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Analyse fonctionnelle/*Functional Analysis*

Zero divisors and $L^2(G)$

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Abstract – Let G be a discrete group, let H be a normal subgroup of G, and let $L^2(G)$ denote the Hilbert space with Hilbert basis the elements of G. Suppose $\alpha\beta\neq 0$ whenever $0\neq\alpha\in\mathbb{C}H$ and $0\neq\beta\in L^2(H)$. If G/H is a free group, then we shall prove $\alpha'\beta'\neq 0$ whenever $0\neq\alpha'\in\mathbb{C}G$ and $0\neq\beta'\in L^2(G)$.

Diviseurs de Zéro et $L^2(G)$

Résumé – *Soient* G *un groupe discret,* H *un sous-groupe normal de* G, *et* $L^2(G)$ *l'espace de Hilbert avec base de Hilbert les éléments de* G. *Supposons* $\alpha\beta\neq 0$ *chaque fois que* $0\neq\alpha\in\mathbb{C}H$ *et* $0\neq\beta\in L^2(H)$. *Si* G/H *est un groupe libre, nous prouverons* $\alpha'\beta'\neq 0$ *chaque fois que* $0\neq\alpha'\in\mathbb{C}G$ *et* $0\neq\beta'\in L^2(G)$.

Version française abrégée – Soient G un groupe, $L^\infty(G)$ l'ensemble de toutes les sommes formelles $\sum_{g\in G} a_g g\,(a_g\in\mathbb{C})$ tel que $\sup_{g\in G}|a_g|<\infty$, $L^2(G)$ l'espace de Hilbert avec base de Hilbert $\{g\,|\,g\in G\}$, et $\mathbb{C}G$ l'algèbre de groupe de G sur $\mathbb{C}$. Ainsi $L^2(G)$ consiste en toutes les sommes formelles $\sum_{g\in G} a_g g$ où $a_g\in\mathbb{C}$ et $\sum|a_g|^2<\infty$, $\mathbb{C}G$ consiste en toutes les sommes formelles $\sum_{g\in G} a_g g$ où $a_g\neq 0$ pour tous sauf un nombre fini des $g\in G$, et

$$\mathbb{C}G\subseteq L^2(G)\subseteq L^\infty(G).$$

Si $\alpha=\sum_{g\in G} a_g g\in L^2(G)$ et $\beta=\sum_{g\in G} b_g g\in L^2(G)$, alors on pose

$$\alpha\beta=\sum_{g,h\in G} a_g b_h gh$$

ce qui définit une multiplication $L^2(G)\times L^2(G)\to L^\infty(G)$. Comme dans [6] nous considérons la conjecture suivante :

CONJECTURE I. – *Si* G *est un groupe sans torsion,* $0\neq\alpha\in\mathbb{C}G$ *et* $0\neq\beta\in L^2(G)$, *alors* $\alpha\beta\neq 0$.

Il y a une description de ce problème dans [2]. J'ai montré dans [6] que la conjecture I est vraie si G est un groupe élémentaire moyennable. Le but de cette Note est la démonstration de la conjecture I dans le cas où G est un groupe libre.

Rappelons que G est ordonnable à droite si G est totalement ordonné par une relation $\leqq$ avec la propriété que $a\leqq b$ implique $ag\leqq bg$ $(a, b, g\in G)$. Par le Theorem 7.3.2 de [7], la classe des groupes ordonnables à droite est fermée sous les opérations d'extension de groupe et de produits libres; en particulier les groupes libres sont ordonnables à droite. Le résultat principal de cette Note est le suivant :

THÉORÈME II. – *Soient* $H\lhd G$ *des groupes tels que* G/H *est ordonnable à droite. Si la conjecture I est vraie pour* H, *alors elle est vraie pour* G.

En prenant $H=1$, on voit que la conjecture I est vraie si G est un groupe ordonnable à droite. On pourrait aussi combiner notre résultat avec le Theorem 2 de [6] pour obtenir, par exemple,

PROPOSITION III. – *Soit* H *un sous-groupe normal élémentaire moyennable d'un groupe* G *tel que* G/H *est un groupe libre. Si* $0\neq\alpha\in\mathbb{C}G$ *et* $0\neq\beta\in L^2(G)$ *alors* $\alpha\beta\neq 0$.

Note présentée par Alain CONNES.

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On peut considérer le théorème II comme une généralisation de [4], exercise 8.15, un résultat sur les séries de Fourier, d'où en effet vient l'idée pour notre démonstration.

Voici une esquisse de la démonstration du théorème II dans le cas $H=1$. Soient $\leqq$ l'ordre total de G et * l'involution $\sum a_g g \mapsto \sum \bar{a}_g g^{-1}$ sur $L^2(G)$. On peut supposer que $\alpha = 1 + \sum_{g>1} a_g g$ et $\beta \neq 0$. Soient U le plus petit sous-espace fermé de $L^2(G)$ contenant $\{g\alpha \mid g>1\}$ et V le complément orthogonal de U. Écrivons $\alpha = \xi + \mu$ où $\xi \in U$ et $\mu \in V$. Alors $g\mu \in U$ et ainsi $(\mu, g\mu) = 0$ pour tout $g>1$. Il s'ensuit (imprécisément) que $(\mu\mu^*, g) = 0$ pour tout $g>1$, et nous en déduisons que $(\mu\mu^*, g) = 0$ pour tout $g \in G \setminus 1$. Cela veut dire que $\mu\mu^* = \lambda$ où $\lambda \in \mathbb{C} \setminus 0$. Si $\alpha\beta = 0$, alors $\mu\beta = 0$, et il existe $\gamma \in L^2(G) \setminus 0$ tel que $\gamma\mu = 0$. Ainsi $\gamma\lambda = 0$, et par conséquent $\gamma = 0$, une contradiction.

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1. Introduction. — Let G be a group, let $L^\infty(G)$ denote the set of all formal sums $\sum_{g\in G} a_g g$ $(a_g \in \mathbb{C})$ such that $\sup_{g\in G} |a_g| < \infty$, let $L^2(G)$ denote the Hilbert space with Hilbert basis $\{g \mid g \in G\}$, and let $\mathbb{C}G$ denote the group ring of G over $\mathbb{C}$. Thus $L^2(G)$ consists of all formal sums $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $\sum |a_g|^2 < \infty$, $\mathbb{C}G$ consists of all formal sums $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $a_g = 0$ for all but finitely many g, and $\mathbb{C}G \subseteq L^2(G) \subseteq L^\infty(G)$. If $\alpha = \sum_{g\in G} a_g g \in L^2(G)$ and $\beta = \sum_{g\in G} b_g g \in L^2(G)$ $(b_g \in \mathbb{C})$, then we set

$$\alpha\beta = \sum_{g,h\in G} a_g b_h gh \tag{1}$$

which defines a multiplication $L^2(G) \times L^2(G) \to L^\infty(G)$. As in [6] we consider the following conjecture:

Conjecture 1. — *If* G *is a torsion free group,* $0 \neq \alpha \in \mathbb{C}G$ *and* $0 \neq \beta \in L^2(G)$, *then* $\alpha\beta \neq 0$.

Some background to this problem is given in [2]. In [6] Conjecture 1 was shown to be true if G is an elementary amenable group (*see* [6], p. 349, for the definition of elementary amenable group). The motivation for this paper was to show that Conjecture 1 is also true when G is a free group.

Recall that G is right orderable if G is totally ordered by a relation $\leqq$ with the property that $a \leqq b$ always implies $ag \leqq bg$ $(a, b, g \in G)$. By Theorem 7.3.2 of [7], the class of right orderable groups is closed under taking extensions and free products; in particular free groups are right orderable. The main result of this paper is

Theorem 2. — *Let* $H \lhd G$ *be groups such that* G/H *is right orderable. If Conjecture* 1 *is true for* H, *then it is true for* G.

Taking $H=1$, we immediately see that Conjecture 1 is true if G is a right orderable group. Alternatively one can combine with Theorem 2 of [6] to obtain, for example,

Proposition 3. — *Let* H *be a torsion free normal elementary amenable subgroup of the group* G *such that* G/H *is a free group. If* $0 \neq \alpha \in \mathbb{C}G$ *and* $0 \neq \beta \in L^2(G)$, *then* $\alpha\beta \neq 0$.

In proving Theorem 2, we derive the following more general result.

Theorem 4. — *Let* $H \lhd G$ *be groups such that* G/H *is right orderable with total order* $\leqq$, *and let* $\nu : G \to G/H$ *denote the natural epimorphism. Let* T *be a transversal for* H *in* G, *let* $\alpha \in L^2(G)$, *and write* $\alpha = \sum_{t\in T} \alpha_t t$ *where* $\alpha_t \in L^2(H)$ *for all* $t \in T$. *Suppose there exists*

$\tau \in T$ *such that* $\alpha_t = 0$ *if* $\nu(t) < \nu(\tau)$. *If* $\alpha_\tau \varphi \neq 0$ *whenever* $0 \neq \varphi \in W(H)$, *then* $\alpha\beta \neq 0$ *whenever* $0 \neq \beta \in L^2(G)$.

For the definition of W(H), the group von Neumann algebra of H, *see* Section 2. What is important here is that $W(H) \subseteq L^2(H)$.

Theorem 4 can be considered as a generalization of exercise 8.15 of [4], a result on Fourier series, and the idea for the proof is taken from there.

2. Notation, Terminology and Assumed Results. — We shall use the notation $\mathbb{C}$ for the complex numbers and $^{-}$ for complex conjugation. A nonzero divisor in a ring R is an element $\alpha \in R$ such that $\beta\alpha \neq 0 \neq \alpha\beta$ whenever $0 \neq \beta \in R$. The identity of a group will be denoted by 1. If V is a Hilbert space, $S \subseteq V$, and $u, v \in V$, then (u, v) will indicate the inner product of u and v, $\| u \|_2$ the norm $\sqrt{(u, u)}$ of u, and $\bar{S}$ the norm closure of S in V. For $\alpha = \sum_{g \in G} a_g g \in L^2(G)$ and $\beta = \sum_{g \in G} b_g g \in L^2(G)$ $(a_g, b_g \in \mathbb{C})$, the inner product (α, β) is defined to be $\sum_{g \in G} a_g \bar{b}_g$.

Let $\mathscr{L}$ denote the set of bounded linear operators considered as acting on the left of $L^2(G)$ and for $\theta \in \mathscr{L}$, let $\| \theta \|$ denote the operator norm; thus $\| \theta \| = \max\{ \| \theta\alpha \|_2 \mid \alpha \in L^2(G) \text{ and } \| \alpha \|_2 = 1 \}$. If $\theta \in \mathbb{C}G$, then we have a bounded linear map defined by $\alpha \mapsto \theta\alpha$ (multiplication by θ) for all $\alpha \in L^2(G)$; thus $\mathbb{C}G$ can be identified as a subring of $\mathscr{L}$. By definition W(G) is the weak closure of $\mathbb{C}G$ in $\mathscr{L}$; thus W(G) is a von Neumann algebra, and $\theta \in W(G)$ if and only if there exists a net $\{\theta_i\}$ in $\mathbb{C}G$ such that $(\theta_i u, v)$ converges to $(\theta u, v)$ for all $u, v \in L^2(G)$. Also we have a monomorphism $W(G) \to L^2(G)$ defined by $\theta \mapsto \theta 1$, so W(G) can be identified with a subspace of $L^2(G)$, and then for $\theta \in W(G)$, $\alpha \in L^2(G)$, $\theta\alpha$ is the same element of $L^2(G)$ whether it is calculated using the multiplication in (1) or by considering θ as an operator in $\mathscr{L}$.

If $\alpha = \sum_{g \in G} a_g g \in L^2(G)$ $(a_g \in \mathbb{C})$, we set $\alpha^* = \sum_{g \in G} \bar{a}_g g^{-1}$. When W(G) is identified with a subspace of $L^2(G)$ and $\theta \in W(G)$, then θ^* is the adjoint of θ. Thus $(\theta u, v) = (u, \theta^* v)$ for all $u, v \in L^2(G)$. Note that we also have $(u\theta, \varphi) = (\theta, u^* \varphi)$ for all $\varphi \in W(G)$, and $(u\theta)^* = \theta^* u^*$.

If $H \leqq G$ are groups and $\theta \in W(H)$, then $\theta \in W(G)$ and $\| \theta \|$ is the same whether we consider θ as an operator on $L^2(H)$ or as an operator on $L^2(G)$. For $\alpha = \sum_{g \in G} a_g g \in L^2(G)$ $(a_g \in \mathbb{C})$ we define $\alpha_H = \sum_{g \in H} a_g g$. Clearly $\alpha_H \in L^2(H)$ and we have

Lemma 5. — *Let* $H \lhd G$ *be groups and let* $\theta \in W(G)$. *Then* $\theta_H \in W(H)$ *and* $\| \theta_H \| \leqq \| \theta \|$.

3. Proofs.

Lemma 6. — *Let* G *be a group and let* $\alpha \in W(G)$. *Then there exists* $\theta \in W(G) \setminus 0$ *such that* $\theta\alpha = 0$ *if and only if there exists* $\varphi \in W(G) \setminus 0$ *such that* $\alpha\varphi = 0$.

Proof. — Use the proofs of Lemmas 1 and 2 of [5].

Lemma 7. — *Let* G *be a group and let* $\alpha \in L^2(G)$. *Then there exist nonzero divisors* $\theta, \theta_1 \in W(G)$ *such that* $\alpha\theta, \theta_1 \alpha \in W(G)$.

Proof. — Using $(\alpha\theta)^* = \theta^* \alpha^*$, it will be sufficient to show the existence of θ. Define an unbounded operator β on the dense subspace W(G) of $L^2(G)$ by $\beta\lambda = \alpha\lambda$ for all $\lambda \in W(G)$. Since $(\beta\lambda, \mu) = (\lambda, \alpha^* \mu)$ for all $\mu \in W(G)$, we see that β has an adjoint β^* defined on W(G) by $\beta^* \mu = \alpha^* \mu$. Therefore β extends to a closed operator $\hat{\alpha}$. Now the commutant $W(G)'$ of W(G) is just the operators of the form $\lambda \mapsto \lambda\gamma$ for $\lambda \in L^2(G)$

and $\gamma \in W(G)$, so $\varphi\hat{\alpha}\varphi^* = \hat{\alpha}$ for all unitary $\varphi \in W(G)'$ and hence $\hat{\alpha}$ is affiliated to W (G) [1], p. 150. Since W (G) is a finite von Neumann algebra by a theorem of Kaplansky, the proof of Theorem 10 of [1] shows that there is a nonzero divisor $\theta \in W(G)$ such that $\hat{\alpha}\theta \in W(G)$. Then $\alpha\theta \in W(G)$ as required.

LEMMA 8. — *Let* $H \triangleleft G$ *be groups and let* $\alpha \in L^2(G)$. *Suppose* $(\alpha, t\alpha) = 0$ *for all* $t \in G \setminus H$ *and* $\varphi\alpha_H \neq 0$ *whenever* $0 \neq \varphi \in W(H)$. *If* $0 \neq \theta \in W(G)$, *then* $\theta\alpha \neq 0$.

Proof. — Suppose $\theta\alpha = 0$. Clearly we may assume that $\theta_H \neq 0$. By Kaplansky's density theorem [3], I.3.5, there exists a net $\{\varphi_i\}$ in $\mathbb{C}G$ such that $\|\varphi_i\| \leqq \|\theta - \theta_H\|$ and $\{\varphi_i\}$ converges to $\theta - \theta_H$ in the weak topology on W (G). Then $\|(\varphi_i)_H\| \leqq \|\theta - \theta_H\|$ by Lemma 5 and $\{(\varphi_i)_H\}$ converges to zero in the weak topology on W (H). Therefore $\{(\varphi_i)_H\}$ converges to zero in the weak topology on W (G), hence $\varphi_i - (\varphi_i)_H$ converges to $\theta - \theta_H$ in the weak topology on W (G). Since $(h\alpha, t\alpha) = 0$ for all $h \in H$ and $t \in G \setminus H$, we deduce that

$$(h\alpha, \theta_H \alpha) = (-h\alpha, (\theta - \theta_H)\alpha) = 0 \quad \text{for all } h \in H.$$

Now $\theta_H \in W(H)$ by Lemma 5 so using Kaplansky's density theorem [3], I.3.5, again, there is a net $\{\theta_i\}$ in $\mathbb{C}H$ such that $\|\theta_i\| \leqq \|\theta_H\|$ and $\{\theta_i\}$ converges to θ_H in the weak topology on W (H). Therefore $\{\theta_i\}$ converges to θ_H in the weak topology on W (G) and we deduce that $(\theta_H\alpha, \theta_H\alpha) = 0$. Thus $\theta_H\alpha_H = 0$, a contradiction.

Proof of Theorem 4. — Clearly we may assume that $1 \in T$ and $\tau = 1$. Define

$$S = \{g \in G \mid \nu(g) > 1\},$$

$$L = \{\sum_{s \in S} a_s s \mid a_s \in \mathbb{C} \text{ and } a_s = 0 \text{ for all but finitely many } s\} \subseteqq \mathbb{C}G.$$

Write $\alpha = \xi + \mu$ where $\xi \in \overline{L\alpha}$ and $(\mu, \lambda) = 0$ for all $\lambda \in L\alpha$. Then $s\mu \in \overline{L\alpha}$ and hence $(\mu, s\mu) = 0$ for all $s \in S$. Therefore

$$(\mu, s^{-1}\mu) = (s\mu, \mu) = \overline{(\mu, s\mu)} = 0 \quad \text{for all } s \in S$$

and we deduce that $(\mu, x\mu) = 0$ for all $x \in G \setminus H$. Note that we may write $\mu = \alpha_1 + \sum_{s \in S} \mu_s s$ where $\mu_s \in \mathbb{C}$ for all $s \in S$. Since $\alpha_1\varphi \neq 0$ whenever $0 \neq \varphi \in W(H)$, it follows from Lemmas 6 and 7 that $\varphi\alpha_1 \neq 0$ whenever $0 \neq \varphi \in W(H)$ and hence $\theta\mu \neq 0$ whenever $0 \neq \theta \in W(G)$ by Lemma 8. Another application of Lemmas 6 and 7 shows that $\mu\theta \neq 0$ and hence $\alpha\theta \neq 0$ whenever $0 \neq \theta \in W(G)$.

Let $0 \neq \beta \in L^2(G)$ and use Lemma 7 to choose a nonzero divisor $\psi \in W(G)$ such that $\beta\psi \in W(G)$. Suppose $\beta\psi = 0$. Then $\psi^*\beta^* = 0$. If $e \in \mathscr{L}$ is the projection from $L^2(G)$ onto $\overline{\beta^*\mathbb{C}G}$, then $e \in W(G)$ by [6], Lemma 5, $e \neq 0$ because $\beta^* \neq 0$, and $\psi^* e = 0$. Hence $e\psi = 0$, a contradiction, so $\beta\psi \neq 0$. Therefore $\alpha\beta\psi \neq 0$ and the result follows.

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