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Knots, von Neumann Signatures, and Grope Cobordism

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Abstract

We explain new developments in classical knot theory in 3 and 4 dimensions, i.e. we study knots in 3-space, up to isotopy as well as up to concordance. In dimension 3 we give a geometric interpretation of the Kontsevich integral (joint with Jim Conant), and in dimension 4 we introduce new concordance invariants using von Neumann signatures (joint with Tim Cochran and Kent Orr). The common geometric feature of our results is the notion of a grope cobordism.

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1. Introduction

A lot of fascinating mathematics has been created when successful tools are transferred from one research area to another. We shall describe two instances of such transfers, both into knot theory. The first transfer realizes commutator calculus of group theory by *embedded versions* in 3- and 4-space, and produces many interesting geometric equivalence relations on knots, called *grope cobordism* in 3-space and *grope concordance* in 4-space. It turns out that in 3-space these new equivalence relations give a geometric interpretation (Theorem 2) of Vassiliev's finite type invariants [21] and that the Kontsevich integral [17] calculates the new theory over $\mathbb{Q}$ (Theorem 3).

In 4-space the new equivalence relations factor naturally through knot concordance, and in fact they organize all known concordance invariants in a wonderful manner (Theorem 5). They also point the way to new concordance invariants (Theorem 6) and these are constructed using a second transfer, from the spectral theory of self-adjoint operators and von Neumann's continuous dimension [20].

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1.1. A geometric interpretation of group commutators

To explain the first transfer into knot theory, recall that every knot bounds a Seifert surface (embedded in 3-space), but only the trivial knot bounds an embedded disk. Thus all of knot theory is created by the difference between a surface and a disk. The new idea is to filter this difference by introducing a concept into knot theory which is the analogue of iterated commutators in group theory. Commutators arise because a continuous map $\phi : S^1 \to X$ extends to a map of a surface if and only if ϕ represents a commutator in the fundamental group $\pi_1 X$. Iterated commutators can similarly be expressed by gluing together several surfaces. Namely, there are certain finite 2-complexes (built out of iterated surface stages) called *gropes* by Cannon [1], with the following defining property: $\phi : S^1 \to X$ represents an element in the k-th term of the *lower central series* of $\pi_1 X$ if and only if it extends to a continuous map of a *grope of class* k. Similarly, there are *symmetric gropes* which geometrically implement the derived series of $\pi_1 X$, see Figures 2. and 3.

Gropes, therefore, are not quite manifolds but the singularities that arise are of a very simple type, so that these 2-complexes are in some sense the next easiest thing after surfaces. Two sentences on the history of the use of gropes in mathematics are in place, compare [11, Sec.2.11]. Their inventor Stan'ko worked in high-dimensional topology, and so did Edwards and Cannon who developed gropes further. Bob Edwards suggested their relevance for topological 4-manifolds, where they were used extensively, see [11] or [12]. It is this application that seems to have created a certain "Angst“ of studying gropes, so we should point out that the only really difficult part in that application is the use of *infinite constructions*, i.e. when the class of the grope goes to infinity.

One purpose of this note is to convince the reader that (finite) gropes are a very simple, and extremely powerful tool in low-dimensional topology. The point is that once one can describe iterated commutators in $\pi_1 X$ by maps of gropes, one might as well study *embedded* gropes in 3-space (respectively 4-space) in order to organize knots up to isotopy (respectively up to concordance). In Section 2. we shall explain joint work with Jim Conant on how gropes embedded in 3-space lead to a geometric interpretation of Vassiliev's knot invariants [21] and of the Kontsevich integral [17].

1.2. Von Neumann signatures and knot concordance

In Section 3. we study symmetric gropes embedded in 4-space, and explain how they lead to a geometric framework for all known knot concordance invariants and beyond. More precisely, we explain our joint work with Tim Cochran and Kent Orr [6], where we define new concordance invariants by inductively constructing representations into certain solvable groups G, and associating a hermitian form over the group ring $\mathbb{Z}G$ to the knot K, which is derived from the intersection form of a certain 4-manifold with fundamental group G and whose boundary is obtained by 0-surgery on K. This intersection form represents an element in the Cappell-Shaneson Γ-group [2] of $\mathbb{Z}G$ and we detect it via the second transfer from a different area of mathematics: The standard way to detect elements in Witt groups like the

Γ-group above is to construct unitary representations of G, and then consider the corresponding (twisted) signature of the resulting hermitian form over $\mathbb{C}$. It turns out that the solvable groups G we construct do not have any interesting finite dimensional representations, basically because they are “too big” (e.g. not finitely generated), a property that is intrinsic to the groups G in question because they are “universal solvable” in the sense that many 4-manifold groups (with the right boundary) must map to G, extending the given map of the knot group.

However, every group G has a fundamental unitary representation given by $\ell^2 G$, the Hilbert space of square summable sequences of group elements with complex coefficients. The resulting (weak) completion of $\mathbb{C}G$ is the *group von Neumann algebra* $\mathcal{N}G$. It is of type II_1 because the map $\sum a_g g \mapsto a_1$ extends from $\mathbb{C}G$ to give a finite faithful trace on $\mathcal{N}G$.

The punchline is that hermitian forms over the completion $\mathcal{N}G$ are much easier to understand than over $\mathbb{C}G$ because they are diagonalizable (by functional calculus of self-adjoint operators). Here one really uses the von Neumann algebra, rather than the C^*-algebra completion of $\mathbb{C}G$ because the functional calculus must be applied to the characteristic functions of the positive (respectively negative) real numbers, which are bounded but *not* continuous.

The subspace on which the hermitian form is positive (respectively negative) definite has a *continuous dimension*, which is the positive real number given by the trace of the projection onto that subspace. As a consequence, one can associate to every hermitian form over $\mathcal{N}G$ a real valued invariant, the *von Neumann signature*. In [6] we use this invariant to construct our new knot concordance invariants, and a survey of this work can be found in Section 3. It is not only related to embedded gropes in 4-space but also to the existence of towers of *Whitney disks* in 4-space. Unfortunately, we won’t be able to explain this aspect of the theory, but see [6, Thm.8.12].

1.3. Noncommutative Alexander modules

In Section 3. we shall hint at how the interesting representations to our solvable groups are obtained. But it is well worth pointing out that the methods developed for studying knot concordance have much simpler counterparts in 3-space, i.e. if one is only interested in isotopy invariants.

A typical list of knot invariants that might find its way into a text book or survey talk on *classical* knot theory, would contain the Alexander polynomial, (twisted) signatures, (twisted) Arf invariants, and maybe knot determinants. It turns out that all of these invariants can be computed from the homology of the infinite cyclic covering of the knot complement, and are in this sense “commutative” invariants.

Instead of the maximal abelian quotient one can use other solvable quotient groups of the knot group to obtain “noncommutative” knot invariants. The canonical candidates are the quotient groups of the derived series $G^{(n)}$ of the knot group (compare Section 3. for the definition). One can thus define the higher order *Alexan-*

der modules of a knot K to be the $\mathbb{Z}[G/G^{(n+1)}]$-modules

$$\mathcal{A}_n(K) := H_1(S^3 \smallsetminus K; \mathbb{Z}[G/G^{(n+1)}]).$$

The indexing is chosen so that $\mathcal{A}_0$ is the classical Alexander module. For $n \geq 1$ these modules are best studied by introducing further algebraic tools as follows. By a result of Strebel the groups $G/G^{(n)}$ are torsionfree. Therefore, the group ring $\mathbb{Z}[G/G^{(n)}]$ satisfies the Ore condition and has a well defined (skew) quotient field. This field is in fact the quotient field of a (skew) polynomial ring $\mathbb{K}_n[t^{\pm 1}]$, with $\mathbb{K}_n$ the quotient field of $\mathbb{Z}[G^{(1)}/G^{(n)}]$ and $G/G^{(1)} = \langle t \rangle \cong \mathbb{Z}$. Thus one is exactly in the context of [6, Sec.2] and one can define explicit noncommutative isotopy invariants of knots. For example, let $d_n(K)$ be the dimension (over the field $\mathbb{K}_{n+1}$) of the *rational* Alexander module

$$\mathcal{A}_n(K) \otimes_{\mathbb{Z}[G/G^{(n+1)}]} \mathbb{K}_{n+1}[t^{\pm 1}].$$

It is shown in [6, Prop.2.11] that these dimensions are finite with the degree of the usual Alexander polynomial being $d_0(K)$. Moreover, Cochran [5] has proven the following non-triviality result for these dimensions.

Theorem: *If K is a nontrivial knot then for $n \geq 1$ one has*

$$d_0(K) \leq d_1(K) + 1 \leq d_2(K) + 1 \leq \cdots \leq d_n(K) + 1 \leq 2 \cdot \textit{genus of } K.$$

Moreover, there are examples where these numbers are strictly increasing up to any given n.

Corollary: *If one of the inequalities in the above theorem is strict then K is not fibered. Furthermore, 0-surgery on K cross the circle is not a symplectic 4-manifold.*

The first statement is clear: For fibered knots the degree of the Alexander polynomial $d_0(K)$ equals twice the genus of the knot K. The second statement follows from a result of Kronheimer [18] who showed that this equality also holds if the above 4-manifold is symplectic.

Recently, Harvey [16] has studied similar invariants for arbitrary 3-manifolds and has proven generalizations of the above results: There are lower bounds for the Thurston norm of a homology class, analogous to $d_i(K)$, that are better than McMullen's lower bound, which is the analogy of $d_0(K)$. As a consequence, she gets new algebraic obstructions to a 4-manifold of the form $M^3 \times S^1$ admitting a symplectic structure.

Just like in the classical case $n = 0$, there is more structure on the rational Alexander modules. By [6, Thm.2.13] there are higher order *Blanchfield forms* which are hermitian and non-singular in an appropriate sense, compare [5, Prop.12.2]. It would be very interesting to know whether the n-th order Blanchfield form determines the von Neumann η-invariant associated to the $G^{(n+1)}$-cover. So far, these η-invariants are very mysterious real numbers canonically associated to a knot.

Only in the bottom case $n = 0$ do we understand this η-invariant well: The L^2-index theorem implies that the von Neumann η-invariant corresponding to the

$\mathbb{Z}$-cover is the von Neumann signature of a certain 4-manifold with fundamental group $\mathbb{Z}$. Moreover, this signature is the integral, over the circle, of all (Levine-Tristram) twisted signatures of the knot [7, Prop.5.1] (and is thus a concordance invariant). For $n \geq 1$ there is in general no such 4-manifold available and the corresponding η-invariants are not concordance invariants.

2. Grope cobordism in 3-space

We first give a more precise treatment of the first transfer from group theory to knot theory hinted at in the introduction. Recall that the fundamental group consists of continuous maps of the circle S^1 into some target space X, modulo homotopy (i.e. 1-parameter families of continuous maps). Quite analogously, classical knot theory studies smooth *embeddings* of a circle into S^3, modulo isotopy (i.e. 1-parameter families of embeddings). To explain the transfer, we recall that a continuous map $\phi : S^1 \to X$ represents the trivial element in the fundamental group $\pi_1 X$ if and only if it extends to a map of the disk, $\tilde{\phi} : D^2 \to X$. Moreover, ϕ represents a commutator in $\pi_1 X$ if and only if it extends to a map of a surface (i.e. of a compact oriented 2-manifold with boundary S^1). The first statement has a straightforward analogy in knot theory: $K : S^1 \hookrightarrow S^3$ is trivial if and only if it extends to an embedding of the disk into S^3. However, every knot "is a commutator" in the sense that it bounds a *Seifert surface*, i.e. an embedded surface in S^3.

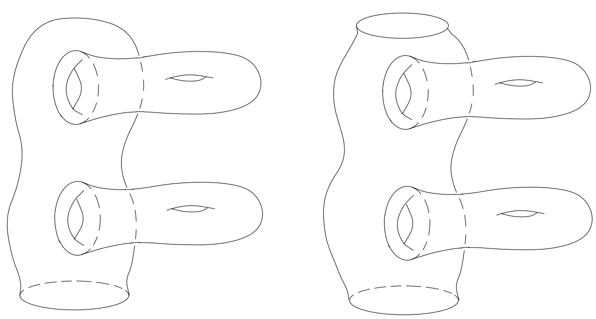

Figure 2: Gropes of class 3, with one respectively two boundary circles.

Recall from the introduction that gropes are finite 2-complexes defined by the following property: $\phi : S^1 \to X$ represents an element in the k-th term $\pi_1 X_k$ of the lower central series of $\pi_1 X$ if and only if it extends to a continuous map of a grope of class k. Here G_k is defined inductively for a group G by the iterated commutators

$$G_2 := [G, G] \text{ and } G_k := [G, G_{k-1}] \text{ for } k > 2.$$

Accordingly, a grope of class 2 is just a surface, and one can obtain a grope of class k by attaching gropes of class $(k-1)$ to g disjointly embedded curves in the *bottom* surface. Here g is the genus of the bottom surface and the curves are assumed to span one half of its homology. This gives gropes of class k with one boundary circle as on the left of Figure 2. It's not the most general way to get gropes because of re-bracketing issues, and we refer to [9, Sec.2.1] for details. The boundary of a grope is by definition just the boundary of the bottom surface, compare Figure 2.

Definition 1 *Two smooth oriented knot types are* grope cobordant *of class k, if there is an embedded grope of class k in S^3 (the* grope cobordism*) such that its boundary represents the given knots.*

An *embedding* of a grope is best defined via the obvious 3-dimensional local model. Since every grope has a 1-dimensional spine, embedded gropes can then be isotoped into the neighborhood of a 1-complex. As a consequence, embedded gropes abound in 3-space! It is important to point out that the two boundary components of a grope cobordism may link in an arbitrary way. Thus it is a much stronger condition on K to assume that it is the boundary of an embedded grope than to say that it cobounds a grope with the unknot. For example, if K bounds an embedded grope of class 3 in S^3 then the Alexander polynomial vanishes. Together with Stavros Garoufalidis, we recently showed [13] that the 2-loop term of the Kontsevich integral detects many counterexamples to the converse of this statement.

In joint work with Jim Conant [9], we show that grope cobordism defines equivalence relations on knots, one for every class $k \in \mathbb{N}$. Moreover, Theorem 2 below implies that the resulting quotients $\mathcal{K}_k$ are in fact finitely generated abelian groups (under the connected sum operation). For the smallest values $k = 2, 3, 4$ and 5, the groups $\mathcal{K}_k$ are isomorphic to

$$\{0\}, \mathbb{Z}/2, \mathbb{Z} \text{ and } \mathbb{Z} \times \mathbb{Z}/2$$

and they are detected by the first two Vassiliev invariants [10, Thm.4.2].

The following theorem is formulated in terms of *clasper surgery* which was introduced independently by Habiro [15] and Goussarov [14], as a geometric answer to finite type invariants a lá Vassiliev [21]. We cannot explain the definitions here but see [9, Thm.2 and 3]. We should say that the notion of a *capped* grope is well known in 4 dimensions, see [11, Sec.2]. In our context, it means that all circles at the "tips" of the grope bound disjointly embedded disks in 3-space which are only allowed to intersect the bottom surface of the grope.

Theorem 2 *Two knots K_0 and K_1 are grope cobordant of class k if and only if K_1 can be obtained from K_0 by a finite sequence of simple clasper surgeries of* grope degree *k (as defined below).*

Moreover, two knots are capped *grope cobordant of class k if and only if they share the same finite type invariants of Vassiliev degree $< k$.*

As a consequence of this result, the invariants associated to grope cobordism are highly nontrivial as well as manageable. For example, we prove the following result in [10, Thm.1.1]:

Theorem 3 *The (logarithm of the) Kontsevich integral (with values in $\mathcal{B}^g_{<k}$), graded by the new* grope degree *k, is an obstruction to finding a grope cobordism of class k between two knots and it gives an isomorphism*

$$\mathcal{K}_k \otimes \mathbb{Q} \cong \mathcal{B}^g_{<k}$$

Here $\mathcal{B}^g_{<k}$ is one of the usual algebras of Feynman diagrams known from the theory of finite type invariants, but graded by the *grope degree*. More precisely, $\mathcal{B}^g_{<k}$ is the $\mathbb{Q}$-vector space generated by connected uni-trivalent graphs of grope degree $i, 1 < i < k$, with at least one univalent vertex and a cyclic ordering at each trivalent vertex. The relations are the usual IHX and AS relations. The grope degree is the Vassiliev degree (i.e. half the number of vertices) *plus* the first Betti number of the graph. Observe that both relations preserve this new degree.

Read backwards, our results give an interpretation of the Kontsevich integral in terms of the geometrically defined equivalence relations of grope cobordism. Note that regrading diagrams from Vassiliev to grope degree is not a technical issue but it removes the caps from the discussion, closing the gap to group theory considerably.

3. Grope concordance

We now turn to the 4-dimensional aspects of the theory. It may look like the end of the story to realize that any knot with trivial Arf invariant bounds a grope of arbitrary big class embedded in D^4, [10, Prop.3.8]. However, group theory has more to offer than the lower central series. Recall that the *derived series* of a group G is defined inductively by the iterated commutators

$$G^{(1)} := [G, G] \text{ and } G^{(h)} := [G^{(h-1)}, G^{(h-1)}] \text{ for } h > 1.$$

Accordingly, we may define *symmetric gropes* with their complexity now measured by *height*, satisfying the following defining property: A continuous map $\phi : S^1 \to X$ represents an element in $\pi_1 X^{(h)}$ if and only if it extends to a continuous map of a symmetric grope of height h. Thus a symmetric grope of height 1 is just a surface, and a symmetric grope of height h is obtained from a bottom surface by attaching symmetric gropes of height $(h-1)$ to a *full* symplectic basis of curves. This defines symmetric gropes of height h with one boundary circle as in Figure 3.

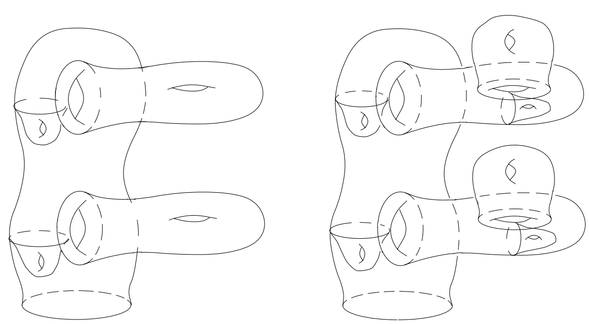

Figure 3: Symmetric gropes of height 2 and 2.5.

Note that a symmetric grope of height h is also a grope of class 2^h, just like in group theory. But conversely, not every grope is symmetric. It should also be clear from Figure 3. how one defines symmetric gropes with half-integer height (even though there is no group theoretic analogue).

In the following definition we attempt to distinguish the terms "cobordant" and "concordant" in the sense that the latter refers to 4 dimensions, whereas the former was used in dimension 3, see Definition 1. Historically, these terms were used interchangeably, but we hope not to create any confusion with our new distinction.

Definition 4 *Two oriented knots in S^3 are* grope concordant *of height $h \in \frac{1}{2}\mathbb{N}$, if there is an embedded symmetric grope of height h in $S^3 \times [0,1]$ such that its boundary consists exactly of the given knots $K_i : S^1 \hookrightarrow S^3 \times \{i\}$.*

Observe that since an annulus is a symmetric grope of arbitrary height we indeed get a filtration of the *knot concordance group*. This group is defined by identifying two knots which cobound an embedded annulus in $S^3 \times [0,1]$, where there are two theories depending on whether the embedding is smooth or just topological (and locally flat). For grope concordance the smaller topological knot concordance group is the more natural setting: a locally flat topological embedding of a grope (defined by the obvious local model at the singular points) can be perturbed to become smooth. This perturbation might introduce many self-intersection points in the surface stages of the grope. However, these new singularities are arbitrarily small and thus they can be removed at the expense of increasing the genus of the surface stage in question but without changing the height of the grope.

In joint work with Tim Cochran and Kent Orr [6], we showed that all known knot concordance invariants fit beautifully into the scheme of grope concordance! In particular, all known invariants turned out to already be invariants of grope concordance of height 3.5:

Theorem 5 *Consider two knots K_i in S^3. Then*

1. *K_i have the same Arf invariant if and only if they are grope concordant of height* 1.5 *(or class 3).*
2. *K_i are* algebraically concordant *in the sense of Levine [19] (i.e. all twisted signatures and twisted Arf invariants agree) if and only if they are grope concordant of height* 2.5.
3. *If K_i are grope concordant of height* 3.5 *then they have the same Casson-Gordon invariants [3].*

The third statement includes the generalizations of Casson-Gordon invariants by Gilmer, Kirk-Livingston, and Letsche. In [6] we prove an even stronger version of the third part of Theorem 5. Namely, we give a weaker condition for a knot K to have vanishing Casson-Gordon invariants: it suffices that K is (1.5)-*solvable*. All the obstruction theory in [6] is based on the definition of (h)-*solvable* knots, $h \in \frac{1}{2}\mathbb{N}$, which we shall not give here. Suffice it to say that this definition is closer to the algebraic topology of 4-manifolds than grope concordance. In [6, Thm.8.11] we show that a knot which bounds an embedded grope of height $(h+2)$ in D^4 is (h)-solvable.

It should come as no surprise that the invariants which detect grope concordance have to do with solvable quotients of the knot group. In fact, the above invariants are all obtained by studying the Witt class of the intersection form of a certain 4-manifold M^4 whose boundary is obtained by 0-surgery on the knot. The different cases are distinguished by the fundamental group $\pi_1 M$, namely

1. $\pi_1 M$ is trivial for the Arf invariant,
2. $\pi_1 M$ is infinite cyclic for algebraic concordance, and
3. $\pi_1 M$ is a dihedral group for Casson-Gordon invariants.

So the previously known concordance invariants stopped at solvable groups which are extensions of abelian by abelian groups. To proceed further in the understanding of grope (and knot) concordance, one must be able to handle more complicated solvable groups. A program for that purpose was developed in [6] by giving an elaborate boot strap argument to construct inductively representations of knot groups into certain *universal solvable* groups. On the way, we introduced Blanchfield duality pairings in solvable covers of the knot complement by using noncommutative localizations of the group rings in question.

The main idea of the boot strap is that a particular choice of "vanishing" of the previous invariant *defines* the map into the next solvable group (and hence the next invariant). In terms of gropes this can be expressed quite nicely as follows: pick a grope concordance of height $h \in \mathbb{N}$ and use it to construct a certain 4-manifold whose intersection form gives an obstruction to being able to extend that grope to height $h.5$. There is an obvious technical problem in such an approach, already present in [3]: to show that there is no grope concordance of height $h.5$, one needs to prove non-triviality of the obstruction *for all* possible gropes of height h. One way around this problem is to construct examples where the grope concordances of small height are in some sense unique. This was done successfully in [6] for the level above Casson-Gordon invariants, and in [7] we even obtain the following infinite generation result. Let $\mathcal{G}_h$ be the graded quotient groups of knots, grope concordant of height h to the unknot, modulo grope concordance of height $h.5$. Then the results of Levine and Casson-Gordon show that $\mathcal{G}_2$ and $\mathcal{G}_3$ are not finitely generated.

Theorem 6 *$\mathcal{G}_4$ is not finitely generated.*

The easiest example of a non-slice knot with vanishing Casson-Gordon invariants is given in [6, Fig.6.5]. As explained in the introduction, the last step in the proof of Theorem 6 is to show that the intersection form of the 4-manifold in question is nontrivial in a certain Witt group. Our new tool is the von Neumann signature which has the additional bonus that it takes values in $\mathbb{R}$, which is not finitely generated as an abelian group. This fact makes the above result tractable. We cannot review any aspect of the von Neumann signature here, but see [6, Sec.5].

Last but not least, it should be mentioned that we now know that for every $h \in \mathbb{N}$ the groups $\mathcal{G}_h$ are nontrivial. This work in progress [8] uses as the main additional input the Cheeger-Gromov estimate for von Neumann η-invariants [4] in order to get around the technical problem mentioned above. It is very likely that non of the groups $\mathcal{G}_h, h \in \mathbb{N}, h \geq 2$, are finitely generated.

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