RESIDUALLY-FINITE RATIONALLY-SOLVABLE GROUPS AND VIRTUAL FIBRING

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ABSTRACT. We show that a non-trivial finitely-generated residually-finite rationally-solvable (or RFRS) group G is virtually fibred, in the sense that it admits a virtual surjection to $\mathbb{Z}$ with a finitely-generated kernel, if and only if the first L^2-Betti number of G vanishes. This generalises (and gives a new proof of) the analogous result of Ian Agol for fundamental groups of 3-manifolds.

1. INTRODUCTION

In 2013, Ian Agol [Ago13] completed the proof of Thurston's Virtually Fibred Conjecture, which states that every closed hyperbolic 3-manifold admits a finite covering which fibres over the circle. The final step consisted of showing that the fundamental group G of a hyperbolic 3-manifold has a finite index subgroup H which embeds into a right-angled Artin group (a RAAG). Agol in [Ago08] showed that in this case H is virtually *residually-finite rationally-solvable* (or *RFRS*); the same article contains the following result.

Theorem (Agol [Ago08]). *Every compact irreducible orientable 3-manifold with trivial Euler characteristic and non-trivial RFRS fundamental group admits a finite covering which fibres.*

Since closed hyperbolic 3-manifolds are irreducible and have vanishing Euler characteristic, it follows that such manifolds are virtually fibred, that is, admit fibring finite coverings.

Our focus here is the above result of Agol – morally, we will generalise this theorem by removing the assumption of the group being the fundamental group of a 3-manifold.

Theorem 5.3. *Let G be an infinite finitely-generated group which is virtually RFRS. Then G is virtually fibred, in the sense that it admits a finite-index subgroup mapping onto $\mathbb{Z}$ with a finitely-generated kernel, if and only if $\beta_1^{(2)}(G) = 0$.*

Here, $\beta_1^{(2)}(G)$ denotes the first L^2-Betti number. The assumption on the vanishing of $\beta_1^{(2)}(G)$ is supposed to be thought of as the analogue of Agol's insistence on the vanishing of the Euler characteristic of the 3-manifold. Also, it is a natural assumption here: the connection between fibring and the vanishing of the L^2-Betti numbers was exhibited by Wolfgang Lück (see Theorem 2.27).

Let us remark that virtually-RFRS groups abound in nature: every subgroup of a RAAG is virtually RFRS (as shown by Agol [Ago08, Corollary 2.3]), and so every virtually-special group (in the sense of Wise) is virtually RFRS.

We show that Theorem 5.3 implies Agol's result; in fact, we prove a slightly stronger, uniform version, see Theorem 6.2 – the same uniform version can in fact be deduced directly from Agol's theorem, see [FV15, Corollary 5.2].

The method. We start by observing that it follows from a result of Thomas Schick [Sch02] that every RFRS group G satisfies the (strong) Atiyah conjecture (over $\mathbb{Q}$), since it is residually {torsion-free solvable}. The group G is also torsion-free, and so Linnell's theorem [Lin93] gives us a skew-field $\mathcal{D}(G)$ which contains $\mathbb{Q}G$. Moreover, the $\mathcal{D}(G)$-dimensions of $H_*(G; \mathcal{D}(G))$ coincide with the L^2-Betti numbers, and so $\beta_1^{(2)}(G) = 0$ if and only if $H_1(G; \mathcal{D}(G)) = 0$. Now the main technical result of the current paper is Theorem 4.13, which says roughly that $\mathcal{D}(G)$ is covered by the Novikov rings of finite index subgroups of G.

Which brings us to the Novikov rings, as defined by Sikorav [Sik87]: For every homomorphism $\phi\colon G \to \mathbb{R}$ there exists the associated Novikov ring $\widehat{\mathbb{Q}G}^{\phi}$, which contains the group ring $\mathbb{Q}G$. One should think of $\widehat{\mathbb{Q}G}^{\phi}$ as a 'completion' of $\mathbb{Q}G$ with respect to the 'point at infinity' determined by ϕ – more precisely, by the ray $\{\lambda\phi \mid \lambda \in (0, \infty)\}$. The key point for us here is the theorem of Sikorav [Sik87] (originally proved for $\mathbb{Z}$-coefficients; we recast it in the setting of $\mathbb{Q}$-coefficients in Theorem 3.11), which states that a character $\phi\colon G \to \mathbb{Z}$ is fibred, that is has a finitely-generated kernel, if and only if $H_1(G; \widehat{\mathbb{Q}G}^{\phi}) = H_1(G; \widehat{\mathbb{Q}G}^{-\phi}) = 0$.

At this point we have made a connection between the vanishing of the first L^2-Betti number of G, equivalent to

$$H_1(G; \mathcal{D}(G)) = 0$$

and fibring of a finite index subgroup H of G, equivalent to

$$H_1(H; \widehat{\mathbb{Q}H}^{\pm\psi}) = 0$$

and these two homologies are related by Theorem 4.13.

The one concept which we have not elucidated so far is that of RFRS groups. The definition is somewhat technical (see Definition 4.1); the way RFRS groups should be thought of for the purpose of this article is as follows: in a RFRS group G, every finite subset of G can be separated in the abelianisation of some finite index subgroup.

Passing to a finite index subgroup is well-understood in terms of group homology; the significance of a finite subset S of G being separated in (the free part of) the abelianisation G^{fab} lies in the following observation: if $x \in \mathbb{Q}G$ has support equal to S, then x becomes invertible in $\widehat{\mathbb{Q}G}^{\phi}$ for a 'generic' $\phi\colon G \to \mathbb{R}$.

To summarise, here is a (rather rough) heuristic behind the proof of Theorem 5.3: We investigate a chain complex C_* of free $\mathbb{Q}G$-modules computing the group homology of G. Since G is a finitely-generated group, the modules C_1 and C_0 are finitely-generated modules. Since we only care about computing $H_1(G; ?)$, we modify C_* by removing C_i for $i \geqslant 3$ and replacing C_2 by a finitely-generated module. Now, to compute $H_1(G; \mathcal{D}(G))$ we can change the bases of C_0, C_1 and C_2 over $\mathcal{D}(G)$ and diagonalise the differentials. Crucially, the change of basis matrices are all finite, and hence by Theorem 4.13 we obtain the same change of basis matrices in the Novikov ring $\widehat{\mathbb{Q}H}^{\psi}$ of some finite index subgroup H of G, for fairly generic ψ. This then implies the equivalence

$$H_1(H; \widehat{\mathbb{Q}H}^{\psi}) = 0 \Leftrightarrow H_1(G; \mathcal{D}(G)) = 0$$

as desired.

We offer also a more general result, Theorem 5.2, showing the equivalence of vanishing of higher L^2-Betti numbers and higher Novikov homology groups $H_i(H; \widehat{\mathbb{Q}H}^{\phi})$.

The tools. The main tools we use are the same as the ones used by the author in [Kie]: twisted group rings, Linnell skew-field, Ore localisations, Malcev–Neumann completions and the Novikov homology. The flavour of the article is therefore rather ring-theoretic. All of the tools are discussed in detail in Section 2.

We have already discussed the Novikov homology; we also already met the skew-field of Linnell. The Ore localisation is a very simple tool allowing one to form fields of fractions from rings satisfying a suitable condition (known as the Ore condition). The elements of an Ore localisation are fractions of ring elements.

The Malcev–Neumann completion creates a skew-field containing $\mathbb{Q}G$ for groups G which are biorderable – the biordering allows us to consistently choose a way of building power series expansions for inverses of elements of $\mathbb{Q}G$, and thus we may use infinite series instead of fractions.

And, last but not least, twisted group rings: the slogan is that group rings are better behaved than groups with respect to short exact sequences. More specifically, start with a short exact sequence of groups

$$H \to G \to Q$$

Choosing a section $s\colon Q \to G$ allows us to write every element $\sum \lambda_g g$ of $\mathbb{Q}G$ as

$$\sum_{q \in Q} \Big(\sum_{h \in H} \lambda_{hs(q)} h \Big) s(q)$$

which clearly is an element of the group ring $(\mathbb{Q}H)Q$ via the identification $s(q) \mapsto q$. The caveat is that the ring multiplication in $(\mathbb{Q}H)Q$ is chosen in such a way that $(\mathbb{Q}H)Q$ is isomorphic as a ring to $\mathbb{Q}G$. The multiplication is given explicitly in Section 2.2; what is important here is that Q acts on $\mathbb{Q}H$ by conjugation, and multiplication in Q is given by the section s, and so there are correction terms appearing due to the fact that s is not necessarily a homomorphism. The resulting ring is a *twisted* group ring, which we will nevertheless denote by $(\mathbb{Q}H)Q$, since it behaves in essentially the same way as an honest group ring. Also, twisted group rings can be easily defined using other coefficients than $\mathbb{Q}$, which we will repeatedly use in the article.

2. Preliminaries

2.1. **Generalities about rings.** Throughout the article, all rings are associative and unital, and the multiplicative unit 1 is not allowed to coincide with the additive unit 0 (this latter point has no bearing on the article; we state it out of principle). Ring homomorphisms preserve the units. Also, we will use the natural convention of denoting the non-negative integers by $\mathbb{N}$.

Before proceeding, let us establish some notation for ring elements.

Definition 2.1. Let R be a ring. A *zero-divisor* is an element x such that $xy = 0$ or $yx = 0$ for some non-zero $y \in R$.

An element x is *invertible* in R if and only if it admits both a right and a left inverse (which then necessarily coincide).

Definition 2.2 (Division closure). Let R be a subring of a ring S. We say that R is *division closed* in S if and only if every element in R which is invertible in S is already invertible in R.

The *division closure* of R is the smallest (with respect to inclusion) division-closed subring of S containing R. It is immediate that the division closure of R coincides with the intersection of all division closed subrings of S containing R.

2.2. **Twisted group rings.** In this section, let R be a ring and let G be a group.

Definition 2.3 (Support). Let R^G denote the set of functions $G \to R$. Since R has the structure of an abelian group, pointwise addition turns R^G into an abelian group as well.

Given an element $x \in R^G$, we will use both the function notation, that is we will write $x(g) \in R$ for $g \in G$, and the sum notation (standard in the context of group rings), that is we will write
$$x = \sum_{g \in G} x(g)g$$
When we use the sum notation, we do not insist on the sum being finite.

Given $x \in R^G$, we define its *support* to be
$$\operatorname{supp} x = \{g \in G \mid x(g) \neq 0\}$$

We define RG to be the subgroup of R^G consisting of all elements of finite support.

For every $H \leqslant G$, we identify R^H with R^G by declaring functions to be zero outside on $G \smallsetminus H$. In particular, this implies that $RH \subseteq RG$ (in fact, it is an abelian subgroup).

Throughout the paper, we will only make three identifications of sets; this was the first one, the second one is discussed in Section 2.3, and the third one in the proof of Proposition 2.20.

We will now endow RG with a multiplication, turning it into a ring.

Definition 2.4 (Twisted group ring). Let $R^\times$ denote the group of units of R. Let $\nu \colon G \to \operatorname{Aut}(R)$ and $\mu \colon G \times G \to R^\times$ be two functions satisfying
$$\nu(g) \circ \nu(g') = c\big(\mu(g, g')\big) \circ \nu(gg')$$
$$\mu(g, g') \cdot \mu(gg', g'') = \nu(g)\big(\mu(g', g'')\big) \cdot \mu(g, g'g'')$$
where $c \colon R^\times \to \operatorname{Aut}(R)$ takes r to the left conjugation by r. The functions ν and μ are the *structure functions*, and turn RG into a *twisted group ring* by setting

$(\star)$ \qquad\qquad\qquad $rg \cdot r'g' = r\nu(g)(r')\mu(g, g')gg'$

and extending to sums by linearity (here we are using the sum notation).

When ν and μ are trivial, we say that RG is *untwisted* (in this case we obtain the usual group ring). We adopt the convention that a group ring with $\mathbb{Z}$ or $\mathbb{Q}$ coefficients is always untwisted.

We will suppress ν from the notation, and instead write r^g for $\nu(g^{-1})(r)$.

Remark 2.5. When $H \leqslant G$ is a subgroup, restricting the structure functions to, respectively, H and $H \times H$ allows us to construct a twisted group ring RH. Observe that, as abelian groups, $RH \leqslant RG$ as H is a subset of G, and thus the structure functions turn RH into a subring of RG.

It is also easy to see that when $H \trianglelefteq G$, then RH is invariant under conjugation by elements of G.

The key source of twisted group rings is the following construction.

Proposition 2.6. *Let Q be a quotient of G with associated kernel H. Let $s\colon Q \to G$ be a set-theoretic section of the quotient map. Let RG be a twisted group ring. The structure functions $\nu\colon Q \to \mathrm{Aut}(RH)$ and $\mu\colon Q \times Q \to (RH)^\times$ given by, respectively,*

$$\nu(q) = c(s(q))$$

and

$$\mu(q,p) = s(q)s(p)s(qp)^{-1}$$

satisfy the requirements of Definition 2.4 and turn $(RH)Q$ into a twisted group ring.

Proof. We need to verify the two conditions of Definition 2.4. Firstly, for every $q, p \in Q$ we have

$$
\begin{aligned}
\nu(q) \circ \nu(p) &= c\big(s(q)\big) \circ c\big(s(p)\big) \\
&= c\big(s(q)s(p)\big) \\
&= c\big(s(q)s(p)s(qp)^{-1}s(qp)\big) \\
&= c\big(\mu(q,p)s(qp)\big) \\
&= c\big(\mu(q,p)\big) \circ \nu(qp)
\end{aligned}
$$

Secondly, for every $p, q, r \in Q$ we have

$$
\begin{aligned}
\mu(p,q) \cdot \mu(pq,r) &= s(p)s(q)s(pq)^{-1} \cdot s(pq)s(r)s(pqr)^{-1} \\
&= s(p)s(q)s(r)s(pqr)^{-1} \\
&= s(p)s(q)s(r) \cdot s(qr)^{-1}s(p)^{-1} \cdot s(p)s(qr) \cdot s(pqr)^{-1} \\
&= s(p)s(q)s(r)s(qr)^{-1}s(p)^{-1} \cdot \mu(p,qr) \\
&= c\big(s(p)\big)\big(s(q)s(r) \cdot s(qr)^{-1}\big) \cdot \mu(p,qr) \\
&= \nu(p)\big(\mu(q,r)\big) \cdot \mu(p,qr) \qquad\qquad \square
\end{aligned}
$$

We will say that the twisted group ring $(RH)Q$ is *induced* by the quotient map $G \to Q$. Note that the ring structure of $(RH)Q$ depends (a priori) on the section $s\colon Q \to G$.

Remark 2.7. Throughout, all sections are set-theoretic sections, and for every section s we always require $s(1) = 1$.

Definition 2.8. We extend the definition of $s\colon Q \to G$ to

$$s\colon (RH)Q \to RG$$

by setting

$$s\Big(\sum_{q \in Q} x_q q\Big) = \sum_{q \in Q} x_q s(q)$$

where $x_q \in RH$ for every $q \in Q$.

Lemma 2.9. *The map s is a ring isomorphism*

$$s\colon (RH)Q \xrightarrow{\cong} RG$$

Proof. The map is injective since $x_q s(q)$ and $x_p s(p)$ for $p \neq q$ are supported on distinct cosets of H in G. The map is surjective since the support of every $x \in RG$ can easily be written as a disjoint union of finite subsets of these cosets.

The fact that the map is a ring homomorphism follows directly from the definition of the twisted multiplication $(\star)$ in $(RH)Q$. $\qquad\square$

The definition of the twisted group ring structure on $(RH)Q$ depends on the choice of s. But the above result shows that for two different sections, the twisted group rings $(RH)Q$ are isomorphic, and the isomorphism restricts to the identity on the coefficients RH.

2.3. **Ore condition.**

Definition 2.10. Let R be a ring, and let T denote the set of its elements which are not zero-divisors. We say that R satisfies the *Ore condition* if and only if for every $r \in R$ and $s \in T$ there exist $p, p' \in R$ and $q, q' \in T$ such that

$$qr = ps \text{ and } rq' = sp'$$

We call R an *Ore domain* if and only if $T = R \smallsetminus \{0\}$ and R satisfies the Ore condition.

It is a classical fact that every ring satisfying the Ore condition embeds into its *Ore localisation* $\mathrm{Ore}(R)$. Elements of such an Ore localisation are expressions of the form r/s (*right fractions*) where $r \in R$ and $s \in T$; formally, they are equivalence classes of pairs (r, s), with the equivalence relation given by multiplying both elements on the right by a non-zero-divisor.

Equivalently, the Ore localisation consists of *left fractions* $s\backslash r$, and the Ore condition allows one to pass from one description to the other. This is needed in order to define multiplication in $\mathrm{Ore}(R)$. A detailed discussion of the Ore localisation is given in Passman's book [Pas85, Section 4.4].

Since the map $r \mapsto r/1$ is an embedding, we will identify R with its image in $\mathrm{Ore}(R)$. This is the second identification of sets, to which we alluded in Section 2.2.

When R is an Ore domain, $\mathrm{Ore}(R)$ is a skew-field, sometimes called the *(classical) skew-field of fractions* of R. It is immediate that if $\mathbb{F}$ is a skew-field, then $\mathbb{F} = \mathrm{Ore}(\mathbb{F})$, as every fraction r/s is equal to rs^{-1}.

Let us state a useful (and immediate) fact about $\mathrm{Ore}(R)$.

Proposition 2.11. *Let R and S be rings satisfying the Ore condition. Let*

$$\rho \colon R \to S$$

be a ring homomorphism, and suppose that non-zero-divisors in R are taken by ρ to non-zero-divisors in S. Then

$$\mathrm{Ore}(\rho) \colon \mathrm{Ore}(R) \to \mathrm{Ore}(S)$$
$$r/s \mapsto \rho(r)/\rho(s)$$

is an injective ring homomorphism (which clearly extends ρ).

We are interested in Ore localisations for the following reason.

Theorem 2.12 (Tamari [Tam57]). *Let RG be a twisted group ring of an amenable group G. If RG is a domain, then it satisfies the Ore condition, and hence $\mathrm{Ore}(RG)$ is a skew-field.*

In fact, Tamari's theorem admits a partial converse: if $\mathbb{Q}G$ is an Ore domain then G is amenable; see the appendix of [Bar19].

2.4. **Atiyah conjecture.** Let G be a countable group.

Definition 2.13. The *von Neumann algebra* $\mathcal{N}(G)$ of G is defined to be the algebra of bounded G-equivariant operators on $L^2(G)$.

Above, we view the operators as acting on the left, and the G action is the action on the right. Observe that a bounded (that is, continuous) G-equivariant operator ζ on $L^2(G)$ is uniquely determined by $\zeta(1) \in L^2(G)$. This allows us to view $\mathcal{N}(G)$ as a subset of $L^2(G)$, which in turn is a subset of $\mathbb{C}^G$. We will adopt this point of view, since it is convenient for us to work in $\mathbb{C}^G$.

The group ring $\mathbb{Q}G$ acts on $L^2(G)$ on the left by bounded operators, commuting with the right G-action. Thus, we have a map $\mathbb{Q}G \to \mathcal{N}(G)$. This map is easily seen to be injective. With the point of view adopted above we can in fact say much more: this map is the identity map $\mathbb{Q}G \leqslant \mathbb{Q}^G < \mathbb{C}^G$. Thus, we have $\mathbb{Q}G$ being an actual subset of $\mathcal{N}(G)$.

We need the von Neumann algebra only to define Linnell's ring $\mathcal{D}(G)$ (which we shall do in a moment), and so readers not familiar with von Neumann algebras should not feel discouraged.

Theorem 2.14 ([Lüc02, Theorem 8.22(1)])**.** *The von Neumann algebra $\mathcal{N}(G)$ satisfies the Ore condition.*

Note that $\mathcal{N}(G)$ is extremely far from being a domain.

Definition 2.15 (Linnell ring)**.** Let G be a group. We define $\mathcal{D}(G)$ (the *Linnell ring*) to be the division closure of $\mathbb{Q}G$ inside $\mathrm{Ore}(\mathcal{N}(G))$.

It is important to note that left multiplication by x for $x \in \mathbb{Q}G$ endows $\mathcal{D}(G)$ with a $\mathbb{Q}G$-module structure.

It is also important to observe that in Lück's book, $\mathcal{D}(G)$ denotes the division closure of $\mathbb{C}G$ rather than $\mathbb{Q}G$; this is a minor point, as all the desired properties are satisfied by our $\mathcal{D}(G)$ as well.

Definition 2.16 (Atiyah conjecture)**.** We say that a torsion-free group G satisfies the *Atiyah conjecture* if and only if $D(G)$ is a skew-field.

Remark 2.17. This is not the usual definition of the Atiyah conjecture; it is however equivalent to the (strong) Atiyah conjecture (over $\mathbb{Q}$) for torsion-free groups by a theorem of Linnell [Lin93].

The Atiyah conjecture has been established for many classes of torsion-free groups. Crucially for us, it is known for residually {torsion-free solvable} groups by a result of Schick [Sch02]. It is also known for virtually {cocompact special} groups [Sch14] and locally-indicable groups [JZLÁ], as well as for large classes of groups with strong inheritance properties.

Lemma 2.18. *Suppose that G is a torsion-free group satisfying the Atiyah conjecture. Every sub-skew-field $\mathbb{K}$ of $\mathcal{D}(G)$ containing $\mathbb{Q}G$ is equal to $\mathcal{D}(G)$.*

Proof. Every non-zero element in $\mathbb{K}$ is invertible in $\mathbb{K}$, and therefore $\mathbb{K}$ is division closed in $\mathrm{Ore}(\mathcal{N}(G))$. Also, $\mathbb{K}$ contains $\mathbb{Q}G$. But $\mathcal{D}(G)$ is the intersection of all such rings, and hence $\mathcal{D}(G) = \mathbb{K}$. $\square$

We will now discuss the behaviour of $\mathcal{D}(G)$ under passing to subgroups of G.

Proposition 2.19 ([Lüc02, Lemma 10.4])**.** *Every subgroup of a group satisfying the Atiyah conjecture also satisfies the Atiyah conjecture.*

Proposition 2.20. *Let G be a group with a subgroup H. The Linnell ring $\mathcal{D}(H)$ is a subring of $\mathcal{D}(G)$. Moreover, if H is a normal subgroup then $\mathcal{D}(H)$ is invariant under conjugation by elements of G.*

Sketch proof. Since the mathematical content of this proposition is standard (see for example [Kie, Proposition 4.6] or [Lüc02, Section 10.2]), we offer only a sketch proof.

Take $\zeta \in \mathcal{N}(H)$. We will now make ζ act on $L^2(G)$. To this end, let P be a set of coset representatives of H in G. Note that $L^2(G)$ is equal to the set of L^2 functions in $\mathbb{C}^{\bigcup_{p \in P} Hp} = \prod_{p \in P} \mathbb{C}^{Hp}$, and hence is a subset of $\prod_{p \in P} L^2(Hp)$. We let ζ act on each $L^2(Hp)$ by $x \mapsto \zeta(xp^{-1})p$. It is immediate that the resulting action of ζ on $\prod_{p \in P} L^2(Hp)$ preserves $L^2(G)$; it is equally clear that the action is continuous and G-equivariant. This procedure produces a ring homomorphism $\mathcal{N}(H) \to \mathcal{N}(G)$, which we denote $\zeta \mapsto \zeta'$.

Recall that we are viewing $\mathcal{N}(H)$ and $\mathcal{N}(G)$ as subsets of $\mathbb{C}^H$ and $\mathbb{C}^G$, respectively. Since we have $\zeta(1) = \zeta'(1)$ by construction, the map $\zeta \mapsto \zeta'$ is actually a restriction of the identity map $\mathbb{C}^H \leqslant \mathbb{C}^G$. Therefore, we have $\mathcal{N}(H) \leqslant \mathcal{N}(G)$.

Now an easy argument (using the decomposition of G into cosets of H) shows that non-zero-divisors of $\mathcal{N}(H)$ are still non-zero-divisors in $\mathcal{N}(G)$. Proposition 2.11 tells us that the identity map id induces an injective map

$$\mathrm{Ore(id)}\colon \mathrm{Ore}(\mathcal{N}(H)) \to \mathrm{Ore}(\mathcal{N}(H))$$

Formally speaking, since Ore localisation consist of equivalence classes, we are not allowed to say that $\mathrm{Ore}(\mathcal{N}(H))$ is a subset of $\mathrm{Ore}(\mathcal{N}(H))$. We never view elements of Ore localisations as equivalence classes however – we view them always as fractions r/s, and such expressions in $\mathrm{Ore}(\mathcal{N}(H))$ are also valid expressions in $\mathrm{Ore}(\mathcal{N}(G))$. We will therefore identify (for the third and final time) the set $\mathrm{Ore}(\mathcal{N}(H))$ with the corresponding subset of $\mathrm{Ore}(\mathcal{N}(G))$. It is now easy to see that we also have $\mathcal{D}(H) \leqslant \mathcal{D}(G)$.

Now suppose that H is a normal subgroup, and let $g \in G$. Conjugation gives us an action of g on $\mathbb{C}^G$, and this action clearly preserves $\mathbb{C}^H$. It is immediate that it also preserves $\mathcal{N}(H)$ and $\mathbb{Q}H$, and hence has to preserve $\mathcal{D}(H)$. $\square$

We will now concentrate on the situation in which H is a normal subgroup of G.

Remark 2.21. The above proposition immediately implies that if H is the kernel of an epimorphism $G \to Q$, we may form a twisted group ring $\mathcal{D}(H)Q$, where the structural functions are precisely as in Proposition 2.6. We will say that $\mathcal{D}(H)Q$ is *induced by $G \to Q$*.

In practice, we will be interested in two cases: Q will be either finite or free-abelian. Let us first focus on the case of Q being finite.

Lemma 2.22. *If Q is finite, and G is torsion free and satisfies the Atiyah conjecture, then the natural map $\mathcal{D}(H)Q \to \mathcal{D}(G)$ given by*

$$\sum_{q \in Q} \lambda_q q \mapsto \sum_{q \in Q} \lambda_q s(q)$$

is a ring isomorphism.

The statement above can be proven completely analogously to [Lüc02, Lemma 10.59]. Note that Lück is only assuming $\mathcal{D}(G)$ to be semisimple, whereas we are working under a stronger assumption of $\mathcal{D}(G)$ (and hence also $\mathcal{D}(H)$) being skew-fields.

As before, we will refer to the isomorphism $\mathcal{D}(H)Q \to \mathcal{D}(G)$ above simply as s.

Now we move to the case of Q being free abelian.

Proposition 2.23. *Suppose that G is finitely generated, torsion free, and satisfies the Atiyah conjecture. Let s_α denote a section of an epimorphism $\alpha\colon G \to Q$ onto a free-abelian group; let $K = \ker \alpha$. The twisted group ring $\mathcal{D}(K)Q$ satisfies the Ore condition, and the map*

$$s_\alpha\colon \mathcal{D}(K)Q \to \mathcal{D}(G)$$

is injective and induces an isomorphism

$$\mathrm{Ore}(s_\alpha)\colon \mathrm{Ore}\big(\mathcal{D}(K)Q\big) \cong \mathcal{D}(G)$$

Sketch proof. The proof is completely analogous to that of [Lüc02, Lemma 10.69], and hence we only offer an outline here.

First we note that, as before, our $\mathcal{D}(G)$ is not the same as Lück's, but it exhibits the same properties. Lück starts with a short exact sequence with G in the middle, and the quotient being virtually finitely-generated abelian. We are looking at a short exact sequence

$$K \to G \to Q$$

where the quotient is finitely generated and free abelian, and so certainly satisfies the assumptions of [Lüc02, Lemma 10.69]. The lemma also requires $\mathcal{D}(K)$ to be semi-simple, but in our case we have the stronger assumption that $\mathcal{D}(K)$ is a skew-field, since K is torsion-free and satisfies the Atiyah conjecture. Lück concludes that the Ore localisation of $\mathcal{D}(K)Q$ exists and is isomorphic to $\mathcal{D}(G)$; the isomorphism produced coincides with our $\mathrm{Ore}(s_\alpha)$. $\qquad\square$

2.5. L^2-homology.

Definition 2.24 (L^2-Betti numbers). Let G be a torsion-free group satisfying the Atiyah conjecture. Let C_* be a chain complex of free left $\mathbb{Q}G$-modules. The nth *L^2-Betti number* of C_*, denoted $\beta_n^{(2)}(C_*)$, is defined to be the dimension of $H_n(\mathcal{D}(G)\otimes_{\mathbb{Q}G} C_*)$ viewed as a (left) $\mathcal{D}(G)$-module. Note that there is no ambiguity here, since dimensions of modules over skew-fields (which are essentially vector spaces) are well-defined.

We say that C_* is L^2-acyclic if and only if $\beta_n^{(2)}(C_*) = 0$ for every n.

Similarly, we define the nth *L^2-Betti number* of G, denoted $\beta_n^{(2)}(G)$, to be the dimension of $H_n(G; \mathcal{D}(G))$ viewed as a $\mathcal{D}(G)$-module; we say that G is L^2-acyclic if and only if $\beta_n^{(2)}(G) = 0$ for every n.

Remark 2.25. Again, this is not the usual definition of L^2-Betti numbers. In the above setting it coincides with the usual definition by [Lüc02, Lemma 10.28(3)].

The following is a version of a celebrated theorem of Lück [Lüc02, Theorem 1.39] (it is stated here in a slightly reworded fashion).

Theorem 2.26 ([Lüc02, Theorem 7.2(5)]). *Let*

$$1 \to H \to G \to K \to 1$$

be an exact sequence of groups such that $\beta_p^{(2)}(H)$ is finite for all $p \leqslant d$. Suppose that K is infinite amenable or suppose that BK has finite d-skeleton and there is an injective endomorphism $K \to K$ whose image is proper and of finite index in K. Then $\beta_p^{(2)}(G) = 0$ for every $p \leqslant d$.

Restricting the above to the situation we will be interested in gives the following.

Theorem 2.27. *If H is a finitely-generated group, and if*

$$1 \to H \to G \to \mathbb{Z} \to 1$$

is an exact sequence of groups, then $\beta_1^{(2)}(G) = 0$.

Proof. We need only observe that $\mathbb{Z}$ is an infinite amenable group, that being finitely generated implies finiteness of the zeroth and first L^2-Betti numbers. $\qquad\square$

2.6. Biorderable groups.

Definition 2.28. A *biordering* of a group G is a total order $\leqslant$ on G such that for every $a, b, c \in G$ we have

$$a \leqslant b \Rightarrow ac \leqslant bc \text{ and } a \leqslant b \Rightarrow ca \leqslant cb$$

A group G is *biorderable* if and only if it admits a biordering.

Definition 2.29 (Malcev–Neumann skew-field)**.** Let $\leqslant$ denote a biordering of a group G. Let R be a ring, and let RG be a twisted group ring. The set

$$\mathbb{F}_{\leqslant}(RG) = \{x \colon G \to R \mid \operatorname{supp} x \text{ is } \leqslant\text{-well ordered}\}$$

becomes a skew-field when endowed with the twisted convolution $(\star)$ – see [Mal48, Neu49]. We call it the *Malcev–Neumann skew-field* of RG with respect to $\leqslant$. (Recall that a totally ordered set is *well-ordered* if and only if every non-empty subset thereof admits a unique minimum.)

3. NOVIKOV RINGS

3.1. Novikov rings.

Definition 3.1 (Novikov rings)**.** Let RG be a twisted group ring, and let

$$\phi \in H^1(G; \mathbb{R}) = \operatorname{Hom}(G; \mathbb{R})$$

(we will say that ϕ is a *character*). We set

$$\widehat{RG}^{\phi} = \left\{ x \colon G \to R \,\middle|\, \left| \operatorname{supp} x \cap \phi^{-1}((\infty, \kappa]) \right| < \infty \text{ for every } \kappa \in \mathbb{R} \right\}$$

The set $\widehat{RG}^{\phi}$ is turned into a ring using the twisted convolution formula $(\star)$. We will call it the *Novikov ring* of RG with respect to ϕ.

Definition 3.2. Let $(-\infty, \infty] = \mathbb{R} \sqcup \{\infty\}$ be endowed with the obvious total ordering and an abelian monoid structure given by addition (with the obvious convention that $x + \infty = \infty$ for every $x \in (-\infty, \infty]$).

Let $\phi \in H^1(G; \mathbb{R})$ be given. We extend the definition of $\phi \colon G \to \mathbb{R}$ to

$$\phi \colon \widehat{RG}^{\phi} \to (-\infty, \infty]$$

by setting

$$\phi(x) = \begin{cases} \min \phi\big(\operatorname{supp}(x)\big) & \text{if} \quad x \neq 0 \\ \infty & \text{if} \quad x = 0 \end{cases}$$

In particular, this also defines $\phi \colon RG \leqslant \widehat{RG}^{\phi} \to (-\infty, \infty]$.

Remark 3.3. Let $(x_i)_i$ be a sequence of elements of $\widehat{RG}^{\phi}$ with $(\phi(x_i))_i$ converging to ∞. It is easy to see that the partial sums

$$\sum_{i=0}^{N} x_i$$

converge pointwise as functions $G \to R$, and hence one can easily define their limit $\sum_{i=0}^{\infty} x_i \in \widehat{RG}^{\phi}$.

Lemma 3.4. *Let RG be a twisted group ring, and let $\phi \in H^1(G; \mathbb{R})$ be a character. For every $x, y \in \widehat{RG}^{\phi}$ we have*

(1) $\phi(x + y) \geqslant \min\{\phi(x), \phi(y)\}$;
(2) $\phi(xy) \geqslant \phi(x) + \phi(y)$;
(3) $\phi(xy) = \phi(x) + \phi(y)$ *if RG has no zero-divisors.*

Proof.   (1) This follows immediately from the definition of $\phi\colon \widehat{RG}^\phi \to (-\infty, \infty]$ and the fact that $\mathrm{supp}(x+y) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$.

(2) Let $a \in \mathrm{supp}(x)$ and $b \in \mathrm{supp}(y)$ be such that

$$\phi(x) = \phi(a) \text{ and } \phi(y) = \phi(b)$$

For every $a' \in \mathrm{supp}(x)$ and $b' \in \mathrm{supp}(y)$ we have

$$\phi(a') \geqslant \phi(a) \text{ and } \phi(b') \geqslant \phi(b)$$

Now, every $c \in \mathrm{supp}(xy)$ can be written as a product $c = a'b'$ of some a' and b' as above. Thus

$$\phi(xy) \geqslant \min_{a',b'} \phi(a'b') = \min_{a',b'} \big(\phi(a') + \phi(b')\big) = \phi(a) + \phi(b) = \phi(x) + \phi(y)$$

(3) If RG has no zero-divisors, we know that $ab \in \mathrm{supp}(xy)$, and so

$$\phi(xy) = \phi(a) + \phi(b) = \phi(x) + \phi(y) \qquad \square$$

Remark 3.5. It follows immediately from (3) above, that if RG has no zero-divisors, then $\phi\colon RG \smallsetminus \{0\} \to \mathbb{R}$ is a monoid homomorphism.

Definition 3.6 (Free abelianisation)**.** Let G be a group. Let G^{fab} denote the image of G in $H_1(G; \mathbb{Q})$ (under the natural homomorphism). When G is finitely generated, G^{fab} is simply the *free part of the abelianisation* of G, that is abelianisation of G divided by the torsion part. We define $\alpha\colon G \to G^{\mathrm{fab}}$ to be the natural epimorphism. We will refer to α as the *free abelianisation map*.

Remark 3.7. Clearly, every character in $H^1(G; \mathbb{R})$ yields a character $G^{\mathrm{fab}} \to \mathbb{R}$ in the obvious way. Conversely, every character in $H^1(G^{\mathrm{fab}}; \mathbb{R})$ yields a character in $H^1(G; \mathbb{R})$. Thus we will not differentiate between such characters, and use the same symbol to denote both.

Proposition 3.8. *Let G be a group, and let $\phi \in H^1(G; \mathbb{R})$. Choose a section $s_\alpha\colon G^{\mathrm{fab}} \to G$, and recall that this choice gives $(\mathbb{Q}K)G^{\mathrm{fab}}$ the structure of a twisted group ring, where $K = \ker \alpha$. The ring isomorphism $s_\alpha\colon (\mathbb{Q}K)G^{\mathrm{fab}} \to \mathbb{Q}G$ extends to a ring isomorphism*

$$\widehat{(\mathbb{Q}K)G^{\mathrm{fab}}}^\phi \cong \widehat{\mathbb{Q}G}^\phi$$

Proof. We define $\iota\colon \mathbb{Q}K^{G^{\mathrm{fab}}} \to \mathbb{Q}^G$ by setting $\iota(x)(g) = x(\alpha(g))(gs_\alpha(\alpha(g))^{-1})$ for every $x\colon G^{\mathrm{fab}} \to \mathbb{Q}K$. Let us unravel this definition: we take $\alpha(g)$, an element of the the group G^{fab}, and evaluate x on it. This yields an element of $\mathbb{Q}K$, which is formally a subset of $\mathbb{Q}^K$, the set of functions from K to $\mathbb{Q}$. Thus, it makes sense to evaluate $x(\alpha(g))$ on elements in $K = \ker \alpha$, and $gs_\alpha(\alpha(g))^{-1}$ is such an element. After this last evaluation we obtain an element of $\mathbb{Q}$, as desired.

When restricted to $(\mathbb{Q}K)G^{\mathrm{fab}}$, the map ι coincides with the map s_α (which is an isomorphism by Lemma 2.9): this is easy to verify, but cumbersome to write down, so we omit it here.

We claim that $\iota(\widehat{(\mathbb{Q}K)G^{\mathrm{fab}}}^\phi)$ lies within $\widehat{\mathbb{Q}G}^\phi$. Indeed, take $x \in \widehat{(\mathbb{Q}K)G^{\mathrm{fab}}}^\phi$. By definition, for every $\kappa \in \mathbb{R}$ the set $\mathrm{supp}\, x \cap \phi^{-1}((-\infty, \kappa]) \subseteq G^{\mathrm{fab}}$ is finite.

Observe that for $a \in G^{\mathrm{fab}}$ we have

$$\mathrm{supp}\,\iota(x) \cap \alpha^{-1}(a) = \{g \in G \mid \alpha(g) = a \text{ and } x(a)(gs_\alpha(\alpha(g))^{-1}) \neq 0\}$$

Since for every $a \in G^{\mathrm{fab}}$ the function $x(a) \in \mathbb{Q}K$ has finite support in K, we have

$$|\mathrm{supp}\,\iota(x) \cap \alpha^{-1}(a)| < \infty$$

for every $a \in G^{\mathrm{fab}}$. Therefore $\mathrm{supp}\,\iota(x) \cap \phi^{-1}((\infty, \kappa]) \subseteq G$ is finite, as it is a finite union of finite sets. This shows that $\iota(x) \in \widehat{\mathbb{Q}G}^\phi$, as claimed.

There is an obvious inverse to ι defined on $\widehat{\mathbb{Q}G}^{\phi}$, namely $x \in \widehat{\mathbb{Q}G}^{\phi}$ being mapped to $x' \colon G^{\mathrm{fab}} \to \mathbb{Q}K$ given by $x'(a)(k) = x(ks(a))$. It is immediate that $x' \in \widehat{(\mathbb{Q}K)G^{\mathrm{fab}}}^{\phi}$. We now verify that this map is an inverse to ι:

$$\iota(x')(g) = x'\big(\alpha(g)\big)\big(gs_\alpha(\alpha(g))^{-1}\big) = x\big(gs_\alpha(\alpha(g))^{-1}s_\alpha(\alpha(g))\big) = x(g)$$

and

$$\iota(x)'(a)(k) = \iota(x)(ks_\alpha(a)) = x(\alpha(ks_\alpha(a))\big(ks_\alpha(a)s_\alpha(\alpha(ks_\alpha(a)))^{-1}\big) = x(a)(k) \quad \square$$

Again, we will denote the isomorphism $\widehat{(\mathbb{Q}K)G^{\mathrm{fab}}}^{\phi} \overset{\cong}{\to} \widehat{\mathbb{Q}G}^{\phi}$ by s_α.

3.2. **BNS invariants and Sikorav's theorem.** Let G be a finitely-generated group, and let X denote the Cayley graph of G with respect to some finite generating set T. Recall that G is the vertex set of X, and each edge comes with an orientation and a label from T.

Definition 3.9. A character $\phi \in H^1(G;\mathbb{R}) \smallsetminus \{0\}$ lies in the *Bieri–Neumann–Strebel* (or *BNS*) *invariant* $\Sigma(G)$ if and only if the full subgraph of X spanned by $\{g \in G \mid \phi(g) \geqslant 0\}$ is connected.

The invariant $\Sigma(G)$ does not depend on the choice of a finite generating set – this can easily be proven directly, but it will also follows from Sikorav's theorem below.

The significance of BNS invariants for us lies in the following result.

Theorem 3.10 (Bieri–Neumann–Strebel [BNS87, Theorem B1]). *Let G be a finitely-generated group, and let $\phi \colon G \to \mathbb{Z}$ be non-trivial. Then $\ker \phi$ is finitely generated if and only if $\{\phi, -\phi\} \subseteq \Sigma(G)$.*

In [Sik87] Sikorav gave a way of computing $\Sigma(G)$ using the Novikov rings. He used the rings $\widehat{\mathbb{Z}G}^{\phi}$, whereas we have to work with $\widehat{\mathbb{Q}G}^{\phi}$. For this reason we will prove Sikorav's theorem in our setting (the proof is identical).

Theorem 3.11 (Sikorav's theorem). *Let G be a finitely-generated group, and let $\phi \in H^1(G;\mathbb{R}) \smallsetminus \{0\}$. The following are equivalent:*

(1) $\phi \in \Sigma(G)$;
(2) $H_1(G; \widehat{\mathbb{Z}G}^{\phi}) = 0$;
(3) $H_1(G; \widehat{\mathbb{Q}G}^{\phi}) = 0$.

Proof. Let us start with some general setup: Since $\phi \neq 0$, there exists a generator $s \in T$ with $\phi(s) \neq 0$. Without loss of generality we may assume that $\phi(s) > 0$.

Let

$$C_1 \to C_0$$

denote the cellular chain complex of X with coefficients in $\mathbb{Z}$. It is immediate that C_1 and C_0 are finitely generated free $\mathbb{Z}G$-modules. In fact, C_0 is 1-dimensional, and we can append the chain complex on the right with the augmentation map $\mathbb{Z}G \to \mathbb{Z}$. We obtain

$$C_1 \to C_0 \to \mathbb{Z}$$

and it is immediate that this complex is exact at C_0. We append the complex again, this time on the left by adding a free $\mathbb{Z}G$-module C_2 in such a way that

$$C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

is exact at C_1 and C_0. We denote the above chain complex by C_*. The differentials will be denoted by ∂.

We pick cellular bases for the free modules C_0 and C_1. For C_0 the basis we pick is the singleton $\{1\}$ – recall that group elements are the vertices of X. For C_1 the basis is the collection of oriented edges emanating from 1. We will denote the basis element corresponding to an edge labelled $t \in T$ by e_t.

Setting the value of ϕ on the basis elements of C_1 to zero gives as a map $\phi \colon C_1 \to (-\infty, \infty]$. Specifically, we have $\phi(g.e_t) = \phi(g)$, which assigns a value to every edge in X.

It is clear that $\{e_t \mid t \in T\}$ forms a basis of $\widehat{\mathbb{Z}G}^{\phi} \otimes_{\mathbb{Z}G} C_1$, and we may view chains in $\widehat{\mathbb{Z}G}^{\phi} \otimes_{\mathbb{Z}G} C_1$ as functions from the edge set E of X to $\mathbb{Z}$.

$(\mathbf{1}) \Rightarrow (\mathbf{2})$ In this part of the proof we adopt the convention that tensoring without specified ring is always over $\mathbb{Z}G$.

Since $\phi(s) \neq 0$, the element $1 - s$ is invertible in $\widehat{\mathbb{Z}G}^{\phi}$ with inverse $\sum_{i \in \mathbb{N}} s^i$. Therefore the space of 1-cycles in $\widehat{\mathbb{Z}G}^{\phi} \otimes C_*$ is spanned by

$$e_t' = e_t - (1 - t)(1 - s)^{-1} e_s$$

with $t \in T \smallsetminus \{s\}$. In fact it is easy to see that these elements form a basis of the space of 1-cycles. We will now define a $\widehat{\mathbb{Z}G}^{\phi}$-linear map c from the 1-cycles to $\widehat{\mathbb{Z}G}^{\phi} \otimes C_2$ by specifying its value on every basis element e_t'.

Take $t \in T \smallsetminus \{s\}$. Since $\phi(s) > 0$, there exists $n \in \mathbb{N}$ such that $\phi(ts^n) > \phi(s)$ and $\phi(s^n) > \phi(s)$. We are assuming that $\phi \in \Sigma(G)$, and the definition tells us that we may connect $s^{-1}ts^n$ to $s^{-1}s^n$ in X by a path which goes only through vertices g with $\phi(g) \geqslant 0$. This implies that the path traverses only edges whose ϕ value is non-negative.

We now act (on the left) by s, and conclude that the existence of a 1-chain p (supported on the image of the path we just constructed) with boundary $\partial p = ts^n - s^n$, and such that ϕ evaluated at the edges in the support of p is always positive. Since C_* is exact at C_1, there exists $c_t \in C_2$ such that

$$(\ddagger\ddagger) \qquad \partial c_t = e_t + t \sum_{i=0}^{n-1} s^i e_s - p - \sum_{i=0}^{n-1} s^i e_s$$

since the right-hand side is a cycle – it is supported on a cycle in X composed of the edge corresponding to e_t, the path underlying p, and two segments, one connecting 1 to s^n and the other connecting t to ts^n. We define $c(e_t') = c_t$.

Now $\partial \circ c(e_t') = \partial c_t$, which we have already computed. Since ∂c_t is a cycle, we can write it using the basis $\{e_t' \mid t \in T \smallsetminus \{s\}\}$ – this is done by forgetting e_s in $(\ddagger\ddagger)$, and replacing e_t by e_t'. We obtain

$$\partial c_t = e_t' - \sum_{r \in T \smallsetminus \{s\}} \lambda_r e_r'$$

where $\lambda_r \in \widehat{\mathbb{Z}G}^{\phi}$ is such that $\phi(\lambda_r) > 0$ for every r – this follows from the properties of p. Writing $\partial \circ c$ as a matrix, with respect to the basis $\{e_t' \mid t \in T \smallsetminus \{s\}\}$, we obtain

$$\mathrm{I} - M$$

where ϕ takes every entry of M to $(0, \infty)$. This matrix is invertible over $\widehat{\mathbb{Z}G}^{\phi}$ with inverse $\sum_{i \in \mathbb{N}} M^i$ (compare Remark 3.3), and therefore $\partial \circ c$ is an epimorphism. Hence the differential

$$\partial \colon \widehat{\mathbb{Z}G}^{\phi} \otimes C_2 \to \widehat{\mathbb{Z}G}^{\phi} \otimes C_1$$

is onto the cycles, and thus $H_1(\widehat{\mathbb{Z}G}^{\phi} \otimes C_*) = 0$, as claimed.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ This is immediate since $\widehat{\mathbb{Q}G}^{\phi} = \widehat{\mathbb{Q}G}^{\phi} \otimes_{\widehat{\mathbb{Z}G}^{\phi}} \widehat{\mathbb{Z}G}^{\phi}$, and tensoring is right-exact.

$(\mathbf{3}) \Rightarrow (\mathbf{1})$ We start by replacing C_* by $\mathbb{Q} \otimes_{\mathbb{Z}} C_*$; in order not to make the notation too cumbersome, we will continue to use C_*. Also, in this part, unspecified tensor products are taken over $\mathbb{Q}G$.

We are assuming that

$$0 = H_1(G; \widehat{\mathbb{Q}G}^{\phi}) = H_1(\widehat{\mathbb{Q}G}^{\phi} \otimes C_*)$$

Take $g, h \in G$ with $\phi(g), \phi(h) \geqslant 0$. We need to show that there exists a path in X going only through vertices $x \in G$ with $\phi(x) \geqslant 0$ connecting g to h.

Since X is connected, there exists a path p in X connecting g to h. We take p to be geodesic. Recall that $\phi(s) > 0$. Consider the function $\xi \colon E \to \mathbb{Q}$ which is the characteristic function of the set of edges of p (taking orientations into account) and the edges of the infinite rays r_1 and r_2 emanating from g and h and using only edges with label s (and only positively orientated).

The function ξ clearly lies in $\widehat{\mathbb{Q}G}^{\phi} \otimes C_1$; by construction, it is actually a cycle there. Since $H_1(\widehat{\mathbb{Q}G}^{\phi} \otimes C_*) = 0$, there exists an element $x \in \widehat{\mathbb{Q}G}^{\phi} \otimes C_2$ with $\partial x = \xi$.

Since C_2 and $\widehat{\mathbb{Q}G}^{\phi} \otimes C_2$ are free modules, there exists a finite-dimensional free submodule $C_2' \leqslant C_2$ such that

$$x \in \widehat{\mathbb{Q}G}^{\phi} \otimes C_2'$$

The module C_2' has a finite basis; the boundary of each of the basis elements is a chain in C_1. The chosen basis for C_2' gives also a basis for $\widehat{\mathbb{Q}G}^{\phi} \otimes C_2'$, and so we may write $x = (x_1, \ldots, x_m)$ for some $m \in \mathbb{N}$ with $x_i \in \widehat{\mathbb{Q}G}^{\phi}$. By taking the restriction of each $x_i \colon G \to \mathbb{Q}$ to suitable finite subset of G we form elements $x_i' \in \mathbb{Q}G$ such that

$$\phi\big(\partial(x_i - x_i')\big) > 0$$

for every i. Let $x' = (x_1', \ldots, x_m') \in C_2' \leqslant C_2$. By construction, the support of $\partial(x - x')$ contains only edges whose value under ϕ is positive. Now

$$\xi = \partial x = \partial(x - x') + \partial x'$$

and so $\xi - \partial x'$ is a cycle in $\widehat{\mathbb{Q}G}^{\phi} \otimes C_1$ supported only on edges of positive ϕ-value.

Note that $\partial(x')$ has finite support, and the rays r_1 and r_2 are both infinite. Thus, $\xi - \partial x'$ contains in its support some edges of r_1 and r_2. Since $\xi - \partial x'$ is a cycle, there exists a path in $\mathrm{supp}(\xi - \partial x')$, say q, connecting some endpoints, say x_1 and x_2, of these edges. Now a path starting at g, following r_1 up to the vertex x_1 (without loss of generality), then following q, and then following the ray r_2 against its orientation to h, is as desired. $\qquad\square$

3.3. Linnell's skew-field and Novikov rings.

Suppose that we have a finitely-generated group G which is torsion free and satisfies the Atiyah conjecture. We have already discussed that, in this case, $\mathbb{Q}G$ embeds into Linnell's skew-field $\mathcal{D}(G)$. We pick a character $\phi \colon G \to \mathbb{R}$. We also pick a section s_α of the free abelianisation map α, and let $K = \ker \alpha$. Recall that we will also treat ϕ as a character $\phi \colon G^{\mathrm{fab}} \to \mathbb{R}$.

The key point of this section is to find a skew-field (depending on a character ϕ) that will simultaneously house $\widehat{\mathbb{Q}G}^{\phi}$ and $\mathcal{D}(G)$. We will use it to define a set of good elements of $\mathcal{D}(G)$, which we will call *representable in* $\widehat{\mathbb{Q}G}^{\phi}$, by taking the intersection of $\mathcal{D}(G)$ and $\widehat{\mathbb{Q}G}^{\phi}$ in the larger object. To achieve this goal, we need to study a multitude of injective maps between objects we have previously constructed. These are summarised in Figure 3.12, which is a commutative diagram.
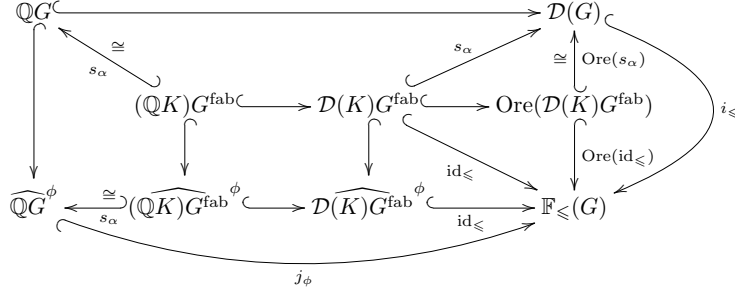
FIGURE 3.12. A commutative diagram of the various embeddings

In the diagram, we see three maps labelled s_α. These are all induced by the set-theoretic section $s_\alpha \colon G^{\mathrm{fab}} \to G$ in a way we had made explicit previously.

All the unlabelled maps are various identity maps.

We pick a biordering $\leqslant$ on the finitely-generated free-abelian group G^{fab} which is *compatible* with ϕ, in the sense that $\phi \colon G^{\mathrm{fab}} \to \mathbb{R}$ is an order-preserving homomorphism. Let $\mathbb{F}_{\leqslant}(G)$ denote the Malcev–Neumann skew-field $\mathbb{F}_{\leqslant}(\mathcal{D}(K)G^{\mathrm{fab}})$ associated to $\leqslant$. By definition, $\mathbb{F}_{\leqslant}(G)$ is a subset of $\mathcal{D}(K)^{G^{\mathrm{fab}}}$. The ring $\widehat{\mathcal{D}(K)G^{\mathrm{fab}}}^{\phi}$ is another such subset, and in fact we have

$$\widehat{\mathcal{D}(K)G^{\mathrm{fab}}}^{\phi} \subseteq \mathbb{F}_{\leqslant}(G)$$

since ϕ is order preserving; we will denote the inclusion map by $\mathrm{id}_{\leqslant}$. This map is located in the bottom-right part of Figure 3.12. Note that $\mathrm{id}_{\leqslant}$ does not in fact depend on $\leqslant$. We are adding the subscript for clarity. We define

$$j_\phi = \mathrm{id}_{\leqslant} \circ \mathrm{id} \circ s_\alpha^{-1} \colon \widehat{\mathbb{Q}G}^{\phi} \to \mathbb{F}_{\leqslant}(G)$$

Composing

$$\mathrm{id}_{\leqslant} \circ \mathrm{id} \colon \mathcal{D}(K)G^{\mathrm{fab}} \to \mathbb{F}_{\leqslant}(G)$$

gives us an inclusion which we will also call $\mathrm{id}_{\leqslant}$; again, $\mathrm{id}_{\leqslant}$ does not depend on $\leqslant$. Now $\mathrm{Ore}(\mathrm{id}_{\leqslant}) \colon \mathrm{Ore}(\mathcal{D}(K)G^{\mathrm{fab}}) \to \mathbb{F}_{\leqslant}(G)$ denotes the map induced by $\mathrm{id}_{\leqslant} \colon \mathcal{D}(K)G^{\mathrm{fab}} \to \mathbb{F}_{\leqslant}(G)$, as in Proposition 2.11; this last map is injective, as it is a non-trivial ring homomorphism from a skew-field. We define

$$i_{\leqslant} = \mathrm{Ore}(\mathrm{id}_{\leqslant}) \circ \mathrm{Ore}(s_\alpha)^{-1} \colon \mathcal{D}(G) \to \mathbb{F}_{\leqslant}(G)$$

Note that $\mathrm{Ore}(\mathrm{id}_{\leqslant})$ and $i_{\leqslant}$ are the only maps in Figure 3.12 which depend on the choice of $\leqslant$.

This completes the description of the embeddings visible in Figure 3.12.

Definition 3.13 (Representable elements). We say that an element $x \in \mathcal{D}(G)$ is *representable* in $\widehat{\mathbb{Q}G}^{\phi}$ if and only if we have $i_{\leqslant}(x) \in j_\phi(\widehat{\mathbb{Q}G}^{\phi})$ for every biordering $\leqslant$ compatible with ϕ.

Lemma 3.14. *Let $x \in \mathcal{D}(G)$ be representable in $\widehat{\mathbb{Q}G}^{\phi}$. If $\leqslant$ and $\leqslant'$ are biorderings on G^{fab} compatible with ϕ, then*

$$i_{\leqslant}(x) = i_{\leqslant'}(x)$$

Proof. Recall that $x = s_\alpha(p)s_\alpha(q)^{-1}$ with $p, q \in \mathcal{D}(K)G^{\mathrm{fab}}$ and $q \neq 0$.

Now,

$$
\begin{aligned}
i_{\leqslant}(x) &= \mathrm{Ore}(\mathrm{id}_{\leqslant})\,\mathrm{Ore}(s_\alpha)^{-1}(s_\alpha(p)s_\alpha(q)^{-1}) \\
&= \mathrm{Ore}(\mathrm{id}_{\leqslant})(p/q) \\
&= \mathrm{id}_{\leqslant}(p)\,\mathrm{id}_{\leqslant}(q)^{-1}
\end{aligned}
$$

and similarly for $\leqslant'$. Since x is representable, both $i_{\leqslant}(x)$ and $i_{\leqslant'}(x)$ are elements of the ring $\mathcal{D}\widehat{(K)G}^{\mathrm{fab}}{}^{\phi}$ (which is a subset of $\mathcal{D}(K)^{G^{\mathrm{fab}}}$). But we also know that

$$
\mathrm{id}_{\leqslant}(p) = \mathrm{id}_{\leqslant'}(p) = \mathrm{id}(p) \in \mathcal{D}\widehat{(K)G}^{\mathrm{fab}}{}^{\phi}
$$

and

$$
\mathrm{id}_{\leqslant}(q) = \mathrm{id}_{\leqslant'}(q) = \mathrm{id}(q) \in \mathcal{D}\widehat{(K)G}^{\mathrm{fab}}{}^{\phi}
$$

and thus

$$
\begin{aligned}
\big(i_{\leqslant}(x) - i_{\leqslant'}(x)\big)\,\mathrm{id}(q) &= \mathrm{id}_{\leqslant}(p)\,\mathrm{id}_{\leqslant}(q)^{-1}\,\mathrm{id}_{\leqslant}(q) - \mathrm{id}_{\leqslant'}(p)\,\mathrm{id}_{\leqslant'}(q)^{-1}\,\mathrm{id}_{\leqslant'}(q) \\
&= \mathrm{id}_{\leqslant}(p) - \mathrm{id}_{\leqslant'}(p) \\
&= 0
\end{aligned}
$$

But the ring $\mathcal{D}\widehat{(K)G}^{\mathrm{fab}}{}^{\phi}$ embeds in a skew-field (via $\mathrm{id}_{\leqslant}$), and hence it does not admit zero-divisors. Since $q \neq 0$, we conclude that

$$
i_{\leqslant}(x) = i_{\leqslant'}(x) \qquad\qquad \square
$$

In view of the above, we will denote by $\iota_\phi(x) \in \widehat{\mathbb{Q}G}^{\phi}$ the unique element such that for any (hence every) suitable $\leqslant$ we have

$$
j_\phi\big(\iota_\phi(x)\big) = i_{\leqslant}(x)
$$

We think of ι_ϕ as a map taking representable elements in $D(G)$ to their representatives in $\widehat{\mathbb{Q}G}^{\phi}$. It is easy to see that representable elements form a subring of $\mathcal{D}(G)$, and ι_ϕ is a ring homomorphism.

Definition 3.15 ($\mathcal{D}(G,S)$)**.** For $S \subseteq H^1(G;\mathbb{R})$ we define $\mathcal{D}(G,S)$ to be the subset of $\mathcal{D}(G)$ consisting of elements representable over $\widehat{\mathbb{Q}G}^{\phi}$ for every $\phi \in S$.

When $S = \{\phi\}$, we will write $\mathcal{D}(G,\phi)$ for $\mathcal{D}(G,\{\phi\})$.

Remark 3.16. It is clear that $\mathcal{D}(G,S)$ is a subring of $\mathcal{D}(G)$, as it is an intersection of subrings.

Lemma 3.17. *Let $S, S' \subseteq H^1(G;\mathbb{R})$ be two subsets with $S' \subseteq S$. Then*

$$
\mathcal{D}(G,S) \leqslant \mathcal{D}(G,S')
$$

is a subring.

Proof. This follows immediately from the definitions. $\qquad\qquad \square$

Since we are working with Novikov rings, it is essential for us to be able to evaluate ϕ at elements in the various rings appearing in Figure 3.12. On all rings but the ones in the right-most column of Figure 3.12 we already know how to evaluate ϕ, since all these rings are group rings or Novikov rings of G or G^{fab}. We will now focus on the right-most column.

Definition 3.18. Let $\phi\colon G \to \mathbb{R}$ be a character, and let $p, q \in \mathcal{D}(K)G^{\mathrm{fab}}$ be given, with $q \neq 0$. We have already defined $\phi(p) \in (-\infty, \infty]$ and $\phi(q) \in \mathbb{R}$. We now set

$$
\phi(p/q) = \phi(p) - \phi(q)
$$

It is immediate that $\phi(pr/qr) = \phi(p/q)$ for $r \neq 0$, and hence this formula defines the evaluation $\phi \colon \mathrm{Ore}\,(\mathcal{D}(K)G^{\mathrm{fab}}) \to (-\infty, \infty]$.

Since $\mathrm{Ore}(s_\alpha)$ is an isomorphism, we define $\phi \colon \mathcal{D}(G) \to (-\infty, \infty]$ by

$$\phi(s_\alpha(p)s_\alpha(q)^{-1}) = \phi(p/q) = \phi(p) - \phi(q)$$

Finally, let $x \in \mathbb{F}_{\leqslant}(G)$. If $x = 0$ we set $\phi(x) = \infty$ as usual. Now suppose that $x \neq 0$. By definition, $x \in \mathcal{D}(K)^{G^{\mathrm{fab}}}$, and $\mathrm{supp}\,x$ is well-ordered. In particular, $\mathrm{supp}\,x$ has a unique $\leqslant$-minimum, say a. We set $\phi(x) = \phi(a)$.

We can now add all the different evaluation maps called ϕ to Figure 3.12. We claim that in doing so, we produce another commutative diagram – in other words, the various evaluations defined above coincide.

The claim is trivial to verify everywhere, except for the maps

$$\mathrm{id}_{\leqslant} \colon \widehat{\mathcal{D}(K)G^{\mathrm{fab}}}^{\phi} \to \mathbb{F}_{\leqslant}(G)$$

and

$$\mathrm{Ore}(\mathrm{id}_{\leqslant}) \colon \mathrm{Ore}(\mathcal{D}(K)G^{\mathrm{fab}}) \to \mathbb{F}_{\leqslant}(G)$$

For the former map, take $x \in \widehat{\mathcal{D}(K)G^{\mathrm{fab}}}^{\phi}$; we may assume that $x \neq 0$. By definition, $\phi(x) = \phi(a)$ where $a \in \mathrm{supp}\,x$ is an element minimising ϕ. But, since $\leqslant$ is ϕ-compatible, we may take a to be the $\leqslant$-minimum of $\mathrm{supp}\,x$.

For the latter map, take $p, q \in \mathcal{D}(K)G^{\mathrm{fab}} \smallsetminus \{0\}$. Let

$$a = \min_{\leqslant} \mathrm{supp}\,p$$

and

$$b = \min_{\leqslant} \mathrm{supp}\,q$$

Arguing as above, we see that

$$\phi(p/q) = \phi(p) - \phi(q) = \phi(a) - \phi(b)$$

On the other hand, the construction of $\mathbb{F}_{\leqslant}(G)$ tells us that

$$\min_{\leqslant} \mathrm{supp}\,\mathrm{id}_{\leqslant}(q)^{-1} = b^{-1}$$

and so

$$\phi\big(\mathrm{Ore}(\mathrm{id}_{\leqslant})(p/q)\big) = \phi\big(\min_{\leqslant} \mathrm{supp}(\mathrm{id}_{\leqslant}\,p\,\mathrm{id}_{\leqslant}(q)^{-1})\big) = \phi(ab^{-1}) = \phi(a) - \phi(b)$$

and we are done.

Remark 3.19. It follows immediately from the Ore condition and Remark 3.5 (noting that $\mathcal{D}(K)G^{\mathrm{fab}}$ has no zero-divisors, as it embeds in a skew-field $\mathbb{F}_{\leqslant}(G)$) that ϕ is now a well-defined group homomorphism $\mathcal{D}(G) \smallsetminus \{0\} \to \mathbb{R}$.

It is also immediate that for $x \in \mathcal{D}(G)$ the map $H^1(G; \mathbb{R}) \to \mathbb{R}$ given by $\phi \mapsto \phi(x)$ is continuous.

3.4. **Groups with finite quotients.** Let G be a finitely-generated torsion-free group satisfying the Atiyah conjecture (so $\mathcal{D}(G)$ is a skew-field). Let Q be a finite quotient of G with the corresponding kernel H. We fix a section $s \colon Q \to G$ of the quotient map. As always, we have $s(1) = 1$.

We will also use $\alpha_H \colon H \to H^{\mathrm{fab}}$ to denote the free abelianisation map of H, and we pick a section s_{α_H} of this map.

Definition 3.20. For every $\psi \in H^1(H; \mathbb{R})$ we define $\psi^q \colon H \to \mathbb{R}$ to be the character given by $\psi^q(x) = \psi(s(q)xs(q)^{-1})$.

Note that the notation ψ^q is reasonable, since

$$
\begin{aligned}
(\psi^q)^p(x) &= \psi^q\big(s(p)xs(p)^{-1}\big) \\
&= \psi\big(s(q)s(p)xs(p)^{-1}s(q)^{-1}\big) \\
&= \psi(\mu(q,p)s(qp)xs(qp)^{-1}\mu(q,p)^{-1}) \\
&= \psi^{qp}(x)
\end{aligned}
$$

since $\mu(q,p) \in H$ and therefore conjugating the argument by $\mu(q,p)$ does not alter the value under ψ.

Note that conjugation by elements $s(q)$ with $q \in Q$ induces an outer action of Q on H. But an outer action descends to an action of Q on $H^1(H;\mathbb{R})$.

Let $S \subseteq H^1(H;\mathbb{R})$ be a subset invariant under the action of Q. Recall that we have a twisted group ring $\mathcal{D}(H)Q$ induced by the quotient $G \to Q$. Since S is Q-invariant, it is immediate that the action of Q on $\mathcal{D}(H)$ preserves $\mathcal{D}(H,S)$. Indeed, take $x \in \mathcal{D}(H,S)$ and $\psi \in S$. By definition, we have $\iota_\psi(x) \in \widehat{\mathbb{Q}H}^\psi$. In particular, $\iota_\psi(x)$ is an element of $\mathbb{Q}^H$. Take $q \in Q$. The element $s(q)^{-1}\iota_\psi(x)s(q)$ is clearly still an element of $\mathbb{Q}^H$, since H is normal in G. It is immediate that in fact $s(q)^{-1}\iota_\psi(x)s(q) \in \widehat{\mathbb{Q}H}^{\psi^q}$, and $\psi^q \in S$ since S is Q-invariant.

Now that we know that $\mathcal{D}(H,S)$ is Q-invariant, the structure functions of $\mathcal{D}(H)Q$ give us a twisted group ring structure for $\mathcal{D}(H,S)Q$.

Interestingly, if S is not necessary Q-invariant, the abelian group $\mathcal{D}(H,S)Q$ whose elements are formal sums of elements in Q with coefficients in $\mathcal{D}(H,S)$, and the addition is pointwise, forms a right $\mathcal{D}(H,S')Q$-module, where $S' = \bigcap_{q \in Q} S^q$. The right action of $\mathcal{D}(H,S')Q$ on $\mathcal{D}(H,S)Q$ is induced by the following rule

$$
x_q q \cdot y_p p = x_q y_p^{q^{-1}} \mu(q,p) qp
$$

where $q, p \in Q$, $x_p \in \mathcal{D}(H,S)$ and $y_p \in \mathcal{D}(H,S')$, and where we treat

$$
y_p^{q^{-1}} \in \mathcal{D}(H,S')
$$

as an element in $\mathcal{D}(H,S)$, and the multiplication $x_p y_p^{q^{-1}}$ is carried out in $\mathcal{D}(H,S)$.

Similarly, we endow the abelian group $\mathbb{F}_{\leqslant}(H)Q$, consisting of formal sums of elements in Q with coefficients in $\mathbb{F}_{\leqslant}(H)$, with the structure of a right $\mathcal{D}(H)Q$ module: the right action is induced by almost the same rule as above, that is,

$$
x_q q \cdot y_p p = x_q i_{\leqslant}(y_p^{q^{-1}}) \mu(q,p) qp
$$

where now x_q is an element of $\mathbb{F}_{\leqslant}(H)$, and $y_p^{q^{-1}} \in \mathcal{D}(H)$ is sent to an element of $\mathbb{F}_{\leqslant}(H)$ by the map $i_{\leqslant} = \mathrm{Ore}(\mathrm{id}_{\leqslant}) \, \mathrm{Ore}(s_{\alpha_H})^{-1}$ (compare Figure 3.12).

Definition 3.21 (Value and defect)**.** Let $\psi \in H^1(H;\mathbb{R})$ be a character. For $x \in \mathcal{D}(H)Q$ we write $x = \sum_{q \in Q} x_q q$ with $x_q \in \mathcal{D}(H)$. We define the *Q-value* of ψ at x to be

$$
\psi_Q(x) = \min_{p,q \in Q} \left\{ \psi^p(x_q) + \frac{1}{|Q|}\psi(s(q)^{|Q|}) \right\} \in (-\infty, \infty]
$$

For a character $\psi \in H^1(H;\mathbb{R})$ we define its *Q-defect*

$$
|\psi|_Q = \max_{p,q \in Q} \left| \psi_Q\big(s(p)s(q)s(pq)^{-1}\big) - \psi_Q(p) - \psi_Q(q) + \psi_Q(pq) \right| \in [0, \infty)
$$

(note that $s(p)s(q)s(pq)^{-1} \in H \subseteq \mathcal{D}(H)$).

In order not to overload notation, we will not differentiate between characters in $H^1(G;\mathbb{R})$ and their restriction to H in $H^1(H;\mathbb{R})$.

Lemma 3.22. *For every $\phi \in H^1(G; \mathbb{R})$, $q \in Q$, and $x \in \mathcal{D}(H)Q$, we have $\phi^q = \phi$, $|\phi|_Q = 0$, and $\phi_Q(x) = \phi(s(x))$.*

Proof. Let $x = \sum_{q \in Q} x_q q$ with $x_q \in \mathcal{D}(H)$. Note that x_1 is an arbitrary element of $\mathcal{D}(H)$. If $x_1 = 0$ then

$$\phi^q(x_1) = \phi(s(q)x_1 s(q)^{-1}) = \phi(0) = \phi(x_1)$$

If $x_1 \neq 0$ then again

$$\phi^q(x_1) = \phi(s(q)x_1 s(q)^{-1}) = \phi(x_1)$$

since ϕ a homomorphism from $\mathcal{D}(G) \smallsetminus \{0\}$ to an abelian group.

We also have

$$\phi_Q(x) = \min_{p,q \in Q} \left\{ \phi^p(x_q) + \frac{1}{|Q|} \phi(s(q)^{|Q|}) \right\} = \min_q \phi(x_q s(q)) = \phi(s(x))$$

Finally, we have

$$|\phi|_Q = \max_{p,q \in Q} \left| \phi_Q\big(s(p)s(q)s(pq)^{-1}\big) - \phi_Q(p) - \phi_Q(q) + \phi_Q(pq) \right| = \max_{p,q \in Q} |\phi(1)| = 0$$

which completes the proof $\qquad\qquad\square$

Let us now investigate the key properties of the Q-value map

$$\psi_Q \colon \mathcal{D}(H)Q \to (-\infty, \infty]$$

Lemma 3.23. *Let $\psi \in H^1(H; \mathbb{R})$, $x, y \in \mathcal{D}(H)$, $q \in Q$, $w, z \in \mathcal{D}(H)Q$ be given. All of the following hold:*

- (1) $\psi_Q(x^{q^{-1}}) = \psi_Q(x)$;
- (2) $\psi_Q(xq) = \psi_Q(q \cdot x) = \psi_Q(q) + \psi_Q(x)$;
- (3) $\psi_Q(xy) \geqslant \psi_Q(x) + \psi_Q(y)$;
- (4) $\psi_Q(x + y) \geqslant \min\{\psi_Q(x), \psi_Q(y)\}$;
- (5) $\psi_Q(z + w) \geqslant \min\{\psi_Q(z), \psi_Q(w)\}$;
- (6) $\psi_Q(zw) \geqslant \psi_Q(z) + \psi_Q(w) - |\psi|_Q$.

Proof. (1) We have

$$\begin{aligned}
\psi_Q(x^{q^{-1}}) &= \psi_Q\big(s(q)xs(q)^{-1}\big) \\
&= \min_{p \in Q} \psi^{pq^{-1}}\big(s(q)xs(q)^{-1}\big) \\
&= \min_p \psi\big(s(pq^{-1})s(q)xs(q)^{-1}s(pq^{-1})^{-1}\big) \\
&= \min_p \psi\big(s(pq^{-1})s(q)s(p)^{-1}s(p)xs(p)^{-1}s(p)s(q)^{-1}s(pq^{-1})^{-1}\big) \\
&\overset{(\dagger)}{=} \min_p \psi\big(s(p)xs(p)^{-1}\big) \\
&= \min_p \psi^p(x) \\
&= \psi_Q(x)
\end{aligned}$$

where ($\dagger$) follows from the fact that $s(pq^{-1})s(q)s(p) \in H \subseteq \mathcal{D}(H) \smallsetminus \{0\}$ and that ψ restricted to $\mathcal{D}(H) \smallsetminus \{0\}$ is a homomorphism with abelian image, and so is stable on conjugacy classes.

(2) By definition,

$$\psi_Q(xq) = \min_p \{\psi^p(x)\} + \frac{1}{|Q|} \psi\big(s(q)^{|Q|}\big) = \psi_Q(x) + \psi_Q(q)$$

Also,

$$\psi_Q(q \cdot x) = \psi_Q(x^{q^{-1}} q) = \psi_Q(x^{q^{-1}}) + \psi_Q(q) \overset{(1)}{=} \psi_Q(x) + \psi_Q(q)$$

(3) We have

$$\psi_Q(xy) = \min_p \psi^p(xy)$$

$$\overset{(\ddagger)}{=} \min_p \left(\psi^p(x) + \psi^p(y) \right)$$

$$\geqslant \min_p \psi^p(x) + \min_p \psi^p(y)$$

$$= \psi_Q(x) + \psi_Q(y)$$

where ($\ddagger$) follows from Remark 3.19.

(4) The Ore condition tells us that we may write $x = s_\phi(x')s_\phi(d)^{-1}$ and $y = s_\phi(y')s_\phi(d)^{-1}$ for $x', y', d \in \mathcal{D}(K)G^{\mathrm{fab}}$. Thus we have

$$\psi_Q(x + y) = \min_p \psi^p(x + y)$$

$$= \min_p \{ \psi^p(x' + y') - \psi^p(d) \}$$

$$\overset{(\dagger\dagger)}{\geqslant} \min_p \min\{ \psi^p(x') - \psi^p(d), \psi^p(y') - \psi^p(d) \}$$

$$= \min_p \min\{ \psi^p(x), \psi^p(y) \}$$

$$= \min\{ \psi^p(x), \psi^p(y) \mid p \in Q \}$$

$$= \min\{ \min_p \psi^p(x), \min_p \psi^p(y) \}$$

$$= \min\{ \psi_Q(x), \psi_Q(y) \}$$

where ($\dagger\dagger$) follows from Lemma 3.4(1).

(5) We write $z = \sum_p z_p p$ and $w = \sum_p w_p p$ with $z_p, w_p \in \mathcal{D}(H)$. Now, $\psi_Q(z+w)$ is defined as the minimum over $p \in Q$ of

$$\psi_Q\big((z_p + w_p)p\big) \overset{(2)}{=} \psi_Q(z_p + w_p) + \psi_Q(p)$$

$$\overset{(4)}{\geqslant} \min\{ \psi_Q(z_p), \psi_Q(w_p) \} + \psi_Q(p)$$

$$\overset{(2)}{=} \min\{ \psi_Q(z_p p), \psi_Q(w_p p) \}$$

$$\geqslant \min\{ \psi_Q(z), \psi_Q(w) \}$$

(6) Again, we write $z = \sum_p z_p p$ and $w = \sum_p w_p p$. Now, for every $p, p' \in Q$ we have

$$\psi_Q(z_p p \cdot w_{p'} p') = \psi_Q(z_p w_{p'}{}^{p^{-1}} s(p)s(p')s(pp')^{-1} pp')$$

$$\overset{(2)}{=} \psi_Q(z_p w_{p'}{}^{p^{-1}} s(p)s(p')s(pp')^{-1}) + \psi_Q(pp')$$

Observe that $s(p)s(p')s(pp')^{-1} \in H$ and hence we may continue:

$$\psi_Q(z_p p \cdot w_{p'} p') \overset{(3)}{\geqslant} \psi_Q(z_p) + \psi_Q(w_{p'}{}^{p^{-1}}) + \psi_Q\big(s(p)s(p')s(pp')^{-1}\big) + \psi_Q(pp')$$

$$\geqslant \psi_Q(z_p) + \psi_Q(w_{p'}{}^{p^{-1}}) - |\psi|_Q + \psi_Q(p) + \psi_Q(p')$$

$$\overset{(1)}{=} \psi_Q(z_p) + \psi_Q(w_{p'}) - |\psi|_Q + \psi_Q(p) + \psi_Q(p')$$

$$\overset{(2)}{=} \psi_Q(z_p p) + \psi_Q(w_{p'} p') - |\psi|_Q$$

There exists $p'' \in Q$ such that

$$\psi_Q(zw) = \psi_Q\Big(\sum_{pp'=p''} z_p p \cdot w_{p'} p' \Big)$$

since $\psi_Q(zw)$ is defined as a minimum of such terms. Now we have

$$\psi_Q\Big(\sum_{pp'=p''} z_p p \cdot w_{p'} p'\Big) \overset{(5)}{\geqslant} \min_{pp'=p''} \psi_Q(z_p p \cdot w_{p'} p')$$

$$\geqslant \min_{pp'=p''} \big(\psi_Q(z_p p) + \psi_Q(w_{p'} p') - |\psi|_Q\big)$$

$$= \min_{pp'=p''} \big(\psi_Q(z_p p) + \psi_Q(w_{p'} p')\big) - |\psi|_Q$$

$$\geqslant \min_p \psi_Q(z_p p) + \min_p \psi_Q(w_p p) - |\psi|_Q$$

$$= \psi_Q(z) + \psi_Q(w) - |\psi|_Q \qquad\qquad \square$$

We are now ready for the key lemma which will allow us to construct inverses for elements in $\mathcal{D}(H, S)Q$.

Definition 3.24. For every $\psi \in H^1(H; \mathbb{R})$ we set $\psi^Q = \{\psi^q \mid q \in Q\}$.

Lemma 3.25. *Let $\psi \in H^1(H; \mathbb{R})$. Let $x, y \in \mathcal{D}(H, \psi^Q)Q$ be given. Suppose that x is invertible in $\mathcal{D}(H, \psi^Q)Q$ with inverse x^{-1}. Suppose further that we have*

$$\psi_Q(y) + \psi_Q(x^{-1}) - 2|\psi|_Q > 0$$

Then the inverse of $s(x+y)$ in $\mathcal{D}(G)$ exists and lies in $s(\mathcal{D}(H, \psi)Q)$.

Proof. Note that the map $s \colon \mathcal{D}(H)Q \to \mathcal{D}(G)$ is an isomorphism by Lemma 2.22.

We start by remarking that $x + y \neq 0$, as otherwise we would have (using Lemma 3.23(6))

$$\psi_Q(y) + \psi_Q(x^{-1}) = \psi_Q(-x) + \psi_Q(x^{-1}) \leqslant \psi_Q(-1) + |\psi|_Q = |\psi|_Q$$

which would contradict the assumed inequality. Therefore $s(x+y)$ is invertible in $\mathcal{D}(G)$, with inverse $s(z)$ where $z = \sum_{q \in Q} z_q q$ with $z_q \in \mathcal{D}(H)$. Our goal is to show that in fact every z_q is representable in $\widehat{\mathbb{Q}H}^\psi$.

We have

$$\psi_Q(x^{-1}y) \geqslant \psi_Q(y) + \psi_Q(x^{-1}) - |\psi|_Q > |\psi|_Q$$

by Lemma 3.23(6) and by assumption. Let us quantify this inequality: there exists $\epsilon > 0$ such that

$$\psi_Q(x^{-1}y) \geqslant |\psi|_Q + \epsilon$$

Now repeated application of Lemma 3.23(6) tells us that for every $i > 0$ we have

$$\psi_Q\big((-x^{-1}y)^i\big) = \psi_Q\big((x^{-1}y)^i\big) \geqslant |\psi|_Q + i\epsilon$$

As $(-x^{-1}y)^i \in \mathcal{D}(H)Q$, we may write $(-x^{-1}y)^i = \sum_{q \in Q} w_{i,q} q$ with $w_{i,q} \in \mathcal{D}(H)$. Now

$$i\epsilon \leqslant \psi_Q\big((-x^{-1}y)^i\big) = \min_{p,q \in Q}\{\psi^p(w_{i,q}) + \frac{1}{|Q|}\psi(s(q)^{|Q|})\} \leqslant \psi(w_{i,q}) + \frac{1}{|Q|}\psi(s(q)^{|Q|})$$

for every $q \in Q$. Since Q is finite and ψ is fixed, this implies that the sequence $(\psi(w_{i,q}))_i$ converges to ∞ for every $q \in Q$. Since $x^{-1}y \in \mathcal{D}(H, \psi^Q)Q$, we have

$$\iota_\psi(w_{i,q}) \in \widehat{\mathbb{Q}H}^\psi$$

which together with the previous observation yields the convergence of the series

$$\sum_{q \in Q}\left(\sum_{i=0}^\infty \iota_\psi(w_{i,q})\right) q$$

in $\widehat{\mathbb{Q}H}^\psi Q$ (compare Remark 3.3); we denote the limit by z'. (We will show later that z' is the element representing zx in $\widehat{\mathbb{Q}H}^\psi Q$.)

Let us pick a biordering $\leqslant$ on H^{fab} compatible with ψ. By definition, we have

$$j_\psi(z') \in \mathbb{F}_{\leqslant}(H)Q$$

(this is a slight abuse of notation, as j_ψ is really defined as a map $\widehat{\mathbb{Q}H}^\psi \to \mathbb{F}_{\leqslant}(H)$; here we extend it by the identity on Q; we will similarly abuse notation for $i_{\leqslant}$).

We have already observed that $\mathbb{F}_{\leqslant}(H)Q$ is a right $\mathcal{D}(H)Q$-module. We claim that in the module $\mathbb{F}_{\leqslant}(H)Q$ we have

$$j_\psi(z') \cdot (1 + x^{-1}y) = 1$$

Indeed, this can be seen by taking the partial sums

$$j_\psi\left(\sum_{q \in Q}\left(\sum_{i=0}^{m} \iota_\psi(w_{i,q})\right)q\right) = \sum_{q \in Q}\left(\sum_{i=0}^{m} i_{\leqslant}(w_{i,q})\right)q = i_{\leqslant}\left(\sum_{i=0}^{m}(-x^{-1}y)^i\right)$$

and multiplying them by $1 + x^{-1}y$ on the right. Observing that the module action happens via the map $i_{\leqslant}$, we obtain

$$i_{\leqslant}\left(\sum_{i=0}^{m}(-x^{-1}y)^i\right) \cdot (1 + x^{-1}y) = i_{\leqslant}\left(\sum_{i=0}^{m}(-x^{-1}y)^i(1 + x^{-1}y)\right)$$
$$= i_{\leqslant}\left(1 - (-x^{-1}y)^{m+1}\right)$$
$$= 1 - i_{\leqslant}\left((-x^{-1}y)^{m+1}\right)$$
$$= 1 - i_{\leqslant}\left(\sum_{q \in Q} w_{m+1,q}q\right)$$

We have already shown that $\psi\left(\sum_{q \in Q} w_{m+1,q}q\right)$ tends to ∞ with m, and hence

$$\lim_{m \longrightarrow \infty} \iota_\psi\left(\sum_{q \in Q} w_{m+1,q}q\right) = 0$$

Therefore we also have

$$\lim_{m \longrightarrow \infty} j_\psi \iota_\psi\left(\sum_{q \in Q} w_{m+1,q}q\right) = 0$$

But $j_\psi \iota_\psi = i_{\leqslant}$ and so

$$j_\psi(z') \cdot (1 + x^{-1}y) = \lim_{m \longrightarrow \infty} i_{\leqslant}\left(\sum_{i=0}^{m}\left(-x^{-1}y\right)^i\right) \cdot (1 + x^{-1}y)$$
$$= 1 - \lim_{m \longrightarrow \infty} i_{\leqslant}\left(\sum_{q \in Q} w_{m+1,q}q\right)$$
$$= 1$$

as claimed.

Recall that we have $z \in \mathcal{D}(H)Q$ which is an inverse of $x + y$ in $\mathcal{D}(H)Q$. We have also

$$i_{\leqslant}(zx) \cdot (1 + x^{-1}y) = i_{\leqslant}\left(z(x + y)\right) = 1$$

We conclude that $\left(j_\psi(z') - i_{\leqslant}(zx)\right) \cdot (1 + x^{-1}y) = 0$. But then

$$j_\psi(z') - i_{\leqslant}(zx) = \left(j_\psi(z') - i_{\leqslant}(zx)\right) \cdot \left((1 + x^{-1}y) \cdot zx\right) = 0$$

and so $i_{\leqslant}(zx) \in \operatorname{im}(j_\psi)$. But this is the precise meaning of zx being representable in $\widehat{\mathbb{Q}H}^\psi Q$. Since $x^{-1} \in \mathcal{D}(H, \psi^Q)Q$, we conclude that z is representable in $\widehat{\mathbb{Q}H}^\psi Q$. $\qquad\square$

4. RFRS

4.1. Generalities.

Definition 4.1 (RFRS). A group G is called *residually-finite rationally-solvable* or *RFRS* if and only if there exists a sequence $(H_i)_{i\in\mathbb{N}}$ of finite index normal subgroups $H_i \trianglelefteq G$ with $H_0 = G$ such that

- $H_{i+1} \leqslant H_i$ (that is, $(H_i)_i$ is a *chain*), and
- $\bigcap H_i = \{1\}$ (that is, $(H_i)_i$ is a *residual chain*), and
- $\ker \alpha_i \leqslant H_{i+1}$ for every i, where $\alpha_i \colon H_i \to H_i^{\mathrm{fab}}$ is the free abelianisation map.

The sequence $(H_i)_i$ is called a *witnessing chain*, and it always comes equipped with set-theoretic sections $s_i \colon Q_i = G/H_i \to G$ of the quotient maps $\beta_i \colon G \to Q_i$ such that $s_i(1) = 1$.

Proposition 4.2. *Every RFRS group G is torsion-free and satisfies the Atiyah conjecture.*

Proof. We will show that G is residually {torsion-free solvable}. This will suffice, as residually torsion-free groups are themselves torsion free, and residually {torsion-free solvable} groups satisfy the Atiyah conjecture by [Sch02, Proposition 1 and Theorem 1].

Let us now proceed with the proof. Set $K_i = \ker \alpha_i \trianglelefteq H_i$. Since $K_i \leqslant H_i$, it is immediate that $\bigcap K_i = \{1\}$. We have $K_i \trianglelefteq H_{i+1}$ by the definition of RFRS. Further, we have $K_{i+1} \leqslant K_i$ since K_{i+1} is the intersection of the kernels of all homomorphisms $H_{i+1} \to \mathbb{Q}$, and every homomorphism $H_i \to \mathbb{Q}$ restricts to a homomorphism $H_{i+1} \to \mathbb{Q}$. The group K_i/K_{i+1} is torsion-free abelian since H_{i+1}/K_{i+1} is. Therefore $(K_i)_{i\in\mathbb{N}}$ is a residual chain witnessing the fact that G is residually {torsion-free solvable}. $\qquad\square$

4.2. Linnell's skew-field of finitely-generated RFRS groups.
Let G be a finitely-generated RFRS group with witnessing chain $(H_i)_{i\in\mathbb{N}}$ and finite quotients $\beta_i \colon G \to Q_i$ with kernels H_i and sections $s_i \colon Q_i \to G$.

Note that $H^1(H_i; \mathbb{R}) = \operatorname{Hom}(H_i, \mathbb{R})$ is naturally a subspace of $H^1(H_j; \mathbb{R})$ for every $j \geqslant i$.

Definition 4.3 (Rich and very rich subsets). A character $\psi \in H^1(H_n; \mathbb{R})$ is *irrational* if and only if it is injective on H_n^{fab}. Note that characters irrational in $H^1(H_n; \mathbb{R})$ are not necessarily irrational in $H^1(H_{n+1}; \mathbb{R})$.

Let $\mathcal{L} = \{L_{-m}, \ldots, L_{-1}\}$ denote a finite chain of linear subspaces of $H^1(G; \mathbb{R})$ with $L_{-m} < \cdots < L_{-1}$. We endow each L_i with a fixed dense subset; we will refer to the characters in this subset as *irrational*. The dense subsets of irrational characters of the subspaces L_i are part of the structure of $\mathcal{L}$.

Suppose that $\mathcal{L}$ is fixed (it might be empty). For notational convenience we set $L_i = H^1(H_i; \mathbb{R})$ for $i \geqslant 0$. A subset of L_i (for $i \geqslant -m$) is *very rich* if and only if it is open and contains all irrational characters in L_i.

Now we define a notion of a rich set inductively: a subset of L_{-m} is *rich* if and only if it is very rich. For $i > -m$, a subset U of L_i is *rich* if and only if $\overline{U}^\circ \cap L_{i-1}$ is rich in L_{i-1}, where $\overline{U}^\circ$ denotes the interior of the closure of U, all taken in L_i.

Clearly, every very rich subset is also rich.

Note that both notions of richness are relative to the chosen witnessing chain of G, and the chain $\mathcal{L}$. The witnessing chain is fixed throughout; we also fix $\mathcal{L}$.

The reason for introducing $\mathcal{L}$ will became apparent in Remark 4.8. Note that we do not require the fixed dense sets inside of L_{-i} to satisfy any additional properties – these dense sets will be coming from the intrinsic notion of irrationality in linear subspaces of $H^1(G; \mathbb{R})$, and they are in general not the intersections of the subspaces L_{-i} with the irrational characters in $H^1(G; \mathbb{R})$.

Lemma 4.4. *The intersection of two (and hence finitely many) rich subsets of L_i is rich. Thus, every rich subset of $H^1(H_i; \mathbb{R})$ contains a rich subset which is Q_i-invariant.*

Proof. Clearly the intersection of two very rich sets is itself very rich. Thus the statement holds for $i = -m$. We now proceed by induction: Let V and V' be two open subsets of L_i such that $\overline{V}^\circ \cap L_{i-1}$ and $\overline{V'}^\circ \cap L_{i-1}$ are rich. By the inductive hypothesis, $\overline{V}^\circ \cap \overline{V'}^\circ \cap L_{i-1}$ is also rich. It is an easy exercise in point-set topology to show that for open sets V, V' we have

$$\overline{V}^\circ \cap \overline{V'}^\circ = \overline{V \cap V'}^\circ$$

This finishes the proof of the first part of the statement.

The second part of the statement follows by intersecting all rich sets in a Q_i-orbit of a given rich set. $\qquad\square$

Lemma 4.5. *Let U and V be two open subsets of L_i such that the intersections of U and V with the set of irrational characters of L_i coincide. If U is rich, then so is V.*

Proof. Since U is open and the set of irrational characters is dense, the closure $\overline{U}$ is equal to the closure of the intersection of the two. Hence $\overline{U} = \overline{V}$, and the result follows. $\qquad\square$

Lemma 4.6. *Suppose that $\mathcal{L} = \emptyset$. If U is a rich subset of $H^1(H_n; \mathbb{R})$, then its closure in $H^1(H_n; \mathbb{R})$ contains $H^1(G; \mathbb{R})$.*

Proof. The proof is an induction on n. When $n = 0$ the set U is in fact very rich, and hence is dense in $H^1(G; \mathbb{R})$.

Now suppose that U is an open subset of $H^1(H_n; \mathbb{R})$ such that $\overline{U}^\circ \cap H^1(H_{n-1}; \mathbb{R})$ is rich. The inductive hypothesis tells us that $H^1(G; \mathbb{R})$ lies in the closure of $\overline{U}^\circ \cap H^1(H_{n-1}; \mathbb{R})$ taken in $H^1(H_{n-1}; \mathbb{R})$. But this closure is clearly contained in $\overline{\overline{U}^\circ}$ taken in $H^1(H_n; \mathbb{R})$, and obviously $\overline{\overline{U}^\circ} \subseteq \overline{U}$. $\qquad\square$

Definition 4.7 ($\mathbb{K}(G, \mathcal{L})$)**.** An element $x \in \mathcal{D}(G)$ is *well representable* with *associated integer n* and *associated rich sets* $\{U_i \mid i \geqslant n\}$ if and only if for every $i \geqslant n$ the set U_i is rich in $H^1(H_i; \mathbb{R})$ and $x \in s_i(\mathcal{D}(H_i, U_i)Q_i)$.

We denote the set of all well-representable elements in $\mathcal{D}(G)$ by $\mathbb{K}(G, \mathcal{L})$. We write $\mathbb{K}(G) = \mathbb{K}(G, \emptyset)$.

Remark 4.8. Every H_n is itself a RFRS group with witnessing chain $(H_i)_{i \geqslant n}$. Since we have $\mathcal{D}(H_n) \leqslant \mathcal{D}(G)$, it is immediate that every element in $\mathbb{K}(H_n, \mathcal{L})$ is automatically an element of $\mathbb{K}(G, \mathcal{L})$ when $\mathcal{L}$ is a chain of subspaces of $H^1(G; \mathbb{R})$, since we may use the same associated integer and the same associated rich sets.

Conversely, every element $x \in \mathbb{K}(G, \mathcal{L}) \cap \mathcal{D}(H_n)$ lies in $\mathbb{K}(H_n, \mathcal{L}')$, where

$$\mathcal{L}' = \mathcal{L} \cup \{H^1(H_j; \mathbb{R}) \mid 0 \leqslant j < n\}$$

(and where the dense subset of irrationals in every $H^1(H_j; \mathbb{R})$ is the obvious one) since it is immediate that when we write $x = s_i\left(\sum_{q \in Q_i} x_q q\right)$ with $x_q \in \mathcal{D}(H_i, U)$ for $i \geqslant n$ and some rich set U, we have $x_q = 0$ for every $q \notin \beta_i(H_n)$ (formally, this follows from the fact that $s_i \colon \mathcal{D}(H_i)Q_i \to \mathcal{D}(G)$ is an isomorphism, as shown in Lemma 2.22).

The above remark is the reason why we introduced $\mathcal{L}$: otherwise, we could not pass from $\mathbb{K}(G) \cap \mathcal{D}(H_1)$ to $\mathbb{K}(H_1)$.

Lemma 4.9. *For every $\mathcal{L}$, the subset $\mathbb{K}(G, \mathcal{L})$ is a subring of $\mathcal{D}(G)$ containing $\mathbb{Q}G$.*

Proof. Take $x, y \in \mathbb{K}(G, \mathcal{L})$, and let n denote the maximum of their associated integers.

By Lemma 4.4, for every $i \geqslant n$ there exists a Q_i-invariant rich set U_i such that

$$x, y \in s_i\big(\mathcal{D}(H_i, U_i)Q_i\big)$$

Since $\mathcal{D}(H_i, U_i)Q_i$ is a ring, we have $x + y, x \cdot y \in \mathcal{D}(H_i, U_i)Q_i$, and hence $x + y$ and $x \cdot y$ are both well representable. This shows that $\mathbb{K}(G, \mathcal{L})$ is a ring.

Every element in $\mathbb{Q}G$ is well representable with associated integer 0 and associated (very) rich sets $U_i = H^1(H_i; \mathbb{R})$. $\square$

Lemma 4.10. *Let $x \in \mathbb{K}(G, \mathcal{L}) \smallsetminus \{0\}$, and let A be a finite indexing set. Suppose that for every $a \in A$ we are given an element $x_a \in \mathbb{K}(G, \mathcal{L}) \smallsetminus \{0\}$ with an inverse $x_a^{-1} \in \mathbb{K}(G, \mathcal{L})$. Let U be a rich subset of $H^1(G; \mathbb{R})$. If for every irrational $\phi \in H^1(G; \mathbb{R})$ which lies in U there exists $a \in A$ with*

$$\phi(x_a) = \phi(x) < \phi(x - x_a)$$

then x is invertible in $\mathbb{K}(G, \mathcal{L})$.

Proof. Take $j \in \mathbb{N}$. Recall that $s_j \colon \mathcal{D}(H_j)Q_j \to \mathcal{D}(G)$ is an isomorphism.

For every $a \in A$ we set

$$V_{j,a} = \left\{ \psi \in H^1(H_j; \mathbb{R}) \,\middle|\, \psi_{Q_j}\big(s_j^{-1}(x - x_a)\big) + \psi_{Q_j}\big(s_j^{-1}(x_a^{-1})\big) > 2|\psi|_{Q_j} \right\}$$

We observe that $V_{j,a}$ is open. We define

$$V_j = \bigcup_{a \in A} V_{j,a}$$

and observe that V_j is an open set as well.

We claim that V_j contains all irrational characters of $H^1(G; \mathbb{R})$ lying in U. Indeed, let ϕ be such a character. Since $\phi \in H^1(G; \mathbb{R})$, we have $|\phi|_{Q_j} = 0$ for all j by Lemma 3.22. By assumption, there exists $a \in A$ such that

$$\phi_{Q_j}\big(s_j^{-1}(x - x_a)\big) = \phi(x - x_a) > \phi(x_a)$$

(here we have used Lemma 3.22 again). Also,

$$\phi_{Q_j}\big(s_j^{-1}(x_a^{-1})\big) = \phi(x_a^{-1}) = -\phi(x_a)$$

These three facts immediately imply that $\phi \in V_{j,a}$, and hence prove the claim. We conclude, using Lemma 4.5, that $V_j \cap H^1(G; \mathbb{R})$ is rich in $H^1(G; \mathbb{R})$, and therefore V_j is rich in $H^1(H_j; \mathbb{R})$ (formally, this requires an easy induction argument, which is left to the reader).

Let V_j' be the intersection of the Q_j-orbit of V_j; the set V_j' is rich by Lemma 4.4. Let n be the maximum of the associated integers of the elements x, x_a and x_a^{-1} for all $a \in A$. For every $j \geqslant n$ there exist a Q_j-invariant rich set W_j such that

$$x, x_a, x_a^{-1} \in s_j\big(\mathcal{D}(H_j, W_j)Q_j\big)$$

for every a (this uses Lemma 4.4). We define $U_j = V_j' \cap W_j$. It is immediate that U_j is rich for every j. We now claim that for every $j \geqslant n$, the element x admits an inverse in $s_j\big(\mathcal{D}(H_j, U_j)Q_j\big)$; clearly this will imply that x is invertible in $\mathbb{K}(G, \mathcal{L})$, since each of the inverses will coincide with the inverse of x in $\mathcal{D}(G)$.

Fix $j \geqslant n$, and let $\psi \in U_j$. Clearly $\psi \in V_{j,a}$ for some a. Also,

$$\psi^{Q_j} = \{\psi^q \mid q \in Q_j\} \subseteq W_j \cap V_j'$$

as both W_j and V_j' are Q_j-invariant. Therefore $x, x_a, x_a^{-1} \in s_j\big(\mathcal{D}(H_j, \psi^{Q_j})Q_j\big)$, and we may apply Lemma 3.25; we conclude that $x = x_a + (x - x_a)$ admits an inverse in $s_j\big(\mathcal{D}(H_j, \psi)Q_j\big)$. Since ψ was arbitrary, and the various inverses coincide in $\mathcal{D}(G)$, we conclude that x is invertible in $s_j\big(\mathcal{D}(H_j, U_j)Q_j\big)$, and so x is invertible in $\mathbb{K}(G, \mathcal{L})$. $\hspace{2cm}\square$

Proposition 4.11. *Every $x \in \mathbb{Q}G \smallsetminus \{0\}$ is invertible in $\mathbb{K}(G, \mathcal{L})$.*

Proof. The proof is an induction on $\kappa = |\operatorname{supp} x|$. By assumption, we have $\kappa \geqslant 1$. The base case $\kappa = 1$ is immediate, since then x is already invertible in $\mathbb{Q}G$.

In the inductive step, we will assume that the result holds for all elements y with $|\operatorname{supp} y| < \kappa$ across all finitely-generated RFRS groups and all chains $\mathcal{L}$. Recall that $\alpha \colon G \to G^{\mathrm{fab}}$ is the free abelianisation map. There are two cases to consider.

Case 1: Suppose that $|\alpha(\operatorname{supp} x)| \geqslant 2$.

We write
$$x = \sum_{a \in \alpha(\operatorname{supp} x)} x_a$$
where for each a we have $x_a \in \mathbb{Q}G$ with $\alpha(\operatorname{supp} x_a) = \{a\}$.

Using the inductive hypothesis we see that every x_a is invertible in $\mathbb{K}(G, \mathcal{L})$. Now we apply Lemma 4.10 with $U = H^1(G; \mathbb{R})$ and $A = \alpha(\operatorname{supp} x)$, and where to each irrational ϕ we associate $a \in A$ on which $\phi|_A$ is minimal (note that such an a is unique, since ϕ is irrational).

Case 2: Suppose that $\kappa > 1$ but $|\alpha(\operatorname{supp} x)| = 1$. In this case there exists $\gamma \in G$ and i such that $\gamma x \in \mathbb{Q}H_i$ but $\gamma' x \notin \mathbb{Q}H_{i+1}$ for every $\gamma' \in G$ (we are using here that $(H_j)_j$ is a residual chain). It is clear that without loss of generality we may assume that $\gamma = 1$.

Let $\alpha_i \colon H_i \to H_i^{\mathrm{fab}}$ be the free abelianisation map. If $\alpha_i(\operatorname{supp} x)$ is supported on a singleton, then there exists γ' such that $\gamma' x$ is supported on $\ker \alpha_i$. But $\ker \alpha \leqslant H_{i+1}$, which contradicts our assumption on i. Thus
$$|\alpha_i(\operatorname{supp} x)| \geqslant 2$$

We now apply the argument of Case 1 to x thought of as an element of $\mathbb{Q}H_i$, and conclude that x admits an inverse in $\mathbb{K}(H_i, \mathcal{L}')$, where $\mathcal{L}' = \mathcal{L} \cup \{H_1(H_j; \mathbb{R}) \mid j < i\}$. But $\mathbb{K}(H_i, \mathcal{L}')$ is a subring of $\mathbb{K}(G, \mathcal{L})$ by Remark 4.8, and this finishes the proof. $\hspace{0.5cm}\square$

Lemma 4.12. *Let $S \subseteq H^1(G; \mathbb{R})$ and let $x \in \mathcal{D}(G, S) \smallsetminus \{0\}$ be given. There exists a very rich subset $V \subseteq H^1(G; \mathbb{R})$ such that*
$$V = \bigsqcup_{a \in A} V_a$$
where A is a finite indexing set, and where for every $a \in A$ there is $x_a \in \mathbb{Q}G \smallsetminus \{0\}$ such that for every $\phi \in V_a \cap S$ we have
$$\phi(x_a) = \phi(x) < \phi(x - x_a)$$

Proof. Recall that $\alpha \colon G \to G^{\mathrm{fab}}$ denotes the free abelianisation map; let K denote its kernel, and let s_α denote a section.

We have $x = s_\alpha(y) s_\alpha(z)^{-1}$ with $y, z \in \mathcal{D}(K)G^{\mathrm{fab}} \smallsetminus \{0\}$ (since $x \neq 0$). We define $A = \operatorname{supp} y \times \operatorname{supp} z$ where the supports are subsets of G^{fab}, and for $a = (b, c)$ we declare V_a to be the set of those characters in $H^1(G; R)$ which attain their minima on $\operatorname{supp} y$ precisely at b, and on $\operatorname{supp} z$ precisely at c. It is immediate that the sets V_a are open and pairwise disjoint. It is also immediate that $V = \bigsqcup_{a \in A} V_a$ is very rich, since it contains all irrational characters in $H^1(G; R)$.

Now let us fix $a = (b, c) \in A$. We write

$$y = y_b + y' \text{ and } z = z_c + z'$$

where $y_b, y', z_c, z' \in \mathcal{D}(K)G^{\text{fab}}$ satisfy $\operatorname{supp} y_b = \{b\}$, $\operatorname{supp} y' = \operatorname{supp} y \smallsetminus \{b\}$, $\operatorname{supp} z_c = \{c\}$, and $\operatorname{supp} z' = \operatorname{supp} z \smallsetminus \{c\}$. Take $\phi \in S \cap V_a$. By the definition of $\mathcal{D}(G, S)$, there exists a biordering $\leqslant$ on G^{fab} compatible with ϕ such that $i_\leqslant(x) \in (\widehat{\mathbb{Q}K)G^{\text{fab}}}^\phi$. Now

$$i_\leqslant(x) = i_\leqslant(s_\alpha(y)) i_\leqslant(s_\alpha(z))^{-1} = y \operatorname{id}_\leqslant(z)^{-1}$$

and so

$$i_\leqslant(x) z = y$$

where both equations hold in $\mathbb{F}_\leqslant(G)$.

Applying ϕ to both sides of the equality tells us that ϕ attains its minimum on $\operatorname{supp} \iota_\leqslant(x)$ at d which satisfies $dc = b$. Note that this forces d to be unique. Let

$$x'_a = i_\leqslant(x)(d)d \in (\mathbb{Q}K)G^{\text{fab}}$$

We immediately see that $\phi(x'_a) < \phi(i_\leqslant(x) - x'_a)$. Setting $x_a = s_\alpha(x'_a) \in \mathbb{Q}G$, we obtain

$$\phi(x - x_a) = \phi(i_\leqslant(x - x_a)) = \phi(i_\leqslant(x) - x'_a) > \phi(x'_a) = \phi(x_a)$$

Note that $\iota_\leqslant(x)(d) = y(b)z(c)^{-1}$ is independent of the choice of $\leqslant$. Therefore x_a depends only on x and $a \in A$, but not on ϕ. □

Theorem 4.13. *Let G be a finitely-generated RFRS group. We have*

$$\mathbb{K}(G, \mathcal{L}) = \mathcal{D}(G)$$

for every chain $\mathcal{L}$.

Proof. Recall that every sub-skew-field of $\mathcal{D}(G)$ containing $\mathbb{Q}G$ is equal to $\mathcal{D}(G)$ by Lemma 2.18. In view of this fact, we need only show that $\mathbb{K}(G, \mathcal{L})$ is a skew-field (since we have already seen in Lemma 4.9 that $\mathbb{K}(G, \mathcal{L})$ contains $\mathbb{Q}G$). To this end, take $x \in \mathbb{K}(G, \mathcal{L}) \smallsetminus \{0\}$. Let n denote the associated integer of x. We proceed by induction on n.

Base case: Suppose that $n = 0$. Let U denote the associated rich set of x in $H^1(G; \mathbb{R})$. We now apply Lemma 4.12 to $x \in \mathcal{D}(G, U)$, which produces for us a very rich set $V = \bigsqcup_{a \in A} V_a$ where A is a finite set.

Let $\phi \in H^1(G; \mathbb{R})$ be an irrational character contained in U. Lemma 4.12 tells us that there exists $a \in A$ and $x_a \in \mathbb{Q}G \smallsetminus \{0\}$ such that

$$\phi(x_a) = \phi(x) < \phi(x - x_a)$$

Also, since $x_a \in \mathbb{Q}G \smallsetminus \{0\}$, it is invertible in $\mathbb{K}(G, \mathcal{L})$ by Proposition 4.11. Now an application of Lemma 4.10 gives us invertibility of x in $\mathbb{K}(G, \mathcal{L})$.

Inductive step: Let

$$x = s_n \left(\sum_{q \in Q_n} x_q q \right)$$

with $x_q \in \mathcal{D}(H_n)$ for every $q \in Q_n$.

Let $B = \{q \in Q_n \mid x_q \neq 0\}$, and consider the map $\xi \colon B \to Q_1$ given by $\xi(q) = \beta_1 s_n(q)$. If ξ is constant on B, then up to multiplying x by an element of G we may assume that $\xi(B) = \{1\}$. This implies that $s_n(q) \in H_1$ for every $q \in B$, and so $x = s_n(\sum x_q q) \in \mathcal{D}(H_1)$. Therefore $x \in \mathbb{K}(H_1, \mathcal{L}')$ (using Remark 4.8), where

$$\mathcal{L}' = \mathcal{L} \cup \{H^1(G; \mathbb{R})\}$$

Treated as an element in $\mathbb{K}(H_1, \mathcal{L}')$, the element x has lower associated integer, and hence is invertible in $\mathbb{K}(H_1, \mathcal{L}')$ (and therefore in $\mathbb{K}(G, \mathcal{L})$) by induction.

Now suppose that ξ is not constant on B. Let $A = \{\xi(q)) \mid q \in B\}$. Clearly, A is a finite set. For every $a \in A$, the element

$$x_a = s_n \left(\sum_{\xi(q)=a} x_q q \right)$$

is invertible in $\mathbb{K}(G, \mathcal{L})$ by the above discussion. Also,

$$x_a s_1(a)^{-1} = \sum_{\xi(q)=a} x_q s_n(q) s_1(a)^{-1} \in \mathcal{D}(H_1)$$

since $x_q \in \mathcal{D}(H_n) \leqslant \mathcal{D}(H_1)$ and $s_n(q) s_1(a)^{-1} \in H_1$.

Consider the restriction $\alpha\colon H_1 \to G^{\mathrm{fab}}$. Its image $\alpha(H_1)$ is a finitely-generated free-abelian group, and its kernel is K (which is also the kernel of $\alpha\colon G \to G^{\mathrm{fab}}$). Thus, Proposition 2.23 tells us that the isomorphism

$$\mathrm{Ore}(s_\alpha)\colon \mathrm{Ore}(\mathcal{D}(G)G^{\mathrm{fab}}) \to \mathcal{D}(G)$$

restricts to an isomorphism

$$\mathrm{Ore}(\mathcal{D}(K)\alpha(H_1)) \to \mathcal{D}(H_1)$$

Hence,

$$\phi(x_a) = \phi\big(x_a s_1(a)^{-1}\big) + \phi(s_1(a)) \in \phi(s_1(a)H_1) \subseteq \mathbb{R}$$

for every $\phi \in H^1(G; \mathbb{R})$.

Let $\phi \in H^1(G; \mathbb{R})$ be irrational, and take distinct a and b in A. Since ϕ is injective as a map $G^{\mathrm{fab}} \to \mathbb{R}$, the set of values that ϕ attains at the coset $s_1(a)H_1$ of H_1 in G is disjoint from the set of values it attains at $s_1(b)H_1$ – here we are using the fact that $H_1 \geqslant \ker \alpha$ in a crucial way. Therefore

$$\phi(x_a) \neq \phi(x_b)$$

Hence, for each irrational ϕ there exists $a \in A$ such that

$$\phi(x) = \phi(x_a) < \phi(x - x_a)$$

We finish the argument by an application of Lemma 4.10. $\square$

5. The main results

Throughout this section, all unspecified tensoring happens over $\mathbb{Q}G$.

Definition 5.1 (Antipodal map)**.** Given a vector space V, we call the map $V \to V$ given by $v \mapsto -v$ the *antipodal map*.

Theorem 5.2. *Let G be a finitely-generated RFRS group, and let $N \in \mathbb{N}$ be an integer. Let C_* denote a chain complex of free $\mathbb{Q}G$-modules such that for every $i \leqslant N$ the module C_i is finitely generated, and such that $H_i(\mathcal{D}(G) \otimes C_*) = 0$. There exists a finite-index subgroup H of G and an open subset $U \subseteq H^1(H; \mathbb{R})$ such that*

(1) *the closure of U contains $H^1(G; \mathbb{R})$;*
(2) *U is invariant under the antipodal map;*
(3) *$H_i\big(\widehat{\mathbb{Q}H}^\psi \otimes_{\mathbb{Q}H} C_*\big) = 0$ for every $i \leqslant N$ and every $\psi \in U$.*

Proof. We fix a free $\mathbb{Q}G$-basis of every C_i with $i \leqslant N$. Since we only care about homology up to degree N, we apply the following procedure to C_*: all modules in degrees above $N + 1$ are set to 0; since $H_N(\mathcal{D}(G) \otimes C_*) = 0$ and C_N is finitely generated, there exists a finitely-generated free submodule $\overline{C_{N+1}}$ of C_{N+1} such that replacing C_{N+1} by $\overline{C_{N+1}}$ still yields vanishing of the Nth homology with coefficients in $\mathcal{D}(G)$. We replace C_{N+1} by $\overline{C_{N+1}}$. We will continue to denote the new chain

complex by C_* – it is now a finite chain complex of finitely-generated free modules, and we still have

$$H_i(\mathcal{D}(G) \otimes C_*) = 0$$

for every $i \leqslant N$. The homology of the new complex C_* with any coefficients agrees with the homology of the old complex in degrees lower than N; in degree N this does not have to be the case, but nevertheless if some homology of the new complex vanishes in degree N, then it must have vanished for the old complex too.

By assumption, we have $H_i(\mathcal{D}(G) \otimes C_*) = 0$ for all $i \leqslant N$. Since $\mathcal{D}(G)$ is a skew-field, this implies the existence of invertible matrices $M_0, \ldots, M_{N+1}$ over $\mathcal{D}(G)$ such that if we change the basis of $\mathcal{D}(G) \otimes C_i$ by M_i, we obtain a chain complex C'_* which can be written as follows: every C'_i splits as $D_i \oplus E_i$, and every differential is the identity matrix taking D_{i+1} isomorphically to E_i, and is trivial on E_{i+1}.

We have shown in Theorem 4.13 that $\mathcal{D}(G) = \mathbb{K}(G)$. Since we are looking at finitely many matrices M_i and their inverses, and each of them is finite, there are only finitely many entries appearing in these matrices. Therefore, there exists n (the maximum of the associated integers) and a rich set $U \subseteq H^1(H_n; \mathbb{R})$ (where $(H_i)_i$ denotes a witnessing chain of G) such that all the matrices M_i and their inverses lie over $\mathcal{D}(H_n, U)Q_n$, where $Q_n = G/H_n$. We additionally assume that U is Q_n-invariant and invariant under the antipodal map (see Lemma 4.4). For notational convenience we set $H = H_n$ and $Q = Q_n$. Note that the closure of U contains $H^1(G; \mathbb{R})$ by Lemma 4.6.

Every entry of a matrix M_i lies in $\mathcal{D}(H, U)Q$. The ring $\mathcal{D}(H, U)Q$ has the structure of a free finitely-generated left $\mathcal{D}(H, U)$-module. Therefore the right action of M_i on the free $\mathcal{D}(G)$-module $\mathcal{D}(G) \otimes C_i$ can be seen as right-multiplication by a matrix M'_i with entries in $\mathcal{D}(H, U)$ on the free $\mathcal{D}(H, U)$-module $\mathcal{D}(H, U) \otimes_{\mathbb{Q}H} C_i$.

Take $\psi \in U$. By the very definition of $\mathcal{D}(H, U)$, we may replace each entry of M'_i by an element of $\widehat{\mathbb{Q}H}^\psi$. Applying this procedure to the inverse of the matrix M_i gives a matrix over $\widehat{\mathbb{Q}H}^\psi$ which is the inverse of M'_i; we will therefore denote it by M'^{-1}_i.

We are now ready to conclude the proof: for every i there exists a matrix M'_i over $\widehat{\mathbb{Q}H}^\psi$ invertible over the same ring, and such that if we change the basis of $\widehat{\mathbb{Q}H}^\psi \otimes C_*$ using these matrices, we obtain a chain complex C''_* such that every C''_i splits as $D'_i \oplus E'_i$, and every differential vanishes on E'_i and takes D'_{i+1} isomorphically to E'_i. The homology of this chain complex obviously vanishes in every degree. $\square$

Theorem 5.3. *Let G be an infinite finitely-generated group which is virtually RFRS. Then G is virtually fibred, in the sense that it admits a finite-index subgroup mapping onto $\mathbb{Z}$ with a finitely-generated kernel, if and only if $\beta_1^{(2)}(G) = 0$.*

Proof. Suppose first that G is virtually fibred; let H denote a finite index subgroup of G which maps onto $\mathbb{Z}$ with a finitely-generated kernel. By Theorem 2.27, the first L^2-Betti number of H vanishes. But [Lüc02, Theorem 1.35(9)] tells us that then $\beta_1^{(2)}(G) = 0$ as well.

Now let us look at the other direction. Suppose that $\beta_1^{(2)}(G) = 0$. Note that the first L^2-Betti number of every finite index subgroup of G vanishes as well (by the same argument as above), and being virtually fibred clearly passes to finite index overgroups, and therefore we may assume that G itself is RFRS, infinite and finitely generated. Since G is infinite, we have $\beta_0^{(2)}(G) = 0$ by [Lüc02, Theorem 6.54(8b)].

We apply Theorem 5.2 where C_* is constructed as follows: take the cellular chain complex of the universal covering of some classifying space of G with finite 1-skeleton, and tensor it with $\mathbb{Q}G$. We take $N = 1$. We conclude the existence of

a finite index subgroup H and a non-empty open subset $U \subseteq H^1(H; \mathbb{R})$ such that

$$H_1(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Q}H} C_*) = 0$$

for every $\psi \in U$. Let us take $\psi \in U$ whose image is $\mathbb{Z}$ – such a ψ exists since U is open, non-empty, and $H^1(H; \mathbb{R})$ is non-trivial. Since $\mathbb{Q}H \otimes_{\mathbb{Q}H} C_*$ is a resolution of the trivial $\mathbb{Q}H$-module $\mathbb{Q}$, the vanishing of the above homology tells us that

$$H_1(H; \widehat{\mathbb{Q}H}^{\psi}) = 0$$

Since U is invariant under the antipodal map, we also have

$$H_1(H; \widehat{\mathbb{Q}H}^{-\psi}) = 0$$

Now Sikorav's Theorem (Theorem 3.11) tells us that $\{\psi, -\psi\} \subseteq \Sigma(H)$, and so $\ker \psi$ is finitely generated by Theorem 3.10. This finishes the proof. $\qquad \square$

Recall that a group G is said to be *of type* $\mathtt{FP}_2$ if and only if there exists a long exact sequence

$$\cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

of projective $\mathbb{Z}G$-modules C_i, where $\mathbb{Z}$ is considered to be a trivial $\mathbb{Z}G$-module, such that C_2, C_1 and C_0 are finitely generated. By standard homological algebra (see for example [Bie81, Remark 1.1(2)]) we may in fact take the modules C_i to be free for all i, and the modules C_2, C_1, and C_0 to be additionally finitely generated.

Note that being of type $\mathtt{FP}_2$ forces G to be finitely generated.

A group G is *of cohomological dimension at most* 2 if and only if there exists an exact sequence

$$0 \to D_2 \to D_1 \to D_0 \to \mathbb{Z}$$

of projective $\mathbb{Z}G$-modules D_i, where $\mathbb{Z}$ is again a trivial $\mathbb{Z}G$-module.

Again, standard homological algebra (or [Bie81, Proposition 4.1(b)]) tells us that if G is both of type $\mathtt{FP}_2$ and of cohomological dimension at most 2, we may take D_2 to be finitely generated and projective, and D_1 and D_0 to be finitely generated and free.

The author is grateful to the referee for pointing out the following application.

Theorem 5.4. *Let G be a non-trivial virtually-RFRS group of type* $\mathtt{FP}_2$ *and of cohomological dimension at most* 2. *If $\beta_2^{(2)}(G) = \beta_1^{(2)}(G) = 0$, then there exists a finite index subgroup H of G and an epimorphism $\psi \colon H \to \mathbb{Z}$ with kernel of type* $\mathtt{FP}_2$.

Proof. Since G is of type $\mathtt{FP}_2$, we have an exact sequence C_* as above. Observe that $H_i(\mathcal{D}(G) \otimes_{\mathbb{Z}G} C_*) = 0$ for all $i \leqslant 2$, since the L^2-Betti numbers of G vanish – note that $\beta_0^{(2)}(G) = 0$ as G is non-trivial and of finite cohomological dimension, which together imply that G is infinite, which suffices by [Lüc02, Theorem 6.54(8b)]. In fact, the higher L^2-Betti numbers also vanish, as can be easily seen using the resolution D_* from above.

Note that being infinite implies that G^{fab} is non-trivial.

Now, the chain complex $C'_* = \mathbb{Q}G \otimes_{\mathbb{Z}G} C_*$ satisfies the assumptions of Theorem 5.2 with $N = 2$. From the theorem we obtain a finite index subgroup H, and a surjective character $\psi \colon H \to \mathbb{Z}$ such that

$$H_i(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Q}H} C'_*) = 0 = H_i(\widehat{\mathbb{Q}H}^{-\psi} \otimes_{\mathbb{Q}H} C'_*)$$

for $i \in \{0, 1, 2\}$.

Since ψ is non-trivial, we have

$$H_0(H; \widehat{\mathbb{Z}H}^{\psi}) = 0 = H_0(H; \widehat{\mathbb{Z}H}^{-\psi})$$

(this is an easy exercise).

By Theorem 3.11, we have

$$H_1\big(H; \widehat{\mathbb{Z}H}^{\psi}\big) = 0 = H_1\big(H; \widehat{\mathbb{Z}H}^{-\psi}\big)$$

We now want to compute the second homology of H with these coefficients. Consider the resolution D_* from above. We have $D_3 = 0$ and

$$H_2(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Z}H} D_*) = H_2(H; \widehat{\mathbb{Q}H}^{\psi}) = 0$$

This implies that $\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Z}H} \partial$ is injective, where $\partial \colon D_2 \to D_1$ denotes the differential.

The module D_2 is finitely generated and projective, and so we have an isomorphism of $\mathbb{Z}H$-modules

$$D_2 \oplus E \cong \mathbb{Z}H^n$$

for some n and some $\mathbb{Z}H$-module E. We immediately see that

$$\left(\widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} D_2\right) \oplus \left(\widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} E\right) \cong \widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} \mathbb{Z}H^n = \left(\widehat{\mathbb{Z}H}^{\psi}\right)^n$$

and

$$\left(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Z}H} D_2\right) \oplus \left(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Z}H} E\right) \cong \left(\widehat{\mathbb{Q}H}^{\psi}\right)^n$$

Since the natural map $\widehat{\mathbb{Z}H}^{\psi} \to \widehat{\mathbb{Q}H}^{\psi}$ is clearly injective, the same is true for the induced map

$$\left(\widehat{\mathbb{Z}H}^{\psi}\right)^n \to \left(\widehat{\mathbb{Q}H}^{\psi}\right)^n$$

and hence for the natural map

$$\widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} D_2 \to \widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Z}H} D_2$$

This implies that $\widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} \partial$ is injective, and so

$$H_2(H; \widehat{\mathbb{Z}H}^{\psi}) = H_2(\widehat{\mathbb{Z}H}^{\psi} \otimes_{\mathbb{Z}H} D'_*) = 0$$

A completely analogous argument shows that

$$H_2(H; \widehat{\mathbb{Z}H}^{-\psi}) = 0$$

also holds. We now conclude that $\ker \psi$ is of type $\mathtt{FP}_2$ from a theorem of Pascal Schweitzer [Bie07, Theorem A.1] (which should be thought of as a higher degree version of Sikorav's theorem). $\qquad\square$

6. Agol's Theorem

In [Ago08], Agol proved the following theorem.

Theorem 6.1 (Agol's theorem)**.** *Let M be a compact connected orientable irreducible 3-manifold with $\chi(M) = 0$ such that $\pi_1(M)$ is RFRS. If $\phi \in H^1(M; \mathbb{Z}) \smallsetminus \{0\}$ is a non-fibred homology class, then there exists a finite-sheeted cover $p \colon M' \to M$ such that $p^*\phi \in H^1(M'; \mathbb{Z})$ lies in the cone over the boundary of a fibred face of $B(M')$.*

Let us explain the notation: χ stands for the Euler characteristic. A non-trivial character $\phi \colon \pi_1(M) \to \mathbb{Z}$ is *fibred* if and only if it is induced by a fibration of M over the circle; equivalently (thanks to a theorem of Stallings [Sta62]), ϕ is fibred if and only if its kernel is finitely generated. The notation $p^*\phi$ corresponds to simply ϕ in our notation, since we identify $H^1(M; \mathbb{R})$ with a subspace of $H^1(M'; \mathbb{R})$.

The notation $B(M)$ stands for the Thurston polytope (defined in [Thu86]). It is a compact rational polytope lying in $H^1(M; \mathbb{R})$, and its key property is that certain maximal open faces of $B(M)$ are called fibred, and a primitive integral character

$\phi \in H^1(M; \mathbb{Z}) \setminus \{0\}$ is fibred if and only if it lies in the cone in $H^1(M; \mathbb{R})$ over a fibred face. (This property of $B(M)$ was shown by Thurston in [Thu86], and then reproved by the author in [Kie] using methods very similar to the ones used in this article).

We will now give a new proof of Agol's theorem; in fact, we give a slightly stronger, more uniform statement, since we will show that the finite cover M' can be chosen independently of ϕ.

Theorem 6.2 (Uniform Agol's theorem). *Let M be a compact connected orientable irreducible 3-manifold with $\chi(M) = 0$ such that $\pi_1(M)$ is RFRS. There exists a finite-sheeted cover $p: M' \to M$ such that for every $\phi \in H^1(M; \mathbb{Z}) \setminus \{0\}$ either ϕ is fibred or $p^*\phi$ lies in the cone over the boundary of a fibred face of $B(M')$.*

Proof. We begin by observing that an irreducible 3-manifold with infinite fundamental group is aspherical. Since the statement is vacuous when $G = \pi_1(M) = \{1\}$, and since G is RFRS, we may assume that G is infinite, and thus that M is aspherical. Now $\chi(M) = 0$ is equivalent to the vanishing of the L^2-Betti numbers by [LL95, Theorem 0.1b].

Since M is compact, G is finitely generated. We may now apply Theorem 5.2 with $N = 3$ to the cellular chain complex C_* of the universal covering of M (with $\mathbb{Q}$-coefficients). We are given a finite index subgroup $H \leqslant G$ and an open set U in $H^1(H; \mathbb{R})$ which is invariant under the antipodal map, and whose closure contains $H^1(G; \mathbb{R})$. Furthermore, we have

$$H_1\big(\widehat{\mathbb{Q}H}^{\psi} \otimes_{\mathbb{Q}H} C_*\big) = 0$$

for every $\psi \in U$.

Let $M' \to M$ be the covering corresponding to $H \leqslant G$. Then $\mathbb{Q}H \otimes_{\mathbb{Q}H} C_*$ coincides with the cellular chain complex of the universal covering of M' (as a $\mathbb{Q}H$-module).

Let $\phi: G \to \mathbb{Z}$ be non-trivial. Since $\phi \in H^1(G; \mathbb{R})$, there exists a sequence $(\psi_i)_i$ of characters in U with

$$\lim_{i \to \infty} \psi_i = \phi$$

Since U is invariant under the antipodal map, we have $-\psi_i \in U$ for every i, and therefore

$$H_1\big(H; \widehat{\mathbb{Q}H}^{\psi_i}\big) = H_1\big(H; \widehat{\mathbb{Q}H}^{-\psi_i}\big) = 0$$

for every i. Sikorav's Theorem (Theorem 3.11) tells us that

$$\{\psi_i, -\psi_i \mid i \in \mathbb{N}\} \subseteq \Sigma(H)$$

and Theorem 3.10 tells us that each ψ_i has a finitely-generated kernel. Stallings's Theorem [Sta62] now implies that every ψ_i is fibred (note that M' is still irreducible). Thus, every ψ_i lies in the cone of a fibred face of $B(M')$.

Since the polytope $B(M')$ has finitely many faces, and so finitely many fibred faces, we may pass to a subsequence and conclude that every ψ_i lies in the cone of the same fibred face of $B(M')$. Hence, ϕ itself must lie in the closure of this cone. Now one of two possibilities occurs: ϕ lies in the cone of a fibred face, and so is itself fibred, or ϕ lies in the cone of the boundary of a fibred face of $B(M')$. $\square$

REFERENCES

[Ago08] I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284. MR2399130 (2009b:57033)

[Ago13] ———, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning. MR3104553

[Bar19] L. Bartholdi, *Amenability of groups is characterized by Myhill's Theorem*, J. Eur. Math. Soc. (2019). With an appendix by Dawid Kielak.

[Bie07] R. Bieri, *Deficiency and the geometric invariants of a group*, J. Pure Appl. Algebra **208** (2007), no. 3, 951–959. With an appendix by Pascal Schweitzer. MR2283437

[Bie81] _____, *Homological dimension of discrete groups*, Second, Queen Mary College Mathematical Notes, Queen Mary College, Department of Pure Mathematics, London, 1981. MR715779

[BNS87] R. Bieri, W. D. Neumann, and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. **90** (1987), no. 3, 451–477. MR914846 (89b:20108)

[FV15] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, J. Reine Angew. Math. **707** (2015), 87–102. MR3403454

[JZLÁ] A. Jaikin-Zapirain and D. López-Álvarez, *The strong Atiyah and Lück approximation conjectures for one-relator groups*, arXiv:1810.12135.

[Kie] D. Kielak, *The Bieri–Neumann–Strebel invariants via Newton polytopes*, arXiv:1802.07049.

[Lin93] P. A. Linnell, *Division rings and group von Neumann algebras*, Forum Math. **5** (1993), no. 6, 561–576. MR1242889

[LL95] J. Lott and W. Lück, L^2-*topological invariants of 3-manifolds*, Invent. Math. **120** (1995), no. 1, 15–60. MR1323981

[Lüc02] W. Lück, L^2-*invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926649 (2003m:58033)

[Mal48] A. I. Mal'cev, *On the embedding of group algebras in division algebras*, Doklady Akad. Nauk SSSR (N.S.) **60** (1948), 1499–1501. MR0025457 (10,8c)

[Neu49] B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. **66** (1949), 202–252. MR0032593 (11,311f)

[Pas85] D. S. Passman, *The algebraic structure of group rings*, Robert E. Krieger Publishing Co., Inc., Melbourne, FL, 1985. Reprint of the 1977 original. MR798076

[Sch02] T. Schick, *Erratum: "Integrality of L^2-Betti numbers"*, Math. Ann. **322** (2002), no. 2, 421–422. MR1894160

[Sch14] K. Schreve, *The strong Atiyah conjecture for virtually cocompact special groups*, Math. Ann. **359** (2014), no. 3-4, 629–636. MR3231009

[Sik87] J.-C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*, Thèse Orsay (1987).

[Sta62] J. Stallings, *On fibering certain 3-manifolds*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), 1962, pp. 95–100. MR0158375

[Tam57] D. Tamari, *A refined classification of semi-groups leading to generalized polynomial rings with a generalized degree concept*, Proceedings of the International Congress of Mathematicians, 1954, vol. 1, Amsterdam, 1957, pp. 439–440.

[Thu86] W. P. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130. MR823443 (88h:57014)

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