

**Uniform non-amenability, cost, and the first  $\ell^2$ -Betti number**

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**Abstract.** It is shown that  $2\beta_1(\Gamma) \leq h(\Gamma)$  for any countable group  $\Gamma$ , where  $\beta_1(\Gamma)$  is the first  $\ell^2$ -Betti number and  $h(\Gamma)$  the uniform isoperimetric constant. In particular, a countable group with non-vanishing first  $\ell^2$ -Betti number is uniformly non-amenable.

We then define isoperimetric constants in the framework of measured equivalence relations. For an ergodic measured equivalence relation  $R$  of type  $\text{II}_1$ , the uniform isoperimetric constant  $h(R)$  of  $R$  is invariant under orbit equivalence and satisfies

$$2\beta_1(R) \leq 2C(R) - 2 \leq h(R),$$

where  $\beta_1(R)$  is the first  $\ell^2$ -Betti number and  $C(R)$  the cost of  $R$  in the sense of Levitt (in particular  $h(R)$  is a non-trivial invariant). In contrast with the group case, uniformly non-amenable measured equivalence relations of type  $\text{II}_1$  always contain non-amenable subtrees.

An ergodic version  $h_e(\Gamma)$  of the uniform isoperimetric constant  $h(\Gamma)$  is defined as the infimum over all essentially free ergodic and measure preserving actions  $\alpha$  of  $\Gamma$  of the uniform isoperimetric constant  $h(R_\alpha)$  of the equivalence relation  $R_\alpha$  associated to  $\alpha$ . By establishing a connection with the cost of measure-preserving equivalence relations, we prove that  $h_e(\Gamma) = 0$  for any lattice  $\Gamma$  in a semi-simple Lie group of real rank at least 2 (while  $h_e(\Gamma)$  does not vanish in general).

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**1. Introduction**

The isoperimetric constant of a graph offers a simple way to capture the isoperimetric behavior of finite sets in this graph. For a finitely generated countable group it reflects isoperimetry in Cayley graphs associated to finite generating sets of this group and is related to amenability. In the present paper, we introduce an analogue of this constant for measured equivalence relations of type  $\text{II}_1$ . As we shall see, it behaves

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in a very different manner from a measured dynamic point of view from the uniform isoperimetric constant of a group.

Let  $R$  be a measured equivalence relation of type  $\text{II}_1$  on a standard (non-atomic) probability space  $(X, \mu)$ . Our main interest lies in two geometric invariants that have recently been attached to  $R$ : the cost and the  $\ell^2$ -Betti numbers. The cost of  $R$  is a real number with values in  $[1, \infty]$  (assuming  $R$  to be ergodic) denoted by  $C(R)$ . From its definition one can readily infer that it is invariant under isomorphism – i.e., orbit equivalence – of  $R$  and the main problem is to compute it. In [9] Damien Gaboriau established an explicit formula relating the cost of an amalgamated free product (over amenable equivalence subrelations) to the costs of their components; this allowed him to solve the long-standing problem of distinguishing the free groups up to orbit equivalence. In [10] he went further and introduced the so-called  $\ell^2$ -Betti numbers of  $R$ . These are non-negative numbers  $\beta_0(R), \beta_1(R), \beta_2(R), \dots$  in  $[0, \infty]$  defined by geometric constructs using an approximation process (as for their analogues for countable groups; see Cheeger and Gromov [6]) and one of the main problems here (solved in [10]) was to show that the resulting numbers only depend on the isomorphism class of  $R$ . The first  $\ell^2$ -Betti number provides another way to distinguish the free groups up to orbit equivalence. The relation between  $\ell^2$ -Betti numbers and the cost is still unclear, but the inequality  $C(R) \geq \beta_1(R) + 1$  is known to hold for any ergodic equivalence relation of type  $\text{II}_1$  [10]. Recall that  $\ell^2$ -Betti numbers were first introduced by Atiyah [3] in 1976 in his work on the index of equivariant elliptic operators on coverings spaces of Riemannian manifolds. In the present paper we consider a new isomorphism invariant for  $R$ , the uniform isoperimetric constant,  $h(R)$ . It takes values in  $[0, \infty]$  and is defined as an infimal value of isoperimetric ratios for ‘finite sets’ in the Cayley graphs of  $R$  (see Section 4). For compact Riemannian manifolds (and their coverings), the isoperimetric constant was considered by Cheeger when he proved his well-known ‘Cheeger inequality’ relating it to the bottom of the spectrum of the Laplacian.

Our main theorem asserts that for any ergodic measured equivalence relation of type  $\text{II}_1$ , one has

$$\beta_1(R) \leq C(R) - 1 \leq \frac{h(R)}{2}.$$

Here the relation  $R$  is assumed to be finitely generated, in which case the uniform isoperimetric constant  $h(R)$  has a finite value. (A natural extension of  $h(R)$  to infinitely generated  $R$  is to set  $h(R) = \sup_{R'} h(R')$ , where  $R'$  runs over all finitely generated subrelations of  $R$ , so that, for instance, a measured equivalence relation  $R$  given by a measure-preserving and essentially free action of the free group  $F_\infty$  on infinitely many generators will satisfy  $\beta_1(R) = h(R) = \beta_1(F_\infty) = h(F_\infty) = +\infty$  [10].)

We start by proving the following weaker inequality in Section 2,

$$\beta_1(\Gamma) \leq h(\Gamma)$$

for a finitely generated group  $\Gamma$  (in particular, a countable group with non-vanishing first  $\ell^2$ -Betti number is uniformly non-amenable). The proof uses only group-theoretic tools. It is inspired by the proof of Cheeger–Gromov’s celebrated vanishing theorem in [6] asserting that when  $\Gamma$  is amenable, the sequence

$$\beta_0(\Gamma), \beta_1(\Gamma), \beta_2(\Gamma), \dots$$

of all  $\ell^2$ -Betti numbers vanishes identically. Note that for an amenable  $\Gamma$ , one has  $h(\Gamma) = 0$ , but there do exist non-amenable groups with  $h(\Gamma) = 0$  (see [22], [2]).

Now the inequality  $\beta_1(\Gamma) \leq h(\Gamma)$  is not optimal in general, but we do not know how to get the optimal inequality using only group-theoretic arguments. In Section 3 we prove that in fact

$$\beta_1(\Gamma) \leq \frac{h(\Gamma)}{2}.$$

Our proof relies on invariant percolation theory. This inequality is an equality for free groups, where  $2\beta_1(F_k) = h(F_k) = 2k - 2$ , and thus is optimal. However, it can be strict as there are groups of cost 1 that have  $h > 0$ , for instance, higher rank lattices in semi-simple Lie groups.

The general inequality relating the cost and the isoperimetric constant of a measured equivalence relation, as stated above, is proved in Section 4. Together with Gaboriau’s results [10], it implies that  $2\beta_1(\Gamma) \leq h(\Gamma)$  as well for every finitely generated group  $\Gamma$ .

We call an ergodic measured groupoid  $G$  (of type II<sub>1</sub>) uniformly non-amenable if its isoperimetric constant  $h(G)$  is non-zero. The class of uniformly non-amenable groups is quite large and has been studied recently by Osin (see, e.g., [22], [23]) and Arzhantseva, Burillo, Lustig, Reeves, Short, Ventura ([2]). Breuillard and Gelander [5] have shown that for an arbitrary field  $K$ , any non-amenable and finitely generated subgroup of  $\mathrm{GL}_n(K)$  is uniformly non-amenable.

The class of uniformly non-amenable measured equivalence relations turns out to be “much smaller” than its corresponding group-theoretic analogue. For instance, if an equivalence relation  $R$  is the partition into the orbits of an essentially free measure-preserving action of a (non-uniform) lattice in a higher rank Lie group, we have  $h(R) = 0$  (Corollary 5.6) and thus  $R$  is not uniformly non-amenable. Note that  $R$  has the property T of Kazhdan (cf. [28] and references) in that case. Also, equivalence relations that are decomposable as a direct product of two infinite equivalence subrelations have trivial uniform isoperimetric constant (Corollary 5.7). These results are derived by establishing a relation between the cost and the uniform isoperimetric constant and by appealing to some of Gaboriau’s results in [9]. Namely, we show (in Section 5.3) that an ergodic equivalence relation with cost 1 has a vanishing isoperimetric constant. The proof is reminiscent of the (non-)concentration of measure property for measured equivalence relations (see [27]) which is implemented here via the Rokhlin Lemma. We also show that uniformly non-amenable

measured equivalence relations have trivial fundamental groups (see Section 5.2; this will alternatively follow from [9]) and always contain a non-amenable subtreeing (see Section 5.1). The latter is related to the measure-theoretic analogue of the Day–von Neumann problem. Recall that in the group case this problem (i.e., is it true that every non-amenable group contains a non-amenable free group?) is well known to have a negative answer, as was proved by Ol’shanskii. Osin [23] showed that the answer was negative as well even within the class of uniformly non-amenable groups. To prove that it is positive in our situation, we combine our results with the corresponding known fact for equivalence relations with cost greater than 1 ([16], [25]).

**Note added in proof.** Our main inequalities can be improved to use the external vertex isoperimetric constant in place of the edge isoperimetric constant. This has the consequence of implying uniform exponential growth when the uniform vertex isoperimetric constant is positive, e.g. when  $\beta_1 > 0$ . We give the corresponding proof of Theorem 3.1 here to illustrate this. Let  $A$  be a finite set of vertices in  $Y$ . Let  $F'$  be the subgraph of  $Y$  spanned by those edges of  $F$  that are incident to some vertex of  $A$ . Since  $F'$  is a forest with no isolated vertices, we have

$$\begin{aligned} 2C(F) &= \frac{1}{|A|} \sum_{x \in A} \mathbf{E}[\deg_F x] = \frac{1}{|A|} \mathbf{E} \left[ \sum_{x \in A} \deg_F x \right] \\ &\leq \frac{1}{|A|} \mathbf{E} \left[ \sum_{x \in V(F')} \deg_F x - |V(F') \setminus A| \right] \leq \frac{\mathbf{E}[2|V(F')| - |V(F') \setminus A|]}{|A|} \\ &\leq \frac{\mathbf{E}[2|A| + |V(F') \setminus A|]}{|A|} \leq 2 + \frac{|\partial_Y^V A|}{|A|}. \end{aligned}$$

Taking the infimum over  $A$  gives the desired inequality.

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## 2. The uniform non-amenability of groups with non-vanishing $\beta_1$

In this section we recall the definitions of the uniform isoperimetric constant  $h(\Gamma)$  and the first  $\ell^2$ -Betti number  $\beta_1(\Gamma)$  for a countable group  $\Gamma$  and demonstrate that  $\beta_1(\Gamma) \leq h(\Gamma)$ . This inequality is not optimal and will be improved in the next sections to  $2\beta_1(\Gamma) \leq h(\Gamma)$  by using ergodic-theoretic arguments.

Let  $Y$  be a locally finite graph and  $A$  be a subset of vertices of  $Y$ . Define the (*edge*) *boundary of  $A$  in  $Y$*  to be the set  $\partial_Y A$  of edges of  $Y$  with one extremity in  $A$  and the

other one in  $Y \setminus A$ . The *isoperimetric constant* of  $Y$  is the non-negative number

$$h(Y) = \inf_{A \subset Y} \frac{|\partial_Y A|}{|A|},$$

where the infimum runs over all finite subsets  $A$  of vertices of  $Y$ .

Let  $\Gamma$  be a finitely generated group and  $S$  be a finite generating set of  $\Gamma$ . Recall that the *Cayley graph* of  $\Gamma$  with respect to  $S$  is the graph  $Y$  whose vertices are the elements of  $\Gamma$  and whose edges are given by right multiplication by elements of  $S$ . The isoperimetric constant of this graph is called the *isoperimetric constant of  $\Gamma$  with respect to  $S$*  and is denoted by  $h_S(\Gamma)$ . Følner's theorem asserts that a finitely generated group  $\Gamma$  is amenable if and only if  $h_S(\Gamma) = 0$  for some (hence every) finite generating set  $S$  ([15], Chapter VII).

By definition the *uniform isoperimetric constant* of  $\Gamma$  is the infimum

$$h(\Gamma) = \inf_S h_S(\Gamma)$$

over all finite generating sets  $S$  of  $\Gamma$ .

**Example 2.1.** Osin [22] has given examples of non-amenable groups with  $h(\Gamma) = 0$ . For instance, he proved (see [22], Example 2.2) that the Baumslag–Solitar groups, with presentation

$$\text{BS}_{p,q} = \langle a, t \mid t^{-1}a^p t = a^q \rangle,$$

where  $p, q > 1$  are relatively prime, have vanishing uniform isoperimetric constant. (Note that the uniform isoperimetric constant considered in [22] is defined in terms of the regular representation of the given group: compare Section 13 in the paper of Arzhantseva et al. [2].)

Finitely generated groups with  $h(\Gamma) > 0$  are called *uniformly non-amenable* (see [2]). In fact our definition differs slightly from the one given in [2] due to a different choice for the boundary of a finite subset of vertices in a graph (basically the present paper deals with the “edge boundary”, while the definition in [2] involves the “internal boundary”; see Appendix A for more details). The Baumslag–Solitar groups have vanishing uniform isoperimetric constant for any reasonable definition of the boundary.

The  $\ell^2$ -Betti numbers of a countable group  $\Gamma$  are non-negative real numbers  $\beta_0(\Gamma)$ ,  $\beta_1(\Gamma)$ ,  $\dots$  coming from  $\ell^2$ -(co-)homology as  $\Gamma$ -dimension (also known as Murray–von Neumann dimension). We refer to [14], [6], [24], [18] for their precise definition. By a well-known theorem due to Cheeger and Gromov [6], the  $\ell^2$ -Betti numbers of a countable amenable group vanish identically. By elaborating on ideas of [6] (see in particular §3 in [6]), in the case of non-amenable groups we obtain the following explicit relation between the first  $\ell^2$ -Betti number and the uniform isoperimetric constant.

**Theorem 2.2.** *Let  $\Gamma$  be a finitely generated group. Then  $\beta_1(\Gamma) \leq h(\Gamma)$ .*

*Proof.* Let  $S$  be a finite generating set of  $\Gamma$  and  $Y$  be the Cayley graph of  $\Gamma$  with respect to  $S$ . Write  $C_i^{(2)}(Y)$ ,  $i = 0, 1$ , for the space of square integrable functions on the  $i$ -cells (vertices and edges) of  $Y$ . Associated to the simplicial boundary  $\partial_Y$  on  $Y$  we have a bounded operator

$$\partial_1^{(2)}: C_1^{(2)}(Y) \rightarrow C_0^{(2)}(Y).$$

Denote by  $Z_1(Y)$  the space of finite 1-cycles and by  $Z_1^{(2)}(Y)$  square integrable 1-cycles on  $Y$ . Thus  $Z_1^{(2)}(Y)$  is the kernel of  $\partial_1^{(2)}$ , while  $Z_1(Y)$  is space of functions with finite support in this kernel. The first  $\ell^2$ -Betti number  $\beta_1(\Gamma)$  of  $\Gamma$  coincides with the Murray–von Neumann dimension

$$\beta_1(\Gamma) = \dim_{\Gamma} \bar{H}_1^{(2)}(Y)$$

of the orthogonal complement  $\bar{H}_1^{(2)}(Y) \subset C_1^{(2)}(Y)$  of the closed subspace  $\overline{Z_1(Y)}$  in  $Z_1^{(2)}(Y)$ , where the closure of  $Z_1(Y)$  is taken with respect to the Hilbert norm. A proof of this fact, together with the basic definitions used here, can be found in [26], Section 3 (in particular, this step takes care of the approximation process involved in Cheeger–Gromov’s definition of  $\ell^2$ -Betti numbers; compare to [6]).

Let  $\Omega \subset Y^{(1)}$  be a finite subset of edges of  $Y$  and consider the space

$$\bar{H}_1^{(2)}(Y)|_{\Omega} = \{\sigma|_{\Omega} \mid \sigma \in \bar{H}_1^{(2)}(Y)\}$$

of restrictions of harmonic chains to  $\Omega$ . This is a linear subspace of the space  $C_1(\Omega)$  of complex functions on  $\Omega$ . Let

$$P: C_1^{(2)}(Y) \rightarrow C_1^{(2)}(Y)$$

be the (equivariant) orthogonal projection on  $\bar{H}_1^{(2)}(Y)$  and

$$\chi_{\Omega}: C_1^{(2)}(Y) \rightarrow C_1^{(2)}(Y)$$

be the orthogonal projection on  $C_1(\Omega)$ .

Given a finite set  $A$  of  $\Gamma$ , we denote by  $A_S$  the set of edges with a vertex in  $A$ . We have  $\partial_S A \subset A_S$ , where  $\partial_S A$  is the boundary of  $A$  in  $Y$  defined at the beginning of this section. Let us now prove that, for every non-empty finite set  $A$  of  $\Gamma$ ,

$$\beta_1(\Gamma) \leq \frac{1}{|A|} \dim_{\mathbb{C}} \bar{H}_1^{(2)}(Y)|_{A_S}.$$

Write  $M_{A_S}$  for the composition  $\chi_{A_S} P$ , considered as an operator from  $C_1(A_S)$  to itself with range  $\bar{H}_1^{(2)}(Y)|_{A_S}$ . We have

$$\begin{aligned} \dim_{\Gamma} \bar{H}_1^{(2)}(Y) &= \sum_{s \in S} \langle P \delta_{(e,s)} \mid \delta_{(e,s)} \rangle \\ &= \frac{1}{|A|} \sum_{a \in A, s \in S} \langle P \delta_{(a,as)} \mid \delta_{(a,as)} \rangle \\ &\leq \frac{1}{|A|} \sum_{u \in A_S} \langle P \delta_u \mid \delta_u \rangle \\ &= \frac{1}{|A|} \operatorname{tr} M_{A_S} \\ &\leq \frac{1}{|A|} \dim_{\mathbb{C}} \bar{H}_1^{(2)}(Y)|_{A_S}, \end{aligned}$$

where the last inequality follows from the fact that  $\|M_{A_S}\| \leq 1$ . This gives the desired inequality.

We now observe that every harmonic 1-chain (i.e., element of  $\bar{H}_1^{(2)}(Y)$ ) that “enters” a subset  $A_S$  has to intersect its boundary  $\partial_S A$ :

**Lemma 2.3.** *Let  $A$  be a finite subset of  $\Gamma$ . The canonical (restriction) map*

$$r: \bar{H}_1^{(2)}(Y)|_{A_S} \rightarrow \bar{H}_1^{(2)}(Y)|_{\partial_S A}$$

*is injective.*

*Proof.* Recall that

$$Z_1^{(2)}(Y) = \bar{H}_1^{(2)}(Y) \oplus_{\perp} \overline{Z_1(Y)}.$$

Let  $\sigma \in \bar{H}_1^{(2)}(Y)$ . If  $\sigma$  vanishes on  $\partial_S A$ , then  $\sigma|_{A_S}$  is a finite 1-cycle as the boundary operator commutes with the restriction to  $A_S$  in that case. Thus  $\sigma|_{A_S}$  vanishes identically.  $\square$

The above lemma gives

$$\dim_{\mathbb{C}} \bar{H}_1^{(2)}(Y)|_{A_S} = \dim_{\mathbb{C}} \bar{H}_1^{(2)}(Y)|_{\partial_S A} \leq |\partial_S A|,$$

which immediately yields Theorem 2.2:

$$\dim_{\Gamma} \bar{H}_1^{(2)}(Y) \leq \frac{|\partial_S A|}{|A|},$$

and thus  $\beta_1(\Gamma) \leq h(\Gamma)$ .  $\square$

**Remark 2.4.** Lück’s generalization of the Cheeger–Gromov theorem to arbitrary module coefficients (Theorem 5.1 in [17]) does not hold for groups with vanishing uniform isoperimetric constant (since these groups may contain non-abelian free groups; compare Remark 5.14 in [17]).

**Remark 2.5.** As mentioned above, the inequality  $\beta_1(\Gamma) \leq h(\Gamma)$  is not optimal in general. Consider, for example, the case of the free group  $F_k$  on  $k$  generators. As is well known, one has  $\beta_1(F_k) = k - 1$ , and the uniform isoperimetric constant  $h(F_k) = 2k - 2$  can be computed by considering large balls with respect to a fixed generating set, e.g., the usual system  $S_k$  of  $k$  free generators (see Appendix C).

### 3. The optimal inequality for countable groups via percolation theory

Percolation theory is concerned with random subgraphs of a fixed graph, usually a Cayley graph. Recently, the notions of cost and  $\ell^2$ -Betti number, as well as the theory of equivalence relations, have found use in percolation theory; see, e.g., [19], Remarks after Conjecture 3.8, [11], or [21], Corollary 3.24. Percolation theory has also been used in the theory of equivalence relations: see [12]. Here, we use percolation theory to give a short proof of the optimal inequality relating uniform isoperimetric constants to  $\ell^2$ -Betti numbers.

We call a graph *transitive* if its automorphism group acts transitively on its vertices. A graph is a *forest* if it contains no cycles. A *percolation* on a graph  $Y$  is a probability measure on subgraphs of  $Y$ ; it is a *foresting* if it is concentrated on forests, while it is *invariant* if it is invariant under all automorphisms of  $Y$ . We denote the degree of a vertex  $x$  in  $Y$  by  $\deg_Y x$ . If  $F$  is an invariant percolation on a transitive graph  $Y$ , then by definition, for every vertex  $x \in Y$ , we have  $2C(F) = \mathbf{E}[\deg_F x]$ . See Definition 2.10 of [11] for the definition of  $\beta_1(Y)$  when  $Y$  is a general transitive graph.

**Theorem 3.1.** *If  $Y$  is a transitive graph and  $F$  is a random invariant foresting of  $Y$ , then*

$$C(F) - 1 \leq \frac{h(Y)}{2}.$$

*Proof.* Let  $A$  be a finite set of vertices in  $Y$ . We have

$$2C(F) = \frac{1}{|A|} \sum_{x \in A} \mathbf{E}[\deg_F x] = \frac{1}{|A|} \mathbf{E} \left[ \sum_{x \in A} \deg_F x \right] \leq \frac{2|A| + |\partial_Y A|}{|A|} = 2 + \frac{|\partial_Y A|}{|A|}.$$

Taking the infimum over  $A$  gives the desired inequality.  $\square$



**Corollary 3.2.** *Let  $Y$  be a transitive graph. Then*

$$\beta_1(Y) \leq \frac{h(Y)}{2}.$$

*Proof.* Apply Theorem 3.1 to the free uniform spanning forest  $F$  of  $Y$ ; see [4] for its definition. The fact that  $C(F) - 1 = \beta_1(Y)$  follows from Theorems 6.4 and 7.8 of [4], an identity first observed in [20] for Cayley graphs.  $\square$

In particular, if  $\Gamma$  is a finitely generated group, then  $\beta_1(\Gamma) \leq h(\Gamma)/2$ .

#### 4. The case of measured equivalence relations

In this section we recall the definition of the first  $\ell^2$ -Betti number and the cost of a measured equivalence relation, define the isoperimetric constant, and show that  $2\beta_1(R) \leq 2C(R) - 2 \leq h(R)$ .

**4.1. Measured equivalence relations.** Let  $(X, \mu)$  be a standard (non-atomic) probability space. An equivalence relation  $R$  with countable classes on  $X$  is called Borel if its graph  $R \subset X \times X$  is a Borel subset of  $X \times X$ . It is called a measured equivalence relation if the  $R$ -saturation of a  $\mu$ -null subset of  $X$  is again  $\mu$ -null. For instance, if  $\Gamma$  is a countable group and  $\alpha$  is a non-singular action of  $\Gamma$  on  $(X, \mu)$ , then the associated equivalence relation  $R_\alpha$ , defined as the partition of  $X$  into  $\Gamma$ -orbits, is a measured equivalence relation on  $X$ . See [8] for more details.

Let  $R$  be a measured equivalence relation on a standard probability space  $(X, \mu)$ . We assume throughout the paper that  $R$  is *ergodic*, i.e., every saturated measurable subset of  $X$  has measure 0 or 1 (in fact our results can be generalized to equivalence relations with *infinite classes*). Endow  $R$  with the horizontal counting measure  $\mathfrak{h}$  given by

$$\mathfrak{h}(K) = \int_X |K^x| d\mu(x),$$

where  $K$  is a measurable subset of  $R$  and  $K^x$  is the subset of  $X \times X$  defined as  $K^x = \{(x, y) \in K\}$ . A partial automorphism of  $R$  is a partial automorphism of  $(X, \mu)$  whose graph is included in  $R$ . One says that  $R$  is of type  $\text{II}_1$  if the measure  $\mu$  is invariant under every partial automorphism of  $R$ . The group of (full) automorphisms of  $R$  is denoted  $[R]$ .

Recall that a *graphing* of a measured equivalence relation can be described in either one the following two ways (cf. [9]):

- (1) a family  $\Phi = \{\varphi_i\}_{i \geq 1}$  of partial automorphisms of  $R$  such that for almost every  $(x, y) \in R$ , there exists a finite sequence  $(\varphi_1, \varphi_2, \dots, \varphi_n)$  of elements of  $\Phi \cup \Phi^{-1}$  with  $y = \varphi_n \dots \varphi_1(x)$ ;

- (2) a measurable subset  $K$  of  $R$  such that  $R$  coincides with  $\bigcup_1^\infty K^n$  up to a negligible (i.e.,  $\mu$ -null) set, where  $K^n$  is the  $n$ -th convolution product of  $K$ .

A graphing  $K$  of  $R$  is said to be *finite* if it can be partitioned into a finite number of partial automorphisms of  $R$ . A type  $\text{II}_1$  equivalence relation  $R$  on  $(X, \mu)$  is said to be *finitely generated* if it admits a finite graphing (this is equivalent to saying that  $R$  has finite cost; see [9]).

**4.2. The first  $\ell^2$ -Betti number.** We sketch the definition of the first  $\ell^2$ -Betti number for a measure-preserving equivalence relation. For more details and proofs, see the paper of Gaboriau [10] (in particular, Section 3.5).

Let  $(X, \mu)$  be a standard probability space and  $\Sigma$  be a measurable field of oriented 2-dimensional cellular complexes. For  $i = 0, 1, 2$ , we denote by  $\mathbb{C}[\Sigma^{(i)}]$  the algebras of functions on the  $i$ -cells of  $\Sigma$  that have uniformly finite support: a function  $f: \Sigma^{(i)} \rightarrow \mathbb{C}$  is in  $\mathbb{C}[\Sigma^{(i)}]$  if and only if there exists a constant  $C_f$  such that for almost every  $x \in X$ , the number of  $i$ -cells  $\sigma$  of  $\Sigma^x$  with  $f(\sigma) \neq 0$  is bounded by  $C_f$ . The completion of  $\mathbb{C}[\Sigma^{(i)}]$  for the norm

$$\|f\|_2^2 = \int_X \sum_{\substack{\sigma \text{ } i\text{-cells} \\ \text{in } \Sigma^x}} |f(\sigma)|^2 d\mu(x)$$

is a Hilbert space, which we denote by  $C_i^{(2)}(\Sigma)$ . If  $\Sigma$  is uniformly locally finite, i.e., if the number of cells attached to a vertex  $\sigma \in \Sigma^{(0)}$  is almost surely bounded by a constant  $C$ , then the natural (measurable fields of) boundary operators  $\partial_i: \mathbb{C}[\Sigma^i] \rightarrow \mathbb{C}[\Sigma^{i-1}]$  coming from the ‘attaching cells maps’ extend to bounded operators  $\partial_i^{(2)}: C_i^{(2)}(\Sigma) \rightarrow C_{i-1}^{(2)}(\Sigma)$ . The *first reduced  $\ell^2$ -homology space* of  $\Sigma$  is then the quotient space

$$\bar{H}_1^{(2)}(\Sigma) = \ker \partial_1^{(2)} / \overline{\text{Im } \partial_2^{(2)}}$$

of the kernel of  $\partial_1^{(2)}$  by the (Hilbert) closure of the image of  $\partial_2^{(2)}$ . It is naturally isometric to the orthogonal complement of  $\overline{\text{Im } \partial_2^{(2)}}$  in  $\ker \partial_1^{(2)}$ .

Now let  $R$  be an ergodic equivalence relation of type  $\text{II}_1$  on  $(X, \mu)$ . Consider a measurable field of oriented 2-dimensional cellular complexes  $\Sigma$  endowed with a measurable action of  $R$  with fundamental domain (see Section 2 in [10]; a fundamental domain  $D$  in  $\Sigma$  is a measurable set of cells of  $\Sigma$  intersecting almost each  $R$ -orbit at a single cell of  $D$ ). Assume that  $\Sigma$  is uniformly locally finite and denote by  $N$  the von Neumann algebra of  $R$  (see, e.g., Section 1.5 in [10]). The first  $\ell^2$ -homology space  $\bar{H}_1^{(2)}(\Sigma)$  is then a Hilbert module and it has a Murray–von Neumann dimension over  $N$ . This dimension is called the first  $\ell^2$ -Betti number of  $\Sigma$  and is denoted by  $\beta_1(\Sigma, R)$ .

Gaboriau [10] has extended this definition beyond the uniformly locally finite case by using an approximation technique in the spirit of Cheeger–Gromov [6]. He then proved that *the associated  $\ell^2$ -Betti number  $\beta_1(\Sigma, R)$  is independent of the choice of  $\Sigma$  provided that  $\Sigma$  is simply connected* (i.e., almost each fiber is simply connected). We refer to this result as Gaboriau’s homotopy invariance theorem (it holds for all  $\ell^2$ -Betti numbers; see [10], Théorème 3.13). The number  $\beta_1(\Sigma, R)$  for a simply connected  $\Sigma$  is called the first  $\ell^2$ -Betti number of  $R$  and is denoted by  $\beta_1(R)$  (note that such a  $\Sigma$  always exists – for instance, one can take the classifying space  $ER$  of  $R$ ; see Section 2.2.1 in [10]).

**4.3. Cost.** Let  $R$  be an ergodic equivalence relation of type  $\text{II}_1$  on a probability space  $(X, \mu)$ . The cost of a partial automorphism  $\varphi: A \rightarrow B$  of  $R$  is the measure of its domain,  $C(\varphi) = \mu(A)$ . The cost of a graphing  $\Phi = \{\varphi_i\}_{i \geq 1}$  of  $R$  is defined to be

$$C(\Phi) = \sum_{i \geq 1} C(\varphi_i),$$

while the *cost of  $R$*  is the infimum

$$C(R) = \inf_{\Phi} C(\Phi),$$

where  $\Phi$  runs among all graphings  $\Phi$  of  $R$ . The *cost of a countable group  $\Gamma$*  is the infimum

$$C(\Gamma) = \inf_{\alpha} C(R_{\alpha}),$$

where  $\alpha$  runs over all ergodic essentially free measure-preserving actions  $\alpha$  of  $\Gamma$  on a probability space. This definition has been introduced by Levitt. See [16] for an exposition.

**4.4. Isoperimetric constants.** Let  $R$  be an ergodic equivalence relation of type  $\text{II}_1$  on a probability space  $(X, \mu)$  and let  $K$  be a graphing of  $R$ . We now define the isoperimetric constant  $h_K(R)$  of  $R$  with respect to  $K$ . Consider the measurable field of graphs  $\Sigma = \bigsqcup_{x \in X} \Sigma^x$  over  $X$  defined as follows (see [10], Section 2): the vertices of  $\Sigma^x$  are the elements of  $R^x$  and the set of edges of  $\Sigma^x$  is the family of pairs  $((x, y), (x, z)) \in R^x \times R^x$  such that  $(y, z) \in K$ .

There is an obvious action of  $R$  on  $\Sigma$  by permutation of fibers. In concordance with group theory, we shall call  $\Sigma$  the *Cayley graph* of  $R$  associated to  $K$ . This is an example of a “quasi-periodic metric space” associated to  $R$  [25].

We write  $\Sigma^{(0)}$  for the set of vertices and  $\Sigma^{(1)}$  for the set of edges of  $\Sigma$  (thus  $\Sigma^{(0)} = R$ ). We define vertices in  $\Sigma$  as follows. This strengthens the corresponding definition of vertices in [25] so as to fit our present purposes (in [25] vertices of  $\Sigma$  were defined to be partially supported *sections of vertices* of the map  $\Sigma \rightarrow X$ ).

**Definition 4.1.** Let  $\Sigma$  be the Cayley graph of  $R$  with respect to a graphing  $K$ . By a *symmetric vertex of  $\Sigma$*  we mean the graph in  $\Sigma^{(0)}$  of an automorphism of the equivalence relation  $R$ . We shall say that two symmetric vertices of  $\Sigma$  are *disjoint* if the corresponding graphs in  $\Sigma^{(0)}$  have  $\mathfrak{h}$ -null intersection.

Given a finite set  $A$  of symmetric vertices of  $\Sigma$ , we denote by  $\partial_K A \subset \Sigma^{(1)}$  the set of edges of  $\Sigma$  with one vertex in  $A$  and the other one outside of  $A$ . We endow  $\Sigma^{(1)}$  with the measure  $\nu^{(1)}$  defined by

$$\nu^{(1)}(E) = \int_X |(E \cap \Sigma^x)| d\mu(x)$$

for a measurable subset  $E$  of  $\Sigma^{(1)}$ .

**Definition 4.2.** The *isoperimetric constant* of  $R$  with respect to a finite graphing  $K$  is the non-negative number

$$h_K(R) = \inf_A \frac{\nu^{(1)}(\partial_K A)}{|A|},$$

where the infimum is taken over all finite sets  $A$  of pairwise disjoint symmetric vertices of  $\Sigma_K$ .

The *uniform isoperimetric constant* of a finitely generated equivalence relation  $R$  is the non-negative number

$$h(R) = \inf_{K \subset R} h_K(R),$$

where the infimum is taken over the finite graphings  $K$  of  $R$ . Note that by definition the uniform isoperimetric constant of an equivalence relation is invariant under isomorphism.

**Remark 4.3.** For every graphing  $K$  of  $R$ , we have

$$h_K(R) \geq \int_X h(\Sigma_K^x) d\mu(x),$$

where  $h(\Sigma_K^x)$  is the isoperimetric constant of the graph  $\Sigma_K^x$ , but strict inequality may occur. The reason is that measured equivalence relations always admit graphings having *vanishing Følner sequences* [27], Définition 7.1, without this having any consequence on their algebraic structure (compare [27], Théorème 7.5). Thus for any type  $\text{II}_1$  equivalence relation  $R$ , one gets by taking the graphing  $K$  of Exemple 7.3 in [27] that

$$\int_X h(\Sigma_K^x) d\mu(x) = 0,$$

and a definition of  $h(R)$  as the infimum over  $K$  of  $\int_X h(\Sigma_K^x) d\mu(x)$  would lead to a trivial invariant. In fact, it is not clear either that our definition of  $h(R)$  can achieve non-trivial numbers. It involves an infimum over all possible (finite) graphings of  $R$ , in the spirit of the cost, so that a proof that  $h(R)$  is a non-trivial invariant requires an homotopy-invariance type of argument. One can give a proof of that relying on Gaboriau’s homotopy invariance theorem [10], Théorème 3.13, for  $L^2$ -Betti numbers (by a straightforward adaptation of Theorem 2.2 to equivalence relations using Definition 4.1). The proof we present below (Theorem 4.5) relies on computations of the cost in [9].

**Definition 4.4.** An ergodic equivalence relation of type  $\text{II}_1$  is called *uniformly non-amenable* if  $h(R) > 0$ .

Note that by Connes–Feldman–Weiss theorem [7] (amenable equivalence relations are singly generated), a uniformly non-amenable ergodic equivalence relation of type  $\text{II}_1$  is not amenable.

#### 4.5. The inequality for measured equivalence relations

**Theorem 4.5.** *Let  $R$  be a finitely generated ergodic equivalence relation of type  $\text{II}_1$ . Then*

$$\beta_1(R) \leq C(R) - 1 \leq \frac{h(R)}{2}.$$

*Proof.* The first inequality is due to Gaboriau [10], Corollaire 3.22, so we concentrate on the second one. As was shown by Pichot (see [25]) and (simultaneously) by Kechris and Miller [16], the equivalence relation  $R$  contains subtrees of cost at least  $C(R)$ . More precisely, by Corollaire 39, p. 25 in [25], for any given graphing  $K$  of  $R$ , there exists a treeing  $F \subset K$  generating an equivalence subrelation  $S$  of  $R$  with  $C(F) = C(S) \geq C(R)$ . Recall that a treeing is a graphing with no simple cycle (see [9], Definition I.2). To prove the theorem, we thus are left to show that

$$2C(F) \leq 2 + h(R).$$

Let  $A$  be a finite set of pairwise disjoint symmetric vertices in  $\Sigma$  in the sense of Definition 4.1, so elements of  $A$  are (graphs of) automorphisms of  $R$ . For  $x \in X$ , let  $\Sigma_F^x$  denote the family of subtrees of  $\Sigma_K^x$  associated to  $F$ . Since  $C(F) = \frac{1}{2} \int_{x \in X} \deg_F(x) d\mu(x)$ , we have

$$2C(F) = \frac{1}{|A|} \sum_{\varphi \in A} \int_{x \in X} \deg_F(\varphi(x)) d\mu(x) = \frac{1}{|A|} \int_{x \in X} \deg_F(A^x) d\mu(x),$$

where  $\text{deg}_F(A^x)$  is the total degree of the points of  $A^x$  in  $\Sigma_F^x$ . Now as  $\Sigma_F^x$  is a family of trees, we get

$$\text{deg}_F(A^x) = 2|E_A^x| + |\partial_F A^x| \leq 2|A^x| + |\partial_F A^x| \leq 2|A| + |\partial_K A^x|,$$

where  $E_A^x$  is the set of the edges in  $\Sigma_F^x$  with both vertices in  $A^x$  and  $\partial_F A^x$  the edges in  $\Sigma_F^x$  with exactly one vertex in  $A^x$ . Hence

$$2C(F) \leq 2 + \frac{1}{|A|} \int_{x \in X} |\partial_K A^x| d\mu(x) = 2 + \frac{v^{(1)}(\partial_K A)}{|A|}.$$

Taking the infimum over  $A$  and then over  $K$ , we get  $2C(F) \leq 2+h(R)$ , as desired.  $\square$

**Remark 4.6.** The proof of Section 2 can also be adapted to give the inequality  $\beta_1(R) \leq h(R)$ , and in fact it gives the same inequality for any  $r$ -discrete measured groupoid [1].

**4.6. Simultaneous vanishing of the invariants.** Theorem 4.5 shows in particular that  $h(R) = 0$  implies  $C(R) = 1$ . We now prove the converse.

**Proposition 4.7.** *Let  $R$  be an ergodic equivalence relation with cost 1. Then  $h(R) = 0$ .*

*Proof.* Let  $R$  be an ergodic equivalence relation with cost 1. Let  $\varphi$  be an ergodic automorphism of  $R$  and for each real number  $\varepsilon > 0$ , let  $\psi_\varepsilon$  be a partial automorphism of  $R$  of cost  $\varepsilon$  such that

$$\Phi_\varepsilon = (\varphi, \psi_\varepsilon)$$

is a graphing of  $R$ . The existence of such graphings of  $R$  is proved in [9], Lemma III.5. Denote by  $R_\varphi$  the equivalence relation generated by  $\varphi$  and fix  $n \in \mathbb{N}$ . By Rokhlin’s Lemma there exists a family  $B_1, \dots, B_n$  of disjoint subsets of  $X$  such that

$$\varphi(B_i) = B_{i+1} \text{ for } i = 1, \dots, n-1 \text{ and } \mu(X \setminus \bigcup_1^n B_i) \leq 1/4.$$

Suppose that  $\varepsilon < \mu(B_1)$  and consider two partial automorphisms  $\theta_{\varepsilon,1}$  and  $\theta_{\varepsilon,2}$  of the equivalence relation  $R_\varphi$  such that  $\theta_{\varepsilon,1}(\text{dom } \psi_\varepsilon) \subset B_1$  and  $\theta_{\varepsilon,2}(\text{Im } \psi_\varepsilon) \subset B_1$ . Set

$$\psi'_\varepsilon = \theta_{\varepsilon,2} \psi_\varepsilon \theta_{\varepsilon,1}^{-1}.$$

It is not hard to check that  $\Phi'_\varepsilon = (\varphi, \psi'_\varepsilon)$  is a graphing of  $R$ . Now letting  $n \rightarrow \infty$  we obtain that for any integer  $n \in \mathbb{N}$  and any  $\varepsilon > 0$  there exists a graphing  $\Phi_{\varepsilon,n}$  of  $R$  of the form  $\Phi_{\varepsilon,n} = (\varphi, \psi_{\varepsilon,n})$  whose cost is less than  $1 + \varepsilon$  and such that for almost every  $x \in X$ , the intersection of either the domain or the image of  $\psi_{\varepsilon,n}$  with the finite set

$$\{x, \varphi(x), \dots, \varphi^n(x)\}$$

is at most one point (this should be compared to the fact that the “concentration of measure” property fails for automorphisms of standard probability spaces by Rokhlin’s Lemma; see [27]). Let us now consider the Cayley graph  $\Sigma_{\varepsilon,n}$  associated to  $\Phi_{\varepsilon,n}$  as in Section 4.4. Let  $A_n$  be the set of (pairwise disjoint) symmetric points of  $\Sigma_{\varepsilon,n}$  given by  $A_n = \{\varphi^i\}_{i=0}^n$ . Then the boundary of  $A_n^x$  in  $\Sigma_{\varepsilon,n}^x$  consists of at most 4 points for almost every  $x \in X$ . It follows that  $h(R) = 0$ .  $\square$

## 5. Some consequences of the main inequality

**5.1. On the Day–von Neumann problem for uniformly non-amenable equivalence relations.** The question of the existence of non-abelian free groups in non-amenable groups is often referred to as the (Day–)von Neumann problem. It was solved negatively by Ol’shanskii in 1980; Adian (1982) showed that the free Burnside groups with large (odd) exponent are non-amenable. Since they do not contain free groups, this gave another negative solution to the Day–von Neumann problem. Osin [23] extended Adian’s result to show that these Burnside groups are uniformly non-amenable (and even that the regular representation has non-vanishing uniform Kazhdan’s constant), whence he deduced the existence of finitely generated groups that are uniformly non-amenable and do not contain any free group on two generators [23], Theorem 1.3.

The Day–von Neumann problem is an open question for measured equivalence relations. It can be formulated in the following way. Let  $(X, \mu)$  be a standard probability space and  $R$  be a non-amenable ergodic equivalence relation of type  $\text{II}_1$  on  $(X, \mu)$ . Is it true that  $R$  contains a non-amenable subtreeing? The result of KeCHRIS–MILLER and of PICHOT recalled in the proof of Theorem 4.5 shows that every ergodic equivalence relation of type  $\text{II}_1$  with cost greater than 1 (and thus non-amenable) contains a non-amenable subtreeing. Combining this with Proposition 4.7, we get the following result.

**Corollary 5.1** (See [23] for the group case). *Let  $R$  be a uniformly non-amenable ergodic equivalence relation of type  $\text{II}_1$ . Then  $R$  contains a non-amenable subtreeing.*

**5.2. Fundamental groups.** Let  $(X, \mu)$  be a probability space and  $R$  be an ergodic equivalence relation of type  $\text{II}_1$  on  $(X, \mu)$ . The so-called *fundamental group* of  $R$  is the multiplicative subgroup of  $\mathbb{R}_+^*$  generated by the measure of all Borel subsets  $Y$  of  $X$  such that the restricted equivalence relation  $R|_Y$  is isomorphic to  $R$ .

The next corollary follows from Proposition 4.7 and the fact that equivalence relations with non-trivial cost have trivial fundamental group, which is proved in [9], Proposition II.6.

**Corollary 5.2.** *A finitely generated ergodic equivalence relation of type  $\text{II}_1$  that is uniformly non-amenable has a trivial fundamental group.*

This corollary can also be proved directly by using the following compression formula.

**Proposition 5.3.** *Let  $R$  be a finitely generated ergodic equivalence relation on  $(X, \mu)$  and  $Y \subset X$  be a non-negligible measurable subset of  $X$ . Let  $S$  be the restriction of  $R$  to  $Y$ . Then*

$$h(R) \leq \mu(Y)h(S).$$

*Proof.* Note that (by definition)  $h(S)$  should be computed with respect to the normalized measure  $\mu_1 = \frac{\mu}{\mu(Y)}$ . Let us prove this inequality (we do not know whether it is an equality).

Fix  $\varepsilon \in (0, \mu(Y))$ . Let  $K$  be a (finite) graphing of  $S$  such that  $h_K(S) \leq h(S) + \varepsilon/4$  and let  $A$  be a finite set of pairwise distinct vertices of  $\Sigma_K$  such that

$$\frac{\nu_1^{(1)}(\partial_K A)}{|A|} < h_K(S) + \varepsilon/4$$

and  $|A| > 12/\varepsilon$  (that one can choose  $A$  arbitrary large follows by *quasi-periodicity*; see the theorem of “repetition of patterns” on p. 56 of [25]), where  $\nu_1^{(1)}$  is the counting measure on  $\Sigma_K^{(1)}$  associated to  $\mu_1$ . Write  $A = \{\psi_1, \dots, \psi_k\}$ , where  $\psi_j \in [S]$ ,  $j = 1, \dots, k$ .

Let  $n$  be an integer greater than  $\max\{k, 4/\varepsilon\}$  and  $\{Y_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$  be a partition of  $X \setminus Y$  into  $n$  measurable subsets of equal measure  $\delta < \varepsilon/4$ . Choose  $Z \subset Y$  such that  $\mu(Z) = \delta$  and consider partial isomorphisms

$$\theta: Z \rightarrow Y_0$$

and

$$\varphi_i: Y_i \rightarrow Y_{i+1}, \quad i \in \mathbb{Z}/n\mathbb{Z},$$

whose graphs are included in  $R$  and such that the automorphism  $\varphi = \coprod_{i \in \mathbb{Z}/n\mathbb{Z}} \varphi_i$  induces an action of  $\mathbb{Z}/n\mathbb{Z}$  on  $X \setminus Y$ . Then

$$K' = K \cup \{\theta\} \cup \{\varphi\}$$

is a graphing of  $R$ . Denote by  $\Sigma_K$  the Cayley graph of  $S$  associated to  $K$  and  $\Sigma_{K'}$  the Cayley graph of  $R$  associated to  $K'$ . For  $j = 1, \dots, k$  consider the automorphism of  $X$  defined by

$$\psi'_j = \varphi_j \coprod \psi_j$$

and observe that the graphs of  $\psi'_j$ ,  $j = 1, \dots, k$ , are pairwise disjoint. Now set  $A' = \{\psi'_j\}_{j=1, \dots, k}$ . Then for  $y \in Y$  one has

$$|(\partial_{K'} A')^y| \leq |(\partial_K A)^y| + |\{\psi \in A \mid \psi(y) \in Z\}|,$$



while for  $y \in X \setminus Y$  one has  $|(\partial_{K'} A')^y| \leq 3$ . Thus

$$v^{(1)}(\partial_{K'} A') \leq \mu(Y)v_1^{(1)}(\partial_K A) + k\delta + 3\mu(X \setminus Y),$$

where  $v^{(1)}$  is the counting measure on  $\Sigma^{(1)}$  associated to  $\mu$ . It follows that

$$h(R) \leq \frac{v^{(1)}(\partial_{K'} A')}{|A'|} \leq \mu(Y)(h(S) + \varepsilon/2) + \varepsilon/2 \leq \mu(Y)h(S) + \varepsilon.$$

So  $h(R) \leq \mu(Y)h(S)$ , as required.  $\square$

**5.3. Ergodic isoperimetric constant of countable groups.** Let  $\Gamma$  be a finitely generated countable group. We define the *ergodic isoperimetric constant* of  $\Gamma$  by the expression

$$h_e(\Gamma) = \inf_{\alpha} h(R_{\alpha}),$$

where the infimum is taken over all ergodic essentially free measure-preserving actions  $\alpha$  of  $\Gamma$  on a probability space,  $R_{\alpha}$  is the partition into the orbits of  $\alpha$ , and  $h(R_{\alpha})$  is the uniform isoperimetric constant of  $R_{\alpha}$ . The following proposition gives another proof of the optimal inequality in the group case (see Section 3).

**Proposition 5.4.** *Let  $\Gamma$  be an infinite countable group and  $\alpha$  be an ergodic essentially free measure-preserving action of  $\Gamma$  on a probability space  $(X, \mu)$ . Let  $R_{\alpha}$  be the orbit partition of  $X$  into the orbits of  $\alpha$ . Then  $2\beta_1(\Gamma) \leq 2C(\Gamma) - 2 \leq h(R_{\alpha}) \leq h(\Gamma)$ .*

*Proof.* The first inequalities are from Corollary 3.22 in [10] and Theorem 4.5, while the last one follows from the definitions (simply note that for any Cayley graph  $Y$  of  $\Gamma$  and any distinct vertices  $\gamma_1, \gamma_2 \in \Gamma = Y^{(0)}$ , the graphs of  $\alpha(\gamma_1^{-1})$  and  $\alpha(\gamma_2^{-1})$  are disjoint vertices in the corresponding Cayley graph of  $R_{\alpha}$  because  $\alpha$  is essentially free).  $\square$

Thus we have  $2\beta_1(\Gamma) \leq 2C(\Gamma) - 2 \leq h_e(\Gamma) \leq h(\Gamma)$ . Examples of groups for which  $h_e(\Gamma) < h(\Gamma)$  are given below.

**Corollary 5.5.** *If a group has cost 1, then its ergodic isoperimetric constant is zero.*

*Proof.* This follows from Proposition 4.7 and from the fact that the infimum over the actions of the group occurring in the definition of the cost is attained [9], Proposition VI.21. Note that the infimum in [9] is taken over all (not necessarily ergodic) measure-preserving essentially free actions, but this infimum is attained (and thus is a minimum) for an ergodic action, as one easily sees by replacing the infinite product measure by an ergodic joining [13] in the proof of Proposition VI.21 in [9].  $\square$

Thus the class of uniformly non-amenable groups appears to be much larger from the geometric point of view than from the ergodic point of view. Breuillard and Gelander proved in [5] that for an arbitrary field  $K$ , any non-amenable and finitely generated subgroup of  $GL_n(K)$  is uniformly non-amenable. This is the case, for instance, for  $SL_3(\mathbb{Z})$ , while this group has cost 1 (see [9]) and thus  $h_e(SL_3(\mathbb{Z})) = 0$ . More generally, if  $\Gamma$  is a lattice in a semi-simple Lie group of real rank at least 2, then  $\Gamma$  has cost 1 by Corollaire VI.30 in [9]. Note that non-uniform lattices have fixed price [9] and in that case any measure-preserving action gives an equivalence relation with trivial uniform isoperimetric constant.

**Corollary 5.6** (Compare [5]). *Lattices in a semi-simple Lie group of real rank at least 2 have trivial ergodic isoperimetric constant.*

For the case of direct product of groups, one gets the following result.

**Corollary 5.7.** *Finitely generated groups that are decomposable as a direct product of two infinite groups have trivial ergodic isoperimetric constant. Finitely generated equivalence relations that are decomposable as a direct product of two equivalence relations with infinite classes have trivial uniform isoperimetric constant.*

*Proof.* This follows from the fact that the cost of a direct product is 1 (see [9], Proposition VI.23, and [16], Proposition 6.21).  $\square$

### A. Comparison with alternative definitions of uniform amenability

In this section, we analyze the differences and analogies between the different notions of uniform non-amenableity. In [2], Arzhantseva, Burillo, Lustig, Reeves, Short and Ventura give a definition of *Følner constants*, which is very close to our definition of isoperimetric constants. The only difference in the definition is the fact that they consider the inner boundary while we consider the edge boundary. For a finitely generated group  $\Gamma$  with generating system  $S$  and  $A$  a finite part of the Cayley graph of  $\Gamma$ , one has

$$F\phi_S(\Gamma, A) = \frac{|\partial_S^{\text{int}} A|}{|A|}$$

with

$$\partial_S^{\text{int}} A = \{a \in A \mid \text{there exists an } x \in S \cup S^{-1} \text{ such that } ax \notin A\},$$

while our boundary is  $\partial_S A = \{(a, ax) \mid x \in S \cup S^{-1}, a \in A, ax \notin A\}$  with  $(a, ax)$  the edge between the vertices  $a$  and  $ax$ .

Considering sets  $A$  without isolated vertices, one immediately obtains that

$$F\phi_S(\Gamma, A) \leq h_S(\Gamma, A) \leq (2|S| - 1) F\phi_S(\Gamma, A), \tag{1}$$

whence one has the following proposition.

**Proposition A.1.** *Let  $F\phi_S(\Gamma)$  be the Følner constant defined in [2]. Then*

$$F\phi_S(\Gamma) \leq h_S(\Gamma) \leq (2|S| - 1) F\phi_S(\Gamma).$$

*In particular,  $F\phi(\Gamma) \leq h(\Gamma)$ .*

This means that a uniformly non-amenable group in the sense of [2] is uniformly non-amenable in our sense. It is unclear whether the converse is true: there might exist groups that are not uniformly non-amenable in the sense of [2] but are uniformly non-amenable in our sense. However, for all known examples of groups that are not uniformly non-amenable in the sense of [2], there exists a maximal bound for the size of the generating systems used to reach the infimum, so these groups are also not uniformly non-amenable in our sense.

Another notion of uniform non-amenability was introduced by Osin in [22], linked with the Kazhdan estimates for the regular representation  $\lambda$ . A group is said to be uniformly non-amenable in the sense of [22] if  $\alpha(\Gamma) = \inf_S \alpha(\Gamma, S) > 0$ , where  $S$  runs over all finite generating systems of  $\Gamma$  and

$$\alpha(\Gamma, S) = \inf\{\max_{x \in S} \|\lambda(x)u - u\| \mid u \in \ell^2(\Gamma), \|u\| = 1\}.$$

(Note that this constant is also presented as a Kazhdan constant for the regular representation in [2] and denoted by  $K(\Lambda_\Gamma, \Gamma)$ .)

Consider now a Cayley graph  $Y$  associated with a system of generators  $S$ , and let  $A$  be a finite subset in  $Y$  and  $u_A$  be the normalized characteristic function  $u = \frac{\chi_A}{\sqrt{|A|}}$ . Then one has, for any  $x \in S$ ,

$$\begin{aligned} \|\lambda(x)u_A - u_A\|^2 &= \frac{1}{|A|} \sum_{g \in \Gamma} [\chi_{A^{-1}}(x^{-1}g) - \chi_{A^{-1}}(g)]^2 \\ &\leq \frac{1}{|A|} \sum_{x \in S, g \in \Gamma} [\chi_A(gx) - \chi_A(g)]^2 = \frac{|\partial_S A|}{|A|}. \end{aligned}$$

This implies that

$$\alpha(\Gamma, S) \leq \|\lambda(x)u_A - u_A\| \leq \sqrt{h_S(\Gamma, A)},$$

whence we have proved the following proposition.

**Proposition A.2.** *Let  $\alpha(\Gamma)$  be the constant introduced by Osin [22]. Then one has*

$$\alpha(\Gamma) \leq \sqrt{h(\Gamma)}.$$

This means that a uniformly non-amenable group in the sense of Osin is still uniformly non-amenable in our sense, while the converse seems to be an open question [2].

## B. Some properties of isoperimetric constants

In this section we restate properties shown in [2] on Følner constants for the isoperimetric constants we introduce. We have omitted the proofs as they are quasi verbatim the ones in [2]. First of all, isoperimetric constants are linked to exponential growth. Recall that the exponential growth rate of a group  $\Gamma$  finitely generated by  $S$  is defined as the limit  $\omega_S(\Gamma) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |B_S(n)|$  where  $B_S(n)$  is the ball of elements in  $\Gamma$  with geodesic distance (as  $S$  words) at most  $n$ .

**Proposition B.1** (Isoperimetric constant and exponential growth). *Let  $\Gamma$  be a finitely generated group and  $S$  a finite generating system. Then*

$$h(\Gamma) \leq h_S(\Gamma) \leq (2|S| - 1) \left[ 1 - \frac{1}{\omega_S(\Gamma)} \right].$$

Note that we do not know in general in our case if uniform non-amenability implies uniform exponential growth. This is the case when we know that the infimum can be attained for systems of generators with uniform bound on the cardinalities of these systems.

Concerning the isoperimetric constants for subgroups and quotients, the theorems of [2] are exactly the same.

**Theorem B.2** (Subgroups). *Let  $\Gamma$  be a finitely generated group and  $S$  a finite system of generators.*

- (1) *Consider the subgroup  $\Gamma'$  generated by a subsystem  $S' \subset S$ . Then it follows that  $h_S(\Gamma) \geq h_{S'}(\Gamma')$ .*
- (2) *Let  $H$  be a subgroup of  $\Gamma$  generated by a system  $T$  with  $m$  elements and such that the length of each element of  $T$  as a word in  $S$  is at most  $L$ . Then  $h_S(\Gamma) \geq \frac{h_T(H)}{1+mL}$ .*

**Theorem B.3** (Quotients). *Let  $\Gamma$  be a finitely generated group and  $S$  a finite system of generators. Denote by  $N$  a normal subgroup and by  $\pi: \Gamma \rightarrow \Gamma/N$  the natural projection. Then  $h_S(\Gamma) \geq h_{\pi(S)}(\Gamma/N)$ , whence  $h(\Gamma) \geq h(\Gamma/N)$ .*

### C. Some bounds on isoperimetric constants

In this section we give an explicit calculation for the free group, as in [2], and derive some bounds for groups by quotient and subgroup theorems.

**Proposition C.1.** *For the free group on  $k$  generators, one has  $h(F_k) = 2k - 2$ .*

*Proof.* By Theorem 3.2, we know that  $h(F_k) \geq 2\beta_1(\Gamma) = 2k - 2$ . On the other hand, by Proposition B.1, we know that

$$h(F_k) \leq (2k - 1)\left(1 - \frac{1}{\omega_S(F_k)}\right)$$

and that  $\omega_S(F_k) = 2k - 1$  for  $S$  a system of  $k$  free generators of  $F_k$ .  $\square$

From this result and the quotients and subgroup theorems, we can, as in [2], derive the following for groups.

**Proposition C.2.** *Let  $\Gamma$  be a finitely generated group that admits a system  $S$  of  $k$  generators. Then  $h(\Gamma) \leq 2k - 2$  with equality if and only if  $\Gamma$  is a free group  $F_k$ .*

**N.B.** The present paper substitutes and extends a short note by the second author circulating during his Ph.D. under the same title, where the group case only (Theorem 2.2) was considered.

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