approximating L^2-invariants, and the Atiyah conjecture*, Preprint Series SFB 478 Münster, Germany. Communications on Pure and Applied Mathematics, vol. 56, no. 7 (2003), 839-873.

[DFL16] J. Dubois, S. Friedl and W. Lück, *The L^2-Alexander torsion of 3-manifolds*, J. Topol. 9, No. 3 (2016), 889–926.

[Fo53] R. H. Fox, *Free differential calculus I, Derivation in the free group ring*, Ann. Math. 57 (1953), 547–560.

[Fr07] S. Friedl, *Reidemeister torsion, the Thurston norm and Harvey's invariants*, Pac. J. Math. 230 (2007), 271–296.

[Fr15] S. Friedl, *Novikov homology and noncommutative Alexander polynomials*, Preprint (2015), to be published the Tim Cochran memorial volume of the J. Knot. Thy. Ram.

[FL16a] S. Friedl and W. Lück, *L^2-Euler characteristics and the Thurston norm*, Preprint (2016), arXiv:1609.07805.

[FL16b] S. Friedl and W. Lück, *Universal L^2-torsion, polytopes and applications to* 3*-manifolds*, Preprint (2016), arXiv:1609.07809.

[FST15] S. Friedl, K. Schreve and S. Tillmann, *Thurston norm via Fox calculus*, preprint (2015), arXiv:1507.05660.

[FSW15] S. Friedl, D. Silver and S. Williams, *Splittings of knot groups*, Mathematische Annalen 362 (2015), 401–424.

[FSW16] S. Friedl, D. Silver and S. Williams, *The Turaev and Thurston norms*, Pac. J. Math. 284 (2016), 365–382.

[FT15] S. Friedl and S. Tillmann, *Two-generator one-relator groups and marked polytopes*, Preprint (2015), arXiv:1501.03489.

[Fu16] F. Funke, *The integral polytope group*, Preprint (2016), arXiv:1605.01217.

[FK16] F. Funke and D. Kielak, *Alexander and Thurston norms, and the Bieri-Neumann-Strebel invariants for free-by-cyclic groups*, preprint (2016), arXiv:1605.09067.

[He01] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology 40 (2001), no. 3, 631–657.

[KLR16] H. Kammeyer, W. Lück and H. Rüping, *The Farrell-Jones Conjecture for arbitrary lattices in virtually connected Lie groups*, Geometry and Topology 20 (2016), 1275–1287.

[Ko88] T. Kobayashi, *Casson-Gordon's rectangle condition of Heegaard diagrams and incompressible tori in 3-manifolds*, Osaka J. Math. 25 (1988), no. 3, 553–573.

[Li93] P. A. Linnell, *Division rings and group von Neumann algebras*, Forum Math. 5 (1993), vol. 6, 561–576.

[LiL16] P. Linnell and W. Lück, *Localization, Whitehead groups, and the Atiyah Conjecture*, preprint (2016)

[LoL95] J. Lott and W. Lück, *L^2-topological invariants of 3-manifolds*, Invent. Math. 120 (1995), no. 1, 15–60.

[Lü02] W. Lück, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 44, Springer, Berlin, 2002.

[LS77] R. Lyndon and P. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89. Springer-Verlag, Berlin-New York, 1977.

[Ma10] J. Maher, *Random Heegaard splittings*, Journal of Topology (2010) 3 (4), 997–1025.

[Mi66] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966), 358–426.

[PW12] P. Przytycki and D. Wise, *Mixed 3–manifolds are virtually special*, preprint (2012).

[Ro94] J. Rosenberg, *Algebraic K -theory and its applications*, Graduate Texts in Mathematics 147, Springer, New York, 1994.

[Sc93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge Univ. Press (1993)

[Sik87] J.–C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*, Thèse Orsay, 1987.

[Sil81] J. Silvester, *Introduction to algebraic K-theory*, Chapman and Hall Mathematics Series (1981)

[Sin77] I. M. Singer, *Some remarks on operator theory and index theory*, In K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pages 128–138. Lecture Notes in Math., Vol. 575. Springer-Verlag, Berlin, 1977.

[St84] R. Strebel, *Finitely presented soluble groups*, in Group Theory: Essays for Philip Hall, Academic Press, London, 1984.

[Th86] W. P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. 59, no. 339 (1986), 99–130.

[Ti70] D. Tischler, *On fibering certain foliated manifolds over S^1*, Topology 9 (1970), 153–154.

[Tu01] V. Turaev, *Introduction to Combinatorial Torsions*, Lectures in Mathematics, ETH Zürich (2001)

[Wa78] F. Waldhausen, *Algebraic K-theory of generalized free products II*, Ann. of Math. 108 (1978), 135–256.

[We15] C. Wegner, *The Farrell-Jones conjecture for virtually solvable groups*, J. Topol. 8 (2015), 975–1016.

[Wi09] D. Wise, *The structure of groups with a quasi-convex hierarchy*, Electronic Res. Ann. Math. Sci. 16 (2009), 44–55.

[Wi12a] D. Wise, *The structure of groups with a quasi-convex hierarchy*, 189 pages, preprint (2012), downloaded on October 29, 2012 from
`http://www.math.mcgill.ca/wise/papers.html`

[Wi12b] D. Wise, *From riches to RAAGs: 3–manifolds, right–angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, 2012.

[Wi14] D. Wise, *Cubular tubular groups*, Trans. Amer. Math. Soc. 366 (2014), 5503–5521.

Fakultät für Mathematik, Universität Regensburg, Germany
E-mail address: `sfriedl@gmail.com`

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `wolfgang.lueck@him.uni-bonn.de`
URL: `http://www.him.uni-bonn.de/lueck`

School of Mathematics and Statistics, The University of Sydney, NSW 2006 Australia
E-mail address: `stephan.tillmann@sydney.edu.au`