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NONCOMMUTATIVE LOCALIZATION IN GROUP RINGS

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Abstract. This paper will briefly survey some recent methods of localization in group rings, which work in more general contexts than the classical Ore localization. In particular the Cohn localization using matrices will be described, but other methods will also be considered.

1. Introduction

Let R be a commutative ring and let $S = \{s \in R \mid sr \neq 0 \text{ for all } r \in R \setminus 0\}$, the set of non-zerodivisors of R. Then, as in the same manner one constructs $\mathbb{Q}$ from $\mathbb{Z}$, we can form the quotient ring RS^{-1} which consists of elements of the form r/s with $r \in R$ and $s \in S$, and in which $r_1/s_1 = r_2/s_2$ if and only if $r_1s_2 = s_1r_2$. We can consider R as a subring RS^{-1} by identifying $r \in R$ with $r/1 \in RS^{-1}$. Then RS^{-1} is a ring containing R with the property that every element is either a zerodivisor or invertible. Furthermore, every element of RS^{-1} can be written in the form rs^{-1} with $r \in R$ and $s \in S$ (though not uniquely so). In the case R is an integral domain, then RS^{-1} will be a field and will be generated as a field by R (i.e. if K is a subfield of RS^{-1} containing R, then $K = RS^{-1}$). Moreover if K is another field containing R which is generated by R, then K is isomorphic to RS^{-1} and in fact there is a ring isomorphism $RS^{-1} \to K$ which is the identity on R.

The question we will be concerned with here is what one can do with a noncommutative ring R; certainly many of the above results do not hold in general. In particular, Malcev [24] constructed domains which are not embeddable in division rings. We will concentrate on the case when our ring is a crossed product $k * G$, where k is a division ring and G is a group [25], and in particular when the crossed product is the group ring kG with k a field. A field will always mean a commutative field, and we shall use the terminology "division ring" for the noncommutative case. Though our main interest is in group rings, often it is a trivial matter to extend results to crossed products. This has the advantage of facilitating induction arguments, because if $H \lhd G$ and $k * G$ is a crossed product, then $k * G$ can also be viewed as a crossed product $(k * H) * (G/H)$ [25, p. 2].

2. Ore Localization

We shall briefly recall the definition of a crossed product, and also establish some notational conventions for this paper. Let R be a ring with a 1 and let G be a group. Then a crossed product of G over R is an associative ring $R * G$ which is also a free left R-module with basis $\{\bar{g} \mid g \in G\}$. Multiplication is given by $\bar{x}\bar{y} = \tau(x, y)\overline{xy}$ where $\tau(x, y)$ is a unit of R for all $x, y \in G$. Furthermore we assume that $\bar{1}$ is the

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identity of $R*G$, and we identify R with $R\bar{1}$ via $r \mapsto r\bar{1}$. Finally $\bar{x}r = r^{\sigma(x)}\bar{x}$ where $\sigma(x)$ is an automorphism of R for all $x \in G$; see [25, p. 2] for further details.

We shall assume that all rings have a 1, subrings have the same 1, and ring homomorphisms preserve the 1. We say that the element s of R is a non-zerodivisor (sometimes called a regular element) if $sr \neq 0 \neq rs$ whenever $0 \neq r \in R$; otherwise s is called a zerodivisor. Let S denote the set of non-zerodivisors of the ring R. The simplest extension to noncommutative rings is when the ring R satisfies the *right Ore condition*, that is given $r \in R$ and $s \in S$, then there exists $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$. In this situation one can form the Ore localization RS^{-1}, which in the same way as above consists of elements of the form $\{rs^{-1} \mid r \in R,\, s \in S\}$. If $s_1 s = s_2 r$, then $r_1 s_1^{-1} = r_2 s_2^{-1}$ if and only if $r_1 s = r_2 r$; this does not depend on the choice of r and s. To define addition in RS^{-1}, note that any two elements can be written in the form $r_1 s^{-1}, r_2 s^{-1}$ (i.e. have the same common denominator), and then we set $r_1 s^{-1} + r_2 s^{-1} = (r_1 + r_2)s^{-1}$. To define multiplication, if $s_1 r = r_2 s$, we set $(r_1 s_1^{-1})(r_2 s_2^{-1}) = r_1 r (s_2 s)^{-1}$. Then RS^{-1} is a ring with $1 = 11^{-1}$ and $0 = 01^{-1}$, and $\{r1^{-1} \mid r \in R\}$ is a subring isomorphic to R via the map $r \mapsto r1^{-1}$. Furthermore RS^{-1} has the following properties:

- Every element of S is invertible in RS^{-1}.
- Every element of RS^{-1} is either invertible or a zerodivisor.
- If $\theta \colon R \to K$ is a ring homomorphism such that θs is invertible for all $s \in S$, then there is a unique ring homomorphism $\theta' \colon RS^{-1} \to K$ such that $\theta'(r1^{-1}) = \theta r$ for all $r \in R$; in other words, θ can be extended in a unique way to RS^{-1}.
- RS^{-1} is a flat left R-module [30, Proposition II.3.5].

Of course one also has the left Ore condition, which means that given $r \in R$ and $s \in S$, one can find $r_1 \in R$ and $s_1 \in S$ such that $s_1 r = r_1 s$, and then one can form the ring $S^{-1}R$, which consists of elements of the form $s^{-1}r$ with $s \in S$ and $r \in R$. However in the case of the group ring kG for a field k and group G, they are equivalent by using the involution on kG induced by $g \mapsto g^{-1}$ for $g \in G$. When a ring satisfies both the left and right Ore condition, then the rings $S^{-1}R$ and RS^{-1} are isomorphic, and can be identified. In this situation, we say that RS^{-1} is a classical ring of quotients for R. When R is a domain, a classical ring of quotients will be a division ring. On the other hand if already every element of R is either invertible or a zerodivisor, then R is its own classical quotient ring. For more information on Ore localization, see [13, §9].

Problem 2.1. *Let k be a field. For which groups G does kG have a classical quotient ring?*

One could ask more generally given a division ring D, for which groups G does a crossed product $D*G$ always have a classical quotient ring? We have put in the "always" because D and G do not determine a crossed product $D*G$. One could equally consider the same question with "always" replaced by "never".

For a nonnegative integer n, let F_n denote the free group on n generators, which is nonabelian for $n \geq 2$. If G is abelian in Problem 2.1, then kG certainly has a classical quotient ring because kG is commutative in this case. On the other hand if G has a subgroup isomorphic to F_2, then $D*G$ cannot have a classical quotient ring. We give an elementary proof of this well-known statement, which is based on [18, Theorem 1].

Proposition 2.2. *Let G be a group which has a subgroup isomorphic to the free group F_2 on two generators, let D be a division ring, and let $D * G$ be a crossed product. Then $D*G$ does not satisfy the right Ore condition, and in particular does not have a classical quotient ring.*

Proof. First suppose G is free on a, b. We prove that $(\bar{a}-1)D*G \cap (\bar{b}-1)D*G = 0$. Write $A = \langle a \rangle$ and $B = \langle b \rangle$. Suppose $\alpha \in (\bar{a}-1)D*G \cap (\bar{b}-1)D*G$. Then we may write

$$\alpha = \sum_i (u_i - 1)\overline{x_i} d_i = \sum_i (v_i - 1)\overline{y_i} e_i \tag{2.1}$$

where $u_i = \bar{a}^{q(i)}$ for some $q(i) \in \mathbb{Z}$, $v_i = \bar{b}^{r(i)}$ for some $r(i) \in \mathbb{Z}$, $d_i, e_i \in D$ and $x_i, y_i \in G$. The general element g of G can be written in a unique way $g_1 \dots g_l$, where the g_i are alternately in A and B, and $g_i \neq 1$ for all i; we shall define the length $\lambda(g)$ of g to be l. Of course $\lambda(1) = 0$. Let L be the maximum of all $\lambda(x_i), \lambda(y_i)$, let s denote the number of x_i with $\lambda(x_i) = L$, and let t denote the number of y_i with $\lambda(y_i) = L$. We shall use induction on L and then on $s+t$, to show that $\alpha = 0$. If $L = 0$, then $x_i, y_i = 1$ for all i and the result is obvious. If $L > 0$, then without loss of generality, we may assume that $s > 0$. Suppose $\lambda(x_i) = L$ and x_i starts with an element from A, so $x_i = a^p h$ where $0 \neq p \in \mathbb{Z}$ and $\lambda(h) = L - 1$. Then

$$(u_i - 1)\overline{x_i} d_i = (\bar{a}^{q(i)} - 1)\bar{a}^p \bar{h} d d_i = (\bar{a}^{q(i)+p} - 1)\bar{h} d d_i - (\bar{a}^p - 1)\bar{h} d d_i$$

for some $d \in D$. This means that we have found an expression for α with smaller $s + t$, so all the x_i with $\lambda(x_i) = L$ start with an element from B. Therefore if $\beta = \sum_i u_i \overline{x_i} d_i$ where the sum is over all i such that $\lambda(x_i) = L$, then each x_i starts with an element of B and hence $\lambda(a^{q(i)} x_i) = L + 1$. We now see from (2.1) that $\beta = 0$. Since $s > 0$ by assumption, the expression for β above is nontrivial and therefore there exists $i \neq j$ such that $a^{q(i)} x_i = a^{q(j)} x_j$. This forces $q(i) = q(j)$ and $x_i = x_j$. Thus $u_i = u_j$ and we may replace $(u_i - 1)\overline{x_i} d_i + (u_j - 1)\overline{x_j} d_j$ with $(u_i - 1)\overline{x_i}(d_i + d_j)$, thereby reducing s by 1 and the proof that $(\bar{a}-1)D*G \cap (\bar{b}-1)D*G = 0$ is complete.

In general, suppose G has a subgroup H which is free on the elements x, y. Then the above shows that $(x-1)D*H \cap (y-1)D*H = 0$, and it follows that $(x-1)D*G \cap (y-1)D*G = 0$. Since $x - 1$ and $y - 1$ are non-zerodivisors in $D * G$, it follows that $D * G$ does not have the right Ore property. □

Recall that the class of elementary amenable groups is the smallest class of groups which contains all finite groups and the infinite cyclic group $\mathbb{Z}$, and is closed under taking group extensions and directed unions. It is not difficult to show that the class of elementary amenable groups is closed under taking subgroups and quotient groups, and contains all solvable-by-finite groups. Moreover every elementary amenable group is amenable, but F_2 is not amenable. Thus any group which has a subgroup isomorphic to F_2 is not elementary amenable. Also Thompson's group F [4, Theorem 4.10] and the Gupta-Sidki group [14] are not elementary amenable even though they do not contain F_2. The Gupta-Sidki has sub-exponential growth [10] and is therefore amenable [26, Proposition 6.8]. The following result follows from [17, Theorem 1.2]

Theorem 2.3. *Let G be an elementary amenable group, let D be a division ring, and let $D * G$ be a crossed product. If the finite subgroups of G have bounded order, then $D * G$ has a classical ring of quotients.*

It would seem plausible that Theorem 2.3 would remain true without the hypothesis that the finite subgroups have bounded order. After all, if G is a locally finite group and k is a field, then kG is a classical quotient ring for itself. However the lamplighter group, which we now describe, yields a counterexample. If A, C are groups, then $A \wr C$ will indicate the Wreath product with base group $B := A^{|C|}$, the direct sum of $|C|$ copies of A. Thus B is a normal subgroup of $A \wr C$ with corresponding quotient group isomorphic to C, and C permutes the $|C|$ copies of A regularly. The case $A = \mathbb{Z}/2\mathbb{Z}$ and $C = \mathbb{Z}$ is often called the lamplighter group. Then [21, Theorem 2] is

Theorem 2.4. *Let $H \neq 1$ be a finite group, let k be a field, and let G be a group containing $H \wr \mathbb{Z}$. Then kG does not have a classical ring of quotients.*

Thus we have the following problem.

Problem 2.5. *Let k be a field. Classify the elementary amenable groups G for which kG has a classical ring of quotients. If $H \leqslant G$ and kG has a classical ring of quotients, does kH also have a classical ring of quotients?*

The obstacle to preventing a classical quotient ring in the case of elementary amenable groups is the finite subgroups having unbounded order, so let us consider the case of torsion-free groups. In this situation it is unknown whether kG is a domain, so let us assume that this is the case. Then we have the following result of Tamari [31]; see [8, Theorem 6.3], also [23, Example 8.16], for a proof.

Theorem 2.6. *Let G be an amenable group, let D be a division ring, and let $D*G$ be a crossed product which is a domain. Then $D*G$ has a classical ring of quotients which is a division ring.*

What about torsion-free groups which do not contain F_2, yet are not amenable? Given such a group G and a division ring D, it is unknown whether a crossed product $D * G$ has a classical quotient ring. Thompson's group F is orderable [4, Theorem 4.11]; this means that it has a total order $\leq$ which is left and right invariant, so if $a \leq b$ and $g \in F$, then $ga \leq gb$ and $ag \leq bg$. Therefore if D is a division ring and $D * F$ is a crossed product, then by the Malcev-Neumann construction [5, Corollary 8.7.6] the power series ring $D((F))$ consisting of elements with well-ordered support is a division ring. It is still unknown whether Thompson's group is amenable. We state the following problem.

Problem 2.7. *Let F denote Thompson's orderable group and let D be a division ring. Does $D * F$ have a classical ring of quotients?*

If the answer is negative, then Theorem 2.6 would tell us that Thompson's group is not amenable. Since Thompson's group seems to be right on the borderline between amenability and nonamenability, one would expect the answer to be in the affirmative.

3. Cohn's Theory

What happens when the ring R does not have the Ore condition, in other words R does not have a classical ring of quotients? Trying to form a ring from R by inverting the non-zerodivisors of R does not seem very useful. The key idea here is due to Paul Cohn; instead of trying to invert just elements, one inverts matrices instead. Suppose Σ is any set of matrices over R (not necessarily square, though in practice

Σ will consist only of square matrices) and $\theta\colon R \to S$ is a ring homomorphism. If M is a matrix with entries $m_{ij} \in R$, then θM will indicate the matrix over S which has entries $\theta(m_{ij})$. We say that θ is Σ-inverting if θM is invertible over S for all $M \in \Sigma$. We can now define the universal localization of R with respect to Σ, which consists of a ring R and a *universal* Σ-inverting ring homomorphism $\lambda\colon R \to R_\Sigma$. This means that given any other Σ-inverting homomorphism $\theta\colon R \to S$, then there is a unique ring homomorphism $\phi\colon R_\Sigma \to S$ such that $\theta = \phi\lambda$. The ring R_Σ always exists by [5, Theorem 7.2.1], and by the universal property is unique up to isomorphism. Furthermore λ is injective if and only if R can be embedded in a ring over which all the matrices in Σ become invertible.

A related concept is the Σ-rational closure. Given a set of matrices Σ over R and a Σ-inverting ring homomorphism $\theta\colon R \to S$, the Σ-rational closure $R_\Sigma(S)$ of R in S consists of all entries of inverses of matrices in $\theta(\Sigma)$. In general $R_\Sigma(S)$ will not be a subring of S. We say that Σ is upper multiplicative if given $A, B \in \Sigma$, then $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma$ for any matrix C of the appropriate size. If in addition permuting the rows and columns of a matrix in Σ leaves it in Σ, then we say that Σ is multiplicative.

Suppose now that Σ is a set of matrices over R and $\theta\colon R \to S$ is a Σ-inverting ring homomorphism. If Σ is upper multiplicative, then $R_\Sigma(S)$ is a subring of S [5, Theorem 7.1.2]. Also if Φ is the set of matrices over R whose image under θ becomes invertible over S, then Φ is multiplicative [5, Proposition 7.1.1]. In this situation we call $R_\Phi(S)$ the rational closure $\mathcal{R}_S(R)$ of R in S. By the universal property of R_Φ, there is a ring homomorphism $R_\Phi \to R_\Phi(S) = \mathcal{R}_S(R)$ which is surjective. A very useful tool is the following consequence of [5, Proposition 7.1.3], which we shall call Cramer's rule; we shall let $\mathrm{M}_n(R)$ denote the $n \times n$ matrices over R.

Proposition 3.1. *Let Σ be an upper multiplicative set of matrices of R and let $\theta\colon R \to S$ be a Σ-inverting ring homomorphism. If $p \in R_\Sigma(S)$, then p is stably associated to a matrix with entries in $\theta(R)$. This means that there exists a positive integer n and invertible matrices $A, B \in \mathrm{M}_n(S)$ such that $A \operatorname{diag}(p, 1, \dots, 1)B \in \mathrm{M}_n(\theta R)$.*

Given a ring homomorphism $\theta\colon R \to S$ and an upper multiplicative set of matrices Σ of R, the natural epimorphism $R_\Sigma \twoheadrightarrow R_\Sigma(S)$ will in general not be isomorphism, even if θ is injective, but there are interesting situations where it is; we describe one of them. Let k be a PID (principal ideal domain), let X be a set, let $k\langle X\rangle$ denote the free algebra on X, let $k\langle\langle X\rangle\rangle$ denote the noncommutative power series ring on X, and let Λ denote the subring of $k\langle\langle X\rangle\rangle$ generated by $k\langle X\rangle$ and $\{(1+x)^{-1} \mid x \in X\}$. Then $\Lambda \cong kF$ where F denotes the free group on X [5, p. 529]. Let Σ consist of those square matrices over Λ with constant term invertible over k, and let $\Sigma' = \Sigma \cap k\langle X\rangle$. If we identify Λ with kF by the above isomorphism, then Σ consist of those matrices over kF which become invertible under the augmentation map $kF \to k$. Since Σ and Σ' are precisely the matrices over Λ and $k\langle X\rangle$ which become invertible over $k\langle\langle X\rangle\rangle$ respectively, we see that $k\langle X\rangle_{\Sigma'}(k\langle\langle X\rangle\rangle) = \Lambda_\Sigma(k\langle\langle X\rangle\rangle) = \mathcal{R}_{k\langle\langle X\rangle\rangle}(k\langle X\rangle) = \mathcal{R}_{k\langle\langle X\rangle\rangle}(\Lambda)$. By universal properties, we have a sequence of natural maps

$$\Lambda \xrightarrow{\alpha} k\langle X\rangle_{\Sigma'} \xrightarrow{\beta} \Lambda_\Sigma \xrightarrow{\gamma} k\langle X\rangle_{\Sigma'}(k\langle\langle X\rangle\rangle).$$

The map $\gamma\beta$ is an isomorphism by [7, Theorem 24]. Therefore the image under α of every matrix in Σ becomes invertible in $k\langle X\rangle_{\Sigma'}$, hence there is a natural map $\phi\colon \Lambda_\Sigma \to k\langle X\rangle_{\Sigma'}$ such that $\beta\phi$ and $\phi\beta$ are the identity maps. We deduce that γ is also an isomorphism. It would be interesting to know if γ remains an isomorphism if k is assumed to be only an integral domain. We state the following problem.

Problem 3.2. *Let X be a set, let F denote the free group on X, and let k be an integral domain. Define a k-algebra monomorphism $\theta\colon kF \to k\langle\langle X\rangle\rangle$ by $\theta(a) = a$ for $a \in k$ and $\theta(x) = 1 + x$ for $x \in X$, let Σ be the set of matrices over kF which become invertible over $k\langle\langle X\rangle\rangle$ via θ, and let $\phi\colon kF_\Sigma \to k\langle\langle X\rangle\rangle$ be the uniquely defined associated ring homomorphism. Determine when ϕ is injective.*

If R is a subring of the ring T, then we define the division closure $\mathcal{D}_T(R)$ of R in T to be the smallest subring $\mathcal{D}_T(R)$ of T containing R which is closed under taking inverses, i.e. $x \in \mathcal{D}_T(R)$ and $x^{-1} \in T$ implies $x^{-1} \in \mathcal{D}_T(R)$. In general $\mathcal{D}_T(R) \subseteq \mathcal{R}_T(R)$, i.e. the division closure is contained in the rational closure [5, Exercise 7.1.1]. However if T is a division ring, then the rational closure is a division ring and is equal to the division closure.

It is clear that taking the division closure is an idempotent operation; in other words $\mathcal{D}_T(\mathcal{D}_T(R)) = \mathcal{D}_T(R)$. It is also true that taking the rational closure is an idempotent operation; we sketch the proof below.

Proposition 3.3. *Let R be a subring of the ring T and assume that R and T have the same 1. Then $\mathcal{R}_T(\mathcal{R}_T(R)) = \mathcal{R}_T(R)$.*

Proof. Write $R' = \mathcal{R}_T(R)$ and let M be a matrix over R' which is invertible over T; we need to prove that all the entries of M^{-1} are in R'. We may assume that $M \in \mathrm{M}_d(R')$ for some positive integer d. Cramer's rule, Proposition 3.1, applied to the inclusion $\mathrm{M}_d(R) \to \mathrm{M}_d(R')$ tells us that M is stably associated to a matrix with entries in $\mathrm{M}_d(R)$. This means that for some positive integer e, there exists a matrix $L \in \mathrm{M}_e(\mathrm{M}_d(R)) = \mathrm{M}_{de}(R)$ of the form $\mathrm{diag}(M, 1, \dots, 1)$ and invertible matrices $A, B \in \mathrm{M}_{de}(R')$ such that ALB is a matrix $X \in \mathrm{M}_{de}(R)$.

Since A, L, B are all invertible in $\mathrm{M}_{de}(T)$, we see that X^{-1} has (by definition of rational closure) all its entries in $\mathrm{M}_{de}(R')$. But $L^{-1} = BX^{-1}A$, which shows that $L^{-1} \in \mathrm{M}_{de}(R')$. Therefore $M^{-1} \in \mathrm{M}_d(R')$ as required. □

We also have the following useful result.

Proposition 3.4. *Let n be a positive integer, let R be a subring of the ring T, and assume that R and T have the same 1. Then $\mathcal{R}_{\mathrm{M}_n(T)}(\mathrm{M}_n(R)) = \mathrm{M}_n(\mathcal{R}_T(R))$.*

Proof. Write $R' = \mathcal{R}_T(R)$ and $S = \mathrm{M}_n(T)$. Suppose $M \in \mathcal{R}_S(\mathrm{M}_n(R))$. Then M appears as an entry of A^{-1}, where $A \in \mathrm{M}_d(\mathrm{M}_n(R))$ for some positive integer d is invertible in $\mathrm{M}_d(S)$. By definition all the entries of A^{-1} (when viewed as a matrix in $\mathrm{M}_{dn}(T)$) are in R', which shows that $M \in \mathrm{M}_n(R')$.

Now let $M \in \mathrm{M}_n(R')$. We want to show that $M \in \mathcal{R}_S(\mathrm{M}_n(R))$. Since $\mathcal{R}_S(\mathrm{M}_n(R))$ is a ring, it is closed under addition, so we may assume that M has exactly one nonzero entry. Let a be this entry. Then a appears as an entry of A^{-1} where A is an invertible matrix in $\mathrm{M}_m(R)$ for some positive integer m which is a multiple of n. By permuting the rows and columns, we may assume that a is the $(1,1)$-entry. Now form the $p \times p$ matrix $B = \mathrm{diag}(1, \dots, 1, A, 1, \dots, 1)$, so that the $(1,1)$-entry of A is in the (n,n)-entry of B (thus there are $n-1$ ones on the main diagonal and then A)

and m divides p. By considering B^{-1}, we see that $\operatorname{diag}(1,\dots,1,a) \in \mathcal{R}_S(\mathrm{M}_n(R))$. Since $\operatorname{diag}(1,\dots,1,0) \in \mathcal{R}_S(\mathrm{M}_n(R))$, it follows that $\operatorname{diag}(0,\dots,0,a) \in \mathcal{R}_S(\mathrm{M}_n(R))$. By permuting the rows and columns, we conclude that $M \in \mathcal{R}_S(\mathrm{M}_n(R))$. □

When one performs a localization, it would be good to end up with a local ring. We now describe a result of Sheiham [28, §2] which shows that this is often the case. For any ring R, we let $\operatorname{Jac}(R)$ indicate the Jacobson radical of R. Let $\theta\colon R \to S$ be a ring homomorphism, let Σ denote the set of all matrices A over R with the property that $\theta(A)$ is an invertible matrix over S, and let $\lambda\colon R \to R_\Sigma$ denote the associated map. Then we have a ring homomorphism $\phi\colon R_\Sigma \to S$ such that $\theta = \phi\lambda$, and Sheiham's result is

Theorem 3.5. *If S is a local ring, then $\phi^{-1}\operatorname{Jac}(S) = \operatorname{Jac}(R_\Sigma)$*

Thus in particular if S is a division ring, then R_Σ is a local ring.

4. Uniqueness of Division Closure and Unbounded Operators

If R is a domain and D is a division ring containing R such that $\mathcal{D}_D(R) = D$ (i.e. R generates D as a field), then we say that D is a division ring of fractions for R. If R is an integral domain and D, E are division rings of fractions for R, then D and E are fields and are just the Ore localizations of R with respect to the nonzero elements of R. In this case there exists a unique isomorphism $D \to E$ which is the identity on R. Furthermore any automorphism of D can be extended to an automorphism of R.

When D and E are not commutative, i.e. it is only assumed that they are division rings, then this is not the case; in fact D and E may not be isomorphic even just as rings. Therefore we would like to have a criterion for when two such division rings are isomorphic, and also a criterion for the closely related property of when an automorphism of R can be extended to an automorphism of D.

Consider now the complex group algebra $R = \mathbb{C}G$. Here we may embed $\mathbb{C}G$ into the ring of unbounded operators $\mathcal{U}(G)$ on $L^2(G)$ affiliated to $\mathbb{C}G$; see e.g. [20, §8] or [23, §8]. We briefly recall the construction and state some of the properties. Let $L^2(G)$ denote the Hilbert space with Hilbert basis the elements of G; thus $L^2(G)$ consists of all square summable formal sums $\sum_{g\in G} a_g g$ with $a_g \in \mathbb{C}$ and inner product $\langle \sum_g a_g g, \sum_h b_h h\rangle = \sum_{g,h} a_g \overline{b_h}$. We have a left and right action of G on $L^2(G)$ defined by the formulae $\sum_h a_h h \mapsto \sum_h a_h gh$ and $\sum_h a_h h \mapsto \sum_h a_h hg$ for $g \in G$. It follows that $\mathbb{C}G$ acts faithfully as bounded linear operators on the left of $L^2(G)$, in other words we may consider $\mathbb{C}G$ as a subspace of $\mathcal{B}(L^2(G))$, the bounded linear operators on $L^2(G)$. The weak closure of $\mathbb{C}G$ in $\mathcal{B}(L^2(G))$ is the group von Neumann algebra $\mathcal{N}(G)$ of G, and the unbounded operators affiliated to G, denoted $\mathcal{U}(G)$, are those closed densely defined unbounded operators which commute with the right action of G. We have a natural injective $\mathbb{C}$-linear map $\mathcal{N}(G) \to L^2(G)$ defined by $\theta \mapsto \theta 1$ (where 1 denotes the element 1_1 of $L^2(G)$), so we may identify $\mathcal{N}(G)$ with a subspace of $L^2(G)$. When $H \leqslant G$, we may consider $L^2(H)$ as a subspace of $L^2(G)$ and using the above identification, we may consider $\mathcal{N}(H)$ as a subring of $\mathcal{N}(G)$. Also given $\alpha \in L^2(G)$, we can define a $\mathbb{C}$-linear map $\hat{\alpha}\colon \mathbb{C}G \to L^2(G)$ by $\hat{\alpha}(\beta) = \alpha\beta$ for $\beta \in L^2(G)$. Since $\mathbb{C}G$ is a dense linear subspace of $L^2(G)$, it yields a densely defined unbounded operator on $L^2(G)$ which commutes with the right action of G, and it is not difficult to see that this defines a unique element of $\mathcal{U}(G)$, which we shall also call $\hat{\alpha}$. We now have $\mathcal{N}(G) \subseteq L^2(G) \subseteq \mathcal{U}(G)$. Obviously if G is finite, then $\mathcal{N}(G) = L^2(G) = \mathcal{U}(G)$, because all terms are equal to $\mathbb{C}G$. This raises the following question.

Problem 4.1. *Let G be an infinite group. Is it always the true that $\mathcal{N}(G) \neq L^2(G) \neq \mathcal{U}(G)$?*

Presumably the answer is yes, but I am not aware of any reference. Some related information on this, as well as results on various homological dimensions of $\mathcal{N}(G)$ and $\mathcal{U}(\mathcal{G})$, can be found in [32].

At this stage it is less important to understand the construction of $\mathcal{U}(G)$ than to know its properties. Recall that R is a von Neumann regular ring means that given $r \in R$, there exists $x \in R$ such that $rxr = r$. All matrix rings over a von Neumann regular ring are also von Neumann regular [12, Lemma 1.6], and every element of a von Neumann regular ring is either invertible or a zerodivisor. We now have that $\mathcal{U}(G)$ is a von Neumann regular ring containing $\mathcal{N}(G)$, and is a classical ring of quotients for $\mathcal{N}(G)$ [2, proof of Theorem 10] or [23, Theorem 8.22(1)]. Thus the embedding of $\mathcal{N}(H)$ in $\mathcal{N}(G)$ for $H \leqslant G$ as described above extends to a natural embedding of $\mathcal{U}(H)$ in $\mathcal{U}(G)$. Also $\mathcal{U}(G)$ is rationally closed in any overing. Furthermore $\mathcal{U}(G)$ is a self injective unit-regular ring which is the maximal ring of quotients of $\mathcal{N}(G)$ [2, Lemma 1, Theorems 2 and 3]. Thus we have embedded $\mathbb{C}G$ in a ring, namely $\mathcal{U}(G)$, in which every element is either invertible or a zerodivisor. In fact every element of any matrix ring over $\mathcal{U}(G)$ is either invertible or a zerodivisor. Of course the same is true for any subfield k of $\mathbb{C}$, that is kG can be embedded in a ring in which every element is either invertible or a zerodivisor. Let us write $\mathcal{D}(kG) = \mathcal{D}_{\mathcal{U}(G)}(kG)$ and $\mathcal{R}(kG) = \mathcal{R}_{\mathcal{U}(G)}(kG)$. Then if $H \leqslant G$, we may by the above identify $\mathcal{D}(kH)$ with $\mathcal{D}_{\mathcal{U}(G)}(kH)$ and $\mathcal{R}(kH)$ with $\mathcal{R}_{\mathcal{U}(G)}(kH)$. More generally, we shall write $\mathcal{D}_n(kG) = \mathcal{D}_{\mathrm{M}_n(\mathcal{U}(G))}(\mathrm{M}_n(kG))$ and $\mathcal{R}_n(kG) = \mathcal{R}_{\mathrm{M}_n(\mathcal{U}(G))}(\mathrm{M}_n(kG))$. Thus $\mathcal{D}_1(kG) = \mathcal{D}(kG)$ and $\mathcal{R}_1(kG) = \mathcal{R}(kG)$. Also, we may identify $\mathcal{D}_n(kH)$ with $\mathcal{D}_{\mathrm{M}_n(\mathcal{U}(G))}(\mathrm{M}_n(kH))$ and $\mathcal{R}(kG)$ with $\mathcal{R}_{\mathrm{M}_n(\mathcal{U}(G))}(\mathrm{M}_n(kH))$.

Often $\mathcal{R}(kG)$ is a very nice ring. For example when G has a normal free subgroup with elementary amenable quotient, and also the finite subgroups of G have bounded order, it follows from [19, Theorem 1.5(ii)] that $\mathcal{R}(\mathbb{C}G)$ is a semisimple Artinian ring, i.e. a finite direct sum of matrix rings over division rings. Thus in particular every element of $\mathcal{R}(\mathbb{C}G)$ is either invertible or a zerodivisor. We state the following problem.

Problem 4.2. *Let G be a group and let k be a subfield of $\mathbb{C}$. Is every element of $\mathcal{R}_n(kG)$ either invertible or a zerodivisor for all positive integers n? Furthermore is $\mathcal{D}_n(kG) = \mathcal{R}_n(kG)$?*

The answer is certainly in the affirmative if G is amenable.

Proposition 4.3. *Let G be an amenable group, let n be a positive integer, and let k be a subfield of $\mathbb{C}$. Then every element of $\mathcal{R}_n(kG)$ is either a zerodivisor or invertible. Furthermore $\mathcal{D}_n(kG) = \mathcal{R}_n(kG)$.*

Proof. Write $R = \mathcal{R}_n(kG)$ and let $A \in R$. By Cramer's rule Proposition 3.1, there is a positive integer d and invertible matrices $X, Y \in \mathrm{M}_d(R)$ such that $B := X \operatorname{diag}(A, 1, \dots, 1) Y \in \mathrm{M}_{dn}(kG)$. Suppose $ZA \neq 0 \neq AZ$ whenever $0 \neq Z \in \mathrm{M}_n(kG)$. Then B is a non-zerodivisor in $\mathrm{M}_{dn}(kG)$. We claim that B is also a non-zerodivisor in $\mathrm{M}_{dn}(\mathbb{C}G)$. If our claim is false, then either $BC = 0$ or $CB = 0$

for some nonzero $C \in \mathrm{M}_{dn}(\mathbb{C}G)$. Without loss of generality, we may assume that $BC = 0$. Then for some positive integer m, we may choose $e_1, \dots, e_m \in \mathbb{C}$ which are linearly independent over k such that we may write $C = C_1e_1 + \dots + C_me_m$, where $0 \neq C_i \in \mathrm{M}_{dn}(kG)$ for all i. The equation $BC = 0$ now yields $BC_1 = 0$, contradicting the fact that B is a non-zerodivisor in $\mathrm{M}_{dn}(kG)$, and the claim is established.

Now B induces by left multiplication a right $\mathbb{C}G$-monomorphism $\mathbb{C}G^{dn} \to \mathbb{C}G^{dn}$. This in turn induces a right $\mathcal{N}(G)$-map $\mathcal{N}(G)^{dn} \to \mathcal{N}(G)^{dn}$, and the kernel of this map has dimension 0 by [22, Theorem 5.1]. It now follows from the theory of [22, §2] that this kernel is 0, consequently B is a non-zerodivisor in $\mathrm{M}_{dn}(\mathcal{N}(G))$. Since $\mathcal{U}(G)$ is a classical ring of quotients for $\mathcal{N}(G)$, we see that B is invertible in $\mathrm{M}_{dn}(\mathcal{U}(G))$ and hence B is invertible in $\mathrm{M}_d(R)$. Therefore A is invertible in R and the result follows. □

One could ask the following stronger problem.

Problem 4.4. *Let G be a group and let k be a subfield of $\mathbb{C}$. Is $\mathcal{R}(kG)$ a von Neumann regular ring?*

Since being von Neumann regular is preserved under Morita equivalence [12, Lemma 1.6] and $\mathcal{R}_n(kG)$ can be identified with $\mathrm{M}_n(\mathcal{R}(kG))$ by Proposition 3.4, we see that this is equivalent to asking whether $\mathcal{R}_n(kG)$ is a von Neumann regular ring. Especially interesting is the case of the lamplighter group, specifically

Problem 4.5. *Let G denote the lamplighter group. Is $\mathcal{R}(\mathbb{C}G)$ a von Neumann regular ring?*

Suppose $H \leq G$ and T is a right transversal for H in G. Then $\bigoplus_{t\in T} L^2(H)t$ is a dense linear subspace of $L^2(G)$, and $\mathcal{U}(H)$ is naturally a subring of $\mathcal{U}(G)$ as follows. If $u \in \mathcal{U}(H)$ is defined on the dense linear subspace D of $L^2(H)$, then we can extend u to the dense linear subspace $\bigoplus_{t\in T} Dt$ of $L^2(G)$ by the rule $u(dt) = (ud)t$ for $t \in T$, and the resulting unbounded operator commutes with the right action of G. It is not difficult to show that $u \in \mathcal{U}(G)$ and thus we have an embedding of $\mathcal{U}(H)$ into $\mathcal{U}(G)$, and this embedding does not depend on the choice of T. In fact it will be the same embedding as described previously. It follows that $\mathcal{R}(kH)$ is naturally a subring of $\mathcal{R}(kG)$. Clearly if $\alpha_1, \dots, \alpha_n \in \mathcal{U}(H)$ and $t_1, \dots, t_n \in T$, then $\alpha_1t_1 + \dots + \alpha_nt_n = 0$ if and only if $\alpha_i = 0$ for all i, and it follows that if $\beta_1, \dots, \beta_n \in \mathcal{R}(kH)$ and $\beta_1t_1 + \dots + \beta_nt_n = 0$, then $\beta_i = 0$ for all i.

The above should be compared with the theorem of Hughes [15] which we state below. Recall that a group is *locally indicable* if every nontrivial finitely generated subgroup has an infinite cyclic quotient. Though locally indicable groups are left orderable [3, Theorem 7.3.1] and thus $k * G$ is certainly a domain whenever k is a division ring, G is a locally indicable group and $k * G$ is a crossed product, it is still unknown whether such crossed products can be embedded in a division ring. Suppose however $k * G$ has a division ring of fractions D. Then we say that D is *Hughes-free* if whenever $N \lhd H \leqslant G$, H/N is infinite cyclic, and $h_1, \dots, h_n \in N$ are in distinct cosets of N, then the sum $\mathcal{D}_D(k * N)h_1 + \dots + \mathcal{D}_D(k * N)h_n$ is direct.

Theorem 4.6. *Let G be a locally indicable group, let k be a division ring, let $k * G$ be a crossed product, and let D, E be Hughes-free division rings of fractions for $k * G$. Then there is an isomorphism $D \to E$ which is the identity on $k * G$.*

This result of Hughes is highly nontrivial, even though the paper [15] is only 8 pages long. This is because the proof given by Hughes in [15] is extremely condensed, and though all the steps are there and correct, it is difficult to follow. A much more detailed and somewhat different proof is given in [6].

Motivated by Theorem 4.6, we will extend the definition of Hughes free to a more general situation.

Definition. Let D be a division ring, let G be a group, let $D * G$ be a crossed product, and let Q be a ring containing $D * G$ such that $\mathcal{R}_Q(D * G) = Q$, and every element of Q is either a zerodivisor or invertible. In this situation we say that Q is *strongly Hughes free* if whenever $N \lhd H \leqslant G$, $h_1, \dots, h_n \in H$ are in distinct cosets of N and $\alpha_1, \dots, \alpha_n \in \mathcal{R}_Q(D * N)$, then $\alpha_1 h_1 + \dots + \alpha_n h_n = 0$ implies $\alpha_i = 0$ for all i (i.e. the h_i are linearly independent over $\mathcal{R}_Q(D * N)$).

Then we would like to extend Theorem 4.6 to more general groups, so we state

Problem 4.7. *Let D be a division ring, let G be a group, let $D * G$ be a crossed product, and let Q be a ring containing $D * G$ such that $\mathcal{R}_Q(D * G) = Q$, and every element of Q is either a zerodivisor or invertible. Suppose P, Q are strongly Hughes free rings for $D * G$. Does there exists an isomorphism $P \to Q$ which is the identity on $D * G$?*

It is clear that if G is locally indicable and Q is a division ring of fractions for $D * G$, then Q is strongly Hughes free implies Q is Hughes free. We present the following problem.

Problem 4.8. *Let G be a locally indicable group, let D be a division ring, let $D * G$ be a crossed product, and let Q be a division ring of fractions for $D * G$ which is Hughes free. Is Q strongly Hughes free?*

It would seem likely that the answer is always "yes". Certainly if G is orderable, then $\mathcal{R}_{D((G))}(D * G)$, the rational closure (which is the same as the division closure in this case) of $D * G$ in the Malcev-Neumann power series ring $D((G))$ [5, Corollary 8.7.6] is a Hughes free division ring of fractions for $D * G$. Therefore by Theorem 4.6 of Hughes, all Hughes free division ring of fractions for $D * G$ are isomorphic to $\mathcal{R}_{D((G))}(D * G)$. It is easy to see that this division ring of fractions is strongly Hughes free and therefore all Hughes free division ring of fractions for $D * G$ are strongly Hughes free.

5. Other Methods

Embedding $\mathbb{C}G$ into $\mathcal{U}(G)$ has proved to be a very useful tool, but what about other group rings? In general we would like a similar construction when k is a field of nonzero characteristic. If D is a division ring, then we can always embed $D * G$ into a ring in which every element is either a unit or a zerodivisor, as follows. Let $V = D * G$ viewed as a right vector space over D, so V has basis $\{\bar{g} \mid g \in G\}$. Then $D * G$ acts by left multiplication on V and therefore can be considered as a subring of the ring of all linear transformations $\operatorname{End}_D(V)$ of V. This ring is von Neumann regular. However it is too large; it is not even directly finite (that is $xy = 1$ implies $yx = 1$) when G is infinite. Another standard method is to embed $D * G$ in its maximal ring of right quotients [11, §2.C]. If R is a right nonsingular ring, then its maximal ring of right quotients $Q(R)$ is a ring containing R which is a right injective von Neumann regular ring, and furthermore as a right R-module, $Q(R)$ is the injective hull of R [11, Corollary 2.31]. By [29, Theorem 4], when k is a field of characteristic zero, kG is right (and left) nonsingular, consequently $Q(kG)$ is a right self-injective von Neumann regular ring. However again it is too large in general. If G is a nonabelian free group, then kG is a domain which by Proposition 2.2 does not satisfy the Ore condition, so we see from [11, Exercise 6.B.14] that $Q(R)$ is not directly finite.

A very useful technique is that of ultrafilters, see [16, p. 76, §2.6] for example. We briefly illustrate this in an example. Let k be a field and let G be a group. Suppose G has a descending chain of normal subgroups $G = G_0 \geqslant G_1 \geqslant \cdots$ such that $k[G/G_n]$ is embeddable in a division ring for all n. Then can we embed kG in a division ring? It is easy to prove that kG is a domain, but to prove the stronger statement that G can be embedded in division ring seems to require the theory of ultrafilters. For most applications (or at least for what we are interested in), it is sufficient to consider ultrafilters on the natural numbers $\mathbb{N} = \{1, 2, \dots\}$. A filter on $\mathbb{N}$ is a subset ω of the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ such that if $X, Y \in \omega$ and $X \subseteq Z \subseteq \mathbb{N}$, then $X \cap Y \in \omega$ and $Z \in \omega$. A filter is proper if it does not contain the empty set $\emptyset$, and an ultrafilter is a maximal proper filter. By considering the maximal ideals in the Boolean algebra on $\mathcal{P}(\mathbb{N})$, it can be shown that any proper filter can be embedded in an ultrafilter (this requires Zorn's lemma), and an ultrafilter has the following properties.

- If $X, Y \in \omega$, then $X \cap Y \in \omega$.
- If $X \in \omega$ and $X \subseteq Y$, then $Y \in \omega$.
- If $X \in \mathcal{P}(\mathbb{N})$, then either X or its complement are in ω.
- $\emptyset \notin \omega$.

An easy example of an ultrafilter is the set of all subsets containing n for some fixed $n \in \mathbb{N}$; such an ultrafilter is called a *principal ultrafilter*. An ultrafilter not of this form is called a non-principal ultrafilter.

Given division rings D_n for $n \in \mathbb{N}$ and an ultrafilter ω on $\mathbb{N}$, we can define an equivalence relation $\sim$ on $\prod_n D_n$ by $(d_1, d_2, \dots) \sim (e_1, e_2, \dots)$ if and only if there exists $S \in \omega$ such that $d_n = e_n$ for all $n \in S$. Then the set of equivalence classes $(\prod_n D_n)/\sim$ is called the ultraproduct of the division rings D_i with respect to the ultrafilter ω, and is a division ring [16, p. 76, Proposition 2.1]. This can be applied when R is a ring with a descending sequence of ideals $I_1 \supseteq I_2 \supseteq \dots$ such that $\bigcap_n I_n = 0$ and R/I_n is a division ring. The set of all cofinite subsets of $\mathbb{N}$ is a filter, so here we let ω be any ultrafilter containing this filter. The corresponding ultraproduct D of the division rings R/I_n is a division ring. Furthermore the natural embedding of R into $\prod_n R/I_n$ defined by $r \mapsto (r + R/I_1, r + R/I_2, \dots)$ induces an embedding of R into D. This proves that R can be embedded in a division ring.

In their very recent preprint [9], Gábor Elek and Endre Szabó use these ideas to embed the group algebra kG over an arbitrary division ring k in a nice von Neumann regular ring for the class of *sofic* groups. The class of sofic groups is a large class of groups which contains all residually amenable groups and is closed under taking free products.

Suppose $\{a_n \mid n \in \mathbb{N}\}$ is a bounded sequence of real numbers and ω is a non-principal ultrafilter. Then there is a unique real number l with the property that given $\epsilon > 0$, then l is in the closure of $\{a_n \mid n \in S\}$ for all $S \in \omega$. We call this the ω-limit of $\{a_n\}$ and write $l = \lim_\omega a_n$.

Now let G be a countable amenable group. Then G satisfies the Følner condition and therefore there exist finite subsets X_i of G ($i \in \mathbb{N}$) such that

- $\bigcup_i X_i = G$.
- $|X_i| < |X_{i+1}|$ for all $i \in \mathbb{N}$.
- If $g \in G$, then $\lim_{i\to\infty} |gX_i \cap X_i|/|X_i| = 1$.

Let k be a division ring and let V_i denote the right k-vector space with basis X_i ($i \in \mathbb{N}$). The general element of $\prod_i \operatorname{End}_k(V_i)$ (Cartesian product) is of the form $\bigoplus_i \alpha_i$ where $\alpha_i \in \operatorname{End}_k(V_i)$ for all i. For $\beta \in \operatorname{End}_k(V_i)$, we define

$$\operatorname{rk}_i(\beta) = \frac{\dim_k(\beta V_i)}{\dim_k V_i},$$

a real number in $[0, 1]$. Now choose a non-principal ultrafilter ω for $\mathbb{N}$. Then for $\alpha \in \operatorname{End}_k(V_i)$, we define $\operatorname{rk}(\alpha) = \lim_\omega \operatorname{rk}_n(\alpha_n)$ and $I = \{\alpha \in \prod_i \operatorname{End}_k(V_i) \mid \operatorname{rk}(\alpha) = 0\}$. It is not difficult to check that I is a two-sided ideal of $\prod_i \operatorname{End}_k(V_i)$. Now set

$$\mathrm{R}_k(G) = \frac{\prod_i \operatorname{End}_k(V_i)}{I}$$

and let $[\alpha]$ denote the image of α in $\mathrm{R}_k(G)$. Since $\operatorname{End}_k(V_i)$ is von Neumann regular and direct products of von Neumann regular rings are von Neumann regular, we see that $\prod_i \operatorname{End}_k(V_i)$ is von Neumann regular and we deduce that $\mathrm{R}_k(G)$ is also von Neumann regular. Next we define $\operatorname{rk}([\alpha]) = \operatorname{rk}(\alpha)$. It can be shown that rk is a well-defined rank function [12, p. 226, Chapter 16] and therefore $\mathrm{R}_k(G)$ is directly finite [12, Proposition 16.11].

For $g \in G$ and $x \in X_i$, we can define $\phi(g)x = gx$ if $gx \in X_i$ and $\phi(g)x = x$ if $gx \notin X_i$. This determines an embedding (which is not a homomorphism) of G into $\prod_i \operatorname{End}_k(V_i)$, and it is shown in [9] that the composition with the natural epimorphism $\prod_i \operatorname{End}_k(V_i) \twoheadrightarrow \mathrm{R}_k(G)$ yields a homomorphism $G \to \mathrm{R}_k(G)$. This homomorphism extends to a ring homomorphism $\theta\colon kG \to \mathrm{R}_k(G)$ and [9] shows that $\ker\theta = 0$. Thus we have embedded kG into $\mathrm{R}_k(G)$; in particular this shows that kG is directly finite because $\mathrm{R}_i(G)$ is. In fact this construction for G amenable can be extended to the case G is a sofic group, consequently kG is directly finite if k is a division ring and G is sofic. The direct finiteness of $k * G$ for k a division ring and G free-by-amenable had earlier been established in [1].

Another type of localization is considered in [27]. Recall that a monoid M is a semigroup with identity, that is M satisfies the axioms for a group except for the existence of inverses. If A is a monoid with identity 1, then M is an A-monoid means that there is an action of A on M satisfying $a(bm) = (ab)m$ and $1m = m$ for all $a, b \in A$ and $m \in M$. In the case A is a ring with identity 1 (so A is a monoid under multiplication) and M is a left A-module, then M is an A-monoid. Let $\operatorname{End}(M)$ denote the monoid of all endomorphisms of the A-monoid M. Given a submonoid S of $\operatorname{End}(M)$, Picavet constructs an A-monoid $S^{-1}M$ with the property that every endomorphism in S becomes an automorphism of M, in other words the elements of S become invertible. To achieve this, he requires that S is a *localizable* submonoid of $\operatorname{End}(M)$. This means that the following Ore type conditions hold:

- For all $u, v \in S$, there exist $u', v' \in S$ such that $u'u = v'v$.
- For all $u, v, w \in S$ such that $uw = vw$, there is $s \in S$ such that $su = sv$.

The construction is similar to Ore localization. We describe this in the case R is ring, M is an R-module and $S = \{\theta^n \mid n \in \mathbb{N}\}$ where θ is an endomorphism of M. Clearly S is localizable. For $m, n \in \mathbb{N}$ with $m \leq n$, we set $M_n = M$ and $\theta_{mn} = \theta^{n-m} \colon M_m \to M_n$. Then (M_n, θ_{mn}) forms a direct system of R-modules, and $S^{-1}M$ is the direct limit of this system. Clearly θ^n induces an R-automorphism on $S^{-1}M$ for all n, so we have inverted θ. In the case R is a division ring, M is finitely generated and θ is a noninvertible nonnilpotent endomorphism of M, the sequence of R-modules $M\theta^n$ eventually stabilizes to a proper nonzero R-submodule of M, which is $S^{-1}M$. It would be interesting to see if this construction has applications to group rings.

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