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Knot concordance, Whitney towers and L^2-signatures

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Abstract

We construct many examples of nonslice knots in 3-space that cannot be distinguished from slice knots by previously known invariants. Using Whitney towers in place of embedded disks, we define a geometric filtration of the 3-dimensional topological knot concordance group. The bottom part of the filtration exhibits all classical concordance invariants, including the Casson-Gordon invariants. As a first step, we construct an infinite sequence of new obstructions that vanish on slice knots. These take values in the L-theory of skew fields associated to certain *universal* groups. Finally, we use the dimension theory of von Neumann algebras to define an L^2-signature and use this to detect the first unknown step in our obstruction theory.

Contents

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1. Introduction

This paper begins a detailed investigation into the group of topological concordance classes of knotted circles in the 3-sphere. Recall that a knot K is topologically *slice* if there exists a locally flat topological embedding of the 2-disk into B^4 whose restriction to the boundary is K. The knots K_0 and K_1 are topologically *concordant* if there is a locally flat topological embedding of the annulus into $S^3 \times [0,1]$ whose restriction to the boundary components gives the knots. The set of concordance classes of knots under the operation of connected sum forms an abelian group $\mathcal{C}$, whose identity element is the class of slice knots.

THEOREM 6.4 (A special case). *The knot of Figure* 6.1 *has vanishing Casson-Gordon invariants but is not topologically slice.*

In fact, we construct infinitely many such examples that cannot be distinguished from slice knots by previously known invariants. The new slice obstruction that detects these knots is an L^2-signature formed from the dimension theory of the von Neumann algebra of a certain rationally universal solvable group. To construct nontrivial maps from the fundamental group of the knot complement to this solvable group, we develop an obstruction theory and for this purpose, we define noncommutative higher-order versions of the classical Alexander module and Blanchfield linking form. We hope that these generalizations are of considerable independent interest.

We give new geometric conditions which lead to a natural filtration of the slice condition "there is an embedded 2-disk in B^4 whose boundary is the knot". More precisely, we exhibit a new geometrically defined filtration of the knot concordance group $\mathcal{C}$ indexed on the half integers;

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C},$$

where for $h \in \frac{1}{2}\mathbb{N}_0$, the group $\mathcal{F}_{(h)}$ consists of all (h)-*solvable* knots. (h)-solvability is defined using intersection forms in certain solvable covers (see Definition 1.2). The obstruction theory mentioned above measures whether a given knot lies in the subgroups $\mathcal{F}_{(h)}$. It provides a bridge from algebra to the topological techniques of A. Casson and M. Freedman. In fact, (h)-solvability has an equivalent definition in terms of the geometric notions of *gropes* and *Whitney towers* (see Theorems 8.4 and 8.8 in part 1.1 of the introduction). Moreover, the tower of von Neumann signatures might be viewed as an algebraic mirror of infinite constructions in topology. Another striking example of this bridge is the following theorem, which implies that the Casson-Gordon invariants obstruct a specific step (namely a second layer of Whitney disks) in the Freedman-Cappell-Shaneson surgery theoretic program to prove that a knot is slice. Thus one of the most significant aspects of our work is to provide a step toward a new and strictly 4-dimensional homology surgery theory.

THEOREM 9.11. *Let $K \subset S^3$ be (1.5)-solvable. Then all previously known concordance invariants of K vanish.*

In addition to the Seifert form obstruction, these are the invariants introduced by A. Casson and C. McA. Gordon in 1974 and further metabelian invariants by P. Gilmer [G1], [G2], P. Kirk and C. Livingston [KL], and C. Letsche [Let]. More precisely, Theorem 9.11 actually proves the vanishing of the Gilmer invariants. These determine the Casson-Gordon invariants and the invariants of Kirk and Livingston. The Letsche obstructions are handled in a separate Theorem 9.12.

The first few terms of our filtration correspond closely to the previously known concordance invariants and we show that the filtration is nontrivial beyond these terms. Specifically, a knot lies in $\mathcal{F}_{(0)}$ if and only if it has vanishing Arf invariant, and lies in $\mathcal{F}_{(0.5)}$ if and only if it is algebraically slice, i.e. if the Levine Seifert form obstructions (that classify higher dimensional knot concordance) vanish (see Theorem 1.1 together with Remark 1.3). Finally, the family of examples of Theorem 6.4 proves the following:

COROLLARY. *The quotient group $\mathcal{F}_{(2)}/\mathcal{F}_{(2.5)}$ has infinite rank.*

In this paper we will show that this quotient group is nontrivial. The full proof of the corollary will appear in another paper.

The geometric relevance of our filtration is further revealed by the following two results, which are explained and proved in Sections 7 and 8.

THEOREM 8.11. *If a knot K bounds a grope of height $(h+2)$ in D^4 then K is (h)-solvable.*

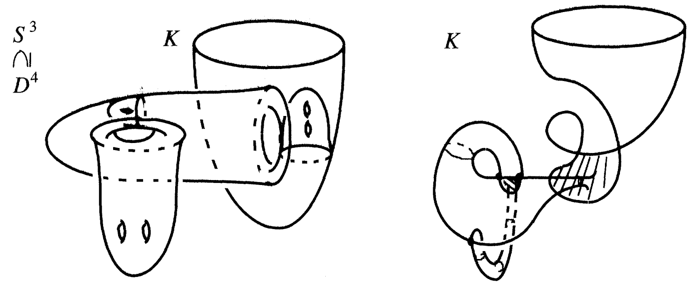

Figure 1.1. A grope of height 2.5 and a Whitney tower of height 2.5.

THEOREM 8.12. *If a knot K bounds a Whitney tower of height $(h+2)$ in D^4 then K is (h)-solvable.*

We establish an infinite series of new knot slicing obstructions lying in the L-theory of large skew fields, and associated to the commutator series of the knot group. These successively obstruct each integral stage of our filtration (Theorem 4.6). We also prove the desired result that the higher-order Alexander modules of an (h)-solvable knot contain submodules that are self-annihilating with respect to the corresponding higher-order linking form. We see no reason that this tower of obstructions should break down after three steps even though the complexity of the computations grows. We conjecture:

CONJECTURE. *For any $n \in \mathbb{N}_0$, there are (n)-solvable knots that are not $(n.5)$-solvable. In fact $\mathcal{F}_{(n)}/\mathcal{F}_{(n.5)}$ has infinite rank.*

For $n=0$ this is detected by the Seifert form obstructions, for $n=1$ this can be established by Theorem 9.11 from examples due to Casson and Gordon, and $n=2$ is the above corollary. Indeed, if there exists a fibered ribbon knot whose classical Alexander module, first-order Alexander module ... and $(n-1)^{\text{st}}$-order Alexander module have *unique* proper submodules (analogous to $\mathbb{Z}_9$ as opposed to $\mathbb{Z}_3 \times \mathbb{Z}_3$), then the conjecture is true for all n. Hence our inability to establish the full conjecture at this time seems to be merely a technical deficiency related to the difficulty of solving equations over noncommutative fields. In Section 8 we will explain what it means for an arbitrary link to be (h)-solvable. Then the following result provides plenty of candidates for proving our conjecture in general.

THEOREM 8.9. *If there exists an (h)-solvable link which forms a standard half basis of untwisted curves on a Seifert surface for a knot K, then K is $(h+1)$-solvable.*

It remains open whether a (0.5)-solvable knot is (1)-solvable and whether a (1.5)-solvable knot is (2)-solvable but we do introduce potentially nontrivial obstructions that generalize the Arf invariant (see Corollary 4.9).

1.1. *Some history, (h)-solvability and Whitney towers.* In the 1960's, M. Kervaire and J. Levine computed the group of concordance classes of knotted n-spheres in S^{n+2}, $n \geq 2$, using ambient surgery techniques. Even-dimensional knots are always slice [K], and the odd-dimensional concordance group can be described by a collection of computable obstructions defined as Witt equivalence classes of linking pairings on a Seifert surface [L1] (see also [Sto]). One modifies the Seifert surface along middle-dimensional embedded disks in the $(n+3)$-ball to create the slicing disk. The obstructions to embedding these middle-dimensional disks are intersection numbers that are suitably reinterpreted as linking numbers of the bounding homology classes in the Seifert surface. This *Seifert form* obstructs slicing knotted 1-spheres as well.

In the mid 1970's, S. Cappell and J. L. Shaneson introduced a new strategy for slicing knots by extending surgery theory to a theory classifying manifolds within a homology type [CS]. Roughly speaking, the classification of higher dimensional knot concordance is the classification of homology circles up to homology cobordism rel boundary. The reader should appreciate the basic fact that a knot is a slice knot if and only if the (n+2)-manifold, M, obtained by (zero-framed) surgery on the knot is the boundary of a manifold that has the homology of a circle and whose fundamental group is normally generated by the meridian of the knot. More generally, for knotted n-spheres in S^{n+2} (n odd), here is an outline of the Cappell-Shaneson surgery strategy. One lets M bound an $(n+3)$-manifold W with infinite cyclic fundamental group. The middle-dimensional homology of the universal abelian cover of W admits a $\mathbb{Z}[\mathbb{Z}]$-valued intersection form. The Cappell-Shaneson obstruction is the obstruction to finding a half-basis of immersed spheres whose intersection points occur in pairs each of which admits an associated immersed Whitney disk. As usual, in higher dimensions, if the obstructions vanish, these Whitney disks may be embedded and intersections removed in pairs. The resulting embedded spheres are then surgically excised resulting in an homology circle, i.e. a slice complement.

These two strategies, when applied to the case $n = 1$, yield the following equivalent obstructions. (See [L1] and [CS] together with Remark 1.3.2.) The theorem is folklore except that condition (c) is new (see Theorem 8.13). Denote by M the 0-framed surgery on a knot K. Then M is a closed 3-manifold and $H_1(M) := H_1(M;\mathbb{Z})$ is infinite cyclic. An orientation of M and a generator of $H_1(M)$ are determined by orienting S^3 and K.

THEOREM 1.1. *The following statements are equivalent*:

(a) (*The Levine condition*) *K bounds a Seifert surface in S^3 for which the Seifert form contains a Lagrangian.*

(b) (*The Cappell-Shaneson condition*) *M bounds a compact spin manifold W with the following properties*:

1. *The inclusion induces an isomorphism $H_1(M) \xrightarrow{\cong} H_1(W)$.*
2. *The $\mathbb{Z}[\mathbb{Z}]$-valued intersection form λ_1 on $H_2(W;\mathbb{Z}[\mathbb{Z}])$ contains a totally isotropic submodule whose image is a Lagrangian in $H_2(W)$.*

(c) *K bounds a grope of height 2.5 in D^4.*

A submodule is *totally isotropic* if the corresponding form vanishes on it. A *Lagrangian* is a totally isotropic direct summand of half rank. Knots satisfying the conditions of Theorem 1.1 are the aforementioned class of *algebraically slice* knots. In particular, slice knots satisfy these conditions, and in higher dimensions, Levine showed that algebraically slice implies slice [L1].

If the Cappell-Shaneson homology surgery machinery worked in dimension four, algebraically slice knots would be slice as well. However, in the mid $1970's$, Casson and Gordon discovered new slicing obstructions proving that, contrary to the higher dimensional case, algebraically slice knotted 1-spheres are not necessarily slice [CG1], [CG2]. The problem is that the Whitney disks that pair up the intersections of a spherical Lagrangian may no longer be embedded, but may themselves have intersections, which might or might not occur in pairs, and if so may have their own Whitney disks. One naturally speculates that the Casson-Gordon invariants should obstruct a second layer of Whitney disks in this approach. This is made precise by Theorem 9.11 together with the following theorem (compare Definitions 7.7, 8.7 and 8.5). Moreover this theorem shows that (h)-solvability filters the Cappell-Shaneson approach to disjointly embedding an integral homology half basis of spheres in the 4-manifold.

THEOREMS 8.4 & 8.8. *A knot is (h)-solvable if and only if M bounds a compact spin manifold W where the inclusion induces an isomorphism on H_1 and such that there exists a Lagrangian $L \subset H_2(W;\mathbb{Z})$ that has the following additional geometric property: L is generated by immersed spheres $\ell_1, \ldots, \ell_k$ that allow a* Whitney tower *of height h.*

We conjectured above that there is a nontrivial step from each height of the Whitney tower to the next. However, even an infinite Whitney tower might not lead to a slice disk. This is in contrast to finding *Casson towers*, which in addition to the Whitney disks have so called *accessory disks* associated to each double point. By Freedman's main result, any Casson tower of height four contains a topologically embedded disk. Thus the ultimate goal is to establish necessary and sufficient criteria to finding Casson towers. Since a Casson tower is in particular a Whitney tower, our obstructions also apply to Casson towers. For example, it follows that Casson-Gordon invariants obstruct finding Casson towers of height two in the above Cappell-Shaneson approach. Thus we provide a proof of the heuristic argument that by Freedman's result the Casson-Gordon invariants must obstruct the existence of Casson towers.

We now outline the definition of (h)-solvability. The reader can see that it filters the condition of finding a half-basis of disjointly embedded spheres by examining intersection forms with progressively more discriminating coefficients, as indexed by the derived series.

Let $G^{(i)}$ denote the i^{th} *derived group* of a group G, inductively defined by $G^{(0)} := G$ and $G^{(i+1)} := [G^{(i)}, G^{(i)}]$. A group G is *(n)-solvable* if $G^{(n+1)} = 1$ ((0)-solvable corresponds to abelian) and G is *solvable* if such a finite n exists. For a CW-complex W, we define $W^{(n)}$ to be the regular covering corresponding to the subgroup $(\pi_1(W))^{(n)}$. If W is an oriented 4-manifold then there is an intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \longrightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}].$$

(see [Wa, Ch. 5], and our §7 where we also explain the self-intersection invariant μ_n). For $n \in \mathbb{N}_0$, an *(n)-Lagrangian* is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of λ_0.

Definition 1.2. A knot is called *(n)-solvable* if M bounds a spin 4-manifold W, such that the inclusion map induces an isomorphism on first homology and such that W admits two *dual* (n)-Lagrangians. This means that the form λ_n pairs the two Lagrangians nonsingularly and that their images together freely generate $H_2(W)$ (see Definition 8.3).

A knot is called *$(n.5)$-solvable*, $n \in \mathbb{N}_0$, if M bounds a spin 4-manifold W such that the inclusion map induces an isomorphism on first homology and such that W admits an $(n+1)$-Lagrangian and a dual (n)-Lagrangian in the above sense. We say that M is *(h)-solvable via* W which is called an *(h)-solution for* M (or K).

Remark 1.3. It is appropriate to mention the following facts:

1. The size of an (h)-Lagrangian L is controlled only by its image in $H_2(W)$; in particular, if $H_2(W) = 0$ then the knot K is (h)-solvable for all $h \in \frac{1}{2}\mathbb{N}$. This holds for example if K is topologically slice. More generally, if K and K' are topologically concordant knots, then K is (h)-solvable if and only if K' is (h)-solvable. (See Remark 8.6.)

2. One easily shows (0)-solvable knots are exactly knots with trivial Arf invariant. (See Remark 8.2.) One sees that a knot is algebraically slice if and only if it is (0.5)-solvable by observing that the definition above for $n = 0$ is exactly condition (b.2) of Theorem 1.1.

3. By the naturality of covering spaces and homology with twisted coefficients, if K is (h)-solvable then it is (h')-solvable for all $h' \leq h$.

4. Given an $(n.5)$-solvable or (n)-solvable knot with a 4-manifold W as in Definition 1.2 one can do surgery on elements in $\pi_1(W^{(n+1)})$, preserving all the conditions on W. In particular, if $\pi_1(W)/\pi_1(W)^{(n+1)}$ is finitely presented then one can arrange for $\pi_1(W)$ to be (n)-solvable. This motivated our choice of terminology. Moreover, since this condition

does hold for $n = 0$, we see that, in the classical case of (0.5)-solvable (i.e., algebraically slice) knots, one can always assume that $\pi_1(W) = \mathbb{Z}$. This is the way that condition (b) in Theorem 1.1 is usually formulated, namely as the vanishing of the Cappell-Shaneson surgery obstruction in $\Gamma_0(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z})$. In particular, this proves the equivalence of conditions (a) and (b) in Theorem 1.1. The equivalence of (b) and (c) will be proved in Section 7.

1.2. *Linking forms, intersection forms, and solvable representations of knot groups.* The Casson-Gordon invariants exploit the observation that linking of 1-dimensional objects in a 3-manifold may be computed via the intersection theory of a homologically simple 4-manifold that it bounds. Thus, 2-dimensional intersection pairings for the 4-manifold are subtly related to the fundamental group of the bounding 3-manifold. Casson and Gordon utilize the $\mathbb{Q}/\mathbb{Z}$, or *torsion linking pairing*, on prime power cyclic knot covers to access intersection data in metabelian covers of 4-manifolds. A secondary obstruction theory results, with vanishing criteria determined by first order choices.

Our obstructions are Witt classes of intersection forms on the homology of higher-order solvable covers, obtained from a sequence of new higher-order linking pairings (see Section 3). We define what we call *rationally universal n-solvable knot groups*, constructed from universal torsion modules, which play roles analogous to $\mathbb{Q}/\mathbb{Z}$ in the torsion linking pairing on a rational homology sphere, and to $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ in the classical Blanchfield pairing of a knot. Representations of the knot group into these groups are parametrized by elements of the higher-order Alexander modules. The key point is that if K is slice (or merely (n)-solvable), then some predictable fraction of these representations extends to the complement of the slice disk (or the (n)-solution W). The Witt classes of the intersection forms of these 4-manifolds then constitute invariants that vanish for slice knots (or merely $(n.5)$-solvable knots).

For any *fixed* knot and any *fixed* (n)-solution W one can show that a signature vanishes by using certain solvable quotients of $\pi_1(W)$, and not using the universal groups. However a general obstruction theory requires the introduction of these universal groups just as the study of torsion linking pairings on all rational homology 3-spheres requires the introduction of $\mathbb{Q}/\mathbb{Z}$.

We first define the rationally universal solvable groups. The metabelian group is a rational analogue of the group used by Letsche [Let]. Let $\Gamma_0 := \mathbb{Z}$ and let $\mathcal{K}_0$ be the quotient field of $\mathbb{Z}\Gamma_0$. Consider a PID $\mathcal{R}_0$ that lies in between $\mathbb{Z}\Gamma_0$ and $\mathcal{K}_0$. For example, a good choice is $\mathbb{Q}[\mu^{\pm 1}]$ where μ generates Γ_0. Note that $\mathcal{K}_0 = \mathbb{Q}(\mu)$. For any choice of $\mathcal{R}_0$, the abelian group $\mathcal{K}_0/\mathcal{R}_0$ is a bimodule over Γ_0 via left (resp. right) multiplication. We choose the right multiplication to define the semi-direct product

$$\Gamma_1 := (\mathcal{K}_0/\mathcal{R}_0) \rtimes \Gamma_0.$$

This is our *rationally universal* metabelian (or (1)-solvable) group for knots in S^3. Inductively, we obtain *rationally universal* $(n+1)$-solvable groups by setting

$$\Gamma_{n+1} := (\mathcal{K}_n/\mathcal{R}_n) \rtimes \Gamma_n$$

for certain PID's $\mathcal{R}_n$ lying in between $\mathbb{Z}\Gamma_n$ and its quotient field $\mathcal{K}_n$. To define the latter we show in Section 3 that the ring $\mathbb{Z}\Gamma_n$ satisfies the so-called *Ore condition* which is necessary and sufficient to construct the (skew) quotient field $\mathcal{K}_n$ exactly as in the commutative case.

Now let M be the 0-framed surgery on a knot in S^3. We begin with a fixed representation into Γ_0 that is normally just the abelianization isomorphism $\pi_1(M)^{ab} \cong \Gamma_0$. Consider $\mathcal{A}_0 := H_1(M; \mathcal{R}_0)$, the ordinary (rational) Alexander module. Denote its dual by

$$\mathcal{A}_0^{\#} := \mathrm{Hom}_{\mathcal{R}_0}(\mathcal{A}_0, \mathcal{K}_0/\mathcal{R}_0).$$

Then the Blanchfield form

$$B\ell_0 : \mathcal{A}_0 \times \mathcal{A}_0 \longrightarrow \mathcal{K}_0/\mathcal{R}_0$$

is nonsingular in the sense that it provides an isomorphism $\mathcal{A}_0 \cong \mathcal{A}_0^{\#}$. Using basic properties of the semi-direct product, we show in Section 3 that there is a one-to-one-correspondence

$$\mathcal{A}_0^{\#} \longleftrightarrow \mathrm{Rep}^*_{\Gamma_0}(\pi_1(M), \Gamma_1).$$

Here $\mathrm{Rep}^*_{\Gamma_n}(G, \Gamma_{n+1})$ denotes the set of representations of G into Γ_{n+1} that agree with some fixed representation into Γ_n, modulo conjugation by elements in the subgroup $\mathcal{K}_n/\mathcal{R}_n$. Hence when $a_0 \in \mathcal{A}_0$ the Blanchfield form $B\ell_0$ defines an action of $\pi_1(M)$ on $\mathcal{R}_1$ and we may define the next Alexander module $\mathcal{A}_1 = \mathcal{A}_1(a_0) := H_1(M; \mathcal{R}_1)$. We prove that a nonsingular Blanchfield form

$$B\ell_1 : \mathcal{A}_1 \xrightarrow{\cong} \mathcal{A}_1^{\#} := \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{A}_1, \mathcal{K}_1/\mathcal{R}_1)$$

exists and induces a one-to-one correspondence

$$\mathcal{A}_1 \longleftrightarrow \mathrm{Rep}^*_{\Gamma_1}(\pi_1(M), \Gamma_2).$$

Iterating this procedure leads to the $(n-1)$-st Alexander module

$$\mathcal{A}_{n-1} = \mathcal{A}_{n-1}(a_0, a_1, \ldots, a_{n-2}) := H_1(M; \mathcal{R}_{n-1})$$

together with the $(n-1)$-st Blanchfield form $B\ell_{n-1} : \mathcal{A}_{n-1} \xrightarrow{\cong} \mathcal{A}_{n-1}^{\#}$ and a one-to-one correspondence

$$\mathcal{A}_{n-1} \longleftrightarrow \mathrm{Rep}^*_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n).$$

We show in Section 4 that for an (n)-solvable knot there exist choices $(a_0, a_1, \ldots, a_{n-1})$ that correspond to a representation $\phi_n : \pi_1(M) \to \Gamma_n$ which extends to a spin 4-manifold W whose boundary is M. We then observe that the intersection form on $H_2(W;\mathcal{K}_n)$ is nonsingular and represents an element $B_n = B_n(M,\phi_n)$ of $L^0(\mathcal{K}_n)$ which is well-defined (independent of W) modulo the image of $L^0(\mathbb{Z}\Gamma_n)$. Here $L^0(R)$, R a ring with involution, denotes the Witt group of nonsingular hermitian forms on finitely generated free R-modules, modulo metabolic forms.

We can now formulate our obstruction theory for (h)-solvable knots. A more general version, Theorem 4.6, is stated and proved in Section 4.

THEOREM 4.6 (A special case). *Let K be a knot in S^3 with 0-surgery M.*

(0): *If K is (0)-solvable then there is a well-defined obstruction $B_0 \in L^0(\mathcal{K}_0)/i(L^0(\mathbb{Z}\Gamma_0))$.*

(0.5): *If K is (0.5)-solvable then $B_0 = 0$.*

(1): *If K is (1)-solvable then there exists a submodule $P_0 \subset \mathcal{A}_0$ such that $P_0^\perp = P_0$ and such that for each $p_0 \in P_0$ there is an obstruction $B_1 = B_1(p_0) \in L^0(\mathcal{K}_1)/i(L^0(\mathbb{Z}\Gamma_1))$.*

(1.5): *If K is (1.5)-solvable then there is a P_0 as above such that for all $p_0 \in P_0$ the obstruction B_1 vanishes.*

$\vdots$

(n): *If K is (n)-solvable then there exists P_0 as above such that for all $p_0 \in P_0$ there exists $P_1 = P_1(p_0) \subset \mathcal{A}_1(p_0)$ with $P_1^\perp = P_1$ and such that for all $p_1 \in P_1$ there exists $P_2 = P_2(p_0,p_1) \subset \mathcal{A}_2(p_0,p_1)$ with $P_2 = P_2^\perp$ and such that $\ldots$ there exists $P_{n-1} = P_{n-1}(p_0,\ldots,p_{n-2})$ with $P_{n-1} = P_{n-1}^\perp$, and such that any $p_{n-1} \in P_{n-1}$ corresponds to a representation $\phi_n(p_0,\ldots,p_{n-1}) : \pi_1(M) \to \Gamma_n$ that extends to some bounding 4-manifold and thus induces a class $B_n = B_n(p_0,\ldots,p_{n-1}) \in L^0(\mathcal{K}_n)/i(L^0(\mathbb{Z}\Gamma_n))$.*

(n.5): *If K is $(n.5)$-solvable then there is an inductive sequence*

$$P_0, P_1(p_0), \ldots, P_{n-1}(p_0,\ldots,p_{n-2})$$

as above such that $B_n = 0$ for all $p_{n-1} \in P_{n-1}$.

Note that the above obstructions depend only on the 3-manifold M. In a slightly imprecise way one can reformulate the integral steps in the theorem as follows. (The imprecision only comes from the fact that we translate the conditions $P_i^\perp = P_i$ into talking about "one-half" of the representations in question.) We try to count those representations of $\pi_1(M)$ into Γ_n that extend to $\pi_1(W)$ for some 4-manifold W.

$$\begin{tikzcd}
\pi_1(M) \arrow[d] \arrow[dr] \arrow[drrr] & & & \\
\Gamma_0 & \Gamma_1 \arrow[l] & \dots \arrow[l] & \Gamma_n \arrow[l]
\end{tikzcd}$$

If the knot K is (0)-solvable, i.e. the Arf invariant vanishes, then the abelianization $\pi_1(M) \to \Gamma_0$ extends to a 4-dimensional spin manifold W. Then B_0 is defined. For (0.5)-solvable (or algebraically slice) knots this invariant vanishes, giving $P_0 \subset \mathcal{A}_0 \cong \mathrm{Rep}_{\Gamma_0}(\pi_1(M), \Gamma_1)$. The corresponding representations to Γ_1 *may not* extend over W. But if the knot K is (1)-solvable via a 4-manifold W, then one-half of the representations to Γ_1 *do extend* to $\pi_1(W)$.

For each such extension p_0 we form the next Alexander module $\mathcal{A}_1(p_0)$, which parametrizes representations into Γ_2, fixed over Γ_1, and consider $B_1 \in L^0(\mathcal{K}_1)$ (which depends on p_0). If K is (1.5)-solvable, this invariant vanishes and gives $P_1 \subset \mathcal{A}_1$. Again the corresponding representations to Γ_2 might not extend to this 4-manifold W. But if K is (2)-solvable, then one *quarter* of the representations to Γ_2 extend to a (2)-solution W. Continuing in this way, we get the following meta-statement:

> *If K is (n)-solvable via W then $\frac{1}{2^n}$ of all representations into Γ_n extend from $\pi_1(M)$ to $\pi_1(W)$.*

To be more precise, the following rather striking statement follows from Lemma 2.12 and Proposition 4.3: For any slice knot for which the degree of the Alexander polynomial is greater than 2 let W be the complement of a slice disk for K. Then, for any n, at least one Γ_n-representation extends from $\pi_1(M)$ to $\pi_1(W)$. Moreover, this representation is nontrivial in the sense that it does *not* factor through Γ_{n-1}.

1.3. *L^2-signatures.* There remains the issue of detecting nontrivial classes in the L-theory of the quotient fields $\mathcal{K}$ of $\mathbb{Z}\Gamma$. Our numerical invariants arise from L^2-homology and von Neumann algebras (see Section 6). We construct an L^2-signature

$$\sigma^{(2)}_\Gamma : L^0(\mathcal{K}) \to \mathbb{R}$$

by factoring through $L^0(\mathcal{U}\Gamma)$, where $\mathcal{U}\Gamma$ is the algebra of (unbounded) operators affiliated to the von Neumann algebra $\mathcal{N}\Gamma$ of the group Γ. We show in Section 5 that this invariant can be easily calculated in a large number of examples. The *reduced* L^2-signature, i.e. the difference of $\sigma^{(2)}_\Gamma$ and the ordinary signature, turns out to be exactly what we need to detect our obstructions B_n from Theorem 4.6. The fact that it does not depend on the choice of an (n)-solution can be proved in three essentially different ways. Firstly, one can show [Ma], [R] that the reduced L^2-signature of a $4k$-manifold with boundary M equals the reduced von Neumann η-invariant of the signature operator (associated to the regular Γ-cover of the $(4k-1)$-manifold M). This so-called von Neumann ρ-invariant was introduced by J. Cheeger and M. Gromov [ChG] who showed in particular that it does not depend on a Riemannian metric on M since it is a difference of η-invariants. It follows that the reduced L^2-signature does not depend on a bounding 4-manifold (which might not even exist) and can thus be viewed as a function of $(M, \phi : \pi_1(M) \longrightarrow \Gamma)$.

In the presence of a bounding 4-manifold, the well-definedness of the invariant can be deduced from Atiyah's L^2-index theorem [A]. This is even true in the topological category (see Section 5). There we also explain the third point of view, namely that for groups Γ for which the analytic assembly map is onto, the reduced L^2-signature actually vanishes on the image of $L^0(\mathbb{Z}\Gamma)$ and thus clearly is well-defined on our obstructions B_n from Theorem 4.6. By a recent result of N. Higson and G. Kasparov [HK] this applies in particular to all torsion-free amenable groups (including our rationally universal solvable groups). This last point of view is the strongest in the sense that it shows that in order to define our obstructions one can equally well work with (n)-solutions W that are finite Poincaré 4-complexes (rather than topological 4-manifolds).

It seems that the invariants of Casson-Gordon should also be interpretable in terms of ρ-invariants (or signature defects) associated to *finite-dimensional* unitary representations of finite-index subgroups of $\pi_1(M)$ [CG1], [KL, p. 661], [Let]. J. Levine, M. Farber and W. Neumann have also investigated finite dimensional ρ-invariants as applied to knot concordance [L3], [N], [FL]. More recently C. Letsche used such ρ-invariants together with a universal metabelian group to construct concordance invariants [Let].

Since the invariants we employ are *von Neumann* ρ-invariants, they are associated to the regular representation of our rationally universal solvable groups on an *infinite* dimensional Hilbert space. These groups have to allow homomorphisms from arbitrary knot (and slice) complement fundamental groups, hence they naturally have to be huge and thus might not allow any interesting finite dimensional representations at all.

The following is the result of applying Theorem 4.6 (just at the level of obstructions to (1.5)-solvability) and the L^2-signature to the case of genus one knots in homology spheres which should be compared to [G2, Th. 4]. The proof, which will appear in another paper, is not difficult. It uses the fact that in the simplest case of an L^2-signature for knots, namely where one uses the abelianization homomorphism $\pi_1(M) \to \mathbb{Z}$, the real number $\sigma^{(2)}_{\mathbb{Z}}(M)$ equals the integral over the circle of the Levine signature function.

THEOREM 1.4 ([COT]). *Suppose K is a (1.5)-solvable knot with a genus one Seifert surface F. Suppose that the classical Alexander polynomial of K is non-*

trivial. Then there exists a homologically essential simple closed curve J on F, with self-linking zero, such that the integral over the circle of the Levine signature function of J (viewed as a knot) vanishes.

1.4. *Paper outline and acknowledgments.* The paper is organized as follows: Section 2 provides the necessary algebra to define the higher-order Alexander modules and Blanchfield linking forms. In Section 3 we construct our rationally universal solvable groups and investigate the relationship between representations into them and higher-order Blanchfield forms. We define our knot slicing obstruction theory in Section 4. Section 5 contains the proof that the L^2-signature may be used to detect the L-theory classes of our obstructions. In Section 6, we construct knots with vanishing Casson-Gordon invariants that are not topologically slice, proving our main Theorem 6.4. Section 7 reviews intersection theory and defines Whitney towers and gropes. Section 8 defines (h)-solvability, and proves our theorems relating this filtration to gropes and Whitney towers. In Section 9 we prove Theorem 9.11, showing that Casson-Gordon invariants obstruct a second stage of Whitney disks.

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2. Higher order Alexander modules and Blanchfield linking forms

In this section we show that the classical Alexander module and Blanchfield linking form associated to the infinite cyclic cover of the knot complement can be extended to torsion modules and linking forms associated to *any* poly-torsion-free abelian covering space. We refer to these as *higher-order Alexander modules* and *higher-order linking forms.* A forthcoming paper will discuss these higher-order modules from the more traditional viewpoint of Seifert surfaces [C].

Consider a tower of regular covering spaces

$$M_n \to M_{n-1} \to \ldots \to M_1 \to M_0 = M$$

such that each $M_{i+1} \to M_i$ has a torsion-free abelian group of deck translations and each $M_i \to M$ is a regular cover. Then the group of deck translations Γ of $M_n \to M$ is a *poly-torsion-free abelian group* (see below) and it is easy to see that such towers correspond precisely to certain normal series for such a group. In this section we use such towers to generalize the Alexander module. We will show that if $\beta_1(M) = 1$ then $H_1(M_n;\mathbb{Z})$ is a torsion $\mathbb{Z}\Gamma$-module.

Definition 2.1. A group Γ is *poly-torsion-free abelian* (PTFA) if it admits a normal series $\langle 1\rangle = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = \Gamma$ such that the factors G_{i+1}/G_i are torsion-free abelian. (In the literature only a subnormal series is required.)

Example 2.2. If G is the fundamental group of a (classical) knot exterior then $G/G^{(n)}$ is PTFA since the quotients of successive terms in the derived series $G^{(i)}/G^{(i+1)}$ are torsion-free abelian [Str]. The corresponding covering space is obtained by taking iterated universal abelian covers.

Remark 2.3. If $A \triangleleft G$ is torsion-free abelian and G/A is PTFA then G is PTFA. Any subgroup of a PTFA group is a PTFA group (Lemma 2.4, p. 421 of [P]). Clearly any PTFA group is torsion-free and solvable (although the converse is false!). The class of PTFA groups is quite large — it contains all torsion-free nilpotent groups [Str, Cor. 1.8].

For us there are two especially important properties of PTFA groups, which we state as propositions. These should be viewed as natural generalizations of well-known properties of the free abelian group. The first is an algebraic generalization of the fact that any infinite cyclic cover of a 2-complex with vanishing H_2 also has vanishing H_2. It holds, more generally, for any locally indicable group Γ.

PROPOSITION 2.4 ([Str, p. 305]). *Suppose* Γ *is a* PTFA *group and* R *is a commutative ring. Any map between projective right* $R\Gamma$*-modules whose image under the functor* $-\otimes_{R\Gamma} R$ *is injective, is itself injective.*

The second important property is that $\mathbb{Z}\Gamma$ has a (skew) quotient field. Recall that if A is a *commutative* ring and S is a subset closed under multiplication, one can construct the *ring of fractions* AS^{-1} of elements as^{-1} which add and multiply like normal fractions. If $S = A - \{0\}$ and A has no zero divisors, then AS^{-1} is called the *quotient field* of A. However, if A is *noncommutative* then AS^{-1} does not always exist (and AS^{-1} is not *a priori* isomorphic to $S^{-1}A$). It is known that if S is a *right divisor set* then AS^{-1} exists ([P, p. 146] or [Ste, p. 52]). If A has no zero divisors and $S = A - \{0\}$ is a right divisor set then A is called an *Ore domain*. In this case AS^{-1} is a skew field, called the *classical right ring of quotients* of A. We will often refer to this merely as the *quotient field* of A. A good reference for noncommutative rings of fractions is Chapter 2 of [Ste]. In this paper we will always use *right* rings of fractions. The following holds more generally for any torsion-free amenable group.

PROPOSITION 2.5. *If* Γ *is* PTFA *then* $\mathbb{Q}\Gamma$ *is a right (and left) Ore domain; i.e.* $\mathbb{Q}\Gamma$ *embeds in its classical right ring of quotients* $\mathcal{K}$, *which is a skew field.*

Proof. For the fact (due to A.A. Bovdi) that $\mathbb{Z}\Gamma$ has no zero divisors see [P, pp. 591-592] or [Str, p. 315]. As we have remarked, any PTFA group is solvable. It is a result of J. Lewin [Le] that for solvable groups such that $\mathbb{Q}\Gamma$ has no zero divisors, $\mathbb{Q}\Gamma$ is an Ore domain (see Lemma 3.6 iii, p. 611 of [P]). $\square$

If $\mathcal{R}$ is an integral domain then a right $\mathcal{R}$-module $\mathcal{A}$ is said to be a *torsion module* if, for each $a \in \mathcal{A}$, there exists some nonzero $r \in \mathcal{R}$ such that $ar = 0$. If $\mathcal{R}$ is an Ore domain then $\mathcal{A}$ is a torsion module if and only if $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{K} = 0$ where $\mathcal{K}$ is the quotient field of $\mathcal{R}$. [Ste, II Cor. 3.3]. In general, the set of torsion elements of A is a submodule.

Remark 2.6. We shall need the following elementary facts about the right skew field of quotients $\mathcal{K}$. It is naturally a right $\mathcal{K}$-module and is a $\mathbb{Z}\Gamma$-bimodule.

Fact 1: $\mathcal{K}$ is flat as a left $\mathbb{Z}\Gamma$-module; i.e. $\cdot \otimes_{\mathbb{Z}\Gamma} \mathcal{K}$ is exact [Ste, Prop. II.3.5].

Fact 2: Every module over $\mathcal{K}$ is a free module [Ste, Prop. I.2.3] and such modules have a well defined rank $\mathrm{rk}_{\mathcal{K}}$ which is additive on short exact sequences [Co1, p. 48].

Homology of PTFA *covering spaces.* Suppose X has the homotopy type of a connected CW-complex, Γ is a group and $\phi : \pi_1(X) \longrightarrow \Gamma$ is a homomorphism. Let X_Γ denote the *regular* Γ-*cover of* X *associated to* ϕ (by pulling back the universal cover of $B\Gamma$). Note that if $\pi = \mathrm{image}(\phi)$ then X_Γ is a disjoint union of Γ/π copies of the connected cover X_π (where $\pi_1(X_\pi) \cong \mathrm{Ker}(\phi)$). Fixing a certain convention (which will become clear in Section 6), X_Γ becomes a right Γ-set. For simplicity, the following are stated for the ring $\mathbb{Z}$, but also hold for $\mathbb{Q}$ and $\mathbb{C}$. Let $\mathcal{M}$ be a $\mathbb{Z}\Gamma$-bimodule (for us usually $\mathbb{Z}\Gamma$, $\mathcal{K}$, or a ring $\mathcal{R}$ such that $\mathbb{Z}\Gamma \subset \mathcal{R} \subset \mathcal{K}$, or $\mathcal{K}/\mathcal{R}$). The following are often called the equivariant homology and cohomology of X.

Definition 2.7. Given X, ϕ, $\mathcal{M}$ as above, let

$$H_*(X; \mathcal{M}) \equiv H_*(C_*(X_\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} \mathcal{M})$$

as a right $\mathbb{Z}\Gamma$ module, and $H^*(X; \mathcal{M}) \equiv H_*\left(\mathrm{Hom}_{\mathbb{Z}\Gamma}(C_*(X_\Gamma; \mathbb{Z}), \mathcal{M})\right)$ as a left $\mathbb{Z}\Gamma$-module.

But these are well-known to be isomorphic (respectively) to the homology (and cohomology) of X with coefficient system induced by ϕ (see Theorems VI 3.4 and 3.4* of [W]).

Remark 2.8. 1. Note that $H_*(X;\mathbb{Z}\Gamma)$ as in Definition 2.7 is merely $H_*(X_\Gamma;\mathbb{Z})$ as a right $\mathbb{Z}\Gamma$-module. Thus if $\mathcal{M}$ is flat as a left $\mathbb{Z}\Gamma$-module then $H_*(X;\mathcal{M}) \cong H_*(X_\Gamma;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} \mathcal{M}$. Hence the homology groups we discuss have an interpretation as homology of Γ-covering spaces. However the cohomology $H^*(X;\mathbb{Z}\Gamma)$ does not have such a direct interpretation, although it can be interpreted as cohomology of X_Γ with compact supports (see, for instance, [Hi, p. 5–6].)

2. Recall that if X is a compact, oriented n-manifold then by Poincaré duality $H_p(X;\mathcal{M})$ is isomorphic to $H^{n-p}(X,\partial X;\mathcal{M})$ which is made into a right $\mathbb{Z}\Gamma$-module using the involution on this ring [Wa].

3. We also have a universal coefficient spectral sequence (UCSS) as in [L2, Th. 2.3]. If R and S are rings with unit, C a free right chain complex over R and $\mathcal{M}$ an $(R-S)$ bimodule, there is a convergent spectral sequence

$$E_2^{p,q} \cong \operatorname{Ext}^q_R(H_p(C),\mathcal{M}) \Longrightarrow H^*(C;\mathcal{M})$$

of left S-modules (with differential d^r of degree $(1-r,r)$). Note in particular that the spectral sequence collapses when $R=S=\mathcal{K}$ is the (skew) field of quotients since $\operatorname{Ext}^n_{\mathbb{Z}\Gamma}(M,\mathcal{K}) \cong \operatorname{Ext}^n_{\mathcal{K}}(M\otimes_{\mathbb{Z}\Gamma}\mathcal{K},\mathcal{K})$ by change of rings [HS, Prop. 12.2], and the latter is zero if $n\ge 1$ since all $\mathcal{K}$-modules are free. Hence

$$H^n(X;\mathcal{K}) \cong \operatorname{Hom}_{\mathcal{K}}(H_n(X;\mathcal{K}),\mathcal{K}).$$

More generally it collapses when $R=S$ is a (noncommutative) principal ideal domain.

Suppose that Γ is a PTFA group and $\mathcal{K}$ is its (skew) field of quotients. We now investigate H_0, H_1 and H_2 of spaces with coefficients in $\mathbb{Z}\Gamma$ or $\mathcal{K}$. First we show that $H_0(X;\mathbb{Z}\Gamma)$ is a torsion module.

PROPOSITION 2.9. *Given X, ϕ as in Definition 2.7, suppose a ring homomorphism $\psi:\mathbb{Z}\Gamma\longrightarrow\mathcal{R}$ defines $\mathcal{R}$ as a $\mathbb{Z}\Gamma$-bimodule. Suppose some element of the augmentation ideal of $\mathbb{Z}[\pi_1(X)]$ is invertible (under $\psi\circ\phi$) in $\mathcal{R}$. Then $H_0(X;\mathcal{R})=0$. In particular, if $\phi:\pi_1(X)\longrightarrow\Gamma$ is a nontrivial coefficient system then $H_0(X;\mathcal{K})=0$.*

Proof. By [W, p. 275] and [Br, p.34], $H_0(X;\mathcal{R})$ is isomorphic to the cofixed set $\mathcal{R}/\mathcal{R}I$ where I is the augmentation ideal of $\mathbb{Z}\pi_1(X)$ acting via $\psi\circ\phi$. □

The following proposition is enlightening, although in low dimensions its use can be avoided by short ad hoc arguments. Here $\mathbb{Q}$ is a $\mathbb{Z}\Gamma$ module via the composition $\mathbb{Z}\Gamma \xrightarrow{\epsilon} \mathbb{Z}\to\mathbb{Q}$ where ϵ is the augmentation homomorphism.

PROPOSITION 2.10. a) *If C_* is a nonnegative $\mathbb{Q}\Gamma$ chain complex which is finitely generated and free in dimensions $0 \le i \le n$ such that $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}) = 0$ for $0 \le i \le n$, then $H_i(C_* \otimes_{\mathbb{Q}\Gamma} \mathcal{K}) = 0$ for $0 \le i \le n$.*

b) *If $f : Y \to X$ is a continuous map, between* CW *complexes with finite n-skeleton which is n-connected on rational homology, and $\phi : \pi_1(X) \longrightarrow \Gamma$ is a coefficient system, then f is n-connected on homology with $\mathcal{K}$-coefficients.*

Proof. Let $\epsilon : \mathbb{Q}\Gamma \to \mathbb{Q}$ be the augmentation and $\epsilon(C_*)$ denote $C_* \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}$. Since $\epsilon(C_*)$ is acyclic up to dimension n, there is a "partial" chain homotopy

$$\{h_i : \epsilon(C_*)_i \to \epsilon(C_*)_{i+1} \mid 0 \le i \le n\}$$

between the identity and the zero chain homomorphisms. By this we mean that $\partial h_i + h_{i-1}\partial = \mathrm{id}$ for $0 \le i \le n$.

Since $C_i \xrightarrow{\epsilon} \epsilon(C_i)$ is surjective, for any basis element σ of C_i we can choose an element, denoted $\widetilde{h}_i(\sigma)$, such that $\epsilon \circ \widetilde{h}_i(\sigma) = h_i(\epsilon(\sigma))$. Since C_* is free, in this manner h can be lifted to a partial chain homotopy $\{\widetilde{h}_i \mid 0 \le i \le n\}$ on C_* between some "partial" chain map $\{f_i \mid 0 \le i \le n\}$ and the zero map. Moreover $\epsilon(f_i)$ is the identity map on $\epsilon(C_*)_i$, and in particular, is injective. Thus, by Proposition 2.4, f_i is injective for each i. Consequently, $\widetilde{h}_i \otimes \mathrm{id}$ is a partial chain homotopy on $C_* \otimes_{\mathbb{Q}\Gamma} \mathcal{K}$ between the zero map and the partial chain map $\{f_i \otimes \mathrm{id}\}$, such that $f_i \otimes \mathrm{id}$ is injective (since $\mathcal{K}$ is flat over $\mathbb{Q}\Gamma$). Any monomorphism between finitely generated, free $\mathcal{K}$-modules of the same rank is necessarily an isomorphism. Therefore a partial chain map exists which is an inverse to $f \otimes \mathrm{id}$. It follows that $C_* \otimes_{\mathbb{Q}\Gamma} \mathcal{K}$ is acyclic up to and including dimension n.

The second statement follows from applying this to the relative cellular chain complex associated to the mapping cylinder of f. □

PROPOSITION 2.11. *Suppose X is a* CW-*complex such that $\pi_1(X)$ is finitely generated, and $\phi : \pi_1(X) \longrightarrow \Gamma$ is a nontrivial coefficient system. Then*

$$\mathrm{rk}_{\mathcal{K}}\, H_1(X; \mathcal{K}) \le \beta_1(X) - 1.$$

In particular, if $\beta_1(X) = 1$ then $H_1(X; \mathcal{K}) = 0$; that is, $H_1(X; \mathbb{Z}\Gamma)$ is a $\mathbb{Z}\Gamma$ torsion module.

Proof. Let Y be a wedge of $\beta_1(X)$ circles. Choose $f : Y \to X$ which is 1-connected on rational homology. Applying Proposition 2.10, one sees that $f_* : H_1(Y; \mathcal{K}) \longrightarrow H_1(X; \mathcal{K})$ is surjective. We claim that $\phi \circ f_*$ is nontrivial on $\pi_1(Y)$. Suppose not. Let G denote the image of ϕ. Note that if $\{x_i\}$ generates $\pi_1(Y)$ then $\{\phi \circ f_*(x_i)\}$ generates $G/G^{(1)} \otimes \mathbb{Q}$, which, under our supposition, would imply that the *nontrivial* PTFA group G had a finite abelianization. But one sees from Definition 2.1 that the abelianization of a PTFA group has a quotient (G_n/G_{n-1} in the language of 2.1) that is a nontrivial torsion-free abelian group and therefore must contain an element of infinite order. This contradiction implies $\phi \circ f_*$ is nontrivial. Finally Lemma 2.12 below shows that

$$\operatorname{rk}_{\mathcal{K}} H_1(Y;\mathcal{K}) = \beta_1(Y) - 1.$$

The claimed inequality follows. If $H_1(X;\mathcal{K}) = 0$ then $H_1(X;\mathbb{Z}\Gamma)$ is a $\mathbb{Z}\Gamma$ torsion module by Remark 2.6.1 and [Ste, II Cor. 3.3]. □

LEMMA 2.12. *Suppose Y is a finite connected 2-complex with $H_2(Y;\mathbb{Z}) = 0$ and $\phi : \pi_1(Y) \longrightarrow \Gamma$ is nontrivial. Then $H_2(Y;\mathbb{Z}\Gamma) = H_2(Y;\mathcal{K}) = 0$ and $\operatorname{rk}_{\mathcal{K}} H_1(Y;\mathcal{K}) = \beta_1(Y) - 1$.*

Proof. Let

$$\mathcal{C} = \left(0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0\right)$$

be the free $\mathbb{Z}\Gamma$ chain complex for the cellular decomposition of Y_Γ (the Γ cover of Y) obtained by lifting cells of Y. Since $H_2(Y;\mathbb{Z}) = 0$, $\partial_2 \otimes_{\mathbb{Z}\Gamma} \mathrm{id}$ is injective, which implies, by Proposition 2.4, that ∂_2 itself is injective. Thus $H_2(Y;\mathbb{Z}\Gamma) = 0$ by Remark 2.8.1 and $H_2(Y;\mathcal{K}) = 0$ by Remark 2.6.1. Since ϕ is nontrivial, Proposition 2.9 implies that $H_0(Y;\mathcal{K}) = 0$. Since the C_i are finitely generated free modules, the Euler characteristic of $\mathcal{C} \otimes \mathcal{K}$ equals the Euler characteristic of $\mathcal{C} \otimes \mathbb{Q}$ (by Remark 2.6.2) and the result follows. □

It is interesting to note that 2.11 and 2.12 are false without the finiteness assumptions (see Section 3 of [C].)

Thus we have shown that the definition of the classical Alexander module, i.e. the torsion module associated to the first homology of the infinite cyclic cover of the knot complement, can be extended to *higher-order Alexander modules* which are $\mathbb{Z}\Gamma$ torsion modules $\mathcal{A} = H_1(M;\mathbb{Z}\Gamma)$ associated to *arbitrary* PTFA covering spaces. Indeed, by Proposition 2.11, this is true for any 3-manifold with $\beta_1(M) = 1$, such as zero surgery on the knot or a prime-power cyclic cover of $S^3 - K$. In this paper we will work with the zero surgery.

Furthermore, we will now show that the Blanchfield linking form associated to the infinite cyclic cover generalizes to linking forms on these higher-order Alexander modules. Under some mild restrictions, we can get a nonsingular linking form in the sense of A. Ranicki. Recall from [Ra2, p. 181–223] that $(\mathcal{A}, \lambda)$ is a *symmetric linking form* if $\mathcal{A}$ is a torsion $\mathcal{R}$-module of (projective) homological dimension 1 (i.e. $\mathcal{A}$ admits a finitely-generated projective resolution of length 1) and

$$\lambda : \mathcal{A} \longrightarrow \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}, \mathcal{K}/\mathcal{R}) \equiv \mathcal{A}^{\#}$$

is an $\mathcal{R}$-module map such that $\lambda(x)(y) = \overline{\lambda(y)(x)}$ (here $\mathcal{K}$ is the field of fractions of $\mathcal{R}$ and $\mathcal{A}^{\#}$ is made into a right $\mathcal{R}$-module using the involution of $\mathcal{R}$.). The linking form is *nonsingular* if λ is an isomorphism. If $\mathcal{R}$ is an integral domain then $\mathcal{R}$ is a (*right*) *principal ideal domain* (PID) if every right ideal is principal.

THEOREM 2.13. *Suppose M is a closed, oriented, connected 3-manifold with $\beta_1(M) = 1$ and $\phi : \pi_1(M) \longrightarrow \Gamma$ a nontrivial* PTFA *coefficient system. Suppose $\mathcal{R}$ is a ring such that $\mathbb{Z}\Gamma \subseteq \mathcal{R} \subseteq \mathcal{K}$. Then there is a symmetric linking form*

$$B\ell : H_1(M;\mathcal{R}) \longrightarrow H_1(M;\mathcal{R})^{\#}$$

defined on the higher-order Alexander module $\mathcal{A} := H_1(M;\mathcal{R})$. *If either $\mathcal{R}$ is a* PID, *or some element of the augmentation ideal of $\mathbb{Z}\pi_1(M)$ is sent (under ϕ) to an invertible element of $\mathcal{R}$, then $B\ell$ is nonsingular.*

Proof. Note that $\mathcal{A}$ is a torsion $\mathcal{R}$-module by Proposition 2.11, since $\mathcal{K}$ is also the quotient field of the Ore domain $\mathcal{R}$. Define $B\ell$ as the composition of the Poincaré duality isomorphism to $H^2(M;\mathcal{R})$, the inverse of the Bockstein to $H^1(M;\mathcal{K}/\mathcal{R})$, and the usual Kronecker evaluation map to $\mathcal{A}^{\#}$. The Bockstein

$$B : H^1(M;\mathcal{K}/\mathcal{R}) \longrightarrow H^2(M;\mathcal{R})$$

associated to the short exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{R} \longrightarrow 0$$

is an isomorphism since $H^2(M;\mathcal{K}) \cong H_1(M;\mathcal{K}) = 0$ by Proposition 2.11, and $H^1(M;\mathcal{K}) = 0$ by Proposition 2.11 and Remark 2.8.3. Under the second hypothesis on $\mathcal{R}$, the Kronecker evaluation map

$$H^1(M;\mathcal{K}/\mathcal{R}) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(H_1(M;\mathcal{R}), \mathcal{K}/\mathcal{R})$$

is an isomorphism by the UCSS since $H_0(M;\mathcal{R}) = 0$ (see Remark 2.8.3 and Proposition 2.9). If $\mathcal{R}$ is a PID then $\mathcal{K}/\mathcal{R}$ is an injective $\mathcal{R}$-module since it is clearly divisible [Ste, I Prop. 6.10]. Thus

$$\operatorname{Ext}^i_{\mathcal{R}}(H_0(M;\mathcal{R}), \mathcal{K}/\mathcal{R}) = 0$$

for $i > 0$ and therefore the Kronecker map is an isomorphism.

We need to show that $\mathcal{A}$ has homological dimension one and is finitely generated. This is immediate if $\mathcal{R}$ is a PID [Ste, p. 22]. Since, in this paper we shall only need this special case we omit the proof of the general case.

We also need to show that $B\ell$ is "conjugate symmetric". The diagram below commutes up to a sign (see, for example, [M, p. 410]), where B' is the homology Bockstein

$$\begin{tikzcd}
H_2(M;\mathcal{K}/\mathcal{R}) \arrow[r, "B'"] \arrow[d, "\cong"', "\text{P.D.}"] & H_1(M;\mathcal{R}) \arrow[d, "\cong"', "\text{P.D.}"] \\
H^1(M;\mathcal{K}/\mathcal{R}) \arrow[r, "B"] \arrow[d, "\kappa"] & H^2(M;\mathcal{R}) \\
\operatorname{Hom}_{\mathcal{R}}(H_1(M;\mathcal{R}), \mathcal{K}/\mathcal{R}) &
\end{tikzcd}$$

and the two vertical homomorphisms are Poincaré duality. Thus our map $B\ell$ agrees with that obtained by going counter-clockwise around the square and thus agrees with the Blanchfield form defined by J. Duval in a noncommutative setting [D, p. 623–624]. The argument given there for symmetry is in sufficient generality to cover the present situation and the reader is referred to it. $\square$

The implications of the following for the higher-order Alexander polynomials of slice knots will be discussed in a forthcoming paper. This is the noncommutative analogue of the result that the classical Alexander polynomial of a slice knot factors as a product $f(t)f(t^{-1})$.

LEMMA 2.14. *If $\mathcal{A}$ is a generalized Alexander module (as in Theorem 2.13) which admits a submodule P such that $P = P^{\perp}$, then the map $h : P \longrightarrow (\mathcal{A}/P)^{\#}$, given by $p \mapsto B\ell(p, \cdot)$, is an isomorphism.*

Proof. Since the Blanchfield form is nonsingular by Theorem 2.13, $h(p)$ is actually a monomorphism if $p \neq 0$ and so h is certainly injective. Since $B\ell : \mathcal{A} \to \mathcal{A}^{\#}$ was shown to be an isomorphism, it is easy to see that h is onto when $P = P^{\perp}$. $\square$

3. Higher-order linking forms and solvable representations of the knot group

We now define and restrict our attention to certain families $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ of PTFA groups that are constructed as semi-direct products, inductively, beginning with $\Gamma_0 \equiv \mathbb{Z}$, and defining $\Gamma_n = A_{n-1} \rtimes \Gamma_{n-1}$ for certain "universal" torsion $\mathbb{Z}\Gamma_{n-1}$ modules A_{n-1}. We then show that if coefficient systems $\phi_i : \pi_1(M) \longrightarrow \Gamma_i$, $i < n$, are defined, giving rise to the higher-order Alexander modules $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$, then any nonzero choice $x_{n-1} \in \mathcal{A}_{n-1}$ corresponds to a nontrivial extension of ϕ_{n-1} to $\phi_n : \pi_1(M) \longrightarrow \Gamma_n$. This coefficient system is then used to define the n^{th} Alexander module $\mathcal{A}_n(x_{n-1})$. Thus, if the ordinary Alexander module $\mathcal{A}_0$ of a knot K is nontrivial, then there exist nontrivial Γ_1-coefficient systems. This allows for the definition of $\mathcal{A}_1$, and if this module is nontrivial there exist nontrivial Γ_2-coefficient systems. In this way, higher Alexander modules and actual coefficient systems are constructed inductively from choices of elements of the lesser modules. Naively stated: if H_1 of a covering space $\widetilde{M}$ of M is not zero then $\widetilde{M}$ itself possesses a nontrivial abelian cover.

We close the section with a crucial result concerning when such coefficient systems extend to bounding 4-manifolds.

Families of universal PTFA *groups.* We now inductively define families $\{\Gamma_n \mid n \geq 0\}$ of PTFA groups. These groups Γ_n are "universal" in the sense that the fundamental group of any knot complement with nontrivial classical

Alexander polynomial admits nontrivial Γ_n-representations, a nontrivial fraction of which extend to the fundamental group of the complement of a slice disk for the knot. These are the groups we shall use to construct our knot slicing obstructions. Our approach elaborates work of Letsche who first used an analogue of the group Γ_1^U [Let].

Let $\Gamma_0 = \mathbb{Z}$, generated by μ. Let $\mathcal{K}_0 = \mathbb{Q}(\mu)$ be the quotient field of $\mathbb{Q}\Gamma_0$ with the involution defined by $\mu \to \mu^{-1}$. Choose a ring $\mathcal{R}_0$ such that $\mathbb{Q}\Gamma_0 \subset \mathcal{R}_0 \subset \mathcal{K}_0$. Note that $\mathcal{K}_0/\mathcal{R}_0$ is a $\mathbb{Z}\Gamma_0$-bimodule. Choose the right multiplication and define Γ_1 as the semidirect product $\mathcal{K}_0/\mathcal{R}_0 \rtimes \Gamma_0$. Note that if, for example, $\mathcal{R}_0 = \mathbb{Q}[\mu^{\pm 1}] = \mathbb{Q}\Gamma_0$ then $\mathcal{K}_0/\mathcal{R}_0$ is a torsion $\mathbb{Q}\Gamma_0$ module that is, in fact, a direct limit of all cyclic torsion $\mathbb{Q}\Gamma_0$ modules.

In general, assuming Γ_{n-1} is defined (a PTFA group), let $\mathcal{K}_{n-1}$ be the quotient field of $\mathbb{Q}\Gamma_{n-1}$ (by Proposition 2.5). Choose any ring $\mathcal{R}_{n-1}$ such that $\mathbb{Q}\Gamma_{n-1} \subset \mathcal{R}_{n-1} \subset \mathcal{K}_{n-1}$. Consider $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$ as a right $\mathbb{Z}\Gamma_{n-1}$-module and define Γ_n as the semi-direct product $\Gamma_n \equiv (\mathcal{K}_{n-1}/\mathcal{R}_{n-1}) \rtimes \Gamma_{n-1}$. Since $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$ is a $\mathbb{Q}$-module, it is torsion-free abelian. Thus Γ_n is PTFA by Remark 2.3. We have the epimorphisms $\Gamma_n \overset{\pi}{\twoheadrightarrow} \Gamma_{n-1}$, and canonical splittings $s_n : \Gamma_{n-1} \to \Gamma_n$. The family of groups depends on the choices for $\mathcal{R}_i$. The larger $\mathcal{R}_i$ is, the more elements of $\mathbb{Z}\Gamma_i$ will be invertible in $\mathcal{R}_i$; hence the more (torsion) elements of $H_1(M;\mathbb{Z}\Gamma_i)$ will die in $H_1(M;\mathcal{R}_i)$; hence the more information will be potentially lost. However, in Proposition 2.9 and Theorem 2.13 we saw that it is useful to have $\mathcal{R}_i \neq \mathbb{Q}\Gamma_i$ if $i > 0$ because it often ensures nonsingularity in higher-order Blanchfield forms.

For the final result of this section, concerning when coefficient systems extend to bounding 4-manifolds, we find it necessary to introduce a rather severe (and hopefully unnecessary) simplification: we take our Alexander modules (as in 2.13) to have coefficients in certain principal ideal domains $\mathcal{R}_0 \dots, \mathcal{R}_{n-1}$ where $\mathbb{Z}\Gamma_i \subseteq \mathcal{R}_i \subseteq \mathcal{K}_i$. In some cases this can have the unfortunate effect of completely killing $H_1(M;\mathbb{Z}\Gamma_i)$, which means that no interesting higher modules can be constructed by the procedure below. However for most knots this does not happen. Because of the importance, in this paper, of the family of groups corresponding to these particular $\mathcal{R}_i$, we give it a specific notation:

Definition 3.1. The family of *rationally universal groups* $\{\Gamma_n^U\}$ is defined inductively as above with $\Gamma_0^U = \mathbb{Z}$, $\mathcal{R}_0^U = \mathbb{Q}[\mu^{\pm 1}]$, for $n \geq 0$,

$$S_n = \mathbb{Q}[\Gamma_n^U, \Gamma_n^U] - \{0\}, \qquad \mathcal{R}_n^U = (\mathbb{Q}\Gamma_n^U) S_n^{-1}$$

and

$$\Gamma_{n+1}^U = \mathcal{K}_n/\mathcal{R}_n^U \rtimes \Gamma_n^U.$$

Here $\mathcal{K}_n$ is the right ring of quotients of Γ_n^U.

Observe that this is quite a drastic localization. To form $\mathcal{R}_n$ we have inverted all the nonzero elements of the rational group ring of the commutator subgroup of Γ_n^U.

Note that $[\Gamma_n^U, \Gamma_n^U]$ is PTFA by Remark 2.3 so that $\mathbb{Q}[\Gamma_n^U, \Gamma_n^U]$ is an Ore domain. Therefore S_n (above) is a right divisor set of $\mathbb{Q}\Gamma_n^U$ by Chapter 13, Lemma 3.5 of [P, p. 609]. One easily shows that Γ_n^U is (n)-solvable.

We will now show that the rings $\mathcal{R}_n^U$ of Definition 3.1 are in fact *skew Laurent polynomial rings* which are (noncommutative) principal right (and left) ideal domains by [Co2, 2.1.1 p. 49] generalizing the case $n = 0$ where $\mathcal{R}_0^U = \mathbb{Q}[\mu^{\pm 1}]$. If $\mathbb{K}$ is a skew field, α is an automorphism of $\mathbb{K}$ and μ is an indeterminate, the *skew (Laurent) polynomial ring in* μ *over* $\mathbb{K}$ *associated with* α, denoted $\mathbb{K}[\mu^{\pm 1}]$, is the ring consisting of all expressions

$$f = \mu^{-m}a_{-m} + \ldots + a_0 + \mu a_1 + \mu^2 a_2 + \ldots + \mu^n a_n$$

where $a_i \in \mathbb{K}$, under "coordinate-wise addition" and multiplication defined by the usual multiplication for polynomials and the rule $a\mu = \mu\alpha(a)$ [Co1, p. 54]. The form above for any element f is unique [Co2, p. 49]. Note also that (for a_{-m} and a_n nonzero), the nonnegative function $\deg f = n+m$ is additive under multiplication of polynomials.

Now if Γ is a PTFA group and G is a normal subgroup such that $\Gamma/G \cong \mathbb{Z}$ is generated by $\mu \in \Gamma$, there is an automorphism of G given by $a \mapsto \mu^{-1}a\mu \equiv a^{\mu}$. It is a rather tedious calculation to show that the abelianizations of our Γ_i^U *are in fact* $\mathbb{Z}$. Thus the G which is relevant for these cases is actually the commutator subgroup. Since this fact is not crucial, we do not include it. In any case, there are other situations where one needs the extra generality of the following argument. Continuing, this automorphism extends to a ring automorphism of $\mathbb{Q}G$ and hence, to one of $\mathbb{K}$, the quotient field of $\mathbb{Q}G$. Let $S = \mathbb{Q}G - \{0\}$ and $\mathcal{R} = (\mathbb{Q}\Gamma)S^{-1}$.

PROPOSITION 3.2. *The embedding* $g : \mathbb{Q}G \longrightarrow \mathbb{K}$ *extends to an isomorphism* $\mathcal{R} \longrightarrow \mathbb{K}[\mu^{\pm 1}]$.

Proof. As an additive group, $\mathbb{Q}\Gamma$ is isomorphic to $\bigoplus_{i=-\infty}^{\infty} \mathbb{Q}G$ since the cosets of G partition Γ. But $\mathbb{K}[\mu^{\pm 1}]$, as a group, is isomorphic to a countable direct sum of copies of $\mathbb{K}$. Therefore g extends in an obvious way to an additive group homomorphism $g : \mathbb{Q}\Gamma \longrightarrow \mathbb{K}[\mu^{\pm 1}]$ such that $g(\mu^i a_i) = \mu^i g(a_i)$ for $a_i \in \mathbb{Q}G$. Since the automorphism $a \mapsto a^{\mu}$ defining $\mathbb{K}[\mu^{\pm 1}]$ agrees with conjugation in Γ, this map is a ring homomorphism. Clearly the nonzero elements of $\mathbb{Q}G$ are sent to invertible elements. Moreover, any element of $\mathbb{K}[\mu^{\pm 1}]$ is of the form $(\Sigma \mu^i g(a_i))\, s^{-1}$ where $a_i \in \mathbb{Q}G$ and $s \in S$. This establishes that $(\mathbb{Q}\Gamma)S^{-1} \cong \mathbb{K}[\mu^{\pm 1}]$, [Ste, p. 50]. $\square$

COROLLARY 3.3. *For each $n \geq 0$ the rings $\mathcal{R}_n^U$ of Definition* 3.1 *are left and right principal ideal domains, denoted $\mathbb{K}_n[\mu^{\pm 1}]$, where $\mathbb{K}_n$ is the right ring of quotients of $\mathbb{Z}[\Gamma_n^U, \Gamma_n^U]$.*

Remark 3.4. Suppose Γ_n^U is one of the rationally universal groups defined by Definition 3.1. Then, if ϕ is nontrivial on $[\pi_1(X), \pi_1(X)]$, Proposition 2.9 applies and $H_0(X; \mathcal{R}_n^U) = 0$ if $n > 0$. However, beware: $H_0(X, \mathcal{R}_0^U)$ is certainly *not* zero.

Suppose M is a closed 3-manifold with $\beta_1(M) = 1$. A choice of a generator of $H_1(M; \mathbb{Z})$ modulo torsion induces an epimorphism $\phi_0 : \pi_1(M, m_0) \to \Gamma_0 = \mathbb{Z}$. In case M is an oriented knot complement this choice is usually made using the knot orientation. Let $\mathcal{A}_0 \equiv H_1(M; \mathcal{R}_0)$ be the *rational Alexander module*, and suppose (inductively) that we are given $\phi_{n-1} : \pi_1(M) \longrightarrow \Gamma_{n-1}$. Then we can define the *higher-order Alexander module* $\mathcal{A}_{n-1} \equiv H_1(M; \mathcal{R}_{n-1})$, using the $\mathbb{Z}\Gamma_{n-1}$ local coefficients induced by ϕ_{n-1}. Varying ϕ_{n-1} by an inner automorphism of Γ_{n-1} changes $H_1(M; \mathcal{R}_{n-1})$ by an isomorphism induced by the conjugating element. Let $\mathrm{Rep}_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n)$ denote the set of homomorphisms from $\pi_1(M, m_0)$ to Γ_n which agree with ϕ_{n-1} after composition with the projection $\Gamma_n \longrightarrow \Gamma_{n-1}$.

Recall that $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$ is a right $\mathbb{Z}\Gamma_{n-1}$ module and hence becomes a right $\mathbb{Z}\pi_1(M)$ module via ϕ_{n-1}. By a universal property of semi-direct products [HS, VI Prop. 5.3], there is a one-to-one correspondence between $\mathrm{Rep}_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n)$ and the set of derivations $d : \pi_1(M) \longrightarrow \mathcal{K}_{n-1}/\mathcal{R}_{n-1}$. One checks that varying by a principal derivation corresponds to varying the representation by a $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$-conjugation (i.e. composing with an inner automorphism of Γ_n given by conjugation with an element of the subgroup $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$). Thus if we let $\mathrm{Rep}^*_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n)$ denote the representations modulo $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$-conjugations, it follows that this set is in bijection with $H^1(M; \mathcal{K}_{n-1}/\mathcal{R}_{n-1})$ (by the well-known identification of the latter with derivations modulo principal derivations [HS, p. 195]). Moreover this bijection is natural with respect to continuous maps. This establishes part (a) of Theorem 3.5 below. Moreover, any choice $x_{n-1} \in \mathcal{A}_{n-1}$ will (together with ϕ_{n-1}) induce ϕ_n under the correspondence (from the proof of Theorem 2.13)

$$\mathcal{A}_{n-1} \equiv H_1(M; \mathcal{R}_{n-1}) \cong H^2(M; \mathcal{R}_{n-1}) \cong H^1(M; \mathcal{K}_{n-1}/\mathcal{R}_{n-1}).$$

We will refer to this as *the coefficient system corresponding to x_{n-1} (and ϕ_{n-1})*. This coefficient system is well-defined up to conjugation. It is also sometimes convenient to think of (the image of) this element x_{n-1} as living in $\mathcal{A}_{n-1}^{\#} = \mathrm{Hom}_{\mathcal{R}_{n-1}}(\mathcal{A}_{n-1}, \mathcal{K}_{n-1}/\mathcal{R}_{n-1})$ under the Kronecker map. This image is called the *character induced by x_{n-1}*. Indeed it is important to note at this point that

$$\phi_n : \pi_1(M) \longrightarrow \Gamma_n = \mathcal{K}_{n-1}/\mathcal{R}_{n-1} \rtimes \Gamma_{n-1}$$

induces a map from $\pi_1(M_{n-1})$, the Γ_{n-1} cover defined by ϕ_{n-1}, to $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$, and that the abelianization of this map $H_1(M_{n-1}) \longrightarrow \mathcal{K}_{n-1}/\mathcal{R}_{n-1}$ is precisely the character induced by x_{n-1} as above. This is true by construction. Finally, given ϕ_n we can define the n^{th} Alexander module $\mathcal{A}_n \equiv H_1(M;\mathcal{R}_n)$. Hence

$$\mathcal{A}_n = \mathcal{A}_n(x_0, x_1, \dots, x_{n-1})$$

is a function of the choices $x_i \in \mathcal{A}_i$.

Of course, $\mathcal{A}_0$ is H_1 of the Γ_0 cover of M. Given $x_0 \in \mathcal{A}_0$, a "$\mathcal{K}_0/\mathcal{R}_0$-cover" of the Γ_0 cover is induced and $\mathcal{A}_1$ is H_1 of this composite Γ_1-cover modulo S_1-torsion where S_1 is the set of elements of $\mathbb{Z}\Gamma_1$ which have inverses in $\mathcal{R}_1$. Generally $\mathcal{A}_n$ is H_1 of the Γ_n-cover of M, modulo S_n-torsion. In summary we have the following:

THEOREM 3.5. *Suppose $\{\Gamma_n \mid n \geq 0\}$ are as in the beginning of Section* 3 (*but not necessarily as in Definition* 3.1). *Suppose M is a compact manifold and $\phi_{n-1} : \pi_1(M) \longrightarrow \Gamma_{n-1}$ is given.*

(a) *There is a bijection $f : H^1(M;\mathcal{K}_{n-1}/\mathcal{R}_{n-1}) \longleftrightarrow \operatorname{Rep}^*_{\Gamma_{n-1}}(\pi_1(M),\Gamma_n)$ which is natural with respect to continuous maps*;

(b) *If M is a closed oriented* 3-*manifold with $\beta_1(M) = 1$ then the isomorphism $H_1(M;\mathcal{R}_{n-1}) \cong H^1(M;\mathcal{K}_{n-1}/\mathcal{R}_{n-1})$ with f gives a natural bijection $\tilde{f} : \mathcal{A}_{n-1} \longleftrightarrow \operatorname{Rep}^*_{\Gamma_{n-1}}(\pi_1(M),\Gamma_n)$.*

(c) *If $x \in \mathcal{A}_{n-1}$ then the character induced by x is given by $y \mapsto B\ell_{n-1}(x,y)$.*

Extension of characters and coefficient systems to bounding 4-*manifolds.* Suppose $M = \partial W$ and $\phi : \pi_1(M) \longrightarrow \Gamma_n$ is given. When does ϕ extend over $\pi_1(W)$? In general this is an extremely difficult question because of our relative ignorance about the types of groups which may occur as $\pi_1(W)$. This problem has obstructed the generalization of the invariants of Casson and Gordon and no doubt blocked many other assaults (for example, see [KL, Cor. 5.3], [N], [L3], [Let]). Our success in this regard is the crucial element in defining concordance invariants. If M is the zero surgery on a slice knot (or more generally an (n)-solvable knot) and W is the 4-manifold which exhibits this (i.e. the complement of the slice disk in the first case) then, under some restrictions on the family Γ_i, $i \leq n$, we will show that (loosely speaking) $1/2^n$ of the possible representations from $\pi_1(M)$ to Γ_n extend to $\pi_1(W)$. In particular as long as the generalized Alexander modules $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{n-1}$ are nonzero there exist nontrivial representations ϕ which do extend. This allows for the construction of an invariant in $L^0(\mathcal{K}_n)/i_*\left(L^0(\mathbb{Z}\Gamma_n)\right)$, which is discussed in Section 4.

For the following, let Γ_{n-1} be an *arbitrary* $(n-1)$-solvable PTFA group and suppose $\Gamma_n = \mathcal{K}_{n-1}/\mathcal{R}_{n-1} \rtimes \Gamma_{n-1}$ as in Section 3. We need not assume that Γ_{n-1} is constructed as in Section 3. We proceed inductively by assuming $\phi_{n-1} : \pi_1(M) \longrightarrow \Gamma_n \longrightarrow \Gamma_{n-1}$ already extends to $\pi_1(W)$.

THEOREM 3.6. *Suppose* $M = \partial W$ *with* $\beta_1(M) = 1$ *and* $\phi_n : \pi_1(M) \longrightarrow \Gamma_n$ *is given, where* $\Gamma_n = \mathcal{K}_{n-1}/\mathcal{R}_{n-1} \rtimes \Gamma_{n-1}$ *as in Section* 3 (*but* Γ_{n-1} *is allowed to be an* arbitrary PTFA *group*). *Assume that the nontrivial map* $\phi_{n-1} : \pi_1(M) \longrightarrow \Gamma_{n-1}$ *extends to a map* $\psi_{n-1} : \pi_1(W) \longrightarrow \Gamma_{n-1}$ *and that* ϕ_n *is a representative of a class in* $\mathrm{Rep}^*_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n)$ *corresponding to* $x \in H_1(M; \mathcal{R}_{n-1})$. *Let*

$$P_{n-1} \equiv \mathrm{Ker}\{j_* : H_1(M; \mathcal{R}_{n-1}) \longrightarrow H_1(W; \mathcal{R}_{n-1})\}.$$

Then 1. *If* $\mathcal{R}_{n-1}$ *is a* PID (*or if* j_* *is surjective*), *then* ϕ_n *extends to* $\pi_1(W)$ *if and only if* $x \in P_{n-1}^{\perp}$. (*Recall that* $P_{n-1}^{\perp} = \{x \in H_1(M; \mathcal{R}_{n-1}) | B\ell_{n-1}(x,p) = 0$ $\forall p \in P_{n-1}\}$.)

2. *If* M *is* (n)*-solvable via* W *then* ϕ_n *extends if and only if* $x \in P_{n-1}$.

The reader will note that Theorem 3.6.2 applies to any slice knot. The difficulty with using Theorem 3.6.2 is that in applications, often W is unknown and one cannot insure that P_{n-1} is nontrivial. In Theorem 4.4 we shall show that if the hypotheses of *both* 3.6.1 and .2 are satisfied then $P_{n-1} = P_{n-1}^{\perp}$. This then is a useful condition which says that 1/2 (in a loose sense) of these ϕ_n extend. The astute reader will note that Theorem 4.4 is a logical consequence of Theorems 3.6.1 and 2. For this reason and because, with our current knowledge, Theorem 3.6.2 is useless without 3.6.1, we shall postpone the proof of Theorem 3.6.2 until after Theorem 4.4.

Proof of Theorem 3.6.1. If $\mathcal{R}_{n-1}$ is a PID then $\mathcal{K}_{n-1}/\mathcal{R}_{n-1}$ is an injective $\mathcal{R}_{n-1}$-module (since it is clearly divisible) [Ste, I Prop. 6.10]. Since $j_* : H_1(M; \mathcal{R}_{n-1})/P_{n-1} \to H_1(W; \mathcal{R}_{n-1})$ is a monomorphism, $j^\# : H_1(W)^\# \to (H_1(M)/P_{n-1})^\#$ is surjective. Therefore the "character" $B\ell_{n-1}(x, \cdot)$ extends to $H_1(W; \mathcal{R}_{n-1})$ if and only if it annihilates P_{n-1}, i.e. if $x \in P_{n-1}^{\perp}$.

Since the bijection between $H^1(\pi_1(M); \mathcal{K}_{n-1}/\mathcal{R}_{n-1})$ and $\mathrm{Rep}^*_{\Gamma_{n-1}}(\pi_1(M), \Gamma_n)$ is functorial and since the Kronecker map

$$H^1(\pi_1(W); \mathcal{K}_{n-1}/\mathcal{R}_{n-1}) \longrightarrow \mathrm{Hom}(\pi_1(W); \mathcal{K}_{n-1}/\mathcal{R}_{n-1})$$

is an isomorphism (as in the proof of Theorem 2.13), the extension of the "character" $B\ell_{n-1}(x, \cdot)$ is equivalent to an extension of ϕ_n on the π_1 level.

A similar argument works if j_* is surjective since this implies that $j^\#$ is an isomorphism. □

4. Linking forms and Witt invariants as obstructions to solvability

In this section we introduce knot invariants that we prove are defined for (n)-solvable knots and vanish for $(n.5)$-solvable knots. This allows us to state our main theorem concerning the existence of higher-order obstructions to a knot's being slice. These invariants lie in Witt groups of hermitian forms and are closely related to the Witt classes of our higher-order linking forms via a localization sequence in L-theory. In this section we also ask what can be said about a higher-order linking form $B\ell$ on M as in Theorem 2.13 if M is the boundary of a certain type of 4-manifold over which the coefficient system extends. A consequence of our answer to this question is that for (n)-solvable knots certain large families of the higher-order linking forms $B\ell_0,\ldots,B\ell_{n-1}$ are hyperbolic.

Suppose M is equipped with a nontrivial PTFA coefficient system $\phi : \pi_1(M) \longrightarrow \Gamma$ that extends to $\pi_1(W)$ where M is the boundary of W and $j_* : H_1(M;\mathbb{Q}) \to H_1(W;\mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism. Then, since $H_*(M;\mathcal{K}) = 0$ by Proposition 2.11 and Remark 2.8.3, the chain complex of the induced cover of W with coefficients in $\mathcal{K}$ is a 4-*dimensional symmetric Poincaré complex* over $\mathcal{K}$, called the *symmetric chain complex* of W, and hence represents an element B in $L^0(\mathcal{K})$, the cobordism classes of such complexes [Ra2, pp.1–24]. Since all $\mathcal{K}$-modules are free, this complex is known to be cobordant to one given by the intersection form λ on $H_2(W;\mathcal{K})$ (which is nonsingular by the above remarks and is discussed in detail in Section 7) [Da, Lemma 4.4 (ii)]. Moreover, in this case $L^0(\mathcal{K})$ is known to be isomorphic to the usual Witt group of nonsingular hermitian forms on finitely-generated $\mathcal{K}$ modules. Let W' be another such 4-manifold and B' the corresponding class. Let V be the closed 4-manifold obtained by taking the union of W and $-W'$ along M and consider the the symmetric complex of V with $\mathbb{Z}\Gamma$ coefficients (the *symmetric signature* of V). Let A denote the image of this element of $L^0(\mathbb{Z}\Gamma)$ under the map $i_* : L^0(\mathbb{Z}\Gamma) \to L^0(\mathcal{K})$. Thus A is the symmetric signature of V with $\mathcal{K}$ coefficients which, as above, is equal to that obtained from the intersection form on $H_2(V;\mathcal{K})$. Since, by a Mayer-Vietoris sequence, the latter is the difference of B and B', we see that $B = B(M,\phi)$ is well defined (independent of W) modulo the image of i_*.

In Section 5 we discuss $L^{(2)}$-signature invariants which can detect the nontriviality of $B(M,\phi)$. Specifically, a homomorphism $\sigma_\Gamma^{(2)} : L^0(\mathcal{K}) \to \mathbb{R}$ is defined which is equal to the ordinary signature σ on the image of $L^{(0)}(\mathbb{Z}\Gamma)$. This L^2-signature has additivity properties similar to σ. Then, given (M,ϕ) as above, one can define the *reduced* L^2-*signature* (von Neumann ρ-invariant) $\rho_\Gamma(M,\phi) = \sigma_\Gamma^{(2)}(B) - \sigma(W)$, a real number independent of W.

In this section the groups Γ are general PTFA groups unless specified otherwise. All 3- and 4-manifolds are compact, connected and oriented. Recall from Section 1 that $W^{(n)}$ denotes the regular cover of W corresponding to the n^{th} derived subgroup $\pi_1(W)^{(n)}$.

Definition 4.1. The manifold M is *rationally* (n)*-solvable via* W if it is the boundary of a compact 4-manifold W such that the inclusion induces an isomorphism on $H_1(\ ;\mathbb{Q})$ and such that W admits a *rational* (n)*-Lagrangian* with *rational* (n)*-duals*; that is, there exist classes $\{\ell_1,\dots,\ell_m\}$ and $\{d_1,\dots,d_m\}$ in $H_2(W^{(n)};\mathbb{Q})$ such that $\lambda_n(\ell_i,\ell_j)=0$ and $\lambda_n(\ell_i,d_j)=\delta_{ij}$, and where the class images (under the covering map) together form a basis of $H_2(W;\mathbb{Q})$. M is *rationally* $(n.5)$*-solvable* if *in addition* there exist classes $\{\ell_1',\dots,\ell_m'\}$ of $H_2(W^{(n+1)};\mathbb{Q})$ which map to ℓ_i as above and such that $\lambda_{n+1}(\ell_i',\ell_j')=0$. It follows that $\sigma(W)=0$. Note that if M is (h)-solvable (see Sections 1 and 8) then M is rationally (h)-solvable.

THEOREM 4.2. *Suppose* Γ *is an* (n)*-solvable group. If* M *is rationally* $(n.5)$*-solvable via a* 4*-manifold* W *over which the coefficient system* ϕ *extends, then* $B(M,\phi)=0$ *and* $\rho_\Gamma^{(2)}(M,\phi)=0$.

Proof. Let

$$L=\{\ell_1,\dots,\ell_m\}\subset H_2(W^{(n+1)};\mathbb{Q})=H_2\left(W;\mathbb{Q}\Big[\pi_1(W)/\pi_1(W)^{(n+1)}\Big]\right)$$

be the rational $(n+1)$-Lagrangian. Since Γ is (n)-solvable, $\psi:\pi_1(W)\longrightarrow\Gamma$ factors through the quotient $\pi_1(W)/\pi_1(W)^{(n+1)}$. Using this we can let L' be the submodule generated by the image of L in $H_2(W;\mathcal{K})$. By naturality, the intersection form with $\mathcal{K}$ coefficients, λ, vanishes on L'. Since all $\mathcal{K}$-modules are free, L' is a free summand of $H_2(W;\mathcal{K})$. It suffices therefore to show that $\operatorname{rk}_{\mathcal{K}} L'$ is one-half that of $H_2(W;\mathcal{K})$. The latter has rank equal to $2m$ by the first part of Proposition 4.3 below. We are given that the image of L in $H_2(W;\mathbb{Q})$ is linearly independent. By the flatness of $\mathcal{K}$, it is sufficient (and necessary) to show that $\{\ell_1,\dots,\ell_m\}$ is linearly independent in $H_2(W;\mathbb{Q}\Gamma)$. Now apply the second part of the proposition below with $n=3$, noting that by assumption $H_3(W;\mathbb{Q})\cong H^1(W,M;\mathbb{Q})=0$. Thus $B(M,\phi)=[\lambda]=0$ and hence $\sigma_\Gamma^{(2)}(B(M,\phi))=0$. Since $\sigma(W)=0$, $\rho_\Gamma^{(2)}(M,\phi)=0$ as well. $\square$

PROPOSITION 4.3. *Suppose* W *is a compact, connected, oriented* 4*-manifold with connected boundary* M *such that* $H_1(M;\mathbb{Q})\longrightarrow H_1(W;\mathbb{Q})$ *is an isomorphism. Suppose* $\phi:\pi_1(W)\longrightarrow\Gamma$ *is a nontrivial* PTFA *coefficient system. Then*

$$\operatorname{rk}_{\mathcal{K}} H_2(W;\mathcal{K})\le\beta_2(W)$$

with equality if $\beta_1(W)=1$.

Now suppose W is a connected (possibly infinite) n-complex such that $H_n(W;\mathbb{Q}) = 0$ and there exist $(n-1)$-dimensional manifolds S_i, continuous maps $f_i : S_i \to W$ and lifts $\tilde{f}_i : S_i \to W_\Gamma$ such that $\{[f_i] \mid i \in I\}$ is linearly independent in $H_{n-1}(W;\mathbb{Q})$. Then $\{[\tilde{f}_i] \mid i \in I\}$ is $\mathbb{Q}\Gamma$ linearly independent in $H_{n-1}(W;\mathbb{Q}\Gamma)$.

Proof. Note that any compact topological 4-manifold has the homotopy type of a finite simplicial complex [KS, Th. 4.1]. Choose a finite, 3-dimensional CW structure for W and let $C_*(W)$ denote the cellular chain complex of W with $\mathbb{Q}$ coefficients. Let $b_i = \operatorname{rk}_{\mathcal{K}} H_i(W;\mathcal{K})$. Then $b_0 = 0$ by Proposition 2.9 and by Proposition 2.11,

$$b_1 \leq \beta_1(W) - 1.$$

Since $H_3(W;\mathbb{Q}) \cong H^1(W,\partial W;\mathbb{Q}) = 0$, the boundary homomorphism $\partial : C_3(W) \to C_2(W)$ is injective. Let $C_*(W_\Gamma)$ be the corresponding $\mathbb{Q}\Gamma$ chain complex free on the cells of W. By Proposition 2.4, $\partial : C_3(W_\Gamma) \to C_2(W_\Gamma)$ is injective so that $H_3(W;\mathbb{Q}\Gamma) \cong H_3(C_*(W_\Gamma)) = 0$. Hence $b_3 = 0$ by Remark 2.6.1. Finally, as noted in the proof of Lemma 2.12, $\chi(W;\mathbb{Q}) = \chi(W;\mathcal{K})$ so that we get $b_2 \leq \beta_2(W)$, with equality if $\beta_1(W) = 1$ (by Proposition 2.11). This completes the proof of the first part of the proposition.

Let X be the one point union of the S_i (using some base paths), and define $f : X \to W$ and $\tilde{f} : X \to W_\Gamma$ to restrict to the given maps on the S_i. After taking mapping cylinders, we may assume $C_*(X)$ is an $(n-1)$-dimensional subcomplex of the n-dimensional $C_*(W)$, and similarly for $C_*(W_\Gamma)$ and the subcomplex $C_*(X_\Gamma)$ where X_Γ is the induced cover of X. Then $C_i(W)$ is naturally identified with $C_i(W_\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}$ and

$$p_\# : C_*(W_\Gamma) \to C_*(W)$$

coincides with the obvious homomorphism defined using the augmentation. The hypothesis is that f_* is injective on $H_{n-1}(\ ;\mathbb{Q})$.

Since $\phi \circ f_*$ is trivial on π_1, X_Γ is the trivial cover consisting of Γ copies of X. Thus $H_{n-1}(X_\Gamma;\mathbb{Q})$ is a free $\mathbb{Q}\Gamma$-module on $\{\tilde{f}_i\}$, and hence to establish the result, we must show that $\tilde{f}_*$ is injective on H_{n-1}. Note that, as in the proof of the first part of the proposition, $H_n(W_\Gamma;\mathbb{Q}) = 0$ (finiteness is not needed), and it follows that $\tilde{f}_*$ is injective on H_{n-1} if and only if $H_n(W_\Gamma, X_\Gamma;\mathbb{Q})$ is zero. The latter is equivalent to the injectivity of

$$\partial^{\mathrm{rel}} : C_n(W_\Gamma, X_\Gamma) \to C_{n-1}(W_\Gamma, X_\Gamma).$$

By Proposition 2.4 it suffices to see that

$$\partial^{\mathrm{rel}} \otimes \mathrm{id} : C_n(W_\Gamma, X_\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathbb{Q} \to C_{n-1}(W_\Gamma, X_\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}$$

is injective. The last statement is equivalent to the vanishing of $H_n(C_*(W_\Gamma, X_\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathbb{Q})$. But $C_*(W_\Gamma, X_\Gamma) \otimes_{\mathbb{Q}\Gamma} \mathbb{Q}$ can be identified with $C_*(W, X; \mathbb{Q})$. Since f_* is injective on H_{n-1} by hypothesis and since $H_n(W;\mathbb{Q}) = 0$, it follows that $H_n(W, X; \mathbb{Q})$ vanishes. □

Now we can show that if K is a slice knot (even in a rational homology ball) and a chosen coefficient system extends to the 4-manifold *and* coefficients are taken in a PID $\mathcal{R}$, then the induced higher-order linking form on the higher-order Alexander module is hyperbolic. (In fact, under the more general conditions of Theorem 2.13, we can show that these forms are *stably* hyperbolic, but this more general result is not needed here). The consequences of this for the higher-order Alexander polynomials will be discussed in a later paper (compare [KL, Cor. 5.3]).

THEOREM 4.4. *Suppose M is rationally (n)-solvable via W, $\beta_1(M) = 1$ and $\phi : \pi_1(M) \longrightarrow \Gamma$ is a nontrivial coefficient system that extends to $\pi_1(W)$ and Γ is an $(n-1)$-solvable* PTFA *group. If $\mathcal{R}$ is a* PID *such that $\mathbb{Q}\Gamma \subset \mathcal{R} \subset \mathcal{K}$ then the linking form $B\ell(M, \phi)$ (as defined in Theorem 2.13) is hyperbolic, and in fact the kernel of $j_* : H_1(M; \mathcal{R}) \longrightarrow H_1(W; \mathcal{R})$ is self-annihilating.*

Proof. Let $P = \mathrm{Ker}\{j_* : H_1(M;\mathcal{R}) \longrightarrow H_1(W;\mathcal{R})\}$. Since all finitely generated modules over a principal ideal domain are homological dimension at most 1, it suffices to show $P = P^\perp$ with respect to $B\ell$ [Ra2, p. 253]. We now need the following crucial lemma.

LEMMA 4.5. *Assume the hypotheses of Theorem 4.4 except that here we do not need that $\mathcal{R}$ is a* PID. *Then $TH_2(W, M; \mathcal{R}) \xrightarrow{\partial} H_1(M, \mathcal{R}) \xrightarrow{j_*} H_1(W; \mathcal{R})$ is exact. (Recall that TH_2 denotes the $\mathcal{R}$-torsion submodule.) Moreover $H_2(W, \mathcal{R})$ is the direct sum of its torsion submodule and a free module.*

Proof. Let $\{\ell_i, d_i \mid i = 1, \ldots, m\}$ denote the classes in

$$H_2(W; \mathbb{Q}[\pi_1(W)/\pi_1(W)^{(n)}])$$

generating the rational (n)-Lagrangian and its duals. Since Γ is $(n-1)$-solvable, the coefficient system $\psi : \pi_1(W) \longrightarrow \Gamma$ (extending ϕ) descends to $\overline{\psi} : \pi_1(W)/\pi_1(W)^{(n)} \longrightarrow \Gamma$. Let $\{\ell_i', d_i'\}$ denote the images of $\{\ell_i, d_i\}$ in $H_2(W;\mathcal{R})$. By naturality, these are still dual and the intersection form λ, with coefficients in $\mathcal{R}$, vanishes on the span of $\{\ell_i'\}$. Consider $\mathcal{R}^m \oplus \mathcal{R}^m$, the free module on $\{\ell_i', d_i'\}$, and the composition

$$\mathcal{R}^m \oplus \mathcal{R}^m \xrightarrow{i_*} H_2(W;\mathcal{R}) \xrightarrow{\lambda} H_2(W;\mathcal{R})^* \xrightarrow{i^*} (\mathcal{R}^m \oplus \mathcal{R}^m)^*.$$

This map is represented by a block diagonal matrix of the form

$$\begin{pmatrix} 0 & I \\ I & X \end{pmatrix}$$

for some $m \times m$ matrix X. This matrix has

$$\begin{pmatrix} -X & I \\ I & 0 \end{pmatrix}$$

as its inverse implying that i^* is a (split) epimorphism and i_* is a monomorphism. Since the $\mathcal{K}$-rank of

$$H_2(W;\mathcal{R}) \otimes_{\mathcal{R}} \mathcal{K} \cong H_2(W;\mathcal{K})$$

is $2m$ by Proposition 4.3, the cokernel C of i_* is a torsion module, and thus $\mathrm{Hom}_{\mathcal{R}}(C,\mathcal{R}) = 0$. Hence, applying the functor $\mathrm{Hom}_{\mathcal{R}}(-,\mathcal{R})$ to the map i_*, we see that its Hom-dual i^* is injective. Therefore i^* is an isomorphism. It follows that λ is surjective (and hence $H_2(W,\mathcal{R})$ is a direct sum of a free module of rank $2m$ and its torsion module.) Now consider the commutative diagram below for (co)-homology with $\mathcal{R}$-coefficients.

$$\begin{tikzcd}
\arrow[r] & H_2(W) \arrow[r, "\pi_*"] \arrow[ddr, "\lambda"'] & H_2(W,\partial W) \arrow[r, "\partial"] \arrow[d, "\cong"', "\mathrm{P.D.}"] & H_1(M) \arrow[r] & {} \\
& & H^2(W) \arrow[d, "\kappa"] & & \\
& & H_2(W)^* & &
\end{tikzcd}$$

Note that κ is a split surjection between modules of the same rank over $\mathcal{K}$ and thus the kernel of $\kappa \circ \mathrm{P.D.}$ is torsion. Now, given $p \in P$, choose x such that $\partial x = p$. Let y be an element of the set $\lambda^{-1}(\kappa \circ \mathrm{P.D.}(x))$. Then $\partial(x - \pi_*(y)) = p$ and $x - \pi_*(y)$ is torsion since it lies in the kernel of $\kappa \circ \mathrm{P.D.}$. This concludes the proof of the lemma. $\square$

Continuing the proof of Theorem 4.4, consider the following diagram, commutative up to sign, where coefficients are in $\mathcal{R}$ unless otherwise specified:

$$\begin{tikzcd}
TH_2(W,\partial W) \arrow[r, "\partial"] \arrow[d, "\cong"', "\mathrm{P.D.}"] & H_1(M) \arrow[r, "j_*"] \arrow[d, "\cong"', "\mathrm{P.D.}"] & H_1(W) \\
TH^2(W) \arrow[r, "j^*"] \arrow[d, "\cong"', "B^{-1}"] & H^2(M) \arrow[d, "\cong"', "B^{-1}"] & \\
H^1(W;\mathcal{K}/\mathcal{R}) \arrow[r, "j^*"] \arrow[d, "\kappa"] & H^1(M;\mathcal{K}/\mathcal{R}) \arrow[d, "\kappa"] & \\
H_1(W)^{\#} \arrow[r, "j^{\#}"] & H_1(M)^{\#} &
\end{tikzcd} \tag{4.1}$$

The vertical homomorphisms above are Poincaré Duality, inverse of the Bockstein B and the Kronecker evaluation map κ. These compositions are denoted β_{rel} and $B\ell$ respectively. To see that this "linking form" β_{rel} exists, examine the sequence

$$H^1(W;\mathcal{K}) \longrightarrow H^1(W;\mathcal{K}/\mathcal{R}) \xrightarrow{B} H^2(W;\mathcal{R}) \longrightarrow H^2(W;\mathcal{K})$$

and note that $H^1(W;\mathcal{K}) = 0$, $H^2(W;\mathcal{K})$ is $\mathcal{R}$-torsion free and all homology with coefficients in $\mathcal{K}/\mathcal{R}$ is $\mathcal{R}$-torsion. It follows that B is an isomorphism onto $TH^2(W;\mathcal{R})$. If $x \in P$ then $x = \partial y$ using Lemma 4.5. Thus $B\ell(x) = j^{\#}(\beta_{\text{rel}}(y))$ and hence, for any $p \in P$, $B\ell(x)(p) = \beta_{\text{rel}}(y)(j_*(p)) = 0$ so that $x \in P^{\perp}$. Hence $P \subset P^{\perp}$.

Finally, we will use the fact that $\mathcal{R}$ is a PID to show that $P^{\perp} \subset P$. Consider the monomorphism $H_1(M;\mathcal{R})/P \xrightarrow{j_*} H_1(W;\mathcal{R})$. Clearly, $\mathcal{K}/\mathcal{R}$ is a divisible $\mathcal{R}$-module which implies it is injective since $\mathcal{R}$ is a PID [Ste, I 6.10]. Therefore the map

$$j^{\#} : H_1(W;\mathcal{R})^{\#} \longrightarrow (H_1(M;\mathcal{R})/P)^{\#}$$

is onto. Now, given $x \in P^{\perp}$, it follows that $B\ell(x)(p) = 0$ for all $p \in P$ so $B\ell(x)$ lifts to an element of $(H_1(M)/P)^{\#}$. Thus $B\ell(x)$ lies in the image of $j^{\#}$. Moreover, the Kronecker map

$$\kappa : H^1(W;\mathcal{K}/\mathcal{R}) \longrightarrow H_1(W;\mathcal{R})^{\#}$$

is an isomorphism since $\mathcal{R}$ is a PID (see proof of Theorem 2.13). By Diagram 4.1, x lies in the image of ∂ and so $x \in P$. Hence $P^{\perp} \subset P$. □

Proof of Theorem 3.6.2. Note that Lemma 4.5 holds. Consider Diagram 4.1. The Kronecker maps may no longer be isomorphisms so ignore them. If $x \in P_{n-1}$ then $x = \partial y$ as above so the image of x in $H^1(M;\mathcal{K}/\mathcal{R})$ is in the image of j^*. Recall that extensions as in Theorem 3.5 correspond fundamentally to these cohomology classes and the proof is finished as in the second paragraph of the proof of Theorem 3.6.1. □

We can now prove our main theorem by combining Theorems 4.2, 4.4, 3.5, and 3.6. This can be applied to the zero surgery on a knot K in a rational homology sphere, or to a prime-power cyclic cover of such a manifold.

If $\beta_1(M) = 1$ then, modulo torsion, $H_1(M) \cong H_1(W) \cong \mathbb{Z}$, and the inclusion induces multiplication by some nonzero integer, whose absolute value we call the *multiplicity*. Note that if $M = \partial W$ with $j_* : H_1(M;\mathbb{Q}) \longrightarrow H_1(W;\mathbb{Q}) \cong \mathbb{Q}$ an isomorphism, then there are precisely two *epimorphisms* $\psi_0 : \pi_1(W) \longrightarrow \Gamma_0^U = \mathbb{Z}$. Let $\phi_0 = \psi_0 \circ j_*$. This map $\phi_0 : \pi_1(M) \longrightarrow \Gamma_0^U$ (up to sign) is canonically associated to M and the multiplicity of $M \longrightarrow W$, and extends to $\pi_1(W)$ by definition. If j_* is an isomorphism on integral homology, as is the case for a slice knot in an integral homology 4-ball, the multiplicity is 1 and ϕ_0 is the canonical epimorphism.

THEOREM 4.6. *Let $\Gamma_0^U, \Gamma_1^U, \dots, \Gamma_n^U$ be the family of universal groups of Definition* 3.1. *Suppose M is a closed, oriented,* 3*-manifold with $\beta_1(M) = 1$. Then*

(0): *If M is rationally* (0)*-solvable via W_0 then either of the two maps $\phi_0 : \pi_1(M) \longrightarrow \Gamma_0^U$ extends to $\pi_1(W_0)$ inducing a class $B_0 = B(M, \phi_0)$ in $L^0(\mathcal{K}_0)$ (modulo the image of $L_0(\mathbb{Z}\Gamma_0^U)$). Moreover ϕ_0 induces an Alexander module $\mathcal{A}_0$ and a nonsingular Blanchfield linking form $B\ell_0$.*

(0.5): *If M is rationally* (0.5)*-solvable via $W_{0.5}$ then, in addition to the above holding for $W_{0.5}$, $B_0 = 0 = \rho^{(2)}_{\Gamma_0}(M, \phi_0)$.*

(1): *If M is rationally* (1)*-solvable via W_1 then, in addition to the above holding for W_1, $P_0 \equiv \operatorname{Ker}\{j_* : \mathcal{A}_0 \longrightarrow \mathcal{A}_0(W_1)\}$ is self-annihilating for $B\ell_0$ and for any $p_0 \in P_0$ a coefficient system $\phi_1(p_0) : \pi_1(M) \longrightarrow \Gamma_1^U$ is induced which extends to $\pi_1(W_1)$ and induces a class $B_1(p_0) = B(M, \phi_1)$ in $L^0(\mathcal{K}_1)$. Here $\mathcal{A}_0(W_1) = H_1(W_1; \mathcal{R}_0^U)$. Moreover ϕ_1 induces the generalized Alexander module $\mathcal{A}_1(p_0)$ and nonsingular linking form $B\ell_1(p_0)$.*

(1.5): *If M is rationally* (1.5)*-solvable via $W_{1.5}$ then, in addition to the above holding for $W_{1.5}$, $B_1(p_0) = 0 = \rho^{(2)}_{\Gamma_1}(M, \phi_1)$.*

$\vdots$

(n): *If M is rationally* (n)*-solvable via W_n then, in addition to the above holding for W_n, $P_{n-1} \equiv \operatorname{Ker}\{j_* : \mathcal{A}_{n-1} \longrightarrow \mathcal{A}_{n-1}(W_n)\}$ is self-annihilating with respect to*

$$B\ell_{n-1}(p_0, \dots, p_{n-2})$$

and for any $p_{n-1} \in P_{n-1}(p_0, \dots, p_{n-2})$ a coefficient system

$$\phi_n(p_0, \dots, p_{n-1}) : \pi_1(M) \longrightarrow \Gamma_n^U$$

is induced which extends to $\pi_1(W_n)$ and induces a class

$$B_n(p_0, \dots, p_{n-1}) = B(M, \phi_n) \in L^0(\mathcal{K}_n)$$

modulo the image of $L^0(\mathbb{Z}\Gamma_n^U)$.

($n.5$): *If M is rationally* ($n.5$)*-solvable via $W_{n.5}$ then, in addition to the above holding for $W_{n.5}$,*

$$B_n(p_0, \dots, p_{n-1}) = 0 = \rho^{(2)}_{\Gamma_n}(M, \phi_n).$$

In particular, if a knot K is $(n.5)$-solvable then, for any choices $(p_0, p_1, \ldots, p_{n-1})$, there exist self-annihilating submodules $P_i \subset \mathcal{A}_i(p_0, \ldots, p_{i-1})$, $0 \leq i < n$, and an induced coefficient system $\phi_n(p_0, \ldots, p_{n-1}) : \pi_1(M) \longrightarrow \Gamma_n^U$ (up to conjugation) such that $B_n(M, \phi_n)$ and $\rho^{(2)}_{\Gamma_n}(M, \phi_n)$ are defined and equal to zero. Here M is zero surgery on K.

Remark 4.7. 1. If one chooses $p_{n-1} = 0$ in Theorem 4.6 (n) then ϕ_n is "trivial" in the sense that it factors through ϕ_{n-1} via the splitting $\Gamma_{n-1} \longrightarrow \Gamma_n$. It follows that B_n, $\mathcal{A}_n$ and $B\ell_n$ are all just B_{n-1}, $\mathcal{A}_{n-1}$ and $B\ell_{n-1}$ "tensored up to $\mathbb{Z}\Gamma_n$" in the appropriate fashion. Therefore there is no additional information and further conclusions are trivial consequences of previous stages. Consequently if $\mathcal{A}_{n-1}(p_0, \ldots, p_{n-2}) = 0$ then no new information can be gleaned just as, if the classical Alexander module $\mathcal{A}_0$ is trivial then Casson-Gordon's invariants give no information. Indeed M. Freedman has shown that a knot with $\mathcal{A}_0 = 0$ is topologically slice [F]. On the other hand, if $\mathcal{A}_{n-1}(p_0, \ldots, p_{n-2})$ is *nontrivial* then the nonsingularity of $B\ell_{n-1}$ guarantees that P_{n-1} is nontrivial. In fact one can show that if $\dim_{\mathbb{Q}} \mathcal{A}_0 > 2$, then $\mathcal{A}_{n-1}$ is *always nontrivial.* This will be shown in a subsequent paper [C].

2. Actually a slightly stronger theorem is true. One need not use the full Γ_i^U but, once M is fixed, can replace these by a family of universal groups (defined inductively as semi-direct products) where $\mathcal{K}_{i-1}/\mathcal{R}_{i-1}$ is replaced by the image of the smallest direct summand of $H_1(W; \mathcal{R}_{i-1})$ which contains the image of $\mathcal{A}_{i-1}$. This leads to a family of much smaller groups Γ_i, depending only on the $\mathcal{A}_i(M)$, which are still of the type from Section 3. Although we shall not here formalize this subtlety further, we will use it to advantage in Section 6.

Proof. Note that all maps on the fundamental group will be nontrivial since ϕ_0 is. By induction, assume Theorem 4.6 $(n-1).5$ holds true. We shall establish 4.6 (n). Suppose M is rationally (n)-solvable via W_n. Then M is rationally (h)-solvable via W_n for any $h < n$. By the induction hypothesis, ϕ_0 extends to $\pi_1(W_n)$, and for any $p_0 \in P_0 \equiv \operatorname{Ker}\{j_* : \mathcal{A}_0 \longrightarrow \mathcal{A}_0(W_n)\}$, $\phi_1(p_0)$ is induced which extends to $\pi_1(W_n)$ (and for any such extension), and $\ldots$ for any

$$p_{n-2} \in P_{n-2}(p_0, \ldots, p_{n-3}) \equiv \operatorname{Ker}\{j_* : \mathcal{A}_{n-2} \longrightarrow \mathcal{A}_{n-2}(W_n))\}$$

$\phi_{n-1}(p_0, \ldots, p_{n-2}) : \pi_1(M) \longrightarrow \Gamma_{n-1}^U$ is induced which extends to $\pi_1(W_n)$ and (for any such extension) induces $\mathcal{A}_{n-1}$, $B\ell_{n-1}$. By Theorem 4.4 we see that P_{n-1} is self-annihilating for $B\ell_{n-1}$. This is the first condition of Theorem 4.6 (n). Now choose $p_{n-1} \in P_{n-1}$. By Theorem 3.5 a coefficient system $\phi_n : \pi_1(M) \longrightarrow \Gamma_n^U$ is induced which extends to $\pi_1(W_n)$ by Theorem 3.6.1

or 2. Then ϕ_n induces $B(M,\phi_n)$ as in Section 4 and with $\mathcal{A}_n$ and $B\ell_n$ as in Theorem 2.13. This establishes 4.6 (n). To establish 4.6 $(n.5)$, merely apply the above to $W_{n.5}$ and then apply Theorem 4.2. □

THEOREM 4.8. *Let $\Gamma_0,\ \Gamma_1,\dots,\Gamma_n$ be a family of rationally universal groups as in Definition* 3.1) *$(\mathcal{R}_i,\ S_i$ variable). Then Theorem* 4.6 *holds with the following changes. Omit all conclusions about P_i being self-annihilating. Replace the conclusion that $B\ell_i$ is nonsingular with the conclusion that $B\ell_i$ is nonsingular if $\mathcal{R}_i$ satisfies the hypothesis of Theorem* 2.13.

Proof. Follow the proof of Theorem 4.6. Apply Theorem 3.6.2 instead of Theorem 3.6.1. □

Bordism invariants generalizing the Arf invariant. The following result could lead to examples of $(n-0.5)$-solvable knots that are not (n)-solvable, but calculations have not been made.

COROLLARY 4.9. *Under the hypotheses of Theorem* 4.6, *suppose K is rationally (n)-solvable (respectively (n)-solvable). Then there exists a submodule $P_0\subseteq\mathcal{A}_0$ which is self-annihilating for $B\ell_0$ and for any $p_0\in P_0$ a coefficient system $\phi_1:\pi_1(M)\longrightarrow\Gamma_1^U$ is induced … such that there exists a submodule $P_{n-1}(p_0,\dots,p_{n-2})\subseteq\mathcal{A}_{n-1}(p_0,\dots,p_{n-2})$ which is self-annihilating for $B\ell_{n-1}$ and for any $p_{n-1}\in P_{n-1}$ a coefficient system $\phi_n:\pi_1(M)\longrightarrow\Gamma_n^U$ is induced such that the element (M,ϕ_n) of $\Omega_3(B\Gamma_n^U)$ (respectively $\Omega_3^{\mathrm{Spin}}(B\Gamma_n^U)$) is zero.*

Proof. This is a direct corollary of Theorem 4.6 (n). □

Note that the obstruction in the case $n=0$ (the spin case) of Corollary 4.9 is well-known to be the Arf invariant of K. Note also that there is a somewhat stronger version along the lines of Remark 4.7.2.

5. L^2-signatures

Given a PTFA group Γ and the quotient field $\mathcal{K}$ of $\mathbb{Z}\Gamma$, the purpose of this section is to construct a homomorphism

$$L^0(\mathcal{K})\longrightarrow\mathbb{R}$$

which detects the slice obstructions from Theorem 4.6. It turns out that such a homomorphism can be found by completing $\mathbb{Z}\Gamma$, or better $\mathbb{C}\Gamma$, to the von Neumann algebra $\mathcal{N}\Gamma$ and also completing $\mathcal{K}$ to the algebra $\mathcal{U}\Gamma$ of unbounded operators affiliated to $\mathcal{N}\Gamma$. Then one can use the dimension theory of von Neumann algebras to define an L^2-signature

$$\sigma_\Gamma:L^0(\mathcal{U}\Gamma)\longrightarrow\mathbb{R}$$

for any group Γ. It agrees with the ordinary signature on the image of $L^0(\mathbb{Z}\Gamma)$ if the analytic assembly map for Γ is onto. This property was recently established by N. Higson and A. Kasparov [HK] for all torsion-free amenable groups, a class of groups which contains our PTFA groups.

The idea that the L^2-signature can be applied to concordance questions originated after discussions with Holger Reich on the extension of the von Neumann dimension to $\mathcal{U}\Gamma$. His Ph.D. thesis [Re] was very helpful for writing this section. Also discussions with Wolfgang Lück and Thomas Schick were very useful for understanding von Neumann algebras in necessary detail. Lück's paper [Lu2] is an excellent survey on the use of von Neumann algebras in topology and geometry.

We claim no originality in the following section, except for the observation that this beautiful theory does relate to classical knot concordance. The section is written for nonexperts in von Neumann algebras.

Let Γ be a countable discrete group and consider the Hilbert space $\ell^2\Gamma$ of square-summable sequences of group elements with complex coefficients. The group Γ acts by left- and right- multiplication on $\ell^2\Gamma$. These operators are obviously isometries and we can consider the embedding

$$\mathbb{C}\Gamma \hookrightarrow \mathcal{B}(\ell^2\Gamma)$$

corresponding to (sums of) *right* multiplications into the space of bounded operators on $\ell^2\Gamma$.

Definition 5.1. The (reduced) C^*-algebra $C^*\Gamma$ is the completion of $\mathbb{C}\Gamma$ with respect to the operator norm on $\mathcal{B}(\ell^2\Gamma)$. The von Neumann algebra $\mathcal{N}\Gamma$ is the completion of $\mathbb{C}\Gamma$ with respect to pointwise convergence in $\mathcal{B}(\ell^2\Gamma)$. In particular we have $\mathbb{C}\Gamma \subseteq C^*\Gamma \subseteq \mathcal{N}\Gamma$.

If follows from von Neumann's double commutant theorem that $\mathcal{N}\Gamma$ is equal to the set of bounded operators which commute with the *left* Γ-action on $\ell^2\Gamma$:

$$\mathcal{N}\Gamma = \mathcal{B}(\ell^2\Gamma)^\Gamma.$$

From this description, the *standard* Γ*-trace*

$$\operatorname{tr}_\Gamma : \mathcal{N}\Gamma \longrightarrow \mathbb{C}$$

is defined by $\operatorname{tr}_\Gamma(a) := \langle (e)a, e\rangle_{\ell^2\Gamma}$ where $e \in \Gamma \subseteq \ell^2\Gamma$ is the unit element. It follows from the left invariance of $a \in \mathcal{N}-$ that for all (isometries) $g \in \Gamma$

$$\langle (e)(ag), e\rangle = \langle (e)a, g^{-1}\rangle = \langle g((e)a), e\rangle = \langle (ge)a, e\rangle = \langle (e)(ga), e\rangle.$$

By the linearity and continuity of the Γ-trace this implies the usual trace property

$$\operatorname{tr}_\Gamma(ab) = \operatorname{tr}_\Gamma(ba) \quad \text{for all } a, b \in \mathcal{N}-.$$

Moreover, it also implies that the Γ-trace is *faithful*, i.e. if $\mathrm{tr}_\Gamma(a^*a) = 0$ then $(e)a = 0$ which implies $a = 0$ by left invariance and continuity. This Γ-trace extends to $(n \times n)$-matrices by sending a matrix to the sum of the Γ-traces of the diagonal entries. For example, if

$$p \in M_n(\mathcal{N}\Gamma) = \mathcal{B}((\ell^2\Gamma)^n)^\Gamma$$

is an orthogonal Γ-equivariant projection onto a subspace $V \subseteq (\ell^2\Gamma)^n$ then we may define the Γ-*dimension* of V by

$$\dim_\Gamma V := \mathrm{tr}_\Gamma(p) \in [0, \infty).$$

The trace property shows that this actually only depends on the Γ-isometry class of the Hilbert space V. Thinking of K-theory as equivalence classes of projections, by implication, we have a map

$$\mathrm{tr}_\Gamma : K_0(\mathcal{N}\Gamma) \longrightarrow \mathbb{R}.$$

If $\mathcal{N}\Gamma$ is a factor, i.e. has center $\mathbb{C}$, then this map is actually an isomorphism. This is the case if and only if for all nontrivial $\gamma \in \Gamma$ the number of elements conjugate to γ is infinite.

To define the L^2-signature, consider a hermitian $(n \times n)$-matrix over $\mathcal{N}\Gamma$, $h \in \mathrm{Herm}_n(\mathcal{N}\Gamma)$, as a bounded, hermitian Γ-equivariant operator on the Hilbert space $(\ell^2\Gamma)^n$. Its spectrum spec(h) is a (compact) subset of the real line and for any bounded measurable function f on spec(h) we may define the bounded Γ-equivariant operator $f(h) \in M_n(\mathcal{N}\Gamma)$ by functional calculus (see e.g. [Pe]). In particular, consider the characteristic functions $p_+, p_- : \mathbb{R} \to \mathbb{R}$ of $(0, +\infty)$, respectively $(-\infty, 0)$.

Definition 5.2. The *signature map* $\sigma_\Gamma : \mathrm{Herm}_n(\mathcal{N}\Gamma) \to K_0(\mathcal{N}\Gamma)$ is defined by sending h to the formal difference $p_+(h) - p_-(h)$ of projections in $M_n(\mathcal{N}\Gamma)$. The L^2-*signature* of $h \in \mathrm{Herm}_n(\mathcal{N}\Gamma)$ is defined to be the real number

$$\sigma_\Gamma^{(2)}(h) := \mathrm{tr}_\Gamma(p_+(h)) - \mathrm{tr}_\Gamma(p_-(h)).$$

As the crucial example, we consider the case $\Gamma = \langle t \rangle \cong \mathbb{Z}$. Then $\mathbb{C}\Gamma$ consists of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ which embed naturally into the space of complex-valued continuous functions on the circle S^1. Indeed, Fourier transformation gives an isomorphism of Hilbert-spaces $\ell^2\mathbb{Z} \cong L^2(S^1)$ and pointwise multiplication by a function induces the isomorphism $C^*\Gamma \cong C(S^1)$. This is a consequence of the Stone-Weierstrass theorem on the density of polynomials in the space of all continuous functions in the supremum norm. Completion in the topology of pointwise convergence then leads to the von Neumann algebra $\mathcal{N}\Gamma$ which turns out to be the space $L^\infty(S^1)$ of complex-valued, bounded, measurable functions on the circle, defined almost everywhere. Finally, the standard Γ-trace is just given by integration.

Now consider $h \in \mathrm{Herm}_n(C(S^1))$ and think of it as a continuous map from S^1 to $\mathrm{Herm}_n(\mathbb{C})$. The ordinary signature $\sigma_0 : \mathrm{Herm}_n(\mathbb{C}) \longrightarrow \mathbb{Z}$ counts the number of positive Eigenvalues minus the number of negative Eigenvalues.

Definition 5.3. The *twisted signature* of h is the step function $\sigma(h) : S^1 \longrightarrow \mathbb{Z}$ which assigns to each $s \in S^1$ the signature $\sigma_0(h(s))$. Moreover, the real number $\sigma^{(2)}(h)$ is defined to be the integral of this function $\sigma(h)$ over the circle (normalized to have total measure 1).

Thus $\sigma^{(2)}(h)$ is the average of all the twisted signatures. It is clear that $\sigma(h)$ makes sense almost everywhere for $h \in \mathrm{Herm}_n(L^\infty(S^1))$ and therefore $\sigma^{(2)}(h)$ is well defined even in this case. As an example, consider the following element in $\mathrm{Herm}_2(\mathbb{C}[t^{\pm 1}])$:

$$h := \begin{pmatrix} t+t^{-1}-2 & 1 \\ 1 & t+t^{-1}-2 \end{pmatrix}.$$

Notice that $\sigma(h)$ is a step function with jumps, at most, at the zeroes of the "Alexander polynomial" $\det(h) \in \mathbb{C}[t, t^{-1}]$. We have $\det(h) = (t+t^{-1}-2)^2 - 1 = (t+t^{-1}-1)(t+t^{-1}-3)$ which has roots on S^1 exactly for the two primitive 6^{th} roots of unity. So we only need to calculate $\sigma(h)$ at two points on the circle which interlace with these two roots, e.g. at ± 1. Clearly $h(1)$ is hyperbolic and one easily checks that the ordinary signature of $h(-1)$ is -2. One therefore gets

$$\sigma^{(2)}(h) = (1/3) \cdot 0 + (2/3) \cdot (-2) = -4/3 \neq 0.$$

LEMMA 5.4. *The average $\sigma^{(2)}(h)$ equals the L^2-signature $\sigma^{(2)}_\Gamma(h)$ for $\Gamma = \mathbb{Z}$.*

Proof. Notice that $\mathrm{tr}_\Gamma(p_+(h))$ is the Γ-dimension of the "positive Eigenspace" of h. In the functional calculus one approximates (say) p_+ by a sequence of real polynomials p_i into which any operator can easily be substituted. Then one takes the pointwise limit to define

$$p_+(h) := \lim_i p_i(h) \quad \text{for } h \in \mathcal{N}\mathbb{Z}.$$

For example, if h is a finite dimensional matrix, then one checks that p_+ is just the projection onto the $(+1)$-Eigenspace of h. This implies that for a point $s \in S^1$, a fancy way to count the number of positive Eigenvalues of $h(s) \in \mathrm{Herm}_n(\mathbb{C})$ is to take the ordinary trace of $p_+(h(s)) := \lim_i p_i(h(s))$. But now one clearly sees that the integral of the function $\sigma(h)$ which associates for each $s \in S^1$ the difference $p_+(h(s)) - p_-(h(s))$ is almost everywhere the same as $\sigma^{(2)}_\Gamma(h)$. $\square$

Remark 5.5. The above proof actually shows that without the integration the twisted signature $\sigma(h) \in L^\infty(S^1; \mathbb{Z})$ as in Definition 5.3 equals the more general *center-valued* L^2-signature map coming from the center-valued trace on $\mathcal{N}\mathbb{Z}$. Since $\mathbb{Z}$ is commutative, this trace is just the identity on $\mathcal{N}\mathbb{Z} = L^\infty(S^1, \mathbb{C})$. Moreover, the elements $p_\pm(h) \in M_n(\mathcal{N}\Gamma)$ in the definition of the L^2-signature are mapped by the corresponding trace on matrices to $L^\infty(S^1, \mathbb{R})$. In fact, the proof above shows that they are equal to step functions almost everywhere. □

Since the functional calculus used above extends to self-adjoint *unbounded* operators [Pe], we can extend the L^2-signature to a super-ring $\mathcal{U}\Gamma$ of $\mathcal{N}\Gamma$. Here $\mathcal{U}\Gamma$ is the algebra of operators *affiliated* to $\mathcal{N}\Gamma$. This is the set of (unbounded) operators $a = (a, \mathrm{dom}(a))$ on $\ell^2\Gamma$ which satisfy the following conditions:

(i) a is densely defined, i.e. $\mathrm{dom}(a)$ is dense in $\ell^2\Gamma$.

(ii) a is closed; i.e., its graph is closed in $\ell^2\Gamma \times \ell^2\Gamma$.

(iii) a is affiliated to $\mathcal{N}\Gamma$; i.e., for every bounded operator b which commutes with all of $\mathcal{N}\Gamma$ we have $ba \subseteq ab$. This means $\mathrm{dom}(ba) \subseteq \mathrm{dom}(ab)$ and on the smaller subset these operators agree.

It takes some work to show that $\mathcal{U}\Gamma$ is indeed a ring with involution, for example to define addition and multiplication one has to close the operators. Then the various associativity and distributivity laws become actual theorems. This was worked out by Murray and von Neumann [MvN] (see also [Re]). It turns out however, that $\mathcal{U}\Gamma$ can also be obtained as the Ore-localization of $\mathcal{N}\Gamma$ with respect to all nonzero divisors. From this point of view the theorem is that the latter set is an Ore-domain. As an example, $\mathcal{U}\mathbb{Z}$ is the set of all measurable functions on S^1, i.e. not necessarily bounded functions. This means that in this example the Γ-trace, or integral, does not extend to a map on $\mathcal{U}\Gamma$. However, one can extend the L^2-signature to a map

$$\sigma^{(2)}_\Gamma : \mathrm{Herm}_n(\mathcal{U}\Gamma) \longrightarrow \mathbb{R}$$

as follows. Observe that a hermitian matrix h with entries in $\mathcal{U}\Gamma$ can be viewed as an (unbounded) self-adjoint operator on $\ell^2(\Gamma)^n$. Since the two projections p_+ and p_- onto the positive respectively negative spectrum are *bounded* functions, it follows that the corresponding projections $p_+(h)$ and $p_-(h)$, obtained via functional calculus, are bounded and thus lie in $M_n(\mathcal{N}\Gamma)$. Therefore their Γ-traces can be defined as before.

LEMMA 5.6. *The L^2-signature only depends on the Γ-isometry class of $h \in \mathrm{Herm}_n(\mathcal{U}\Gamma)$; i.e., it is unchanged under $h \mapsto a^*ha$ for $a \in \mathrm{GL}_n(\mathcal{U}\Gamma)$.*

Proof. We first give an argument for the ring $\mathcal{N}\Gamma$. Consider the Hilbert space $H := (\ell^2\Gamma)^n$ with the bounded Γ-equivariant operators h and a. Let p_0 be the characteristic function of $\{0\} \subset \mathbb{R}$, i.e. $p_0(h)$ is the projection onto the kernel of h. Then $p_0(h), p_+(h), p_-(h)$ are commuting projections which sum up to the identity and therefore their images give an orthogonal decomposition of Hilbert spaces

$$H = H_0 \perp H_+ \perp H_-.$$

For a vector v in one of the three summands above, one has by definition that

$$\langle h(v), v\rangle = 0, > 0 \text{ respectively } < 0.$$

It follows that depending on whether v is in $a^{-1}H_0$, $a^{-1}H_+$, respectively $a^{-1}H_-$, one has

$$\langle (a^*ha)(v), v\rangle = \langle h(av), av\rangle = 0, > 0, \text{ respectively, } < 0.$$

Therefore, the three orthogonal projections

$$a^{-1}H_\dagger \longrightarrow p_\dagger(a^*ha)H \quad \text{ for } \dagger \in \{0,+,-\}$$

are monomorphisms and thus

$$\dim_\Gamma H_\dagger = \dim_\Gamma a^{-1}H_\dagger \leq \dim_\Gamma p_\dagger(a^*ha)H \quad \text{ for } \dagger \in \{0,+,-\}.$$

But the three dimensions on both sides must sum up to the total dimension n of H and therefore the inequalities are actually equalities.

To extend this argument to the ring $\mathcal{U}\Gamma$ of unbounded operators affiliated to $\mathcal{N}\Gamma$ one has to use some of their properties (see [Re, §11]). Namely, define a subspace L of the Hilbert space H to be *essentially dense* if it contains a sequence of closed affiliated subspaces whose Γ-dimension tends to one. Here a closed subspace is called affiliated if the corresponding projection is affiliated to $\mathcal{N}\Gamma$. Then the above proof applies to $\mathcal{U}\Gamma$ because for all $a \in \mathcal{U}\Gamma$,

- $\mathrm{dom}(a)$ is essentially dense, and
- $a^{-1}(L)$ is essentially dense if L is essentially dense. □

If P is a finitely generated projective $\mathcal{U}\Gamma$-module, we may choose a $\mathcal{U}\Gamma$-module Q such that $P \oplus Q \cong (\mathcal{U}\Gamma)^n$. If, moreover,

$$h : P \longrightarrow P^* := \mathrm{Hom}_{\mathcal{U}\Gamma}(P, \mathcal{U}\Gamma)$$

is a hermitian form in the sense that $h = h^* : P \cong P^{**} \to P^*$, then we can extend it by three blocks of zeroes to an element in $\mathrm{Herm}_n(\mathcal{U}\Gamma)$. Using Lemma 5.6, it is easily verified that the L^2-signature $\sigma^{(2)}_\Gamma(h)$ is well defined, i.e. independent of the choice of Q. The following result follows easily now from the observation that metabolic forms on $P \oplus P^*$, which represent zero in Witt groups, have trivial L^2-signature.

COROLLARY 5.7. *The L^2-signature is a well-defined real valued homomorphism on the Witt group of hermitian forms on finitely generated projective $\mathcal{U}\Gamma$-modules. Restricting this homomorphism to nonsingular forms on free modules gives*

$$\sigma_\Gamma^{(2)} : L^0(\mathcal{U}\Gamma) \longrightarrow \mathbb{R}.$$

As a great example, consider a finite CW-complex X together with a homomorphism $\varphi : \pi_1 X \to \Gamma$. Then all the twisted homology groups $H_q(X;\mathcal{U}\Gamma)$ are finitely generated projective $\mathcal{U}\Gamma$-modules. This follows from the fact that $\mathcal{U}\Gamma$ is a *von Neumann regular* ring. In particular, every finitely presented $\mathcal{U}\Gamma$-module is projective and these modules form an abelian category [G] (see also [Re]).

If X is, in addition, an oriented Poincaré complex of dimension $4k$, possibly with boundary, then the intersection form

$$h_X : H_{2k}(X;\mathcal{U}\Gamma) \longrightarrow H_{2k}(X,\partial X;\mathcal{U}\Gamma) \cong H^{2k}(X;\mathcal{U}\Gamma) \cong H_{2k}(X;\mathcal{U}\Gamma)^*$$

is a hermitian form. The first isomorphism above is Poincaré duality and the second comes from the universal coefficient spectral sequence

$$\mathrm{Ext}^p_{\mathcal{U}\Gamma}(H_q(X;\mathcal{U}\Gamma),\mathcal{U}\Gamma) \Longrightarrow H^{p+q}(X;\mathcal{U}\Gamma)$$

which degenerates at $p = 0$ by the projectivity of the twisted homology groups.

Definition 5.8. The *L^2-signature* $\sigma^{(2)}(X,\varphi)$ is defined to be the real number $\sigma_\Gamma^{(2)}(h_X)$.

LEMMA 5.9. *The L^2-signature has the following properties*:

1. *If (X^{4k},φ) is the boundary of a $(4k+1)$-dimensional Poincaré complex (with the homomorphism to Γ extending) then $\sigma^{(2)}(X,\varphi) = 0$.*

2. *The resulting homomorphism from the bordism group of oriented Poincaré complexes*
$$\sigma^{(2)} : \Omega_{4k}^{PC}(B\Gamma) \longrightarrow \mathbb{R}$$
is equal to the ordinary signature σ_0 on the image of the bordism group $\Omega_{4k}^{\mathrm{TOP}}(B\Gamma)$ of oriented topological manifolds.

3. *If (X,φ) and (X',φ') have the same boundary (and the homomorphisms to Γ agree on it) then*
$$\sigma^{(2)}(X \cup_{\partial X} X', \varphi \cup \varphi') = \sigma^{(2)}(X,\varphi) + \sigma^{(2)}(X',\varphi').$$

4. *The* reduced *L^2-signature $\sigma^{(2)}(X,\varphi) - \sigma_0(X)$ of a topological $4k$-manifold only depends on the boundary $(\partial X, \varphi_\partial)$.*

Proof. The proofs of 1. and 3. are exactly as for the ordinary signature. One uses the homological properties of $\mathcal{U}\Gamma$ mentioned above as well as the usual additivity properties of the Γ- dimension. Property 2 for smooth manifolds is exactly Atiyah's L^2-Index theorem [A] applied to the L^2-signature operator S. One needs to check that the definition of $\sigma_\Gamma^{(2)}(X, \varphi)$ given above agrees with Atiyah's definition involving the L^2-index of S. This follows from the L^2-Hodge theorem together with the fact that the von Neumann dimension can be read off after tensoring with $\mathcal{U}\Gamma$. A detailed argument will be given in a forthcoming paper by Lück and Schick.

The statement for topological manifolds follows from the fact that the cokernel of the map from smooth to topological bordism is a torsion group and we are mapping into the torsion-free group $\mathbb{R}$. Finally, it is clear that Property 4 is a direct consequence of 2 and 3. □

Remark 5.10. If ∂X has a smooth structure then one can pick a Riemannian metric g and define the η-invariant $\eta(\partial X, g)$ of the signature operator. By lifting g and the operator to the Γ-cover, Cheeger and Gromov [ChG] also define the von Neumann η-invariant $\eta^{(2)}(\partial X, \varphi_\partial, g)$. They show that the difference $\eta^{(2)} - \eta$ is independent of the metric g. This difference is referred to as the von Neumann ρ-invariant.

Moreover, if X is smooth then the Index and L^2-Index theorems for manifolds with boundary imply that the reduced L^2-signature of (X, φ) equals the von Neumann ρ-invariant of $(\partial X, \varphi_\partial)$. By an argument similar to that in Lemma 5.9, Part 2, it follows that this equality holds true if only ∂X is smooth.

As an example, consider a knot $K : S^{4k-3} \hookrightarrow S^{4k-1}, k > 1$. Then surgery on K leads to a closed $(4k-1)$-dimensional manifold M, together with the abelianization map $\varphi : \pi_1 M \to \mathbb{Z}$. It follows from Remark 5.5 that the corresponding center-valued von Neumann ρ-invariant detects the concordance group modulo torsion.

LEMMA 5.11. *Let $R\sigma$ be Ranicki's symmetric signature map. Then the composition*

$$\Omega_{4k}^{PC}(B\Gamma) \xrightarrow{R\sigma} L^0(\mathbb{Z}\Gamma) \longrightarrow L^0(\mathcal{U}\Gamma) \xrightarrow{\sigma_\Gamma^{(2)}} \mathbb{R}$$

is equal to the L^2-signature from Lemma 5.9, *Part* 3.

Proof. The result follows from the fact that for von Neumann regular rings R the bordism class of a chain complex gives the same element in $L^0(R)$ as the corresponding intersection form on the middle homology. This follows from the chain homotopy invariance of algebraic Poincaré cobordism (Chapter 1 of [Ra1]) and algebraic surgery below the middle dimension (Chapter 4 of [Ra1]). □

The next result is not strictly necessary for the definition of our invariants but it seems appropriate to mention it at this point. In addition to reproving that the L^2-signature gives a well-defined slice obstruction via Theorem 4.6 it also shows that in order to define our obstructions one can equally well work with (n)-solutions W which are finite Poincaré complexes (rather than topological 4-manifolds).

Note that by a theorem of Higson-Kasparov [HK] the following assumption is satisfied for torsion-free amenable groups, and hence in particular for PTFA groups.

PROPOSITION 5.12. *If Γ is torsion-free and the analytic assembly map*

$$A_\Gamma : K_0(B\Gamma) \to K_0(C^*\Gamma)$$

is onto then $\sigma_\Gamma^{(2)}(h) = \sigma_0(\varepsilon_* h)$ for all $h \in L^0(C^*\Gamma)$. In particular, the L^2-signature from Lemma 5.11 *equals the ordinary signature on all of $\Omega_{4k}^{PC}(B\Gamma)$.*

Proof. Since h is invertible, it follows that $0 \notin \operatorname{spec}(h)$ and since $\operatorname{spec}(h)$ is compact, it actually has a gap around 0. Therefore, the characteristic functions p_+ and p_- are *continuous* functions on $\operatorname{spec}(h)$ and $p_+(h), p_-(h) \in M_n(C^*\Gamma)$. By taking the difference we get the signature map $\sigma_\Gamma : L^0(C^*\Gamma) \to K_0(C^*\Gamma)$ which is an isomorphism for any C^*-algebra. Recall that $\sigma_\Gamma^{(2)}(h)$ is given by composing with

$$K_0(C^*\Gamma) \longrightarrow K_0(\mathcal{N}\Gamma) \xrightarrow{\operatorname{tr}_\Gamma} \mathbb{R}.$$

Moreover, by Atiyah's L^2-Index theorem applied to all twisted Dirac operators, the two compositions

$$K_0(B\Gamma) \xrightarrow{A_\Gamma} K_0(C^*\Gamma) \xrightarrow{\operatorname{tr}_\Gamma} \mathbb{R},$$

$$K_0(B\Gamma) \longrightarrow K_0(*) = K_0(\mathbb{C}) \xrightarrow[\cong]{\operatorname{tr}_1} \mathbb{Z}$$

are the same (compare [BCH, 7.15]). The claim now follows from surjectivity of the analytic assembly map and the naturality of the signature and assembly maps in the following commutative diagram.

$$\begin{tikzcd}
L^0(C^*\Gamma) \arrow[r] \arrow[d, "\sigma_\Gamma"'] & L^0(\mathbb{C}) \arrow[d, "\sigma_1"', "\cong"] \\
K_0(C^*\Gamma) \arrow[r] & K_0(\mathbb{C}) \\
K_0(B\Gamma) \arrow[r] \arrow[u, "A_\Gamma", "\cong"'] & K_0(*). \arrow[u, "A_1", "\cong"']
\end{tikzcd}$$

□

We next show how to define the L^2-signature for forms over the quotient field $\mathcal{K}$ of $\mathbb{Z}\Gamma$, if it exists. More generally, for any group Γ we can consider the following diagram of inclusions of rings with involution:

$$\begin{tikzcd} \mathbb{C}\Gamma \arrow[r] \arrow[d] & \mathcal{N}\Gamma \arrow[d] \\ \mathcal{D}\Gamma \arrow[r] & \mathcal{U}\Gamma. \end{tikzcd}$$

Here the *division closure* $\mathcal{D}\Gamma$ of $\mathbb{C}\Gamma$ in $\mathcal{U}\Gamma$ is the smallest intermediate ring which is division closed. This means that if $r \in \mathcal{D}\Gamma$ is invertible in $\mathcal{U}\Gamma$ then the inverse r^{-1} already lies in $\mathcal{D}\Gamma$. For the case $\Gamma = \mathbb{Z}$ we obtain $\mathcal{D}\Gamma = \mathbb{C}(t)$, the quotient field of rational functions on S^1. In fact, if $\mathbb{C}\Gamma$ satisfies the Ore condition, then $\mathcal{D}\Gamma$ is the Ore localization of $\mathbb{C}\Gamma$ [Re] and we have constructed the L^2-signature

$$\sigma_\Gamma^{(2)} : L^0(\mathcal{K}) \longrightarrow L^0(\mathcal{D}\Gamma) \longrightarrow L^0(\mathcal{U}\Gamma) \longrightarrow \mathbb{R}.$$

This applies in particular to PTFA groups for which this L^2-signature equals the ordinary signature σ_0 on the image of $L^0(\mathbb{Z}\Gamma)$ in $L^0(\mathcal{K})$ by Proposition 5.12.

We conclude this section with an innocent looking but extremely useful property.

PROPOSITION 5.13. *For a subgroup $\Gamma_1 \subseteq \Gamma_2$, there are commutative diagrams*

$$\begin{tikzcd} \mathcal{N}\Gamma_1 \arrow[rr] \arrow[dr, "\mathrm{tr}_{\Gamma_1}"'] & & \mathcal{N}\Gamma_2 \arrow[dl, "\mathrm{tr}_{\Gamma_2}"] \\ & \mathbb{C} & \end{tikzcd}, \qquad \begin{tikzcd} L^0(\mathcal{U}\Gamma_1) \arrow[rr] \arrow[dr, "\sigma^{(2)}_{\Gamma_1}"'] & & L^0(\mathcal{U}\Gamma_2) \arrow[dl, "\sigma^{(2)}_{\Gamma_2}"] \\ & \mathbb{R} & \end{tikzcd}$$

Proof. The most difficult part is to construct the homomorphism $\mathcal{U}\Gamma_1 \longrightarrow \mathcal{U}\Gamma_2$. A homomorphism $\mathcal{N}\Gamma_1 \to \mathcal{N}\Gamma_2$ is given by completing

$$a \otimes \mathrm{id} \in \mathrm{End}(\ell^2\Gamma_1 \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2)$$

to a bounded operator on $\ell^2\Gamma_2$ for any $a \in \mathcal{N}_{-\infty}$. Since

$$\langle (e_1 \otimes e_2)(a \otimes \mathrm{id}), e_1 \otimes e_2 \rangle = \langle (e_1)a, e_1 \rangle \cdot \langle e_2, e_2 \rangle = \langle (e_1)a, e_1 \rangle$$

it follows that the first diagram commutes. For details see [Lu1, Thm. 3.3]. This reference also contains the statement that tensoring an $\mathcal{N}\Gamma_1$-module with $\mathcal{N}\Gamma_2$ is a flat functor. In particular, the map $\mathcal{N}\Gamma_1 \to \mathcal{N}\Gamma_2$ sends nonzero divisors to nonzero divisors and thus induces a homomorphism $\mathcal{U}\Gamma_1 \to \mathcal{U}\Gamma_2$. To see that the second diagram above commutes just observe that "diagonalizing" a hermitian matrix over $\mathcal{U}\Gamma_1$ and then tensoring up the ±1-eigenspaces to $\mathcal{U}\Gamma_2$ diagonalizes the induced matrix over $\mathcal{U}\Gamma_2$. Thus the commutativity of the first diagram proves the claim. $\square$

We should warn the reader that the von Neumann algebra $\mathcal{N}\Gamma$ is not functorial in Γ. This has to do with the specific choice of the Hilbert-space $\ell^2\Gamma$. Proposition 5.13 gives the best possible functoriality which is valid for all groups. If Γ is amenable, then the equality of the reduced and maximal C^*-algebras (which are functorial!) implies that the projection $\Gamma \twoheadrightarrow \{1\}$ induces a homomorphism of C^*-algebras $\varepsilon : C^*\Gamma \to \mathbb{C}$. For example, if $\Gamma = \mathbb{Z}$ then this is given by evaluating a continuous function at $1 \in S^1$. This clearly does *not* extend to $\mathcal{N}\mathbb{Z} = L^\infty(S^1)$.

Remark 5.14. The above proposition is fundamental to all our calculations! We will construct our knots in such a way that the relevant intersection form over $\mathbb{Z}\Gamma$ will contain as its entries only linear combinations of powers of a single nontrivial group element $\eta \in \Gamma$. Since our groups Γ are torsion-free, this gives an inclusion of groups

$$\mathbb{Z} \cong \langle \eta \rangle \subset \Gamma$$

to which Proposition 5.13 can be applied. Thus the L^2-signature for Γ can be calculated for this particular hermitian form as an integral over the circle of certain twisted signatures. The concrete example to be used can be found after Definition 5.3.

6. Nonslice knots with vanishing Casson-Gordon invariants

In this chapter we give examples of knots which are (2)-solvable but not (2.5)-solvable. In particular, by Theorem 9.11 these are the first examples of knots that have vanishing metabelian concordance invariants, including Casson-Gordon invariants, but are not slice knots even in the topological category. Only the highly technical Proposition 6.1 prevents us from exhibiting K_n, for each $n \geq 2$, which is (n)-solvable but not $(n.5)$-solvable. We focus on one example and indicate how this may be modified to produce an infinite family of such.

Consider the knot K in Figure 6.1. The rectangles containing integers symbolize full twists between the two strands which pass vertically through the rectangles. Thus the rectangle labeled -2 symbolizes two left-handed full twists (see [Ki]). The rectangle labeled by J^* symbolizes the four component string link obtained by taking four untwisted parallel copies of the knotted arc $J*$, which is shown at the bottom of Figure 6.1 . We shall show that K is (2)-solvable but that it fails to satisfy Theorem 4.6 for (2.5)-solvability, as detected by the reduced L^2-signature of Section 5. The same proof will show

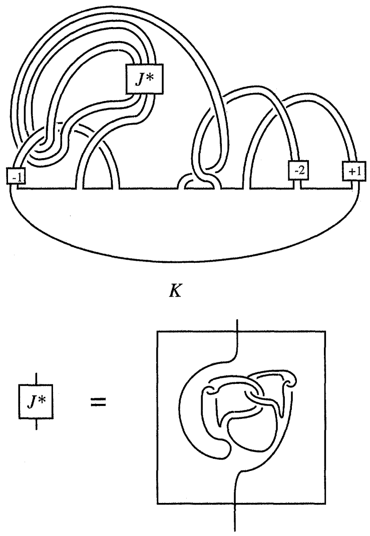
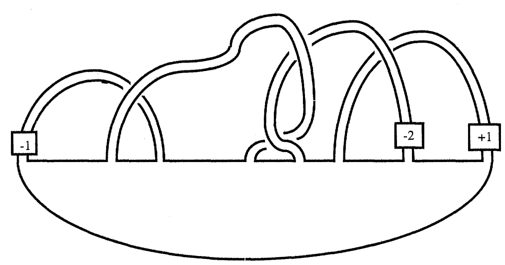


Figure 6.1



K_r

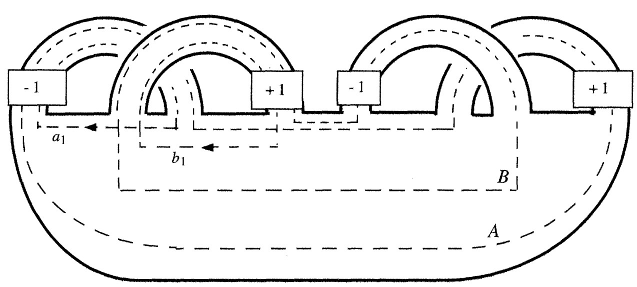
Figure 6.2

that M, the zero surgery on K, is not rationally (2.5)-solvable with multiplicity 1 (see the definition above Theorem 4.6). Consequently, K is not slice in any rational homology ball wherein the meridian generates the free part of H_1 of the slice disk complement.

First we sketch the argument that K is (2)-solvable but not (2.5)-solvable. The bulk of the work is to find a fibered ribbon knot K_r (see Figure 6.2) which is (2)-solvable "in only one way"; i.e., for which $\mathcal{A}_0$ and $\mathcal{A}_1$ have unique self-annihilating submodules. For example we ask the reader to check that the ordinary Alexander module of K_r is cyclic of order $p(t)^2$ where $p(t) = t^{-1}-3+t$, the Alexander polynomial of the figure 8 knot. Since $p(t)$ is irreducible it follows that this module contains a *unique proper submodule*. Thus the rational Alexander module $\mathcal{A}_0$ certainly contains a unique submodule P_0 which is self-annihilating with respect to $B\ell_0$. Then we form the knot K by modifying the ribbon knot by surgeries on two circles, in such a subtle way that $\mathcal{A}_0$ and $\mathcal{A}_1$ are unaffected and K remains (2)-solvable, but such that the (2)-solution for K has some nontrivial second homology. Our proof will proceed by contradiction. Suppose K were (2.5)-solvable via W'. Let W_r denote the complement of the ribbon disk K_r and Y denote the cobordism from M_r, the zero surgery on K_r, to M, which consists of two relative 2-handles. Let W denote the union of W_r and Y along M_r. We will show that W is a (2)-solution for M. Of course W' is also a (2)-solution for M. By Theorem 4.6 applied to each, there exist representations of $\pi_1(M)$ into Γ_2 which extend to $\pi_1(W)$ and $\pi_1(W')$ respectively. We use the fact that there are unique self-annihilating submodules to show that these representations coincide. Let this common map be denoted ϕ_2. Since W' is a (2.5)-solution, $B(M,\phi_2) = 0$ by Theorem 4.2. But W may also be used to calculate $B(M,\phi_2)$. The intersection form of W is represented by a simple (2×2)-matrix whose L^2-signature was calculated to be nonzero in Section 5. This contradiction will then finish the proof.

As stated in the introduction, if there exists a fibered genus two ribbon knot for which $\mathcal{A}_0,\ldots,\mathcal{A}_{n-1}$ all have unique self-annihilating submodules, then the same procedure we discuss herein creates a knot which is (n)-solvable but not $(n.5)$-solvable.

Now we describe in detail the construction of K. Consider $K' = J\#(-J)$ where J is the figure-eight knot (as shown in Figure 6.3). This is a well-known fibered ribbon knot. We summarize the argument. Consider a knotted ball pair $J' = (B^3, B^1)$ representing the figure-eight knot, and cross this with $[0,1]$. The result is a knotted 2-disk $\Delta = J'\times[0,1]$ in B^4 whose boundary is the ribbon knot K'. Now, $S^3 - J = B^3 - J'$ is known to be fibered with fiber the standard Seifert surface (a punctured torus T). It follows that $B^3\times[0,1] - J'\times[0,1]$ is fibered with fiber $T\times[0,1]$, a genus 2 handlebody H, and hence that K' is a genus 2 fibered knot.



$K' = J\# - J$

Figure 6.3

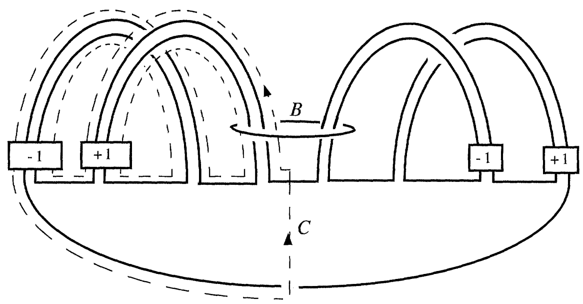


Figure 6.4

Moreover from this point of view it is easy to see that the loops labeled A and B in Figure 6.3 bound embedded disks in H, and that the loops labeled a_1 and b_1 map to generators x and y of the fundamental group $F\langle x, y\rangle$ of H. Let $f : T \to T$ be the monodromy homeomorphism for J, f' that of K' and $\tilde{f}' : H \longrightarrow H$ that of Δ. Then we may assume that f preserves ∂T. It follows that f' preserves the "sub-longitude" loop labeled C in Figure 6.4.

Note that B is unknotted in S^3 and has self-linking zero on the obvious Seifert surface. Therefore if we perform $+1$ surgery on B, K' will be transformed to a new knot K_r as shown in Figure 6.2. This may be seen by pushing B off the Seifert surface, as shown in Figure 6.4, and "blowing down" B by one application of Kirby's calculus [Ki]. It is known that the result, K_r, of such a modification is again a fibered knot whose monodromy f_r equals $D_B \circ f'$ where D_B is a Dehn-Twist along B ([H], [Sta]). Since B bounds a disk in H, D_B extends to $\widetilde{D}$ on H and f_r extends to $\tilde{f}_r : H \to H$. Therefore K_r is also a fibered ribbon knot with fibered ribbon disk Δ_r and fiber H. Moreover $\widetilde{D}$ is homotopic to the identity so $\tilde{f}_r$ is homotopic to $\tilde{f}' : H \to H$. Thus $B^4 - \Delta_r$ is homotopy equivalent to $B^4 - \Delta$. In particular note that the element $[x^{-1}, y]$ of $\pi_1(H)$ is the image of the sub-longitude $C = [b_1^{-1}, a_1]$ and thus

$$(\tilde{f}_r)_*([x^{-1}, y]) = (\tilde{f}')_*([x^{-1}, y]) = [x^{-1}, y]$$

in $\pi_1(H)$. Moreover $(f_r)_*(C)$ is represented by the image of C under D_B.

Finally we will modify K_r by two surgeries, resulting in K. The effect of these surgeries is subtle enough that $\mathcal{A}_0$ and $\mathcal{A}_1$ as well as the Casson-Gordon invariants are unchanged (as we shall see). Consider an embedded circle η in the complement of the obvious Seifert surface for K_r. The specific example we wish to consider is shown in Figure 6.5, but to find examples of knots which are (n)-solvable but not $(n.5)$-solvable one would choose η to represent a nontrivial class in the n^{th} derived group of $\pi_1(W_r)$. This η was also chosen so that $j_*(\eta) = C$, which will later be shown to generate $\mathcal{A}_1(W_r)$. Note that $\{A, B, \eta\}$ is the Borromean ring.

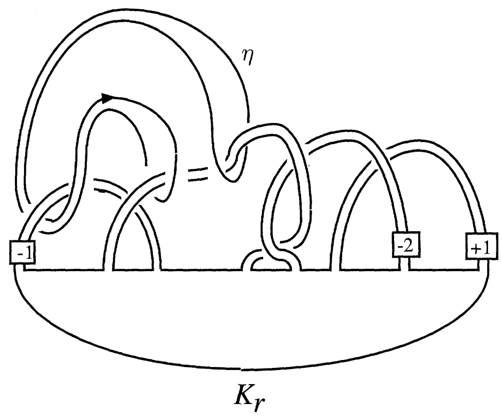


Figure 6.5

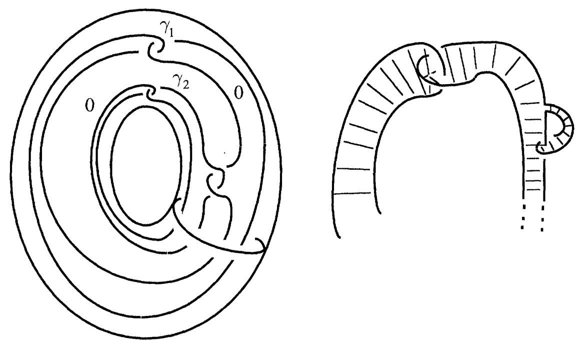


Figure 6.6

Now replace a solid torus neighborhood of η by the 3-manifold shown in Figure 6.6. This manifold is the result of two Dehn surgeries on $S^1 \times D^2$. Since $\{\gamma_1, \gamma_2\}$ forms a zero-framed Hopf link in S^3 (ignoring K_r), and the result of such a surgery is known to be homeomorphic to S^3, the image of K_r under this homeomorphism is a new knot K in S^3 which is shown in Figure 6.1 . The reader proficient in Kirby's calculus may confirm the accuracy of Figure 6.1 by first isotoping $\{\gamma_1, \gamma_2\}$ until it looks more like a Hopf link, next "sliding" all strands of K_r which "pass through γ_1" over γ_2, then "sliding" strands of K_r over γ_2 until the Hopf link becomes split off from the knot [Ki]. We remark that the 3-manifold of Figure 6.6 is a knot complement, namely the complement of the knot J^* obtained by closing up the knotted arc shown at the bottom of Figure 6.1 . Using the trefoil knot in place of J^* would lead to a simpler example of a knot with vanishing Casson-Gordon invariants which is not (2.5)-solvable, but it is difficult to see if it is (2)-solvable. However, in place of J^*, we could use any Arf invariant zero knot, such that the integral of the Levine signature function is nonzero, and reach an identical conclusion. This will be detailed in a forthcoming paper.

We will now show that K is (2)-solvable, using the fact that K is obtained from the ribbon knot K_r by performing two surgeries. Let W_r denote the exterior in B^4 of the fibered ribbon disk for K_r. Let W denote the 4-manifold obtained from W_r by adding 2-handles along the zero-framed circles $\{\gamma_1, \gamma_2\}$. Then W is an H_1-bordism for M, the zero surgery on K, and we will show that it is in fact a (2)-solution. Note that since $\{\gamma_1, \gamma_2\}$ are null-homotopic in M, $W \simeq W_r \vee S^2 \vee S^2$ and so $\pi_1(W) \cong \pi_1(W_r)$ and $H_2(W;\mathbb{Z}) \cong \mathbb{Z}^2$ are generated by $\{\ell_i\}$. The integer-valued intersection form with respect to this basis is the standard hyperbolic form. The circles γ_i bound obvious immersed disks D_i in a neighborhood of η. It is desirable to introduce a local kink in each, as shown in Figure 6.6, so that the push-off of γ_i into D_i has linking number zero with γ_i. These disks, together with the cores of the 2-handles form immersed 2-spheres ℓ_1 and ℓ_2. Being 1-connected, these surfaces lift to any cover. If $[\eta] \in \pi_1(W_r)^{(n)}$, we shall show that $L = \{\ell_1\}$ generates an n-Lagrangian and ℓ_2 is its n-dual. Since the particular η of Figure 6.5 lies in $\pi_1(M_r)^{(2)}$ (it bounds a surface in the complement of a Seifert surface for K_r), this will show that K is (2)-solvable. But we show the more general fact to illustrate how easy it is to generalize K to an (n)-solvable example. It suffices to show that, with $\pi_1(W) \cong \pi_1(W_r)$ coefficients, $\mu(\ell_1) = j_*(\eta) - 1$ and $\lambda(\ell_1, \ell_2) = 1$ where $j_* : \pi_1(M) \longrightarrow \pi_1(W_r)$ because then, with $\pi_1(W_r)/\pi_1(W_r)^{(n)}$ coefficients, $\mu(\ell_1) = \lambda(\ell_1, \ell_1) = 0$. But this is clear from an analysis of the two points of self-intersection of D_1 and the one point of intersection of D_1 with D_2. Section 7 contains a detailed explanation of how to calculate μ and λ. Thus M is (n)-solvable via W ((2)-solvable in our special case).

Now suppose that K is (2.5)-solvable via a 4-manifold W'. By Theorem 4.6 we can find a representation $\phi_2 : \pi_1(M) \longrightarrow \Gamma_2^U$ such that ϕ_2 extends to W' and $B(M, \phi_2) = 0 = \sigma(W')$. (Actually we shall use Remark 4.7.3 to restrict the image of ϕ_2 to a certain subgroup Γ_2 of our universal group Γ_2^U.) If we can show that ϕ_2 also extends to W and that $\phi_2(\eta) \neq 1$, we can quickly reach a contradiction as follows. Since ϕ_2 extends to W, we can calculate $B(M, \phi_2)$ using W. Note that $H_2(W_r; \mathcal{K}_2) \cong H_1(W_r; \mathcal{K}_2) = 0$ by Proposition 4.3 and Proposition 2.11, so that $H_2(W; \mathcal{K}_2) \cong H_2(W, W_r; \mathcal{K}_2)$ is free on $\{\ell_1, \ell_2\}$. (In fact, Lemma 2.12 implies $H_2(W_r; \mathbb{Z}\Gamma_2) = 0$ so that even $H_2(W; \mathbb{Z}\Gamma_2)$ is free.) The intersection and self-intersection forms with $\mathbb{Z}\Gamma_2$ coefficients may be computed, by naturality, from the intersection and self-intersection forms with $\mathbb{Z}\pi_1(W)$ coefficients derived above. Let ψ_2 denote the extension to $\pi_1(W)$ and let $t = \phi_2(\eta) = \psi_2 \circ j_*(\eta)$. Note that $\mu(\ell_1) = \psi_2 \circ j_*(\eta) - 1 = t - 1$ determines that $\lambda(\ell_1, \ell_1) = t + t^{-1} - 2$ by Property 5 of Definition 7.5 below (alternatively this may be computed directly as above). Thus $B(M, \phi_2) \in L^0(\mathcal{K}_2)$ is represented by the matrix

$$\begin{pmatrix} t + t^{-1} - 2 & 1 \\ 1 & t + t^{-1} - 2 \end{pmatrix}.$$

We claim that the reduced L^2-signature of $B(M, \phi_2)$ (see Lemma 5.9 and the discussion above Definition 4.1) is nonzero. For, since $t \neq 1$, the subgroup of Γ_2 generated by t is infinite cyclic. Since the matrix above also represents an element of $L^0(\mathbb{C}(t))$, applying Proposition 5.13, we see that $\sigma^{(2)}_{\Gamma_2}$ agrees with $\sigma^{(2)}_{\mathbb{Z}}$. But the latter was calculated to be nonzero below Definition 5.3. Finally, note that the ordinary signature of the above matrix is zero so the reduced and unreduced L^2-signatures agree. Thus $B(M, \phi_2) \neq 0$. This contradiction will complete the proof that K is not (2.5)-solvable. The remainder of this section is devoted to verifying that if ϕ_2 is the (essentially unique) map guaranteed by Theorem 4.6 (applied to W'), then $\phi_2(\eta) \neq 1$ and ϕ_2 extends to W. Note that since W is also a (2)-solution, Theorem 4.6 applied to W implies that *certain* maps extend to W. From this point of view we must show that one of these can be chosen to coincide with ϕ_2.

Let $\phi_0 : \pi_1(M) \longrightarrow \mathbb{Z} \equiv \Gamma_0$ be the unique homomorphism sending a meridian to 1. Since both W' and W are rational H_1-bordisms with multiplicity 1, ϕ_0 extends uniquely to ψ'_0 and ψ_0 respectively. By Theorem 4.4 with $n = 1$ and $\mathcal{R}_0 = \mathbb{Q}[t^{\pm 1}]$, since M is (1)-solvable via W' and W, $B\ell_0$ is hyperbolic and the kernels of the inclusion maps

$$H_1\left(M; \mathbb{Q}[t^{\pm 1}]\right) \xrightarrow{i_*} H_1\left(W'; \mathbb{Q}[t^{\pm 1}]\right)$$

and

$$H_1\left(M; \mathbb{Q}[t^{\pm 1}]\right) \xrightarrow{j_*} H_1\left(W; \mathbb{Q}[t^{\pm 1}]\right)$$

are self-annihilating. But, as mentioned earlier, $\mathcal{A}_0(K_r)$ has a unique self-annihilating submodule P_0. Now, it is easy to see that $\mathcal{A}_0(K) \cong \mathcal{A}_0(K_r)$ by observation that, since any loop on the obvious Seifert surface for K_r has zero linking number with the γ_i, the Seifert matrix is unaffected by the surgeries. Thus $\operatorname{Ker} i_* = \operatorname{Ker} j_*$. Choose a nonzero element $p_0 \in P_0$ inducing $\phi_1 : \pi_1(M) \longrightarrow \Gamma^U_1$ by Theorem 3.5. By Theorem 3.6 ($n = 1$), ϕ_1 extends to ψ'_1 and ψ_1 on $\pi_1(W')$ and $\pi_1(W)$ respectively. Before proceeding, we want to replace Γ^U_1 by a much smaller group in order to simplify a subsequent calculation (6.1). The basic point is that we can replace Γ^U_1 by a subgroup containing the images of ψ'_1 and ψ_1 and proceed with the argument. In fact we can do slightly better. Let S be the smallest direct summand of $H_1\left(W'; \mathbb{Q}[t^{\pm 1}]\right)$ which contains the image of i_*. Since

$$H_1\left(M;\mathbb{Q}[t^{\pm 1}]\right) \cong \mathbb{Q}[t^{\pm 1}]/p(t)^2$$

and $P_0 = \langle p(t) \rangle$, and the kernel of i_* is P_0, the image of i_* is cyclic of order $p(t)$. Since $p(t)$ is irreducible, it follows that $S \cong \mathbb{Q}[t^{\pm 1}]/p(t)^m$ for some positive integer m, and we can choose the isomorphism so that $i_*(1) = p(t)^{m-1}$. Since

$$(\phi_1)_* : H_1\left(M;\mathbb{Q}[t^{\pm 1}]\right) \longrightarrow \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$$

has kernel *precisely* P_0 (by Theorem 3.5 and since p_0 generates P_0),

$$(\psi_1')_* : S \longrightarrow \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$$

is an embedding. Let S also denote the image of this map. Therefore, if

$$H_1\left(W';\mathbb{Q}[t^{\pm 1}]\right) = S \oplus T$$

we can replace $\mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}]$ by the subgroup S, replace Γ_1^U by $\Gamma_1 = S \rtimes \Gamma_0$, and replace $(\psi_1')_*$ by the projection onto S. The map $(\phi_1)_*$ is replaced by i_* followed by projection, still denoted $(\phi_1)_*$. It remains to note that the image of $(\psi_1)_*$ also lies in S. This is clear because $\pi_1(W) \cong \pi_1(W_r)$ and W_r is a ribbon disk exterior so that j_* is surjective on π_1 and hence on first homology. Moreover we can identify $H_1\left(W;\mathbb{Q}[t^{\pm 1}]\right)$ with $\mathbb{Q}[t^{\pm 1}]/p(t)$ in such a way that $(\psi_1)_*$ is the standard embedding.

By Theorem 3.5, these compatible "characters" induce actual compatible homomorphisms ϕ_1, ψ_1', ψ_1 from $\pi_1(M)$, $\pi_1(W')$, $\pi_1(W)$ respectively to Γ_1. Set

$$\mathcal{R}_1 = \left(\mathbb{Q}[S] - \{0\}\right)^{-1}\mathbb{Q}\Gamma_1 = \mathbb{K}_1[\mu^{\pm 1}]$$

as in Definition 3.1 and Corollary 3.3. The first of these coefficient systems defines $\mathcal{A}_1 = H_1(M;\mathbb{K}_1[\mu^{\pm 1}])$. It will now suffice to prove the following facts about $\mathcal{A}_1$. Note that since $\eta \in \pi_1(M)^{(2)}$, it lifts to the Γ_1-cover and hence represents a class in $\mathcal{A}_1$.

PROPOSITION 6.1. *$\mathcal{A}_1$ contains a unique proper submodule and hence a unique submodule P_1 such that $P_1^{\perp} = P_1$. Moreover there exists $p_1 \in P_1$ such that $B\ell_1(p_1, \eta) \neq 0$.*

Before embarking on the proof of Proposition 6.1, we will show how it completes the proof that K is not (2.5)-solvable. Since M is (2)-solvable via W and via W', Theorem 4.4 applies with $n = 2$, $\Gamma = \Gamma_1$, $\mathcal{R} = \mathcal{R}_1$ to show that the kernels of the maps

$$j_* : H_1(M;\mathcal{R}_1) \longrightarrow H_1(W;\mathcal{R}_1)$$

and

$$i_* : H_1(M;\mathcal{R}_1) \longrightarrow H_1(W';\mathcal{R}_1)$$

are self-annihilating and hence each equal to P_1 by Proposition 6.1. Choose $p_1 \in P_1$ as guaranteed by Proposition 6.1. This induces $(\phi_2)_* : \mathcal{A}_1 \longrightarrow \mathcal{K}_1/\mathcal{R}_1$ and

$$\phi_2 : \pi_1(M) \longrightarrow \Gamma_2 \equiv (\mathcal{K}_1/\mathcal{R}_1) \rtimes \Gamma_1$$

where $\mathcal{K}_1$ is the quotient field of $\mathcal{R}_1$. Apply Theorem 3.6.1 to both W and W' for $n = 2$ and $x = p_1$ to conclude that ϕ_2 extends to both $\pi_1(W)$ and $\pi_1(W')$. Since W' is assumed to be a (2.5)-solution, Theorem 4.2 with $n = 2$ implies that $B(M, \phi_2) = 0$. Moreover we claim that $\phi_2(\eta) \neq 1$ since $(\phi_2)_*(\eta) = B\ell_1(p_1, \eta) \neq 0$. Thus ϕ_2 is the desired coefficient system which leads to a contradiction as explained above.

To prove Proposition 6.1, we must compute $\mathcal{A}_1$. We shall do this in two independent ways — first by using a few general principles and second by explicitly computing the monodromy of the fibered knot K_r. Recall that $\Gamma_1 = S \rtimes \mathbb{Z}$ and $\mathbb{K}_1$ is the (commutative) quotient field of the group ring $\mathbb{Q}S$. If $z \in \mathbb{K}_1$, let z^μ denote the image of z under the automorphism μ of $\mathbb{K}_1$ (see Section 3).

Upon first glance at the form of $\mathcal{A}_1$ in part b below, one might conclude that it had at least two proper submodules if $k \neq 1$. However remember that, although $\mathbb{K}_1$ is commutative, the ring $\mathbb{K}_1[\mu^{\pm 1}]$ is not.

PROPOSITION 6.2. *There are isomorphisms of right $\mathbb{K}_1[\mu^{\pm 1}]$-modules as follows*:

a:

$$H_1(W; \mathbb{K}_1[\mu^{\pm 1}]) \cong H_1(W_r; \mathbb{K}_1[\mu^{\pm 1}]) \cong \frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu - 1)\mathbb{K}_1[\mu^{\pm 1}]}$$

and $j_*(\eta)$ is sent to the generator 1.

b:

$$\mathcal{A}_1 = H_1(M; \mathbb{K}_1[\mu^{\pm 1}]) \cong H_1(M_r; \mathbb{K}_1[\mu^{\pm 1}]) \cong \frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu - 1)(\mu - k)\mathbb{K}_1[\mu^{\pm 1}]}$$

for a certain $k = z^\mu z^{-1}$ such that $z \in \mathbb{Q}S$ and, in the expression for z the coefficient of the additive identity of S is nonzero.

Moreover, under these identifications, the inclusion induced map j_* sends 1 *to* 1.

Proof of Proposition 6.1, *assuming Proposition* 6.2. The kernel P_1 of j_* is of rank 1 over $\mathbb{K}_1$ since any module of the form $\mathbb{K}_1[\mu^{\pm 1}]/g\mathbb{K}_1[\mu^{\pm 1}]$ has $\mathbb{K}_1$-rank equal to the degree of g. So there exists a nonzero *generator* $p_1 \in P_1$. Since $j_*(\eta)$ is not zero, η does not lie in $P_1 = P_1^\perp$ and hence $B\ell_1(p_1, \eta) \neq 0$. This establishes one claim of Proposition 6.1.

Now we will show that the submodule generated by $\mu - 1$ is the unique proper submodule P of $\mathcal{A}_1$. Such a submodule P would have rank 1 over $\mathbb{K}_1$ and thus would be isomorphic to $\mathbb{K}_1[\mu^{\pm 1}]/(\mu - b)\mathbb{K}_1[\mu^{\pm 1}]$ for some $b \in \mathbb{K}_1$. Since the degree of $(\mu - 1)(\mu - k)$ is two, we may assume that $i : P \longrightarrow \mathcal{A}_1$ sends 1 to a degree 1 polynomial. Although $i(1)$ need not be monic, there is some $p \in P$ such that $i(p)$ is monic. But P is cyclic, generated by any nonzero element, so that $P \cong \mathbb{K}_1[\mu^{\pm 1}]/(\mu - b)\mathbb{K}_1[\mu^{\pm 1}]$ (for a different b) where $p \mapsto 1$. Thus we may assume $i(1)$ is monic, say $\mu - d$ for some $d \in \mathbb{K}_1$. This necessitates $(\mu - d)(\mu - b) = (\mu - 1)(\mu - k)$ for some d, $b \in \mathbb{K}_1$. The uniqueness of P is therefore implied by the following lemma which is a purely algebraic statement about the skew polynomial ring $\mathbb{K}_1[\mu^{\pm 1}]$. This lemma completes the proof of Proposition 6.1 and that K is not (2.5)-solvable, modulo the proof of Proposition 6.2. □

LEMMA 6.3. *If $k \in \mathbb{K}_1$ satisfies the algebraic properties from Proposition 6.2.b and $d, b \in \mathbb{K}_1$ are arbitrary then the equation in $\mathbb{K}_1[\mu^{\pm 1}]$*

$$(\mu - d)(\mu - b) = (\mu - 1)(\mu - k)$$

implies that $d = 1$.

Proof. Equating coefficients, using $d\mu = \mu d^{\mu}$, eliminating the variable b, using $k = z^{\mu} z^{-1}$ and setting $\gamma = d - 1$, we are led to:

$$\gamma z^{\mu} = (\gamma + 1)\gamma^{\mu} z \qquad \text{for some } \; \gamma \neq 0 \; \text{ in } \; \mathbb{K}_1. \tag{6.1}$$

The solution $\gamma = 0$ corresponds to the known solution $d = 1$. We will show there are no other solutions. Recall the polynomial $p(t) = t^{-1} - 3 + t$ and the abelian group

$$S = \mathbb{Q}[t^{\pm 1}]/p(t)^m$$

introduced earlier, where $\mathbb{K}_1$ is the quotient field of the group ring $\mathbb{Q}S$. Suppose there is a nonzero solution $\gamma = p/q$ to Equation 6.1 where p, $q \in \mathbb{Q}S$, $pq \neq 0$ and p and q are relatively prime. We may assume that, for p, the coefficient of e, the identity element in the group S, is nontrivial, by absorbing a unit into q. Note that S is *locally* free abelian since it is torsion-free. Thus p, q, p^{μ}, q^{μ}, z and z^{μ} lie in a subring isomorphic to $\mathbb{Q}[\mathbb{Z}^n]$ for some n. In particular this ring is a unique factorization domain and has only trivial units of the form rs where $r \in \mathbb{Q}$ and $s \in \mathbb{Z}^n$. Equation 6.1 then becomes

$$pq^{\mu} z^{\mu} = (p + q)p^{\mu} z, \tag{6.2}$$

an equation in $\mathbb{Q}[\mathbb{Z}^n]$.

Case I. p and z are relatively prime.

Then any factor of p (on the left-hand side of Equation (6.2)) must divide p^{μ} (on the right-hand side). Thus $p = rsp^{\mu}$ for some unit rs ($r \in \mathbb{Q}$, $s \in S$).

Suppose $p = \sum r_i s_i$ for nonzero rationals r_i, $s_i \in S$ and $i \in \mathcal{C}$, a finite index set. Then

$$\sum r_i s_i = \sum (r r_i) s s_i^{\mu}$$

so that, for each $i \in \mathcal{C}$, $s s_i^{\mu} = s_{f(i)}$ for some $f(i) \in \mathcal{C}$. The permutation $f : \mathcal{C} \to \mathcal{C}$ is of finite order since $\mathcal{C}$ is finite, so there exists a positive integer ℓ such that

$$s s^{\mu} s^{\mu_2} \dots s^{\mu^{\ell-1}} s_i^{\mu^{\ell}} = s_i$$

for each i. Note that this is a statement entirely in S (not $\mathbb{Q}S$): the group operation here is from the abelian group structure on S and the action of μ comes from the group automorphism μ. Recall that S is actually the *additive* group of the ring $\mathbb{Q}[t^{\pm 1}]/p(t)^m$, and μ acts by multiplication by t. Switching to additive notation and setting

$$-s' = s + s^{\mu} + \dots + s^{\mu^{\ell-1}}$$

we have $(t^{\ell} - 1)s_i = s'$ for each $i \in \mathcal{C}$. If $\mathcal{C}$ contains two distinct elements s_0 and s_1, say, then $s_0 - s_1$ is annihilated by $t^{\ell} - 1$. This is impossible since $t^{\ell} - 1$ and $t^{-1} - 3 + t$ are relatively prime. Therefore $\mathcal{C}$ contains only one element and $p = r_0 s_0$ for some $r_0 \in \mathbb{Q}$ and $s_0 \in S$. Hence p is a unit and can be assumed to be 1 by absorbing the unit into q. Now Equation (6.2) reduces to:

$$q^{\mu} z^{\mu} = (1 + q)z. \tag{6.3}$$

Let $w = zq$. Then Equation (6.3) becomes $w^{\mu} = z + w$, in $\mathbb{Q}S$. Let r_0 be the coefficient of the additive identity $e \in S$ in the expression for w and similarly let c_0 be the coefficient of e for z. Note that μ is a *group* automorphism of S and as such preserves the identity. By equating coefficients of e one sees that $a_0 = c_0 + a_0$, implying $c_0 = 0$, an obvious contradiction to Proposition 6.2.b. Thus Case I is not possible.

Case II. p and z have greatest common factor f in $\mathbb{Q}[\mathbb{Z}^n]$.

Suppose $p = f\tilde{p}$, and $z = f\tilde{z}$. Then after dividing out $f f^{\mu}$ from Equation (6.2), repeat the argument of Case I to conclude $\tilde{p}$ is a unit which may be assumed to be 1. Setting $w = \tilde{z}q$, we reach the same equation $w^{\mu} = z + w$ and arrive at the same contradiction. □

Proof of Proposition 6.2. Let Y be the cobordism between M_r and M. It will be convenient to refer to the following commutative diagram. Here ψ_1^r is defined using ψ_1 to make the diagram commute and ϕ_1^r is induced by ψ_1^r.

$$
\begin{tikzcd}
\pi_1(M_r) \arrow[r] \arrow[d, "\cong"] \arrow[drr, "\phi_1^r"] & \pi_1(W_r) \arrow[dr, "\psi_1^r"] \arrow[dd, "\cong"] & \\
\pi_1(Y) & & \Gamma_1 \\
\pi_1(M) \arrow[u] \arrow[r, "j_*"'] \arrow[urr, "\phi_1"] & \pi_1(W) \arrow[ur, "\psi_1"'] &
\end{tikzcd}
$$

Given any such compatible Γ_i-coefficient systems (we need the case $i = 1$ pictured above and also the case $i = 0$ which has a corresponding diagram), we claim that one can compare $H_1(M_r; \mathbb{Z}\Gamma_i)$ and $H_1(M; \mathbb{Z}\Gamma_i)$ by considering the linking matrix with $\mathbb{Z}\Gamma_i$ coefficients. To see this, consider the commutative diagram below with $\mathbb{Z}\Gamma_i$ coefficients.

$$
\begin{tikzcd}
H_2(Y) \arrow[r, "\pi_*"] \arrow[ddr, "\lambda"'] & H_2(Y,M) \arrow[r, "\partial"] \arrow[d, "\cong", "\text{P.D.}"'] & H_1(M) \arrow[r] & H_1(Y) \arrow[r] & 0 \\
 & H^2(Y,M_r) \arrow[d, "\cong", "\kappa"'] & & & \\
 & H_2(Y,M_r)^* & & &
\end{tikzcd}
$$

Since $Y \simeq M_r \vee S^2 \vee S^2$, the $\mathbb{Z}\Gamma_i$-modules $H_2(Y, M_r)$ and $H^2(Y, M_r)$ are free of rank two, and $H_1(M_r) \cong H_1(Y)$. Just as it is with untwisted coefficients, the map λ is given by the linking matrix of the attaching circles of the two 2-handles. This linking matrix has been calculated earlier to be

$$
\begin{pmatrix} t + t^{-1} - 2 & 1 \\ 1 & t + t^{-1} - 2 \end{pmatrix}
$$

where $t = \phi_i^r(\eta)$ and $\phi_i^r : \pi_1(M_r) \longrightarrow \Gamma_i$. But since $\eta \in \pi_1(M_r)^{(2)}$, and Γ_i is i-solvable, for the cases $i = 0, 1$ this is the standard hyperbolic matrix. Alternatively, note that each γ_i admits a Seifert surface S_i which is a punctured torus lying inside the tubular neighborhood of η and which lifts to the Γ_1-cover (since $\phi_1^r(\eta) = 1$), and use these to compute the linking matrix. Since this matrix is invertible, the above sequence implies that M and M_r have isomorphic integral Alexander modules and $\mathcal{A}_1 \cong H_1(M_r; \mathbb{K}_1[\mu^{\pm 1}])$. Since $(\Gamma_1)^{(2)} = \{e\}$, the maps ϕ_1 and ϕ_1^r factor through $\pi_1(M)/\pi_1(M)^{(2)}$ and $\pi_1(M_r)/\pi_1(M_r)^{(2)}$

respectively. Therefore the images of ϕ_1 and ϕ_1^r are completely dictated by the images of the induced maps

$$(\phi_1)_* : H_1(M;\mathbb{Z}[t^{\pm 1}]) \longrightarrow S \text{ and } (\phi_1^r)_* : H_1(M_r;\mathbb{Z}[t^{\pm 1}]) \longrightarrow S$$

on the *integral* Alexander modules. But the integral Alexander modules of M and M_r are isomorphic and thus images of the two maps above are identical. We have noted previously that the kernel of ϕ_1 on the *rational* $\mathcal{A}_0$ is P_0 so that the image of $\pi_1(M_r)$ and $\pi_1(M)$ in Γ_1 is

$$\mathbb{Z}[t^{\pm 1}]/p(t) \rtimes \mathbb{Z} \cong (\mathbb{Z}\times\mathbb{Z}) \rtimes \mathbb{Z}.$$

But the latter is precisely $\pi_1(W_r)/\pi_1(W_r)^{(2)}$, where W_r is the ribbon disk complement, so that ψ_1 and ψ_1^r induce a monomorphism modulo the second derived subgroup. Hence

$$H_1(W_r;\mathbb{K}_1[\mu^{\pm 1}]) \cong H_1(W;\mathbb{K}_1[\mu^{\pm 1}]) \cong H_1(W_r^{(2)};\mathbb{Z}) \otimes \mathbb{K}_1[\mu^{\pm 1}]$$

where $W_r^{(2)}$ is the universal abelian cover of the infinite cyclic cover and the tensor product is over $\mathbb{Z}[(\mathbb{Z}\times\mathbb{Z})\rtimes\mathbb{Z}]$ (see Remark 2.8.1). The infinite cyclic cover of the *fibered* ribbon disk complement W_r is $H\times\mathbb{R}$ and thus is homotopy equivalent to a wedge of two circles corresponding to x and y. Hence $W_r^{(2)}$ is homotopy equivalent to the usual planar grid

$$\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1 \text{ or } x_2 \text{ is an integer}\}$$

and so $H_1(W_r^{(2)};\mathbb{Z})$ is free on one element (we choose $[x^{-1},y]$, the image of C) as a $\mathbb{Z}[\mathbb{Z}\times\mathbb{Z}]$-module. Therefore it is certainly a cyclic module over $\mathbb{Z}[(\mathbb{Z}\times\mathbb{Z})\rtimes\mathbb{Z}]$ and the action of μ on $[x^{-1},y]$ is given by the monodromy $\tilde{f}_r : H \longrightarrow H$. But we have previously observed that $(\tilde{f}_r)_*([x^{-1},y]) = [x^{-1},y]$. Consequently if we denote the generator by $\overline{C}$ then $\overline{C}(\mu-1)=0$ so that, as a *right* $\mathbb{K}_1[\mu^{\pm 1}]$-module, $H_1(W_r;\mathbb{K}_1[\mu^{\pm 1}])$ is as claimed in Proposition 6.2.a. It is not difficult to check (using a presentation as in Figure 6.7) that the loop η maps to $[x^{-1},y]$ under the inclusion j_* (indeed that was how η was chosen) and so $j_*(\eta) = j_*(C)$. Thus $j_*(\eta)$ generates.

We can apply Lemma 2.14 to the case at hand where

$$H_1(M_r;\mathbb{K}_1[\mu^{\pm 1}])/P_1 \cong H_1(W_r;\mathbb{K}_1[\mu^{\pm 1}])$$

to conclude that

$$P_1 \cong H_1(W_r;\mathbb{K}_1[\mu^{\pm 1}])^{\#} \cong \frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu^{-1}-1)\mathbb{K}_1[\mu^{\pm 1}]} \cong \frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu-1)\mathbb{K}_1[\mu^{\pm 1}]}$$

and in particular $rk_{\mathbb{K}_1}(P_1)=1$. Now we can make use of a theorem that any finitely-generated $\mathbb{K}_1[\mu^{\pm 1}]$-module is cyclic [Co2, Prop. 2.2.8 and Th. 1.5.5]. (We will also shortly derive the fact that $\mathcal{A}_1$ is cyclic by explicit computation.) Since $rk_{\mathbb{K}_1}\mathcal{A}_1 = 2rk_{\mathbb{K}_1}P_1 = 2$, we have that

$$\mathcal{A}_1 \cong H_1(M_r;\mathbb{K}_1[\mu^{\pm 1}]) \cong \mathbb{K}_1[\mu^{\pm 1}]/g\mathbb{K}_1[\mu^{\pm 1}]$$

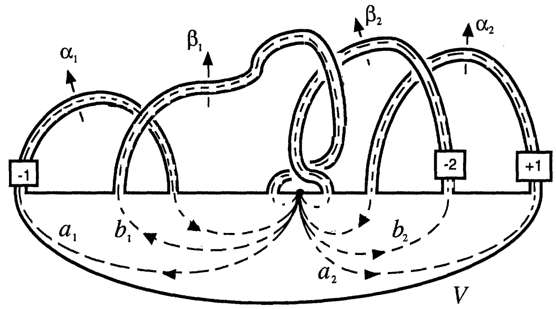


Figure 6.7

where g is a monic degree 2 polynomial in μ. Since this module admits an epimorphism to

$$\mathbb{K}_1[\mu^{\pm 1}]/(\mu-1)\mathbb{K}_1[\mu^{\pm 1}] \quad (\cong \mathbb{K}_1)$$

it admits such an epimorphism sending 1 to 1. The image of g lies in $(\mu-1)\mathbb{K}_1[\mu^{\pm 1}]$ so $g=(\mu-1)(\mu-k)$ for some $k \in \mathbb{K}_1$. The kernel of this epimorphism, P_1, is clearly generated by $\mu-1$ and hence is itself cyclic of order $\mu-k$. But above we saw that P_1 is cyclic of order $\mu-1$. Hence

$$\frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu-k)\mathbb{K}_1[\mu^{\pm 1}]} \cong \frac{\mathbb{K}_1[\mu^{\pm 1}]}{(\mu-1)\mathbb{K}_1[\mu^{\pm 1}]}.$$

This does not imply that $k=1$ since $\mathbb{K}_1[\mu^{\pm 1}]$ is noncommutative. However it can be shown that this is equivalent to the fact that there exists some nonzero $z \in \mathbb{K}_1$ such that $k = z^{\mu}z^{-1}$ [Co2, p. 112 Lemma 3.4.2].

In summary, we have demonstrated Proposition 6.2.a and b except for showing that z has the required properties. For this purpose we are forced to move to a second, more explicit, calculation of $\mathcal{A}_1$ — including a determination of the constant k via a computation of the π_1 monodromy of K_r. This calculation constitutes the remainder of this section.

First we show that it is quite easy to determine an explicit presentation of $\mathcal{A}_1$ as a $\mathbb{K}_1$-module. We have already identified $\mathcal{A}_1$ with $H_1(M_r; \mathbb{K}_1[\mu^{\pm 1}])$. Recall that we argued in the proof of Proposition 6.2.a that the image of ϕ_1 in Γ_1 is $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}$. Let Γ_1^{ℓ} denote this subgroup. The infinite cyclic cover of $S^3 - K_r$ is homotopy equivalent to a wedge of four circles whose fundamental group is free on $\{A, B, a_1, b_1\}$ whereas π_1 of the infinite cyclic cover of W_r is free on $\{j_*(a_1), j_*(b_1)\}$. It is then immediate that, as a $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$-module, $H_1(S^3 - K_r; \mathbb{Z}\Gamma_1^{\ell})$ is isomorphic to the direct sum of $H_1(W_r; \mathbb{Z}\Gamma_1^{\ell})$ and a free $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}]$-module of rank 2 with basis $\{A, B\}$. The same holds for coefficients in $\mathbb{K}_1[\mu^{\pm 1}]$ *as* $\mathbb{K}_1$*-modules.* The inverse of the longitude of K_r, ℓ^{-1}, equals $[b_1^{-1}, a_1][a_2, b_2^{-1}]$ (we use the convention $[a,b] = aba^{-1}b^{-1}$). Since $A = a_1 a_2^{-1}$

and $B = b_1 b_2^{-1}$, the element ℓ^{-1} represents

$$A(x-1) + B(xy^{-1} - x) \in H_1(S^3 - K_r; \mathbb{Z}\Gamma_1^\ell)$$

where $\{x, y\}$ are used as the basis of the $\mathbb{Z} \times \mathbb{Z}$ action. This can be demonstrated by rewriting ℓ^{-1} as

$$[b_1^{-1}, a_1]A^{-1}[a_1, b_1^{-1}]\left(b_1^{-1}a_1Ba_1^{-1}b_1\right)\left(b_1^{-1}Ab_1\right)\left(b_1^{-1}B^{-1}b_1\right),$$

where we recall that $j_*(b_1) = x$ and $j_*(a_1) = y$. Recall also our convention that if $\widetilde{X} \xrightarrow{P} X$ is a regular cover, then a right $\mathbb{Z}[\pi_1(X)/p_*(\pi_1(\widetilde{X}))]$-module structure on $H_1(\widetilde{X})$ (viewed as the abelianization of $p_*(\pi_1(\widetilde{X}))$) is given by $[\gamma]\mu_* = [\mu^{-1}\gamma\mu]$ where $[\gamma] \in H_1(\widetilde{X})$ and $\mu = \mu p_*(\pi_1(\widetilde{X}))$ is a right coset. Therefore we arrive at the *longitudinal relation*:

$$B = A(x^{-1} - 1)_*(y^{-1} - 1)_*^{-1}. \tag{6.4}$$

Since $y^{-1} - 1$ is a unit in $\mathbb{K}_1$, $\mathcal{A}_1$, as a $\mathbb{K}_1$-module, is free on $\{A, C\}$. Since $H_1(W_r; \mathbb{K}_1[\mu^{\pm 1}]) \cong \mathbb{K}_1$, generated by $j_*(C)$, and since A bounds a disk in W_r, this shows that the kernel of j_* in Proposition 6.2.b is indeed the $\mathbb{K}_1$ subspace spanned by A. Note that this implies that the subspace is invariant under the action of μ, a fact we shall presently confirm by direct calculation.

To calculate the structure of $\mathcal{A}_1$ as a $\mathbb{K}_1[\mu^{\pm 1}]$-module, we must derive the action of μ on A and C. Since $\mathcal{A}_1 \cong H_1(M_r; \mathbb{K}_1[\mu^{\pm 1}])$, it is certainly sufficient to know the monodromy (on π_1) of K_r (indeed it would suffice to know it for K'). Let V be the Seifert surface for K_r and $F = \pi_1(V, *)$ the free group on $\{a_1, b_1, a_2, b_2\}$ as in Figure 6.7. We will use the dual basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ of $\pi_1(S^3 - V)$ to help calculate. The basepoint is on the boundary of a tubular neighborhood of K_r, on the negative side of V as shown in Figure 6.7.

Refer to Figure 6.8, which is a schematic picture of the infinite cyclic cover of $S^3 - K_r$. If x is a based loop then $x\mu_* = \mu^{-1}x\mu$ is obtained by traveling from the basepoint halfway around the meridian (in the negative direction) until reaching the positive side of V, traversing x^+, then returning to the basepoint along the same path. This must then be written in terms of the chosen basis, which will be $\{a_1, b_1, a_2, b_2\}$ or $\{a_1^-, b_1^-, a_2^-, b_2^-\}$ which are identified. The initial calculations follow.

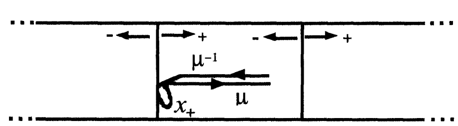


Figure 6.8

These were accomplished using a 25-foot extension cord on a living room floor to simulate K_r. Here $\delta = [\beta_1, \beta_2^{-1}]$. These are essentially the positive push-offs of a_i and b_i, but based at the basepoint on the negative side of V as described above.

$$\begin{aligned} a_1\mu_* &= \delta\alpha_1^{-1}\delta^{-1}, \\ a_2\mu_* &= \delta\alpha_2^{-1}\delta^{-1}, \\ b_1\mu_* &= \delta\alpha_1^{-1}\beta_2\delta^{-1}, \\ b_2\mu_* &= \delta\alpha_2^{-1}\beta_2\beta_1^{-1}\beta_2\delta^{-1}. \end{aligned} \tag{6.5}$$

Now we must translate $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ into $\{a_1, b_1, a_2, b_2\}$ using the negative push-offs:

$$\begin{aligned} a_1 &= a_1^- = \alpha_1^{-1}\beta_1^{-1}, \\ a_2 &= a_2^- = \alpha_2^{-1}\beta_2^{-1}, \\ b_1 &= b_1^- = \beta_1\beta_2\beta_1^{-1}, \\ b_2 &= b_2^- = \beta_2^2\beta_1^{-1}. \end{aligned} \tag{6.6}$$

These enable us to solve: $\beta_2 = b_2b_1b_2^{-1}$, $\beta_1 = b_1^2b_2^{-1}$, $\alpha_1 = b_2b_1^{-2}a_1^{-1}$, $\alpha_2 = b_2b_1^{-1}b_2^{-1}a_2^{-1}$. The "monodromy" equations 6.5 then become:

$$\begin{aligned} a_1\mu_* &= \delta a_1b_1^2b_2^{-1}\delta^{-1} \quad \text{where } \delta = [b_1^{-1}, b_2], \\ a_2\mu_* &= \delta a_2b_2b_1b_2^{-1}\delta^{-1}, \\ b_1\mu_* &= \delta a_1b_1^3b_2^{-1}\delta^{-1}, \\ b_2\mu_* &= \delta a_2b_2b_1. \end{aligned} \tag{6.7}$$

Using these one may calculate the $\mathbb{K}_1[\mu^{\pm1}]$-module relations:

$$A\mu = A + B(y^{-1}), \tag{6.8}$$

$$B\mu = A + B(y^{-1} + x^{-1}y^{-1}), \tag{6.9}$$

$$C\mu = C + B(x - 1). \tag{6.10}$$

Let us clarify the meaning of these relations. It is easy to think of taking the infinite cyclic cover of M_r and then the $\mathbb{Z}\times\mathbb{Z}$ cover of that (this is the boundary of the universal abelian cover of the infinite cyclic cover W_∞ of W_r.) The group of deck translations of this regular cover is $\Gamma_1^\ell = (\mathbb{Z}\times\mathbb{Z}) \rtimes \mathbb{Z}$ where μ generates $\Gamma_0 = \mathbb{Z}$ and x and y generate $\mathbb{Z}\times\mathbb{Z}$ (corresponding to the chosen generators of $H_1(W_\infty)$). The above module relations do constitute a presentation of $H_1(M_r)$ with coefficients in the ring $\mathbb{Z}\Gamma_1^\ell$ with the subset $\mathbb{Z}[x^{\pm1}, y^{\pm1}] - \{0\}$ inverted. However, this ring is a subring of $\mathbb{K}_1[\mu^{\pm1}]$ and it is this larger ring which is our intended coefficient ring. Recall that $\Gamma_1 = S \rtimes \Gamma_0$ where $S = \mathbb{Q}[t^{\pm1}]/p(t)^m$ and where

$$H_1(W_r; \mathbb{Q}[t^{\pm1}]) \cong \mathbb{Q}[t^{\pm1}]/p(t)$$

embeds in S in the standard fashion, and the map $H_1(M_r;\mathbb{Q}[t^{\pm 1}]) \longrightarrow S$ factors through the inclusion to $H_1(W_r;\mathbb{Q}[t^{\pm 1}])$. The latter fact justifies our replacing things like $a_1^{-1}\gamma a_1$ with γy in the above calculations, since $j_*(a_1) = y$ and $j_*(b_1) = x$. Therefore the reader sees that Γ_1^ℓ is naturally the subgroup $H_1(W_r;\mathbb{Z}[t^{\pm 1}]) \rtimes \mathbb{Z}$ of Γ_1, and the elements x, y in the above relations are to be viewed in this way as elements of $\mathbb{K}_1[\mu^{\pm 1}]$, which, as you recall, is $\mathbb{Z}\Gamma_1$ with $\mathbb{Z}S - \{0\}$ inverted. Finally, if $s \in S$ then s^μ will denote the element $\mu^{-1}s\mu$ of Γ_1 or more precisely the image of s under the action of Γ_0 on S. What is this action? Note that under our identifications it agrees with the ordinary action of μ on the Alexander module

$$H_1(W_r;\mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z} \times \mathbb{Z} \cong \langle x, y\rangle \,.$$

The element x^μ may be calculated from the formula above for $\mu^{-1}b_1\mu = b_1\mu_*$ by setting $b_1 = b_2 = x$ and $a_1 = a_2 = y$ and abelianizing. Thus we get:

$$x^\mu = yx^2, \quad y^\mu = yx. \tag{6.11}$$

The relation (6.9) is not needed since B was eliminated using the longitudinal relation; however it can be used as a "consistency check". The relation (6.10) could have been derived more easily. Recall that we argued that C was fixed under the monodromy of K' (connected sum of figure eight with itself) and so the monodromy of K_r sent C to the image of C under a Dehn twist along B. Since C intersects B in two points, relation (6.10) can be easily deduced. From relation (6.10) and the longitudinal relation (6.4) we obtain

$$C(\mu - 1)(x-1)^{-1}w = A \tag{6.12}$$

where $w = (y^{-1} - 1)(x^{-1} - 1)^{-1}$, showing that $\mathcal{A}_1$ is cyclic, generated by C and thus completing the proof of Proposition 6.2.b. Using relation (6.8) and $B = Aw^{-1}$ we get $A(\mu - 1 - w^{-1}y^{-1}) = 0$. Combining this with (6.12) yields that C is annihilated by

$$(\mu - 1)(x-1)^{-1}w(\mu - 1 - w^{-1}y^{-1}).$$

Let $s = (x-1)^{-1}w$ and let $r = 1 + w^{-1}y^{-1}$. Note that

$$s(\mu - r) = \mu s^\mu - sr = (\mu - sr(s^\mu)^{-1})s^\mu.$$

Hence C is annihilated by $(\mu - 1)(\mu - k)$ where $k = sr(s^\mu)^{-1}$. This simplifies to

$$k = \frac{(x-1)^\mu}{x-1}\,\frac{(x^{-1}-1)^\mu}{(x^{-1}-1)}. \tag{6.13}$$

This provides the specific k of Proposition 6.2.b. Note that $k = z^\mu z^{-1}$ where $z = (x-1)(x^{-1}-1)$ and the coefficient of the identity is 2. This concludes the proof of Proposition 6.2.b. The only extra information we have obtained is the specific value of z, which we needed in the proof of Proposition 6.1. Probably there is a way to deduce this property of z without explicit calculation, in which case the 25-foot extension cord is not needed! $\square$

THEOREM 6.4. *The zero surgery on the knot K of Figure* 6.1 *is* (2)*-solvable but not rationally* (2.5)*-solvable with multiplicity* 1 (*see the discussion above Theorem* 4.6). *In particular*, *the knot K of Figure* 6.1 *is not slice in any rational homology ball wherein the meridian of K generates the free part of H_1 of the exterior of the slice disk.*

Proof. Repeat the proof that K is not (2.5)-solvable. The only place where we used solvability was in claiming that ϕ_0 was the usual epimorphism. The point is that when the multiplicity is 1, $\mathcal{A}_0$ is the usual Alexander module of K and our calculations are correct. If it is not 1 then $\mathcal{A}_0$ is larger and new calculations would need to be made. This has not been attempted. $\square$

7. (n)-surfaces, gropes and Whitney towers

Consider a regular covering $X_N \to X$ of smooth connected oriented 4-manifolds, where $\pi_1(X_N) \cong N$ is a normal subgroup of $\pi_1(X)$. To be precise, all the spaces are equipped with a base point (which will be suppressed from the notation).

Definition 7.1. Let F be a closed oriented surface. An *N-surface* in X is a generic immersion $f : F \looparrowright X$ such that $f_*(\pi_1(F)) \leq N$. In addition, the surface is equipped with a *whisker*, i.e. an arc in X from the base point of X to the image of the base point of F.

By covering space theory, an N-surface lifts uniquely to a generic immersion $f_N : F \looparrowright X_N$ leading to the induced homology class $[f] := (f_N)_*[F] \in H_2(X_N)$. Clearly any class in $H_2(X_N)$ is represented in this way. The group of deck transformations $\pi_1(X)/N$ acts on X_N and thus on $H_2(X_N)$. On lifts of N-surfaces and their homology classes, this action is given by pre-composing the whisker with a loop in $\pi_1(X)$. Moreover, addition in $H_2(X_N)$ corresponds to a connected sum of N-surfaces along their whiskers.

LEMMA 7.2. *An N-surface f has a Wall self-intersection invariant*

$$\mu(f) \in \mathbb{Z}[\pi_1(X)/N]/\{a - \bar{a}\}.$$

Here there is the usual involution $a \mapsto \bar{a}$ on elements a in the group ring $\mathbb{Z}[\pi_1(X)/N]$ which is induced by $\bar{g} := g^{-1}$ for group elements.

If $\mu(f) = 0$ then the induced homology class $[f] \in H_2(X_N)$ is represented by an embedded *N-surface whose image can be chosen to be arbitrarily close to the image of f.*

Proof. To define $\mu(f)$ recall that by definition $f : F \looparrowright X$ only has transverse double points p. Choose two arcs in $f(F)$ leaving p on different sheets and ending at the base point of F, missing all other double points on their way. Together with the whisker for f this gives an element $g_p \in \pi_1(X)$, the so-called *double point loop* at p. Since there is no preferred order of the two sheets, we have to identify g_p with $g_p^{-1} = \bar{g}_p$.

Moreover, $f_*(\pi_1(F)) \leq N$ implies that the choice of the two arcs only changes g_p by elements in N. This implies that

$$\mu(f) := \sum_p \epsilon_p \cdot g_p$$

is well-defined in the quotient of the group ring above. Here $\epsilon_p \in \{\pm 1\}$ is the usual sign of the double point p, coming from the orientations of F and X.

Now assume that $\mu(f) = 0$. Then the double points of f can be paired up with signs and double point loops. Consider such a pair p, p' with $\epsilon_{p'} = -\epsilon_p$ and $g_{p'} = g_p^{\pm 1} \in \pi_1(X)/N$. Consider a loop α in $f(F)$ which leaves p on one sheet, changes sheets at p' and returns on the other sheet to p. There are two ways of making α into a loop at the base point of X, and by assumption (at least) one of them lies in the subgroup $N \leq \pi_1(X)$. Consider the corresponding subarc α_0 of α leading from p to p'. Let T be the normal bundle of f restricted to α_0. By construction, $f_*(\pi_1(F \cup T)) \leq N$ and we may use T to do surgery on f: Remove two small disks around p and p' (on the correct sheets) and replace them by the annulus which is the normal circle bundle corresponding to T. This procedure of *adding a tube* along α_0 removes the pair of double points p, p', stays in the same homology class in $H_2(X_N)$, and can be done arbitrarily close to the image of f. A finite number of such tube additions produces the desired embedded N-surface. $\square$

Let $w_2(f) \in \mathbb{Z}/2$ be the second Stiefel-Whitney number of the normal bundle of an N-surface f. Note the identities

$$w_2(f) = \langle f^* w_2(X), [F] \rangle = \langle f_N^* w_2(X_N), [F] \rangle = \langle w_2(X_N), [f] \rangle.$$

In particular, $w_2(f)$ only depends on the induced homology class $[f] \in H_2(X_N)$ and vanishes if X_N is a spin manifold.

The self-intersection invariant $\mu(f)$ is clearly unchanged under isotopies, finger moves and Whitney moves, i.e., under a regular homotopy of immersions. As usual, it is not invariant under an arbitrary homotopy: A local kink changes $\mu(f)$ by adding $\pm$ the trivial group element. This move also changes the Euler number $e(f)$ of the normal bundle of f by ± 2. Therefore, if $w_2(f) = 0 \in \mathbb{Z}/2$ thenthere is a well-defined number of kinks one has to have to make the normal bundle of f trivial. More precisely, we define a *homotopy invariant*

$$\mu(f) := \sum_p \epsilon_p \cdot g_p - \frac{e(f)}{2} \cdot 1$$

and we will use $\mu(f)$ in this sense in the rest of the paper.

> *We will also assume that all surfaces f with $w_2(f) = 0$ are represented by framed immersions.*

The same proof as for Lemma 7.2 above shows if that $\mu(f) = 0$ (in the modified definition) then $[f] \in H_2(X_N)$ is represented by a *framed, embedded* N-surface, i.e. an N-surface with trivial normal bundle. One just has to observe that the double point from a local kink can be removed by the following procedure (which stays within the class of N-surfaces and does not change the normal Euler number): A neighborhood around the double point p can be identified with $D^2 \times D^2$ with the two sheets of f being $D^2 \times 0$ and $0 \times D^2$. On the boundary of this 4-ball we see a Hopf-link. It bounds a twice-twisted band in S^3. If we cut out the local sheets around p and replace them with this annulus we get the desired result.

The following result is an exercise in Morse theory and will not be proved here. We will not need this result and only state it for the sake of completeness.

LEMMA 7.3. *If $w_2(f) = 0$ then the homotopy invariant $\mu(f)$ only depends on $[f] \in H_2(X_N)$. Moreover, $\mu(f) = 0$* if and only if *$[f]$ is represented by a framed embedded N-surface.*

We next turn from self-intersection to intersection numbers. Let f, g be N-surfaces meeting in general position. Then their (Wall) intersection number $\lambda(f,g) \in \mathbb{Z}[\pi_1(X)/N]$ is defined by the formula

$$\lambda(f,g) = \sum_p \epsilon_p \cdot g_p$$

where p runs through the intersection points of f and g. Signs ϵ_p and group elements g_p are defined similarly to the self-intersection invariant.

It is well-known that $\lambda(f,g)$ only depends on $[f],[g] \in H_2(X_N)$. In fact, under the isomorphism $H_2(X_N) \cong H_2(X; \mathbb{Z}[\pi_1(X)/N])$ the pairing λ corresponds to the composition of the following three obvious maps (where $\Lambda := \mathbb{Z}[\pi_1(X)/N]$):

$$H_2(X;\Lambda) \longrightarrow H_2(X,\partial;\Lambda) \xrightarrow{\cong} H^2(X;\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(H_2(X;\Lambda),\Lambda).$$

LEMMA 7.4. *Let f, g be N-surfaces with*

$$\lambda(f,g) = \sum_{i=1}^r \epsilon_i \cdot g_i.$$

Then there exists an N-surface f' with $[f'] = [f] \in H_2(X_N)$ which intersects g geometrically in exactly r points p_i with group elements $g_{p_i} = g_i$ and signs ϵ_i for $i = 1, \dots, r$.

The proof of this lemma proceeds exactly as the proof of Lemma 7.2: Algebraically, canceling pairs of intersection points can be removed by adding tubes to f. We leave the details to the reader. The algebraic properties of the intersection invariants λ, μ on $H_2(X_N)$ can be summarized as follows.

Definition 7.5. Let R be a ring with involution and M a left R-module. A *quadratic form* on M consists of a $\mathbb{Z}$-bilinear map $\lambda : M \times M \longrightarrow R$ together with a map $\mu : M \longrightarrow \overline{R} := R/\{r - \bar{r} \mid r \in R\}$ satisfying the following properties:

1. $\lambda(a,b) = \overline{\lambda(b,a)}$.
2. $\lambda(r \cdot a, b) = r \cdot \lambda(a,b)$ for all $r \in R$.
3. $\mu(r \cdot a) = r \cdot \mu(a) \cdot \bar{r} \in \overline{R}$ for all $r \in R$.
4. $\mu(a+b) = \mu(a) + \mu(b) + \lambda(a,b) \in \overline{R}$.
5. $\lambda(a,a) = \mu(a) + \overline{\mu(a)} \in R$.

One also calls λ a *hermitian form* on M and μ its *quadratic refinement.* The form is *nonsingular* if the homomorphism $M \to \mathrm{Hom}_R(M,R)$ induced by λ is an isomorphism. It is *non-degenerate* if this map is a monomorphism.

The algebraic intersection and self-intersection numbers λ, μ above define a quadratic form on the module $\mathrm{Ker}\{w_2 : H_2(X_N) \longrightarrow \mathbb{Z}/2\}$ over the ring $\mathbb{Z}[\pi_1(X)/N]$. In general, this form may be degenerate.

A simple example of a nonsingular quadratic form is the hyperbolic form on a free R-module of rank $2g$: On a basis $e_1, \dots, e_g, f_1, \dots, f_g$ one has by definition

$$\lambda(e_i, f_j) = \delta_{i,j} \text{ and } \lambda(e_i, e_j) = 0 = \lambda(f_i, f_j) \text{ and } \mu(e_i) = 0 = \mu(f_i).$$

It follows from Lemma 7.4 that if one has classes $e_i, f_j \in \mathrm{Ker}\{w_2 : H_2(X_N) \longrightarrow \mathbb{Z}/2\}$ satisfying the above equations, then there are disjointly embedded N-surfaces E_i (with trivial normal bundle) representing e_i and disjointly embedded N-surfaces F_i representing f_i. Moreover, E_i and F_i intersect (transversely) in exactly one point, whereas for $i \neq j$ the surfaces E_i and F_j are disjoint.

Remark 7.6. Let λ, μ be a nonsingular quadratic form on a free R-module which has a Lagrangian. More precisely, there is a half-basis $e_1, \dots, e_g$ satisfying

$$\lambda(e_i, e_j) = 0 \text{ and } \mu(e_i) = 0.$$

Then this quadratic form is hyperbolic. The proof is by induction on g and proceeds as follows: Since λ is nonsingular and e_i are basis vectors, there exists a vector f_1' such that $\lambda(e_i, f_1') = \delta_{i,1}$. Define

$$f_1 := f_1' - \mu(f_1') \cdot e_1.$$

Then we still have $\lambda(e_i, f_1) = \delta_{i,1}$ but in addition $\mu(f_1) = 0$. This implies that the sub-form $\langle e_1, f_1 \rangle$ is hyperbolic with $e_2, \dots, e_g$ lying in the orthogonal complement. By induction, the form λ, μ is hyperbolic, too.

We should also remark that a quadratic form with a Lagrangian $\langle e_1, \dots, e_g \rangle$ is nonsingular if and only if the vectors e_i have *duals*. That is to say, there are vectors $d_1, \dots, d_g$ such that $\lambda(e_i, d_j) = \delta_{i,j}$. The above remark shows how to improve the d_i (by summing with linear combinations of e_j) to a hyperbolic basis f_i.

We finish this section by explaining Whitney towers and gropes.

Definition 7.7. Let γ be a framed circle in the boundary M of a 4-manifold W. A *Whitney tower of height* 1 is an immersed disk Δ in W which bounds γ and such that the unique framing on the normal bundle of Δ restricts to the given framing on γ. If the double points of Δ can be paired up (with signs and double point loops) then the choice of Whitney circles enables one to iterate the construction. Recall that a Whitney circle is framed by a vector field which is tangent along one sheet and normal along the other. By convention, a Whitney disk (which bounds a Whitney circle) is allowed to have (transverse) double points but it is always assumed to be *framed* in the sense that the above vector field on the Whitney circle extends to a nonvanishing normal vector field on the Whitney disk (see [FQ, p.17]).

For $n \in \mathbb{N}$, a *Whitney tower of height* n on γ is a sequence $\mathcal{C}_j = \{\Delta_{j,k}\}_k$, $j = 1, \dots n$, of collections of Whitney disks $\Delta_{j,k}$ in general position (where $\mathcal{C}_1$ is the Whitney disk with boundary γ) with the following property:

- For $j = 2, \dots, n$ the collection $\mathcal{C}_j$ pairs up all $\mathcal{C}_{j-1}$-(self)-intersections and has interiors disjoint from $\mathcal{C}_1, \dots, \mathcal{C}_{j-1}$.

A Whitney tower of height $(n.5)$ has an additional collection $\mathcal{C}_{n+1}$ of framed immersed Whitney disks such that

- $\mathcal{C}_{n+1}$ pairs up all $\mathcal{C}_n$-(self)-intersections and has interiors disjoint from $\mathcal{C}_1, \dots, \mathcal{C}_{n-1}$ (but $\mathcal{C}_{n+1}$ is allowed to intersect the previous collection $\mathcal{C}_n$).

Finally, we define the notion of a Whitney tower in a slightly different situation: a *Whitney tower of height* 0 is a collection $\mathcal{C}_0$ of 2-spheres $S_i \looparrowright W^4$. For $n \in \mathbb{N}$, a *Whitney tower of height n on* $\mathcal{C}_0$ is a sequence $\mathcal{C}_j = \{\Delta_{j,k}\}_k$, $j = 1, \dots n$, of collections of framed immersed Whitney disks $\Delta_{j,k}$ in general position with the following property:

- For $j = 1, \dots, n$ the collection $\mathcal{C}_j$ pairs up all $\mathcal{C}_{j-1}$-(self)-intersections and has interiors disjoint from $\mathcal{C}_0, \dots, \mathcal{C}_{j-1}$.

A Whitney tower of height $(n.5)$ has an additional collection $\mathcal{C}_{n+1}$ of framed immersed Whitney disks such that

- $\mathcal{C}_{n+1}$ pairs up all $\mathcal{C}_n$-(self)-intersections and has interiors disjoint from $\mathcal{C}_0, \dots, \mathcal{C}_{n-1}$ (but $\mathcal{C}_{n+1}$ is allowed to intersect the previous collection $\mathcal{C}_n$).

Remark 7.8. By definition, a Whitney tower of height (0.5) on $\mathcal{C}_0$ exists if and only if the algebraic (self)-intersection numbers λ and μ vanish on the 2-spheres S_i.

The following definition and lemma are taken from [FT].

Definition 7.9. A *grope* is a special pair (2-complex, base circle). A grope has a *height* $n \in \mathbb{N}$. For $n = 1$ a grope is precisely a compact oriented surface Σ with a single boundary component which is the base circle. A grope of height $(n+1)$ is defined inductively as follows: Let $\{\alpha_i, i = 1, \dots, 2g\}$ be a standard symplectic basis of circles for Σ, the *bottom* stage of the grope. Then a grope of height $(n+1)$ is formed by attaching gropes of height n to each α_i along the base circles. Finally, a grope of height $(n.5)$, $n \in \mathbb{N}$, has a bottom surface Σ which on one half basis of curves bounds gropes of height $(n-1)$ and on the dual half basis of curves bounds gropes of height n.

Thus a grope of height n has n surface stages and its fundamental group is freely generated by the circles of the symplectic basis for all the surfaces in the top stage. For example, if all the surfaces in the grope have genus 1 then there are $2^{(n-1)}$ top stage surfaces each giving 2 free generators.

LEMMA 7.10. *For a space X, a loop γ lies in $\pi_1(X)^{(n)}$ if and only if γ bounds a map of a grope of height n (i.e. γ becomes the base circle of that grope). Moreover, the height of a grope (g, γ) is the maximal $n \in \mathbb{N}$ such that $\gamma \in \pi_1(g)^{(n)}$.*

As one can see from Figure 1.1 every grope (g, γ) embeds properly (i.e. boundary goes to boundary) into $\left(\mathbb{R}^3_+, \mathbb{R}^2 \times \{0\}\right)$ mapping γ to the unit circle in $\mathbb{R}^2$. This determines a framing of the grope or an "untwisted" thickening. Restricted to each surface stage this framing is a nonvanishing normal vector field which on the boundary restricts to a vector field tangent to the lower surface stage. In particular, the framing does not depend on the embedding into $\mathbb{R}^3$.

Given a 4-manifold W with boundary M and a framed circle γ in M, we say that γ bounds a *grope* in W if γ extends to an embedding of a grope with its untwisted framing. Knots in S^3 always are equipped with the linking number zero framing.

8. H_1-bordisms

We fix a closed oriented 3-manifold M and consider the following class of 4-manifolds.

Definition 8.1. An H_1*-bordism* is a 4-dimensional spin manifold W with boundary M such that the inclusion map induces an isomorphism $H_1(M) \cong H_1(W)$.

Note that any spin structure on M extends to a spin structure on an H_1-bordism W because the affine spaces of spin structures are isomorphic via the isomorphism $H^1(W, \mathbb{Z}/2) \cong H^1(M, \mathbb{Z}/2)$.

Remark 8.2. If M is the 0-surgery on a knot K in S^3 then an H_1-bordism exists if and only if the Arf invariant of K vanishes. This fact is well-known and follows from the computation of the bordism group $\Omega_3^{\text{spin}}(S^1) \cong \Omega_2^{\text{spin}} \cong \mathbb{Z}/2$.

Recall that $W^{(n)}$ denotes the regular covering of W which corresponds to the n^{th} term $\pi_1(W)^{(n)}$ of the derived series of $\pi_1(W)$. An (n)*-surface* is by definition a $\pi_1(W)^{(n)}$-surface in the sense of Definition 7.1. In Section 7, we explained the quadratic form λ_n, μ_n on $H_2(W^{(n)})$ in terms of intersection and self-intersection numbers of (n)-surfaces in W.

Definition 8.3. Let W be an H_1-bordism such that λ_0 is a hyperbolic form.

1. A *Lagrangian* for λ_0 is a direct summand of $H_2(W)$ of half rank on which λ_0 vanishes.
2. An (n)*-Lagrangian* is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of the hyperbolic form λ_0 on $H_2(W)$.
3. A *spherical Lagrangian* is a submodule $L \subset \pi_2(W)$ on which λ, μ vanish and which maps onto a Lagrangian of λ_0.
4. Let $k \leq n$. We say that an (n)-Lagrangian L admits (k)*-duals* if L is generated by (n)-surfaces $\ell_1, \dots, \ell_g$ and there are (k)-surfaces $d_1, \dots, d_g$ such that $H_2(W)$ has rank $2g$ and
$$\lambda_k(\ell_i, d_j) = \delta_{i,j}.$$
Similarly, *spherical duals* for L are classes $d_1, \dots, d_g \in \pi_2(W)$ satisfying the above equation for $k = n$.

THEOREM 8.4. *Let M be a closed oriented* 3*-manifold and $n \in \mathbb{N}_0$. Then the following statements are equivalent: There is an H_1-bordism ...*

(i) *... which contains an $(n+1)$-Lagrangian with (n)-duals.*

(ii) *... which contains a spherical Lagrangian with (n)-duals.*

(iii) *... which contains a spherical Lagrangian admitting a Whitney tower of height $(r.5)$ and with $(n-r)$-duals for some $r \in \{0, \dots, n\}$.*

(iv) *... which contains a spherical Lagrangian admitting a Whitney tower of height $(n.5)$.*

Definition 8.5. The 3-manifold M is called $(n.5)$*-solvable* if the conditions above are satisfied. If M is the 0-surgery on a knot or a link then the corresponding knot or link is called $(n.5)$-solvable (and the link has trivial linking numbers, so that H_1 is a free abelian group on the number of components of the link).

This agrees with the definition given in the introduction.

Remark 8.6. It is clear that this notion is invariant under homology cobordisms. More precisely, assume that M and M' form the boundary of a 4-manifold W such that the two inclusions induce isomorphisms on H_*. Then M is $(n.5)$-solvable if and only if M' is $(n.5)$-solvable. For the proof one glues together the obvious 4-manifolds.

Proof of Theorem 8.4. (ii) $\Rightarrow$ (i) is trivially true.

(i) $\Rightarrow$ (ii) By Lemma 7.4 we may assume that we have disjointly embedded framed $(n+1)$-surfaces $\ell_1, \dots, \ell_g$. Moreover, the *geometric* intersections with the (n)-duals $d_1, \dots, d_g$ are $\delta_{i,j}$ and the duals d_i may be assumed to have trivial normal bundle since W is spin. Consider a *standard collection* of simple closed curves $\alpha_{r,s}$ on ℓ_s. By definition, these are simple closed curves which represent a basis of $H_1(\ell_s)$ such that the algebraic and geometric (self)-intersections agree. By assumption, there are (n)-surfaces $A_{r,s}$ whose boundaries are the curves $\alpha_{r,s}$. Note that the orientations of the curves $\alpha_{r,s}$ give a nonvanishing vector field in the normal bundle of $\alpha_{r,s}$ in ℓ_s. After some boundary twists (see [FQ, p.16]) we may assume that this vector field extends to a nonvanishing vector field for the normal bundle of $A_{r,s}$ in W. In this case we may refer to the surfaces $A_{r,s}$ as *framed*. By tubing into the (n)-duals d_i we may achieve that the interiors of the $A_{r,s}$ are disjoint from all ℓ_j. This preserves the framing on the $A_{r,s}$.

Now consider tangential push-offs $\alpha'_{r,s} \subset A_{r,s}$ of $\alpha_{r,s}$. These circles have a normal 2-frame on them, one vector field pointing into $A_{r,s}$, the other being the nonvanishing normal vector field on $A_{r,s}$ restricted to $\alpha'_{r,s}$. We do surgery on all $\alpha'_{r,s}$ such that the 2-frames extend over the new 2-disks $b_{r,s}$. More precisely, we cut out small neighborhoods of $\alpha'_{r,s}$ homeomorphic to $S^1 \times D^3$ (disjoint from d_i and ℓ_j) and add copies of $D^2 \times S^2$ using the 2-frames to identify the boundaries $S^1 \times S^2$. Denote by $S_{r,s}$ the disjointly embedded framed 2-spheres $0 \times S^2$. Every surgery changes $H_2(W)$ by the orthogonal sum with a hyperbolic form on $S_{r,s}$ and

$$B_{r,s} := A_{r,s} \cup_{\alpha'_{r,s}} b_{r,s}.$$

Denoting by W' the result of all these surgeries, we see that it still is an H_1-bordism. Moreover, we claim that W' has a spherical Lagrangian: We may use two parallels of the disks $b_{r,s}$ to do symmetric surgery on the $(n+1)$-surfaces ℓ_s. This operation is also called a *contraction* in [FQ, p.34]. Call the resulting disjointly embedded 2-spheres $L_1, \dots, L_g$. Then the collection of 2-spheres $L_j, S_{r,s}$ form a spherical Lagrangian because the only geometric intersections among these 2-spheres are two points of intersection between L_s and $S_{r,s}$ for each $s = 1, \dots, g$ and each r. But these intersections are algebraically trivial because they can be paired up by small ribbon Whitney disks (see the figure in [FQ, p.35]). By construction, the (n)-surfaces $d_j, B_{r,s}$ form (geometric) duals for these 2-spheres and it is clear that they together generate $H_2(W')$.

Note that statement (ii) is the case $r = 0$ and statement (iv) is the case $r = n$ in statement (iii). Therefore, to prove the equivalence of (ii), (iii) and (iv), it suffices to prove two induction steps for statement (iii), one increasing r, the other decreasing r.

The induction step $r \mapsto r-1$. Applying Lemma 7.4 to the $(n-r)$-duals $d_1, \dots, d_g$ we may assume that their *geometric* intersection with the framed immersed 2-spheres $\ell_1, \dots, \ell_g$ is $\delta_{i,j}$. In fact, by pushing down intersections between d_i and Whitney disks in the tower (introducing many algebraically canceling pairs of intersections between d_i and ℓ_j) we may assume that each d_i intersects the whole tower in a single point. Let α_k be parallels of the bottom stage Whitney circles such that α_k lie on the interior of the Whitney disks Δ_k of the first collection $\mathcal{C}_1$ in the Whitney tower. Picking one of the two double points that correspond to Δ_k, we get Clifford tori T_k that are disjoint from all d_i and ℓ_j and intersect the Whitney tower in exactly one point an Δ_k. Both standard circles on T_k are by definition the meridians of the sheets that are intersecting at that point. By construction, these sheets have $(n-r)$-duals in $W \smallsetminus \cup_j \ell_j$ and thus the T_k are disjoint $(n-r+1)$-surfaces in this 4-manifold. We now do surgeries on the curves α_k. As above these produce disks b_k which are useful in two respects: They can be used to do

Whitney moves of ℓ_j which make these disjointly embedded spheres which can be thus surgered away. Call the resulting 4-manifold W'. Then the unions $\Delta_k \cup_{\alpha_k} b_k$ form a spherical Lagrangian in W' which admits a Whitney tower of height $(r-0.5)$ (formed from the upper stages of the original Whitney tower). Moreover, these 2-spheres have $(n-r+1)$-duals T_k. Note that the T_k form an $(n-r+1)$-Lagrangian.

The induction step $r \mapsto r+1$. By Remark 7.6 we may assume that the $(n-r)$-duals satisfy $\lambda(d_i, d_j) = \mu(d_i) = 0$. More precisely, this involves summing the original $(n-r)$-duals with combinations of the framed 2-spheres ℓ_j. This preserves the property that the d_i are $(n-r)$-surfaces (and also the property $\lambda(\ell_i, d_j) = \delta_{i,j}$). Applying Lemma 7.4 to the new $(n-r)$-duals $d_1, \dots, d_g$ we may assume that each d_i intersects the Whitney tower in exactly one point on ℓ_i and that the d_i are represented by disjointly embedded framed $(n-r)$-surfaces. Let $\alpha_{r,s}$ be a standard collection of simple closed curves for d_s. By assumption, there are $(n-r-1)$-surfaces $A_{r,s}$ with boundary $\alpha_{r,s}$. As in the proof for (i) $\Rightarrow$ (ii) we can arrange that the $A_{r,s}$ are framed and have interiors disjoint from d_i. We again do surgeries on tangential push-offs $\alpha'_{r,s} \subset A_{r,s}$ of $\alpha_{r,s}$. Then we do symmetric surgery on the d_i to obtain disjointly embedded framed 2-spheres $D_1, \dots, D_g$. As before there are disjoint 2-spheres $S_{r,s}$ resulting from each surgery. They have geometric $(n-r-1)$-duals $B_{r,s}$ made by closing off the $A_{r,s}$ with the cores of the 2-disks attached. Recall that the intersections between D_g and $S_{r,s}$ are paired up by ribbon Whitney disks. This time we actually do the corresponding Whitney moves to make D_j disjoint from $S_{r,s}$ (and keep them disjoint from $B_{r,s}$). The cost of these last Whitney moves is that the 2-spheres $S_{r,s}$ now intersect in pairs, corresponding to the intersections $\alpha_{r,s} \cap \alpha_{r',s}$. But these intersections again occur in pairs with disjointly embedded Whitney disks $\Delta_{k,s}$ (see Figure 8.1). Each Whitney disk $\Delta_{k,s}$ intersects the contraction D_s in a single point (on the central square) which we remove by summing into the original Whitney tower (which is dual to d_i and hence to D_i). Finally, we do surgery on the 2-spheres $D_1, \dots, D_g$ to obtain our 4-manifold W'. By construction, $H_2(W')$ is generated by the 2-spheres $S_{r,s}$, which admit a Whitney tower of height $(r+1.5)$, and their $(n-r-1)$-duals $B_{r,s}$. $\square$

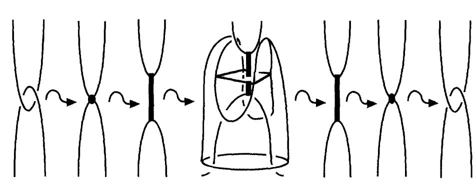


Figure 8.1. $\Delta_{k,s}$ is the union of the thick arcs in this display.

Looking back at the four statements in the definition of $(n.5)$-solvable 3-manifolds, we see that there is an obvious candidate for what an (n)-solvable 3-manifold should be.

Definition 8.7. A 3-manifold M is (0)-solvable if it bounds an H_1-bordism W such that $(H_2(W), \lambda_0)$ is hyperbolic. A 3-manifold M is (n)-solvable, $n > 0$, if any of the conditions of Theorem 8.8 below are satisfied. A link is (n)-solvable if 0-surgery on the link is an (n)-solvable 3-manifold.

THEOREM 8.8. *Let M be a closed oriented* 3-*manifold and $n \in \mathbb{N}$. Then the following statements are equivalent*: *There is an H_1-bordism ...*

(i) ... *which contains an (n)-Lagrangian with (n)-duals.*

(ii) ... *which contains an (n)-Lagrangian with spherical duals.*

(iii) ... *which contains a spherical Lagrangian admitting a Whitney tower of height (r) and with $(n-r)$-duals for some $r \in \{1, \ldots, n\}$.*

(iv) ... *which contains a spherical Lagrangian admitting a Whitney tower of height (n).*

Proof. The arguments that (ii), (iii) and (iv) are equivalent are exactly as in Theorem 8.4. One only needs to make sure that statement (ii) is really equivalent to the $r = 1$ case of statement (iii). In one direction one uses the (n)-Lagrangian from (ii) as the starting point in the induction step $r \Rightarrow r+1$ in the proof of Theorem 8.4 (there one actually turns (n)-duals into an (n)-Lagrangian first, which is not needed here). The output is exactly the $r = 1$ case of statement (iii). Conversely, we already observed at the end of the induction step $r \Rightarrow r-1$ in the proof of Theorem 8.4 that the Clifford tori T_k form an (n)-Lagrangian if one begins with $(n-1)$-duals. Thus this step gives exactly (ii).

(i) *implies* (ii). Let $\ell_1, \ldots, \ell_g$ be an (n)-Lagrangian in W with (n)-duals $d_1, \ldots, d_g$. By Remark 7.6 and Lemma 7.4 we may assume that all ℓ_i, d_j are represented by framed embeddings and that the only geometric intersections among these (n)-surfaces are single points of intersections $p_i = \ell_i \cap d_i$ for $i = 1, \ldots, g$. Now this is a perfectly symmetric setup and thus we also do a symmetric construction. We do abstract surgery on standard collections of simple closed curves $\alpha_{r,s}$ on ℓ_i *and* d_j. Then we contract to get a geometrically hyperbolic collection of 2-spheres $L_1, \ldots, L_g, D_1, \ldots, D_g$. We push the 2-spheres $S_{r,s}$ off the contraction, introducing pairs of double points with Whitney disks which intersect L_i or D_j in a single point. We remove this point by summing into the dual 2-sphere D_i respectively L_j. This introduces many intersections among the Whitney disks which will not be relevant. Finally, we do surgery on the 2-spheres (say) $L_1, \ldots, L_g$ to obtain a 4-manifold W'. It contains a spherical Lagrangian $S_{r,s}$ with Whitney disks disjoint from these spheres. Therefore, we have actually constructed a Whitney tower of height 1. As discussed in the proof of Theorem 8.4 the $S_{r,s}$ have geometric $(n-1)$-duals $B_{r,s}$ (using the fact that we started out with (n)-surfaces). We have thus shown that statement (i) implies statement (iii) with $r = 1$. But this is equivalent to statement (ii). □

We next show that there are many (h)-solvable knots.

THEOREM 8.9. *If there exists an (h)-solvable link L which forms a standard half basis of untwisted curves on a Seifert surface F for a knot K, then K is $(h+1)$-solvable.*

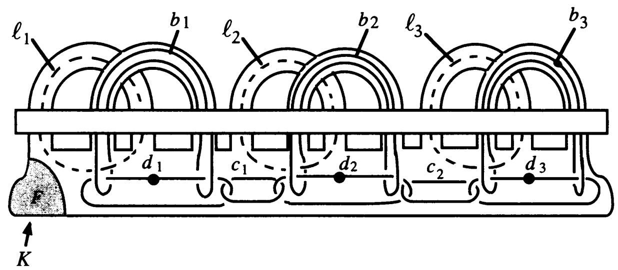


Figure 8.2. The cobordism C.

Proof. In Figure 8.2 we have drawn the Seifert surface F in the case of genus $g = 3$. It shows a box containing an arbitrary string link of (possibly twisted) bands for F. The dashed lines ℓ_i denote the link L whose meridians are called m_i. In addition, we drew g dotted unlinked circles d_i (whose meridians we call t_i) and g solid circles b_i going around the dual bands to L. Finally, there are solid circles $c_1, \ldots, c_{g-1}$ connecting pairs of dotted circles. The figure determines a 4-manifold C in the following way: Start with the lower boundary $\partial_- C = \mathcal{S}^0 L$, which is also the boundary of a 0-handle together with 0-framed 2-handles on the ℓ_i. To $\partial_- C \times I$ we attach g 1-handles corresponding to the dotted circles and $(2g-1)$ 0-framed 2-handles along b_i and c_j. Thus the relative chain-complex $C_*(C, \partial_- C)$ has only terms for $* = 1, 2$ and the boundary map

$$\mathbb{Z}^{2g-1} \cong \langle b_i, c_j \rangle = C_2 \xrightarrow{\partial} C_1 = \langle t_i \rangle \cong \mathbb{Z}^g$$

satisfies $\partial(b_i) = 0, \partial(c_j) = t_{j+1} - t_j$. Therefore, $H_1(C, \partial_- C) \cong \mathbb{Z}$ is generated by any of the t_j and there is an isomorphism

$$H_2(C, \partial_- C) = H_2(C, \mathcal{S}^0 L) \xrightarrow{\cong} H_1(\mathcal{S}^0 L) = \langle m_i \rangle \cong \mathbb{Z}^g.$$

The upper boundary $\partial_+ C$ is given by 0-framed surgery on all the circles in Figure 8.2, i. e. the ℓ_i, b_i, c_j, d_i. In Lemma 8.10 below we show that $\partial_+ C \cong$

$\mathcal{S}^0K$. But already from the figure above it follows that the inclusion induced map $H_1(\partial_+C) \longrightarrow H_1(C, \partial_-C)$ is an isomorphism.

Summarizing the above construction, we have a cobordism C between $\mathcal{S}^0L$ and $\mathcal{S}^0K$ which has the following properties

1. $\partial_- : H_2(C, \mathcal{S}^0L) \longrightarrow H_1(\mathcal{S}^0L)$ is an isomorphism.

2. $i_+ : H_1(\mathcal{S}^0K) \longrightarrow H_1(C, \mathcal{S}^0L)$ is an isomorphism.

Now recall that L is (h)-solvable and let V be the H_1-bordism for $\mathcal{S}^0L$ which contains a (k)-Lagrangian with (n)-duals. Here $k = n$ if $h = n \in \mathbb{N}$ and $k = n+1$ if $h = n.5$ (see Definition 8.3). Define $W := V \cup_{\mathcal{S}^0L} C$ which is a 4-manifold with boundary $\mathcal{S}^0K$. Consider the long exact sequence for the pair (W, V), noticing that by excision $H_*(W, V) \cong H_*(C, \mathcal{S}^0L)$.

$$0 \to H_2V \to H_2W \to H_2(C, \mathcal{S}^0L) \to H_1V \to H_1W \to H_1(C, \mathcal{S}^0L) \to 0.$$

By assumption, $H_1(\mathcal{S}^0L) \cong H_1V$ and therefore the boundary-map $H_2(C, \mathcal{S}^0L) \longrightarrow H_1V$ is an isomorphism by 1. above. This implies that we have isomorphisms.

$$H_2V \cong H_2W \text{ and } H_1W \cong H_1(C, \mathcal{S}^0L).$$

By 2. above this shows that $H_1(\mathcal{S}^0K) \cong H_1W$. Since $H_1V \longrightarrow H_1W$ is the zero map any (r)-surface in V is actually an $(r+1)$-surface when considered in W. Therefore, we actually have a $(k+1)$-Lagrangian with $(n+1)$-duals in W, using the isomorphism $H_2V \cong H_2W$. But this shows that K is $(h+1)$-solvable. $\square$

LEMMA 8.10. *In the above setting, there is a diffeomorphism* $\partial_+C \cong \mathcal{S}^0K$.

Proof. We first isotope Figure 8.2 into the position of Figure 8.3, keeping the dashed, solid and dotted convention even though all circles are considered as 0-framed 2-handles. Now we slide each d_i twice over its partner ℓ_i leading to the handle diagram in which b_i are geometric duals for ℓ_i. Thus we may cancel these handles in pairs, effectively erasing them from the diagram (see Figure 8.4). But now it is clear that $(g-1)$ more cancellations involving the c_j lead to $\mathcal{S}^0K$, by the fact that the ℓ_i were untwisted. $\square$

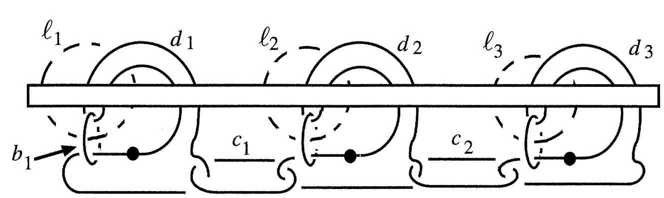


Figure 8.3. A 0-framed handle decomposition for $\mathcal{S}^0K$.

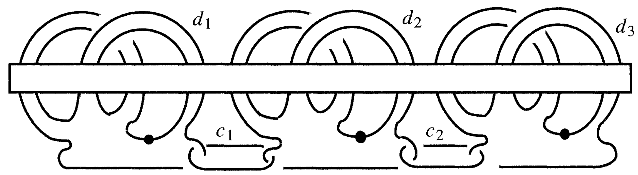


Figure 8.4. Before the last $(g-1)$ cancellations.

We next show that the slightly abstract notion of (h)-solvability is implied by very concrete geometric conditions in the 4-ball. The first condition is in terms of gropes and the second in terms of Whitney towers.

THEOREM 8.11. *If a knot K bounds a grope of height $(h+2)$ in D^4 then K is (h)-solvable.*

Proof. Let $\alpha_1, \ldots, \alpha_{2g}$ be a standard collection of simple closed curves on the bottom stage F of the grope G of height $(n+2)$ which bounds our knot K. (Note that in this proof the α_i form a *full* basis of curves rather than just a half-basis as in the previous proof.) As in Theorem 8.4 we do surgery on tangential push-offs $\alpha_i' \subset A_i$ of α_i, where A_i are the second surface stages of our grope G. As before this leads to 2-disks b_i, 2-spheres S_i and (n)-duals $B_i = A_i \cup b_i$. Again we use two parallels of the b_i to do symmetric surgery on F to obtain a disk D bounding our knot K. Finally, we push the S_i off the contraction D and remove the interior of a thickening of D to obtain a 4-manifold W. By construction, ∂W is 0-surgery on K and $H_2(W)$ is freely generated by S_i and their geometric (n)-duals B_i. Therefore, W satisfies condition (ii) of Theorem 8.8 and we are done for gropes of integral height.

If the height of the grope is a half integer $h = n + 2.5$ then we may pick a half basis $\alpha_1, \ldots, \alpha_g$ which bounds gropes of height $(n+2)$ and such that the dual half basis $\alpha_{g+1}, \ldots, \alpha_{2g}$ bounds gropes of height $(n+1)$. Then the above construction gives a 4-manifold W with framed embedded 2-spheres $S_1, \ldots, S_{2g}$ with geometric $(n+1)$-duals $B_1, \ldots, B_g$, respectively (n)-duals $B_{g+1}, \ldots, B_{2g}$. By construction, the surfaces B_i are framed and embedded disjointly. The deficiencies in this family of surfaces are pairs of intersection points between S_i and S_{i+g} coming from pushing these spheres off the contraction D. However, we may remove all of these intersections by tubing each S_{i+g} twice into parallel copies of B_i for each $i = 1, \ldots, g$. Since these B_i are $(n+1)$-surfaces, we obtain an $(n+1)$-Lagrangian $S_1, \ldots, S_g, S'_{g+1}, \ldots, S'_{2g}$ with (n)-duals $B_1, \ldots, B_{2g}$. Thus condition (i) from Theorem 8.4 is satisfied. □

THEOREM 8.12. *If a knot K bounds a Whitney tower of height $(h+2)$ in D^4 then K is (h)-solvable.*

Proof. Let T be a Whitney tower of height $(h+2)$ in D^4 which bounds our knot K. Consider the Whitney circles α_i for the immersed 2-disk Δ that bounds the knot K. We do surgery on tangential push-offs $\alpha_i' \subset A_i$ of α_i, where A_i denote the next stages of the Whitney tower T. This leads to external 2-disks b_i which can be used in two ways. We may do Whitney moves along b_i to change Δ into an embedded 2-disk D giving us a 4-manifold W which is the surgered D^4 minus an open neighborhood of D. Then $\partial W = \mathcal{S}^0 K$ and $H_2 W$ has a Lagrangian which is generated by 2-spheres $S_i := A_i \cup b_i$ in W which allow a Whitney tower of height h, formed from the upper stages of the Whitney tower T. □

The last proof still owed is of the fact that a knot is algebraically slice if and only if it bounds a grope of height 2.5 in D^4 (see Theorem 1.1). Recall from Remark 1.3 that a knot K is algebraically slice if and only if it is (0.5)-solvable. So by Theorem 8.11 it suffices to prove the following result.

THEOREM 8.13. *If a knot K is algebraically slice then it bounds a grope of height* 2.5 *in D^4.*

Proof. Using the Levine condition from Theorem 1.1, we may start with a Seifert surface F for K in S^3 with a half-basis of curves $\alpha_1, \dots, \alpha_g$ with vanishing linking numbers. This implies the existence of disjointly embedded framed surfaces A_i in D^4 with $\partial A_i = \alpha_i$. Let $\beta_1, \dots, \beta_g$ be a dual half-basis of curves on F. By a base change as in Remark 7.6 we may assume that the self-linking numbers of all b_j are even. Then there are framed embedded surfaces B_j in D^4 with $\partial B_j = \beta_j$.

Let $\gamma_{r,s}$ be a full basis of curves on A_s. Then, after some boundary twists, there are framed embedded surfaces $G_{r,s}$ in the interior of D^4 with $\partial G_{r,s} = \gamma_{r,s}$. We may assume that A_i, B_j and $G_{r,s}$ only intersect in isolated points away from their boundaries. Note also that all these surfaces have interiors disjoint from the Seifert surface F. We now push the Seifert surface slightly into D^4, or more precisely, we add a small collar $S_3 \times I$ to D^4 with an annulus which connects the original knot K to the new boundary S^3. In any case, we now see that K bounds a framed grope of height 2.5 in D^4 whose bottom stage F is disjoint from all other surfaces stages. We shall show next that such a grope can be improved to a framed embedded grope of the same height (then thickening this framed grope leads to the desired grope). The first step is to push down all intersections among $G_{r,s}$ and with B_j into A_s (see [FQ, §2.5]). This makes the (interiors of) $G_{r,s}$ disjoint, and also disjoint from B_j. Let T_{α_j} denote the 2-tori which are the normal circle bundles to F restricted to α_j. They are disjointly embedded, framed and may be assumed to be disjoint from all other surfaces, except a single point of intersection $T_{\alpha_j} \cap B_j$. But this means that we may use tubes into the T_{α_j} to remove all intersections among the B_i. Note that this increases the genus of each B_i but we do not care.

Finally, consider normal tori T_{β_i} to β_i. Again we may assume that they are disjointly embedded, framed and disjoint from everything else (including the T_{β_i}), except a single intersection point $T_{\beta_i} \cap A_i$. Thus tubing into T_{β_i} removes the last intersections, namely those between B_j, respectively $G_{r,s}$ and A_i. Again this procedure increases the genera of the surfaces B_j and $G_{r,s}$ but since they form the top stage of the grope, this is irrelevant. $\square$

Remark 8.14. The above procedure can be used to show that any knot with trivial Arf invariant bounds a framed embedded grope of height 2 in D^4. The difficulty in increasing the height by 0.5 lies in the fact that if one has to use the tori T_{β_i} to remove intersections *among* the A_i then one cannot find the next stage surfaces $G_{r,s}$: One of the curves on each T_{β_i} is by construction the meridian to the pushed-in Seifert surface F and is therefore not null-homologous in $D^4 \smallsetminus F$.

9. Casson-Gordon invariants and solvability of knots

In this section we review the Casson-Gordon invariants, and show they vanish on (1.5)-solvable knots. Throughout this section, all chain and cochain complexes are cellular, with the cellular structure obtained from lifting a cellular structure on the base.

Seifert pairings, linking pairings and (.5)-*solvability.* We recall the definition of the Seifert pairing and the classical knot slicing obstructions due to Levine [L1]. Let $F \subset S^3$ be a Seifert surface for the knot K. The Seifert pairing on $H_1(F)$ is defined by

$$\theta(x,y) = \ell k(\iota_+ x, y)$$

where ℓk is the usual linking in S^3 and ι_+ is the positive push-offs in the normal direction from F. Following Kervaire [K] and Stoltzfus [Sto], we define an isometric structure

$$s : H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$$

by the equation $\theta(x,y) = \langle sx, y\rangle_F$ for all $y \in H_1(F;\mathbb{Z})$, where $\langle\ ,\ \rangle_F$ is the intersection pairing on $H_1(F;\mathbb{Z})$.

Definition 9.1. A metabolizer for the isometric structure on $H_1(F;\mathbb{Z})$ is an s-invariant direct summand $H \subset H_1(F;\mathbb{Z})$ such that

$$H = H^{\perp} = \{y \in H_1(F;\mathbb{Z}) \mid \langle x,y\rangle_F = 0 \text{ for all x } \in H\}.$$

Levine shows [L1] that if K is slice there exists a summand $H \subset H_1(F;\mathbb{Z})$ such that $rk_{\mathbb{Z}}(H) = \frac{1}{2} rk_{\mathbb{Z}}(H_1(F;\mathbb{Z}))$ and $\theta(H \times H) = 0$. It follows that H is a metabolizer for the isometric structure on $H_1(F;\mathbb{Z})$ defined above. (See [K, p. 95].)

If M is 0-framed surgery on K, $k = p^r \in \mathbb{Z}$ is any prime power, and M_k is the k-fold cyclic cover of M, then $H_1(M_k) \cong \mathbb{Z} \times TH_1(M_k)$, where the second summand is the $\mathbb{Z}$-torsion subgroup. This latter summand has all torsion relatively prime to p (see [CG1], for instance). The sequence of isomorphisms

$$TH_1(M_k) \to TH^2(M_k) \to \mathrm{Ext}^1_{\mathbb{Z}}(TH_1(M_k);\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(TH_1(M_k);\mathbb{Z}_{(p)}/\mathbb{Z})$$

defines a nonsingular $\mathbb{Z}_{(p)}/\mathbb{Z}$-valued linking pairing on the torsion subgroup of $H_1(M_k)$. The first isomorphism is Poincaré duality, the second is Universal Coefficients, and the last follows from the long exact $\mathrm{Ext}^*_{\mathbb{Z}}$ sequence associated to the short exact coefficient sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}/\mathbb{Z} \to 0.$$

This pairing may be computed by the usual formula.

PROPOSITION 9.2. *Let K be a $(.5)$-solvable knot via W and F a closed Seifert surface for K, i.e., a surface union a disk at the core of the added 2-handle.*

1. *There exists a choice of oriented 3-manifold $R \subset W$ with boundary F such that*
$$H = \mathrm{Ker}\,(H_1(F) \to H_1(R)/TH_1(R))$$
is a metabolizer for the Seifert pairing on F.

2. *If $k = p^r$ is a prime power, the $\mathbb{Z}_{(p)}/\mathbb{Z}$-linking pairing on M_k, has a self-annihilating subgroup $P \subset H_1(M_k;\mathbb{Z})$, i.e., a subgroup P such that*
$$P = P^{\perp} = \{y \in H_1(M_k;\mathbb{Z}) \mid \ell k(x,y) = 0 \text{ for all } x \in P\}.$$

Proof. The second statement follows from the first, and as it is not needed in this paper the proof is omitted. Assuming (1)-solvability we will prove Statement 2, and more, in Proposition 9.7.

To prove the first statement, we begin by defining R. By Lemma 7.4, a basis for the $(.5)$-Lagrangian of W can be represented by disjoint (1)-surfaces $\{F_i\}$. We will show R may be chosen to be disjoint from the surfaces F_i. Since W is an H_1-bordism, and by transversality, there is an oriented 3-manifold $R' \subset W$ with $\partial R' = F$. Now, R' intersects F_1 in a 1-manifold. This 1-manifold is nulhomologous in F_1 since F_1 lifts to the universal abelian cover, and since R' is dual to the meridian generating $H_1(W)$. A nulhomologous 1-manifold bounds a nested collection of subsurfaces on F_1, which, having boundary, have trivial normal bundles in W. Perform ambient surgeries on R', using these subsurfaces, to remove the intersections of R' with F_1. More precisely, one successively removes the trivial regular neighborhood, in $\mathcal{R}'$, of each circle of intersection, and replaces it with the circle bundle, in W, over the subsurface.

If we continue in this manner, the resulting oriented 3-manifold R has the same boundary as R' but does not intersect any surface F_i.

We now show the Seifert pairing vanishes on H, i.e., $\theta(H \times H) = 0$. Given $[x], [y] \in H$, there exist $c_x \in C_2(R) \subset C_2(W)$ and $d_x \in C_2(M)$ such that $\partial c_x = \partial d_x = \lambda x$ for some $\lambda \in \mathbb{Z} - \{0\}$. Since $(R \cup M) \cap F_i = \emptyset$ for all i, the homomorphism $H_2(R \cup M; \mathbb{Z}) \to H_2(W; \mathbb{Z})$ factors through the dual of the (.5)-Lagrangian $L^\perp = L \subset H_2(W; \mathbb{Z})$. Thus, after adding copies of the F_i's to $c_x \in C_2(W)$, we may assume $(c_x - d_x) = 0 \in H_2(W; \mathbb{Z})$.

Now choose $c_y \in C_2(R)$ such that $\partial c_y = \mu y$, for some $\mu \in \mathbb{Z} - \{0\}$. Since c_y may be pushed off R and is disjoint from all surfaces F_i, the intersection number $c_x \bullet c_y = 0$. Thus

$$0 = c_x \bullet c_y = d_x \bullet c_y = d_x \bullet \lambda y = \lambda\mu \cdot \theta([x], [y]),$$

and therefore $\theta([x], [y]) = 0$.

Finally, H is a summand of $H_1(F)$ since $H_1(F)/H$ is torsion-free by construction. Also, H has $\frac{1}{2}$-rank by the usual duality arguments (see, for instance, [L1]). Thus H is a metabolizer for K. □

Casson-Gordon invariants. Now recall the definitions and fundamental theorems regarding the Casson-Gordon invariants of knots. Let $k = p^r$ and $\ell = q^s$, where p and q are distinct primes. As before, for $K \subset S^3$ a knot, let M_k denote the k-fold cyclic cover of M, where M denotes 0-framed surgery on K. Suppose we are given a representation

$$\rho : \pi_1(M_k) \to \mathbb{Z}_\ell \times \mathbb{Z} \tag{9.1}$$

such that projection to $\mathbb{Z}$ is onto and such that projecting to $\mathbb{Z}_\ell$ sends to zero the cycle in M_k whose image in M is k times the meridian of K. Using standard bordism tools, Casson and Gordon observe there is an oriented 4-manifold W and a representation

$$\psi : \pi_1(W) \to \mathbb{Z}_{\ell'} \times \mathbb{Z}$$

such that ∂W is a disjoint union of copies of M_k, ℓ' is a possibly greater power of q, and such that the restriction of ψ to any component of ∂W is the representation ρ. Now, W can be chosen so that the number of boundary components is relatively prime to p.

Let $\Bbbk = \mathbb{Q}(\zeta_\ell)(t)$, where ζ_ℓ is a primitive ℓ^{th} root of unity, and t is an indeterminant. When we use the $\mathbb{Z}_\ell \times \mathbb{Z}$ cover of W, there is a Hermitian intersection pairing on the middle dimensional homology

$$\lambda : H_2(W; \Bbbk) \otimes H_2(W; \Bbbk) \to \Bbbk$$

via the composition of ring homomorphisms

$$\mathbb{Z}[\mathbb{Z}_\ell \times \mathbb{Z}] \to \mathbb{Z}[\zeta_\ell][\mathbb{Z}] \to \Bbbk.$$

The first homomorphism is the quotient homomorphism, and the second is inclusion to the quotient field. Since $H_2(M_k;\mathbb{k}) \cong H_1(M_k;\mathbb{k}) = 0$ [CG2] this pairing represents an element $[\lambda] \in L_0(\mathbb{k})$. If $\partial W = mM_k$, $(m,p) = 1$, then

$$\sigma(K,\rho) = ([\lambda] - [\lambda_0]) \otimes \frac{1}{m} \in L_0(\mathbb{k}) \otimes \mathbb{Z}_{(p)}$$

where $[\lambda_0]$ is the class of the intersection pairing on $H_2(W;\mathbb{Q})$ viewed as an element of $L_0(\mathbb{k})$ via the obvious inclusion of rings with unit $\mathbb{Q} \to \mathbb{k}$.

Definition 9.3. [CG2] $\sigma(K,\rho) \in L_0(\mathbb{k}) \otimes \mathbb{Z}_{(p)}$ is called a *Casson-Gordon invariant of K.*

The genius of Casson and Gordon's work is revealed through the concordance invariance of their obstructions. We need a definition.

Since $H^1(M_k;\mathbb{Z}_{(p)}/\mathbb{Z}) \cong \mathbb{Z}_{(p)}/\mathbb{Z} \oplus \mathrm{Hom}_{\mathbb{Z}}(TH_1(M_k), \mathbb{Z}_{(p)}/\mathbb{Z})$, the Mayer-Vietoris sequence

$$H^1(M_k;\mathbb{Z}_{(p)}/\mathbb{Z}) \to \oplus^k H^1(M-F;\mathbb{Z}_{(p)}/\mathbb{Z}) \to \oplus^k H^1(F;\mathbb{Z}_{(p)}/\mathbb{Z}) \tag{9.2}$$

together with the Alexander duality isomorphism

$$H^1(M-F;\mathbb{Z}_{(p)}/\mathbb{Z}) \cong H_1(F;\mathbb{Z}_{(p)}/\mathbb{Z})$$

identifies the $\mathbb{Z}_{(p)}/\mathbb{Z}$ valued characters on $TH_1(M_k)$ with a subgroup of

$$\oplus^k H_1(F;\mathbb{Z}_{(p)}/\mathbb{Z}).$$

By a change of basis, Gilmer identifies this subgroup of $\oplus^k H_1(F;\mathbb{Z}_{(p)}/\mathbb{Z})$ with a subgroup $A^k \subset (H_1(F) \otimes \mathbb{Z}_{(p)}/\mathbb{Z})$. (Gilmer prefers to work with the branched cover of K whose homology is canonically identified with $TH_1(M_k)$ [G2].)

Definition 9.4. We call the character χ_x associated to an element $x \in A^k$ the *character Gilmer associated to x*. Note that χ_x determines a character we also denote $\chi_x : \pi_1(M_k) \to \mathbb{Z}_{(p)}/\mathbb{Z} \times \mathbb{Z}$ defined by sending the meridian to the element $(0,1) \in \mathbb{Z}_{(p)}/\mathbb{Z} \times \mathbb{Z}$. We denote the associated ring homomorphism $\mathbb{Z}\pi_1(M_k) \to \mathbb{Z}[\mathbb{Z}_{(p)}/\mathbb{Z} \times \mathbb{Z}] \to \mathbb{k}$ by ρ_x.

Theorem 9.5 below, due to P. Gilmer [G2], is the most general result about concordance invariance we know, extending Casson and Gordon's original idea and results.

Added in proof: Gilmer has informed us that his proof of 9.5 has a serious gap. For any fixed slice disk, the theorem remains true for almost all primes q. These comments also apply to our 9.9. Theorem 9.11 remains valid in this same sense and, in any case, is valid for the original Casson-Gordon invariants (using 9.7 and the proof of 9.11 minus the first and third paragraphs).

THEOREM 9.5 (Gilmer [G2]). *If K is slice, then for any Seifert surface F for K there is a metabolizer $H \subset H_1(F)$ for the isometric structure on $H_1(F)$ having the following* additional *property: For any prime powers $k = p^r$ and $\ell = q^s$, $(p,q) = 1$, for any $x \in A^k \cap (H \otimes (\mathbb{Z}_{(p)}/\mathbb{Z}))$ of order ℓ, and for $\rho_x : \mathbb{Z}\pi_1(M_k) \to \mathbb{k}$, the character Gilmer associated to x, the Casson-Gordon obstruction*

$$\sigma(K, \rho_x) \in L_0(\mathbb{k}) \otimes \mathbb{Z}_{(p)}$$

vanishes.

The aim of this section is to replace the slice hypothesis of Theorem 9.5 with (1.5)-solvability.

We next do an important dimension count.

LEMMA 9.6. *Let W be an H_1-bordism for a knot in S^3, and let W_k be its k-fold cyclic cover ($k = p^r$). Let $\pi_1(M_k) \xrightarrow{\rho} \mathbb{Z}_\ell \times \mathbb{Z} \to \mathbb{k}$ be a character as in (9.1) with extension $\pi_1(W_k) \to \mathbb{k}$. Then*

$$\dim_{\mathbb{Q}} H_2(W_k; \mathbb{Q}) = \dim_{\mathbb{k}} H_2(W_k; \mathbb{k}).$$

Proof. We show the following equalities where $\chi^{\mathbb{F}}(W)$ is the Euler characteristic of W with coefficients in a field $\mathbb{F}$:

$$\begin{aligned}\dim_{\mathbb{Q}} H_2(W_k;\mathbb{Q}) &= \Sigma(-1)^i \dim_{\mathbb{Q}} H_i(W_k;\mathbb{Q}) = \chi^{\mathbb{Q}}(W_k)\\ &= \chi^{\mathbb{k}}(W_k) = \Sigma(-1)^i \dim_{\mathbb{k}} H_i(W_k;\mathbb{k}) = \dim_{\mathbb{k}} H_2(W_k;\mathbb{k}).\end{aligned}$$

The first follows by an easy computation. The second, third and fourth equalities are by definition. The last equality follows from the observation that $H_*(M_k, \mathbb{k}) = 0$ [CG2]. In fact, every 4-manifold with boundary has the homotopy type of a 3-dimensional CW-complex, so $H_{\geq 4}(W_k) = 0$ with any coefficients. Also, $H_0(W_k; \mathbb{k}) \cong H_1(W_k; \mathbb{k}) = 0$ by Lemma 4.5 of [CG1] and the proof of the corollary to Lemma 4 of [CG2]. Also, $H_3(W_k; \mathbb{k}) \cong H^1(W_k, M_k; \mathbb{k}) \cong \mathrm{Hom}_{\mathbb{k}}(H_1(W_k, M_k; \mathbb{k}); \mathbb{k}) = 0$, since $H_1(W_k, M_k; \mathbb{k}) = 0$. □

(1)-*solvability and extending characters.* Similarly, to the $\mathbb{Z}_{(p)}/\mathbb{Z}$ pairing on $TH_1(M_k)$, there are nonsingular relative homology linking pairings defined for a (1)-solution W as follows:

$$TH_2(W_k, M_k) \cong TH^2(W_k) \cong \mathrm{Ext}^1_{\mathbb{Z}}(TH_1(W_k); \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(TH_1(W_k); \mathbb{Z}_{(p)}/\mathbb{Z})$$

and

$$\begin{aligned}TH_1(W_k) \cong TH^3(W_k, M_k) &\cong \mathrm{Ext}^1_{\mathbb{Z}}(TH_2(W_k, M_k); \mathbb{Z})\\ &\cong \mathrm{Hom}_{\mathbb{Z}}(TH_2(W_k, M_k); \mathbb{Z}_{(p)}/\mathbb{Z}).\end{aligned}$$

Recall from [CG1] that $H_1(W_k; \mathbb{Z})$ has no p-torsion which is used in the above isomorphisms. By Poincaré duality and universal coefficients, $H_2(W_k, M_k; \mathbb{Z})$ has also no p-torsion.

PROPOSITION 9.7. *Let K be (1)-solvable via W and let*

$$P = \mathrm{Ker}\,(TH_1(M_k) \to TH_1(W_k)).$$

Then $P = P^{\perp}$, and a character $TH_1(M_k) \to \mathbb{Z}_{(p)}/\mathbb{Z}$ given by $x \mapsto \ell k(\cdot, x)$ factors through $H_1(W_k)$ if and only if $x \in P$.

Note that this implies the character $\chi_x : \pi_1(M_k) \to \mathbb{Z}_\ell \times \mathbb{Z}$ factors through $\pi_1(W_k)$, and similarly for $\rho_x : \pi_1(M_k) \to \Bbbk$. Also, note that the second statement of the proposition depends on the particular choice of self-annihilator P given by the first statement in the proposition.

Proof. We only outline the proof, as much of it follows earlier arguments given in the paper. Consider the following commutative diagram of groups and homomorphisms:

$$\begin{array}{ccccc}
TH_2(W_k, M_k) & \xrightarrow{\partial} & TH_1(M_k) & \xrightarrow{\iota} & TH_1(W_k) \\
\cong \Big\downarrow \ell k & & \cong \Big\downarrow \ell k & & \cong \Big\downarrow \ell k \\
\mathrm{Hom}_{\mathbb{Z}}(TH_1(W_k); \mathbb{Z}_{(p)}/\mathbb{Z}) & \xrightarrow{\iota^*} & \mathrm{Hom}_{\mathbb{Z}}(TH_1(M_k); \mathbb{Z}_{(p)}/\mathbb{Z}) & \xrightarrow{\partial^*} & \mathrm{Hom}_{\mathbb{Z}}(TH_2(W_k, M_k); \mathbb{Z}_{(p)}/\mathbb{Z})
\end{array}$$

That the top horizontal row is exact uses the same argument as in Lemma 4.5 but with $\mathcal{R} = \mathbb{Z}[\mathbb{Z}_k]$, and the fact that $rk_{\mathbb{Z}} H_2(W; \mathcal{R}) = rk_{\mathbb{Z}} H_2(W_k; \mathbb{Z}) = 2km$ (see Lemma 9.6). This equality follows since the Euler characteristic multiplies in covers. The vertical arrows are the nonsingular linking pairings mentioned above.

Assume $x \in P$. Then $\ell k(\cdot, \iota(x)) = \ell k(\cdot, 0) = 0$. Thus, $\ell k(\cdot, x)$ vanishes on $\mathrm{Image}(\partial) = \ker(\iota) = P$, and $P \subset P^{\perp}$. If $x \in P^{\perp}$, then $\partial^* \circ \ell k(\cdot, x) = 0$, and since the vertical pairings are nonsingular, $\iota(x) = 0$. Thus $P^{\perp} \subset P$, and equality follows.

Furthermore, $\ell k(\cdot, x)$ extends over $H_1(W_k)$ if and only if

$$(\ell k(\cdot, x) \in \mathrm{Image}(\iota^*)) \Leftrightarrow (\partial^* \circ \ell k(\cdot, x) = 0) \Leftrightarrow (\iota(x) = 0) \Leftrightarrow (x \in P). \qquad \square$$

COROLLARY 9.8. *Let $\mathcal{S}$ be the set of characters $TH_1(M_k) \to \mathbb{Z}_{(p)}/\mathbb{Z}$ that extend over $H_1(W_k)$. Then the order of $\mathcal{S}$, $|\mathcal{S}| = \frac{1}{2}\,|TH_1(M_k)|$.*

Proof. By Proposition 9.7, the characters that extend lie in one-to-one correspondence to elements in a self-annihilator $P = P^{\perp}$. This yields an exact sequence

$$0 \to P \to TH_1(M_k) \to \mathrm{Hom}_{\mathbb{Z}}(P; \mathbb{Z}_{(p)}/\mathbb{Z}) \to 0$$

where the latter homomorphism takes an element x to the homomorphism given by linking with x in M_k. This is onto since $\mathbb{Z}_{(p)}/\mathbb{Z}$ is injective for p-torsion-free finite $\mathbb{Z}$-modules. Since $P \cong \mathrm{Hom}_{\mathbb{Z}}(P; \mathbb{Z}_{(p)}/\mathbb{Z})$ for any finite p-torsion free $\mathbb{Z}$-module, $|TH_1(M_k)| = 2|P| = 2|\mathcal{S}|$. $\square$

PROPOSITION 9.9. *Let K be (1)-solvable via W. Let H be chosen as in Proposition 9.2. A character $TH_1(M_k) \to \mathbb{Z}_{(p)}/\mathbb{Z}$ extends over $TH_1(W_k)$ if and only if it is a character Gilmer associated to some $x \in A^k \cap (H \otimes \mathbb{Z}_{(p)}/\mathbb{Z})$.*

Proof. As in Gilmer [G2, pp. 5, 6], there are isomorphisms (with $\mathbb{Z}_{(p)}/\mathbb{Z}$-coefficients) where the first is excision and the second is Poincaré duality;

$$H^2(W, W-R) \cong H^2(R \times I, R \times S^0) \cong H_2(R \times I, F \times I) \cong H_2(R, F). \tag{9.3}$$

Now consider the Mayer-Vietoris sequence

$$H^1(W_k; \mathbb{Z}_{(p)}/\mathbb{Z}) \to \oplus^k H^1(W-R; \mathbb{Z}_{(p)}/\mathbb{Z}) \to \oplus^k H^1(R; \mathbb{Z}_{(p)}/\mathbb{Z}).$$

Pre-composing the isomorphism (9.3) with the coboundary homomorphism

$$H^1(W-R; \mathbb{Z}_{(p)}/\mathbb{Z}) \to H^2(W, W-R; \mathbb{Z}_{(p)}/\mathbb{Z})$$

we get a commutative diagram as in [G2], as follows:

$$\begin{array}{ccccccc}
\mathbb{Z}_{(p)}/\mathbb{Z} & \to & H^1(W_k; \mathbb{Z}_{(p)}/\mathbb{Z}) & \to & \oplus^k H_2(R, F; \mathbb{Z}_{(p)}/\mathbb{Z}) & & \\
\downarrow & & \downarrow \iota^* & & \downarrow \oplus\partial & & \\
\mathbb{Z}_{(p)}/\mathbb{Z} & \to & H^1(M_k; \mathbb{Z}_{(p)}/\mathbb{Z}) & \xrightarrow{j} & \oplus^k H_1(F; \mathbb{Z}_{(p)}/\mathbb{Z}) & \xrightarrow{\alpha} & \oplus^k H_1(F; \mathbb{Z}_{(p)}/\mathbb{Z}). \\
 & & & & \downarrow \oplus\mu & & \\
 & & & & \oplus^k H_1(R; \mathbb{Z}_{(p)}/\mathbb{Z}) & &
\end{array}$$

Note that the bottom horizontal sequence is exact, but the top may not be. The characters on $TH_1(M_k)$ that extend over $TH_1(W_k)$ lie in one-to-one correspondence with image$(j \circ \iota^*)$. As in [G2], a diagram chase reveals that

$$\text{Image}(j \circ \iota^*) \subset \oplus^k (H \otimes \mathbb{Z}_{(p)}/\mathbb{Z}) \cap \ker(\alpha).$$

As mentioned following the exact sequence (9.2), these are precisely the characters Gilmer associated to elements in $A^k \cap (H \otimes \mathbb{Z}_{(p)}/\mathbb{Z})$. $\square$

(1.5)-*solvability and vanishing Casson-Gordon invariants.* The following lemma is a straightforward application of Proposition 3.3 of [Let].

LEMMA 9.10. *Let p and q be distinct primes. Let $\psi : C \to D$ be a homomorphism of finitely generated free $\mathbb{Z}G$-modules where $G = \mathcal{Q} \rtimes \mathbb{Z}$, $\mathcal{Q}$ a finite abelian q-group, such that the projection homomorphism $G = \mathcal{Q} \rtimes \mathbb{Z} \to \mathbb{Z}$ is abelianization. Let N be the index p^r subgroup of G (unique since $H_1(G) \cong \mathbb{Z}$) and let $N \to \mathbb{Z}_\ell \times \mathbb{Z}$, be a group homomorphism with finite kernel. Consider the composition*

$$\mathbb{Z}N \to \mathbb{Z}[\mathbb{Z}_\ell \times \mathbb{Z}] \to \Bbbk.$$

If $\psi \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}$ is a split monomorphism, then $\psi \otimes_{\mathbb{Z}N} \mathrm{id}_{\Bbbk}$ is a split monomorphism.

Proof. By hypothesis, $K = \operatorname{coker}(\psi \otimes \mathrm{id}_{\mathbb{Z}})$ is isomorphic to a summand of $D \otimes_{\mathbb{Z}G} \mathbb{Z}$ having $\mathbb{Z}$-rank equal to $d = \operatorname{rank}_{\mathbb{Z}G}(D) - \operatorname{rank}_{\mathbb{Z}G}(C)$. We can extend ψ to $\psi' : C \oplus (\mathbb{Z}G)^d \to D$ so that $\psi' \otimes_{\mathbb{Z}G} \mathrm{id}_{\mathbb{Z}}$ is an isomorphism.

Let $K' = \operatorname{coker}(\psi')$. By the right exactness of the tensor product, $K' \otimes_{\mathbb{Z}G} \mathbb{Z} = 0$. By [Let, Prop. 3.3] $\mathbb{Q}K' = K' \otimes_{\mathbb{Z}N} \mathbb{Q}N$ is a finite dimensional rational vector space. Since the rational vector space image$(\mathbb{Q}N \to \Bbbk)$ is infinite dimensional, and since $\mathbb{Q}K' = K' \otimes_{\mathbb{Z}N} \mathbb{Q}N$ is finite dimensional over $\mathbb{Q}$, it follows that

$$\mathbb{Q}K' \otimes_{\mathbb{Q}N} \Bbbk = 0.$$

In fact, given $x \in \mathbb{Q}K'$ there is an element $\hat{\gamma} \in \mathbb{Q}N$ with nonzero image $\gamma \in \Bbbk$ such that $x\hat{\gamma} = 0$. Thus

$$x \otimes 1 = x \otimes \gamma \cdot \gamma^{-1} = x\hat{\gamma} \otimes \gamma^{-1} = 0.$$

Again by right exactness of the tensor product,

$$\psi' \otimes_{\mathbb{Z}N} \mathrm{id}_{\Bbbk} : (C \oplus (\mathbb{Z}G)^d) \otimes_{\mathbb{Z}N} \Bbbk \to D \otimes_{\mathbb{Z}N} \Bbbk$$

is an epimorphism of finite dimensional $\Bbbk$-vector spaces of the same rank (N has finite index in G) and so is an isomorphism. In particular, the restriction to $C \otimes_{\mathbb{Z}N} \Bbbk$ given by $\psi \otimes_{\mathbb{Z}N} \mathrm{id}_{\Bbbk}$ is a monomorphism. □

THEOREM 9.11. *Let $K \subset S^3$ be a* (1.5)*-solvable knot. Then all Casson-Gordon invariants of K vanish. That is, the conclusions of Gilmer's Theorem* 9.5 *hold.*

Proof. Let W be a (1.5)-solution for K. By Proposition 9.9 and the fact that W is a (1)-solution, the character Gilmer associated to an element character $x \in A^k \cap (H \otimes \mathbb{Z}_{(p)}/\mathbb{Z})$

$$\pi_1(M_k) \to \mathbb{Z}_{(p)}/\mathbb{Z} \times \mathbb{Z} \to \Bbbk$$

factors through a homomorphism $\pi_1(W_k) \to \Bbbk$.

Consider $H_1(W_k;\mathbb{Z})$ as a $\mathbb{Z}[\mathbb{Z}]$-module, where $\mathbb{Z}$ acts through its quotient group $\mathbb{Z}_k$ as the group of deck transformations of W_k. $TH_1(W_k)$ is a $\mathbb{Z}[\mathbb{Z}]$-submodule. The maximal abelian q-group $\mathcal{Q} \subset TH_1(W_k)$ is a split submodule, so there is an epimorphism $H_1(W_k;\mathbb{Z}) \to \mathcal{Q} \times \mathbb{Z}$.

Now suppose $x \in A^k \cap (H \otimes (\mathbb{Z}_{(p)}/\mathbb{Z}))$ is an element of order $\ell = q^s$. Then, since $\mathcal{Q}$ is a split $\mathbb{Z}[\mathbb{Z}]$-module summand of $TH_1(W_k)$, $\chi_x : \pi_1(W_k) \to \mathbb{Z}_{q^t} \times \mathbb{Z} \subset (\mathbb{Z}_{(p)}/\mathbb{Z}) \times \mathbb{Z}$ factors through $\mathcal{Q} \times \mathbb{Z}$, where t is an integer possibly bigger than s. Hence we have a commutative diagram of ring homomorphisms as in the following diagram. Here the vertical homomorphisms are induced by the inclusions of the index p^r normal subgroups, and the top horizontal homomorphism extends the Gilmer associated character $\rho_x : \pi_1(M_k) \to \Bbbk$. Of course, the bottom horizontal homomorphism factors as described since $\mathcal{Q} \rtimes \mathbb{Z}$ is 1-solvable.

$$\begin{array}{ccccccccc}
\mathbb{Z}\pi_1(W_k) & \longrightarrow & \mathbb{Z}H_1(W_k) & \longrightarrow & \mathbb{Z}[\mathcal{Q}\times\mathbb{Z}] & \longrightarrow & \mathbb{Z}[\mathbb{Z}_\ell\times\mathbb{Z}] & \longrightarrow & \Bbbk \\
\downarrow & & & & \downarrow & & & & \\
\mathbb{Z}\pi_1(W) \to H_1(W;\mathbb{Z}[\pi/\pi^{(2)}]) & & & \longrightarrow & \mathbb{Z}[\mathcal{Q}\rtimes\mathbb{Z}]. & & & &
\end{array}$$

The Casson-Gordon obstruction is the difference of L-theory classes of the intersection forms on the second homology of the following chain complexes

$$C_*(W_k;\Bbbk) \quad \text{and} \quad C_*(W_k;\mathbb{Q}).$$

The intersection form $\lambda_0 \otimes \mathrm{id}_{\mathbb{Q}}$ on $H_2(W_k;\mathbb{Q})$ is trivial in $L_0(\Bbbk)$. Indeed, one easily checks that the lifts into W_k of a basis for the image of the Lagrangian $L \subset H_2(W)$ forms a basis for a Lagrangian of the intersection form $\lambda_0 \otimes \mathrm{id}_{\mathbb{Q}}$ on $H_2(W_k;\mathbb{Q})$. Thus it remains to show the form on $H_2(W_k;\Bbbk)$ is trivial in $L_0(\Bbbk)$ to show the Casson-Gordon invariant $\sigma(K,\rho_x) = 0$.

Let $\{\ell_1,\dots,\ell_m\}$ be a set of immersed spheres in $W^{(2)}$ spanning a (2)-Lagrangian $L \subset H_2(W^{(2)})$ and whose projection to W forms a basis for a Lagrangian in $H_2(W;\mathbb{Z})$. The $\mathcal{Q}\times\mathbb{Z}$ cover of W_k is a metabelian cover of W, and so is a quotient space of $W^{(2)}$. Hence the intersection pairing with $\Bbbk$ and $\mathbb{Z}[\mathcal{Q}\times\mathbb{Z}]$-coefficients vanishes on L.

By Lemma 9.6, $\dim_{\Bbbk} H_2(W_k;\Bbbk) = \dim_{\mathbb{Q}} H_2(W_k;\mathbb{Q}) = 2km$. The last equality follows from a dimension count, and the fact that the Euler characteristic multiplies in covers. Thus, it suffices to show that the image of the Lagrangian in $H_2(W_k;\Bbbk)$ has dimension km.

Let $\mathcal{L} \to W$ be an immersion of $\vee^m S^2$ obtained by basing the ℓ_i. Let $\mathcal{L}_k$ be the induced k-fold cover, $\mathcal{L}_k = (\bigvee^m S^2)^k$. Since $H_2(\mathcal{L}_k;\Bbbk) \cong \Bbbk^{km}$, it suffices to show that $H_2(\mathcal{L}_k;\Bbbk) \to H_2(W_k;\Bbbk)$ is one-to-one. Since $H_3(W_k;\Bbbk) = 0$, we must show $H_3(W_k,\mathcal{L}_k;\Bbbk) = 0$.

But W_k has the homotopy type of a 3-dimensional CW complex, so this is equivalent to showing the boundary homomorphism $\partial\otimes\mathrm{id}_{\Bbbk}$ below is one-to-one, where $N = \mathcal{Q}\times\mathbb{Z} \subset \mathcal{Q}\rtimes\mathbb{Z}$.

$$C_3(W_k,\mathcal{L}_k)\otimes_{\mathbb{Z}N}\Bbbk \xrightarrow{\partial\otimes\mathrm{id}_{\Bbbk}} C_2(W_k,\mathcal{L}_k)\otimes_{\mathbb{Z}N}\Bbbk.$$

Since $H_3(W,\mathcal{L};\mathbb{Z}) = 0$, this follows from Lemma 9.10. $\square$

Letsche obstructions. Recall the recently defined Letsche obstructions to slicing a knot. Our treatment is brief, and we refer the reader to [Let] and [L3] for more details on the η-invariant and the Letsche obstructions. Letsche constructs a homomorphism

$$\eta_K : H_1(M;\mathbb{Z}[\mathbb{Z}]) \times R_*(\Gamma) \to \mathbb{R}$$

where $R_*(\Gamma)$ is the representation ring of $\Gamma = (S^{-1}\mathbb{Z}[\mathbb{Z}]/\mathbb{Z}[\mathbb{Z}]) \rtimes \mathbb{Z}$ by

$$\eta_K(x,\theta) = \tilde{\eta}_{\theta\circ B\ell_x}(M) \in \mathbb{R}$$

for any $\theta \in R_k(\Gamma)$ and for any k. Here $B\ell_x : H_1(M;\mathbb{Z}[\mathbb{Z}]) \to S^1\mathbb{Z}[\mathbb{Z}]/\mathbb{Z}[\mathbb{Z}]$ is the homomorphism defined by $B\ell_x(y) = B\ell(x,y)$, $B\ell$ the Blanchfield pairing for the knot K. Now, $\tilde{\eta}_{\theta\circ B\ell_x}(M)$ is the reduced η-invariant associated to the representation

$$\theta \circ B\ell_x : \pi_1(M) \to U_k,$$

and U_k is the space of k-dimensional unitary representations of the group Γ. Letsche defines a special subclass of representation $\mathcal{P}_k(\pi_1(M))$ as those representations $\theta : \pi_1(M) \to U_k$ that factor through a nonabelian group of the form $\mathcal{Q} \rtimes \mathbb{Z}$ where $\mathcal{Q}$ is a finite abelian p-group and such that the image of the meridian of the knot group, $\theta(\mu)$ has eigenvalues that are transcendental over $\mathbb{Q}$. He proves the following theorem, predating our methods.

LETSCHE'S THEOREM. *If K is slice, then there is a $P \subset H_1(M;\mathbb{Z}[\mathbb{Z}])$ such that $P = P^\perp$ with respect to the Blanchfield pairing, and such that for all $x \in P$ and $\theta \in R_k(\Gamma)$ such that $\theta \circ \alpha \in \mathcal{P}_k(\pi_1(M))$, $\eta_K(x,\theta) = 0$.*

THEOREM 9.12. *If $K \subset S^3$ is (1.5)-solvable, then the conclusions from Letsche's theorem above also hold.*

The proof is omitted.

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