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On the zero-in-the-spectrum conjecture

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Abstract

We prove that the answer to the "zero-in-the-spectrum" conjecture, in the form suggested by J. Lott, is negative. Namely, we show that for any $n \geq 6$ there exists a closed n-dimensional smooth manifold M^n, so that zero does not belong to the spectrum of the Laplace-Beltrami operator acting on the L^2 forms of all degrees on the universal covering $\tilde{M}$.

1. The main results

M. Gromov formulated the following conjecture (cf. [5, p. 120] and also [6, pp. 21 and 238]):

Conjecture A. *Let M be a closed aspherical manifold; is it true that zero is always in the spectrum of the Laplace-Beltrami operator Δ_p, acting on the square integrable p-forms on the universal covering $\tilde{M}$, for some p?*

If the Strong Novikov Conjecture (see [15]) holds for the fundamental group $\pi_1(M)$, then $0 \in \mathrm{Spec}(\Delta_p)$ for some p (see [12, p. 371] and also [7, p. 166], both of which rely on the fundamental work of Kasparov [10]). Hence a counterexample to Conjecture A would also be a counterexample to the Strong Novikov Conjecture.

J. Lott raised a more general "zero-in-the-spectrum" question. We refer the reader to survey articles [12] and [13].

Conjecture B. *Is it true, that for any complete Riemannian manifold M zero is always in the spectrum of the Laplace-Beltrami operator Δ_p, acting on the square integrable p-forms on M, for some p?*

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J. Dodziuk and J. Lott showed that the answer to this question is positive for manifolds of low dimension and also for some classes of higher dimensional manifolds. G. Yu obtained in [17], [18] results, confirming Conjecture B under some additional geometrical and topological assumptions.

In this article we shall disprove Conjecture B. We construct counterexamples, which are universal covers of compact Riemannian manifolds of arbitrary dimension $n \geq 6$. Conjecture A remains unsettled.

THEOREM 1. *For any $n \geq 6$ there exists a closed n-dimensional smooth manifold M, so that for any $p = 0, 1, \dots, n$ the zero does not belong to the spectrum of the Laplace-Beltrami operator*

$$\Delta_p : \Lambda^p_{(2)}(\tilde{M}) \to \Lambda^p_{(2)}(\tilde{M}),$$

acting on the space of L^2-forms $\Lambda^p_{(2)}(\tilde{M})$ on the universal covering $\tilde{M}$ of M.

Our proof of Theorem 1 will be based on the fact that it can be restated in an equivalent form using the notion of extended L^2-homology, introduced in [3]:

THEOREM 2. *For any $n \geq 6$ there exists a closed orientable smooth n-dimensional manifold M, so that extended L^2-homology $\mathcal{H}_p(M; \ell^2(\pi)) = 0$ vanishes for all p. Here π denotes the fundamental group $\pi = \pi_1(M)$, and $\ell^2(\pi)$ denotes the L^2-completion of the group ring $\mathbf{C}[\pi]$.*

Equivalence between Theorem 1 and Theorem 2 can be established as follows. Zero not in the spectrum of the Laplacian $\Delta_p : \Lambda^p_{(2)}(\tilde{M}) \to \Lambda^p_{(2)}(\tilde{M})$ for all p is equivalent to vanishing of the extended L^2-cohomology $\mathcal{H}^*(M; \ell^2(\pi))$ (cf. [3]), according to the de Rham Theorem for extended cohomology (cf. §7 of [4] and also [16]). Vanishing of the extended L^2-cohomology is equivalent to vanishing of the extended L^2-homology $\mathcal{H}_*(M; \ell^2(\pi))$, because of Poincaré duality (cf. [3, Th. 6.7]).

The proof of Theorem 2 is based on the following theorem:

THEOREM 3. *There exists a finite 3-dimensional polyhedron Y with fundamental group $\pi_1(Y) = \pi = F \times F \times F$, where F denotes a free group with two generators, such that the extended L^2-homology $\mathcal{H}_p(Y; \ell^2(\pi)) = 0$ vanishes for all $p = 0, 1, \dots$.*

The strategy of our proofs of Theorems 2 and 3 is similar to the method used by M. A. Kervaire [11], who constructed smooth homology spheres with prescribed fundamental groups. Our proof uses an L^2-analogue of the Hopf exact sequence.

Next we state a result in the direction of Conjecture A, which is essentially a Corollary of Theorem 3.

THEOREM 4. *There exist an aspherical* 3*-dimensional finite polyhedron* Z *and a normal subgroup* $H \subset \pi = \pi_1(Z)$ *such that the extended* L^2*-homology* $\mathcal{H}_p(Z; \ell^2(\pi/H))$ *vanishes for all* $p = 0, 1, \dots$.

Since this paper was written, N. Higson, J. Roe, and T. Schick [8] have simplified and generalized our construction of L^2-invisible manifolds. They proved that any finitely presented group π with vanishing extended L^2-homology in dimensions 0, 1, and 2 is the fundamental group of a closed n-dimensional M (for any $n \geq 6$) with $\mathcal{H}_p(M; \ell^2(\pi)) = 0$ for all p. The result of [8] is a very satisfying analogue of the theorem of Kervaire [11].

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2. Proofs of Theorems 2, 3 and 4

A. Let π be a discrete group given by a finite presentation

$$\pi = |x_1, x_2, \dots, x_n : r_1 = 1, r_2 = 1, \dots, r_m = 1|$$

by generators and relations. We will assume that:

(a) The extended L^2-homology of π in dimensions $0, 1$ and 2 vanishes; i.e.

$$\mathcal{H}_0(\pi; \ell^2(\pi)) = \mathcal{H}_1(\pi; \ell^2(\pi)) = \mathcal{H}_2(\pi; \ell^2(\pi)) = 0.$$

(b) Let X be a finite cell complex with fundamental group $\pi_1(X) = \pi$, having one zero-dimensional cell, n cells of dimension 1 and m cells of dimension two, constructed in the usual way out of the given presentation of π. Then the second homotopy group $\pi_2(X)$ of X, viewed as a $\mathbf{Z}[\pi]$-module, is free and finitely generated.

Our purpose is to show that there exists a 3-dimensional cell complex Y, obtained from X by first taking a bouquet with finitely many copies of S^2 and then adding a finite number of 3-dimensional cells, so that

$$\mathcal{H}_i(Y; \ell^2(\pi)) = 0 \quad \text{for any} \quad i = 0, 1, \dots \ . \tag{2.1}$$

B. *The* L^2*-Hopf exact sequence.* First we will calculate the extended L^2-homology of X using the spectral sequence constructed in Theorem 9.7 of [3]. We will work in the von Neumann category $\mathcal{C}_\pi$ of Hilbert representations of π (cf. [4, §2, ex. 5]). We will denote by $\mathcal{E}(\mathcal{C}_\pi)$ the corresponding extended abelian category (cf. [4, §1]). Let $\tilde{X}$ be the universal covering of X. We will use the functors

$$\mathrm{Tor}_p^\pi(\ell^2(\pi), H_q(\tilde{X}; \mathbf{C}))$$

with values in the extended abelian category $\mathcal{E}(\mathcal{C}_\pi)$, which are defined in [3, p. 660] under the assumption that the homology modules $H_q(\tilde{X};\mathbf{C})$ of the universal covering admit finite free resolutions. In our case only two of these homology modules can be nonzero (for $q=0$ and $q=2$), and (since $H_2(\tilde{X};\mathbf{C}) = \mathbf{C}\otimes\pi_2(X)$) our assumption (b) guarantees this finiteness condition for $q=2$. The functor $\mathrm{Tor}_0^\pi(\ell^2(\pi), H_q(\tilde{X};\mathbf{C}))$ can be denoted by

$$\ell^2(\pi)\tilde{\otimes}_\pi H_q(\tilde{X};\mathbf{C}). \tag{2.2}$$

It is an analogue of the tensor product (cf. [4, §6]). Note that in general it takes values in the extended category $\mathcal{E}(\mathcal{C}_\pi)$; i.e., it may have a nontrivial torsion part.

By Theorem 9.7 of [3], there exists a spectral sequence $E^r_{p,q}$, where $r\geq 2$, in the abelian category $\mathcal{E}(\mathcal{C}_\pi)$ with the following properties:

- The initial term of the spectral sequence is $E^2_{p,q} = \mathrm{Tor}_p^\pi(\ell^2(\pi), H_q(\tilde{X};\mathbf{C}))$.
- It converges to the extended L^2-homology $\mathcal{H}_{p+q}(X;\ell^2(\pi))$.

For $q=0$ we have $H_0(\tilde{X};\mathbf{C}) = \mathbf{C}$, and $\mathrm{Tor}_q^\pi(\ell^2(\pi),\mathbf{C})$ can also be understood as the extended L^2-homology of the Eilenberg-MacLane space $K(\pi,1)$. We will use notation

$$\mathrm{Tor}_q^\pi(\ell^2(\pi),\mathbf{C}) = \mathcal{H}_q(\pi;\ell^2(\pi)). \tag{2.3}$$

It is an analogue of the group homology.

Since X is two-dimensional, the spectral sequence contains only two rows ($q=0$ and $q=2$) and may have only one nontrivial differential. Hence we obtain the following isomorphisms:

$$\mathcal{H}_0(X;\ell^2(\pi)) \simeq \mathcal{H}_0(\pi;\ell^2(\pi)) \quad \text{and} \quad \mathcal{H}_1(X;\ell^2(\pi)) \simeq \mathcal{H}_1(\pi;\ell^2(\pi)). \tag{2.4}$$

These are Hurewicz-type isomorphisms for extended L^2-homology. The first nontrivial differential of the E^2-term is $d_2 : E^2_{3,0} \to E^2_{0,2}$. Here $E^2_{3,0} = \mathcal{H}_3(\pi;\ell^2(\pi))$ and $E^2_{0,2} = \ell^2(\pi)\tilde{\otimes}_\pi H_2(\tilde{X};\mathbf{C})$. Using the Hurewicz isomorphism $H_2(\tilde{X}) \simeq \pi_2(\tilde{X}) \simeq \pi_2(X)$, we may write

$$E^2_{0,2} = \ell^2(\pi)\tilde{\otimes}_\pi\pi_2(X)$$

and the above differential is

$$d_2 : \mathcal{H}_3(\pi;\ell^2(\pi)) \to \ell^2(\pi)\tilde{\otimes}_\pi\pi_2(X). \tag{2.5}$$

Note also that this differential must be a monomorphism (viewed as a morphism of the abelian category $\mathcal{E}(\mathcal{C}_\pi)$), since $\mathcal{H}_3(X;\ell^2(\pi)) = 0$ (recall that X is two-dimensional). The spectral sequence above yields the following exact sequence

$$0 \to \mathcal{H}_3(\pi;\ell^2(\pi)) \xrightarrow{d_2} \ell^2(\pi)\tilde{\otimes}_\pi\pi_2(X) \xrightarrow{h} \mathcal{H}_2(X;\ell^2(\pi)) \to \mathcal{H}_2(\pi,\ell^2(\pi)) \to 0.$$

It is an L^2 analogue of the *Hopf exact sequence.*

We conclude (using (2.4) and our assumptions (a)) that

$$\mathcal{H}_0(X;\ell^2(\pi)) = \mathcal{H}_1(X;\ell^2(\pi)) = 0$$

and $\mathcal{H}_2(X;\ell^2(\pi))$ can be found from the exact sequence

$$0 \to \mathcal{H}_3(\pi;\ell^2(\pi)) \to \ell^2(\pi)\tilde{\otimes}_\pi \pi_2(X) \xrightarrow{h} \mathcal{H}_2(X;\ell^2(\pi)) \to 0. \tag{2.6}$$

C. We will now specialize our discussion to the following group

$$\pi = F \times F \times F,$$

where F is a free group with two generators. We will denote the free generators of the factor number r (where $r = 1, 2, 3$) by a_1^r, a_2^r. We will fix the presentation of π given by six generators $a_1^1, a_2^1, a_1^2, a_2^2, a_1^3, a_2^3$ and the following twelve relations

$$(a_i^k, a_j^l) = 1, \quad \text{for} \quad k \neq l, \quad k, l \in \{1, 2, 3\}, \quad i, j \in \{1, 2\},$$

where $(v, w) = vwv^{-1}w^{-1}$ denotes the commutator.

Note that $\mathcal{H}_j(F;\ell^2(F))$ is nonzero only for $j = 1$ and has no torsion. Indeed, the Eilenberg-MacLane space $K(F, 1) = S^1 \vee S^1$ is one-dimensional and so the extended L^2-homology vanishes in dimensions $j > 1$ and $\mathcal{H}_1(F;\ell^2(F))$ is torsion-free (the top dimensional extended L^2-homology of any complex is torsion-free). $\mathcal{H}_0(F;\ell^2(F))$ vanishes by a theorem of Brooks [2] since F is not amenable.

Using the previous remark we see that π satisfies condition (a) above, as follows from the Künneth theorem for the extended L^2-cohomology; cf. Appendix, Theorem 7 (the terms containing the periodic product in formula (3.9), vanish; cf. Proposition 5, statement (b)).

Let us show that this group π, together with its specified presentation, satisfies condition (b). The two-dimensional complex X constructed out of this presentation will have one zero-dimensional cell e^0, six 1-dimensional cells e_i^1, e_i^2, e_i^3 and twelve two-dimensional cells $e_{ij}^{12}, e_{ij}^{13}, e_{ij}^{23}$. Here e_i^k denotes the 1-cell corresponding to the generator a_i^k and e_{ij}^{kl} denotes the 2-cell corresponding to the relation $(a_i^k, a_j^l) = 1$.

Let $0 \to C_2 \to C_1 \to C_0 \to 0$ denote the chain complex of the universal covering $\tilde{X}$. The boundary homomorphism acts as follows

$$\begin{aligned}
\partial e_i^k &= (a_i^k - 1)e^0, \\
\partial e_{ij}^{kl} &= (a_i^k - 1)e_j^l - (a_j^l - 1)e_i^k.
\end{aligned}$$

Using the Hurewicz isomorphisms $\pi_2(X) = \pi_2(\tilde{X}) = H_2(\tilde{X})$, we may compute the group $\pi_2(X)$, viewed as a $\mathbf{Z}[\pi]$-module, as the kernel of $\partial : C_2 \to C_1$. Let

$$x \in C_2, \quad x = \sum_{ij} \lambda_{ij}^{12} e_{ij}^{12} + \sum_{ij} \lambda_{ij}^{13} e_{ij}^{13} + \sum_{ij} \lambda_{ij}^{23} e_{ij}^{23}, \quad \lambda_{ij}^{kl} \in \mathbf{Z}[\pi],$$

be an element with $\partial x = 0$. Then the following equations hold:

$$\sum_i \lambda_{ij}^{12}(a_i^1 - 1) = \sum_i \lambda_{ji}^{23}(a_i^3 - 1),$$

$$\sum_j \lambda_{ij}^{12}(a_j^2 - 1) + \sum_j \lambda_{ij}^{13}(a_j^3 - 1) = 0,$$

$$\sum_i \lambda_{ij}^{13}(a_i^1 - 1) + \sum_i \lambda_{ij}^{23}(a_i^2 - 1) = 0.$$

Hence we may write

$$\lambda_{ij}^{12} = \sum_k \mu_{ijk}^{12}(a_k^3 - 1), \quad \mu_{ijk}^{12} \in \mathbf{Z}[\pi],$$

$$\lambda_{ij}^{23} = \sum_k \mu_{ijk}^{23}(a_k^1 - 1), \quad \mu_{ijk}^{23} \in \mathbf{Z}[\pi],$$

$$\lambda_{ij}^{13} = \sum_k \mu_{ijk}^{13}(a_k^2 - 1), \quad \mu_{ijk}^{13} \in \mathbf{Z}[\pi].$$

Therefore we obtain

$$\mu_{ijk}^{12} = \mu_{jki}^{23} = -\mu_{ikj}^{13}. \tag{2.7}$$

Conversely, any system $\mu_{ijk}^{rs} \in \mathbf{Z}[\pi]$ satisfying (2.7) determines a cycle $x \in C_2$, $\partial x = 0$. This proves that $\pi_2(X)$ *is a free* $\mathbf{Z}[\pi]$*-module of rank 8 with the basis*

$$x_{ijk} = (a_i^1 - 1)e_{jk}^{23} - (a_j^2 - 1)e_{ik}^{13} + (a_k^3 - 1)e_{ij}^{12}, \quad i, j, k \in \{1, 2\}. \tag{2.8}$$

Note that the Eilenberg-MacLane complex $K = K(\pi, 1)$ is $B \times B \times B$, where B is the bouquet of two circles; K is obtained from X by adding eight three-dimensional cells e_{ijk}, where $i, j, k \in \{1, 2\}$, which correspond to different triple products of 1-dimensional cells of B. It is easy to see that the boundary of e_{ijk} is given by

$$\partial e_{ijk} = x_{ijk} \in \pi_2(X).$$

The chain complex of the universal covering $\tilde{K}$ is

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0,$$

where C_3 is the free $\mathbf{Z}[\pi]$-module generated by the cells e_{ijk} and the rest is the same as the chain complex of $\tilde{X}$.

For a discrete group π we will denote by $C^*_{\mathbf{R}}(\pi) \subset C^*_r(\pi)$ the real part of the reduced C^*-algebra, i.e. the norm closure of the real group ring $\mathbf{R}[\pi] \subset \mathbf{C}[\pi]$.

D. Proposition. *Let F be the free group with generators a_1, a_2. Then there exist $u_1, u_2 \in C^*_{\mathbf{R}}(F) \subset C^*_r(F)$ such that*

(i) $u_1(a_1 - 1) + u_2(a_2 - 1) = 0$,

(ii) *for any pair $v_1, v_2 \in \ell^2(\pi)$ with*

$$v_1(a_1 - 1) + v_2(a_2 - 1) = 0 \tag{2.9}$$

there exists a unique $w \in \ell^2(\pi)$ such that

$$v_1 = wu_1, \quad v_2 = wu_2.$$

Here we consider F as a subgroup of $\pi = F \times F \times F$, identifying it with one of the factors. The reduced C^*-algebra $C^*_r(F) \subset C^*_r(\pi)$ acts in the usual way on $\ell^2(\pi)$.

Proof. For convenience, we will assume in the proof that F is the third factor in π. Consider the standard complex

$$\ell^2(F) \oplus \ell^2(F) \xrightarrow{d} \ell^2(F), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1), \tag{2.10}$$

calculating the extended L^2-homology of the bouquet $S^1 \vee S^1$ of two circles with coefficients in $\ell^2(F)$. The operator d is onto; this claim is equivalent to $\mathcal{H}_0(S^1 \vee S^1; \ell^2(F)) = 0$ (the latter was observed above in **C** using a theorem of Brooks [2] and the fact that F is not amenable). Since $\chi(S^1 \vee S^1) = -1$ and $\mathcal{H}_j(S^1 \vee S^1; \ell^2(F)) = 0$ for $j \neq 1$, it follows by the Euler-Poincaré principle (in the extended category of $\ell^2(F)$-modules) that $\ker d = \mathcal{H}_1(S^1 \vee S^1; \ell^2(F))$ is one dimensional; i.e., it is isomorphic to $\ell^2(F)$. Here we use the fact that the von Neumann algebra $\mathcal{N}(F)$ is a factor.

Let $P : \ell^2(F) \oplus \ell^2(F) \to \ell^2(F) \oplus \ell^2(F)$ be the orthogonal projection onto $\ker d$. We claim that the element $P(1, 0)$ belongs to

$$C^*_{\mathbf{R}}(F) \oplus C^*_{\mathbf{R}}(F) \subset C^*_r(F) \oplus C^*_r(F) \subset \ell^2(F) \oplus \ell^2(F). \tag{2.11}$$

Let d^* be the adjoint of d. Then $\ker d = \ker(d^*d)$. Moreover, the image of d^*d is closed and thus zero is an isolated point in the spectrum of d^*d. Hence we may use the holomorphic functional calculus (Cauchy's formula) in order to express the projector P as

$$P = \frac{1}{2\pi i} \int_\Gamma (z - d^*d)^{-1} dz,$$

where Γ is a small circle around the origin. This explains that $P(v_1, v_2)$ belongs to $C^*_r(F) \oplus C^*_r(F)$ (cf. (2.11)), when we assume that that v_1, v_2 lie in the reduced C^*-algebra $C^*_r(F)$. Moreover, since the operator d^*d is real, we obtain that $P(v_1, v_2) \in C^*_{\mathbf{R}}(F) \oplus C^*_{\mathbf{R}}(F)$, for $v_1, v_2 \in C^*_{\mathbf{R}}(F)$.

We will now set

$$(u_1, u_2) = P(1, 0).$$

Then (i) is clearly satisfied.

We want to show that the restriction of P on the first summand $\ell^2(F)$ in (2.10) gives an isomorphism $P : \ell^2(F) \to \ker d$. Since both $\ker d$ and $\ell^2(F)$ have von Neumann dimension one, and the spectrum of P contains only 0 and 1, we conclude that it is enough to show that $P(v, 0) = 0$ for $v \in \ell^2(F)$ implies $v = 0$. If $P(v, 0) = 0$; i.e. $(v, 0) \in (\ker d)^\perp$, then $\langle v, \ker d\rangle = 0$, i.e. v is orthogonal to the projection of $\ker d$ on the first summand $\ell^2(F)$. From this we will obtain that necessarily $v = 0$ if we show that the projection of $\ker d$ on the first summand is dense.

Let $f_i : \ell^2(F) \to \ell^2(F)$ be operator $x \mapsto x(a_i - 1)$, where $i = 1, 2$. It is clear that f_1 and f_2 are injective and hence their images are dense. We claim that $f_1^{-1}(\operatorname{im} f_2)$ is dense. If not, let H denote the orthogonal complement to the closure of $f_1^{-1}(\operatorname{im} f_2)$. Then we may apply Proposition 2.4 from [4]; it implies that H must intersect $\operatorname{im} f_2$, which is impossible. Hence it follows that the projection of $\ker d$ on the first summand $\ell^2(F)$ (which coincides with $f_1^{-1}(\operatorname{im} f_2)$) is dense.

As a result we obtain from the above arguments that for any pair $(v_1, v_2) \in \ker d$ (i.e. which is a solution of (2.9)) there exists $w \in \ell^2(F)$, so that $P(w, 0) = (v_1, v_2)$, i.e. $v_1 = wu_1$ and $v_2 = wu_2$. This is in fact a part of our statement (ii).

In order to prove (ii) in full generality, observe that

$$\ell^2(\pi) = \ell^2(F)\hat{\otimes}\ell^2(F)\hat{\otimes}\ell^2(F) \tag{2.12}$$

(cf. Appendix), and thus (by the Künneth theorem for extended L^2-homology, cf. Theorem 6) we find that the kernel of the operator

$$d : \ell^2(\pi) \oplus \ell^2(\pi) \to \ell^2(\pi), \quad (v_1, v_2) \mapsto v_1(a_1 - 1) + v_2(a_2 - 1),$$

equals $\ell^2(F)\hat{\otimes}\ell^2(F)\hat{\otimes}\mathcal{H}_1(S^1 \vee S^1; \ell^2(F))$; (ii) now follows. □

E. Now we describe the kernel of the Hurewicz homomorphism

$$h : \ell^2(\pi)\tilde{\otimes}_\pi\pi_2(X) \to \mathcal{H}_2(X; \ell^2(\pi)).$$

Let $u_i^s \in C_r^*(\pi)$, where $s = 1, 2, 3$ and $i = 1, 2$, denote the element given by Proposition D applied to the factor $F \subset \pi$, $s = 1, 2, 3$. Here we consider $C_r^*(F)$ as being canonically embedded into the von Neumann algebra $\mathcal{N}(\pi)$.

We claim that *the kernel of the Hurewicz homomorphism h is generated by the element*

$$y = \sum_{ijk} u_i^1 u_j^2 u_k^3 x_{ijk} \in C^*_{\mathbf{R}}(\pi)\tilde{\otimes}_\pi\pi_2(X). \tag{2.13}$$

More precisely, any element $x \in \ell^2(\pi)\tilde{\otimes}_\pi \pi_2(X)$ *with* $h(x) = 0$ *has the form* $x = \mu y$ *for some* $\mu \in \ell^2(\pi)$.

Note that the product μy makes sense because the coefficients of y in the basis x_{ijk} belong to $C^*_{\mathbf{R}}(\pi) \subset C^*_r(\pi)$.

First, it is easy to check (using (2.8) and (i) of Proposition D) that indeed $h(y) = 0$.

Let

$$x = \sum_{ijk} \mu_{ijk} x_{ijk} \in \ell^2(\pi)\tilde{\otimes}_\pi \pi_2(X), \quad h(x) = 0,$$

be an arbitrary element of $\ker h$, where $\mu_{ijk} \in \ell^2(\pi)$. Using (2.8), we obtain (equating to zero the coefficients of the cells e^{23}_{jk}) the fact that for any pair of indices j, k,

$$\sum_{i=1}^{2} \mu_{ijk}(a^1_i - 1) = 0.$$

Hence, applying Proposition D, we conclude that there exist $\mu_{jk} \in \ell^2(\pi)$ such that

$$\mu_{ijk} = \mu_{jk} u^1_i. \tag{2.14}$$

We write again $h(x) = 0$, equating to zero the coefficients of the cells e^{13}_{ik} and using (2.14). We obtain that for any pair of indices i, k,

$$\sum_j \mu_{jk} u^1_i (a^2_j - 1) \;=\; \left[\sum_j \mu_{jk}(a^2_j - 1)\right] u^1_i \;=\; 0. \tag{2.15}$$

Note that $wu^s_1 = 0$ for $w \in \ell^2(\pi)$ implies $wu^s_2 = 0$ (by (2.9)) and from the uniqueness statement in Proposition D, (ii), we obtain that $w = 0$. Therefore (2.15) implies

$$\sum_j \mu_{jk}(a^2_j - 1) = 0$$

and hence by Proposition D,

$$\mu_{jk} = \mu_k u^2_j, \quad \text{where} \quad \mu_{\mathrm{k}} \in \ell^2(\pi).$$

Substitute again $\mu_{ijk} = \mu_k u^1_i u^2_j$ into $h(x) = 0$ and equating to zero the coefficients of the cells e^{13}_{ik} we obtain

$$\left[\sum_k \mu_k(a^3_k - 1)\right] u^1_i u^2_j = 0, \quad \text{and hence} \quad \sum_k \mu_k(a^3_k - 1) = 0. \tag{2.16}$$

Using Proposition D as above we finally obtain

$$\mu_k = \mu u^3_k, \quad \text{where} \quad \mu \in \ell^2(\pi).$$

Therefore, we find that $\mu_{ijk} = \mu u^1_i u^2_j u^3_k$ and $x = \mu y$. $\square$

F. Our goal is to show that *one may add eight cells of dimension* 3 *to the bouquet* $X \vee S^2$ *such that the obtained* 3*-dimensional complex* Y *will have trivial extended* L^2*-homology*

$$\mathcal{H}_j(Y; \ell^2(\pi)) = 0, \quad j = 0, 1, \dots .$$

For the proof, we examine again the exact sequence (2.6):

$$0 \to \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{\phi} \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X) \xrightarrow{h} \mathcal{H}_2(X; \ell^2(\pi)) \to 0. \tag{2.17}$$

As we know, ϕ maps the generator y of $\mathcal{H}_3(\pi; \ell^2(\pi))$ according to formula (2.13); i.e., ϕ is given by a matrix with entries in $C^*_{\mathbf{R}}(\pi) \subset C^*_r(\pi)$. Let

$$Q : \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X) \to \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X)$$

denote the orthogonal projection onto $(\operatorname{im} \phi)^\perp$, the orthogonal complement of the image of ϕ. Since X is two-dimensional, $\mathcal{H}_2(X; \ell^2(\pi))$ has no torsion and therefore $\operatorname{im} \phi$ is closed. Note that $(\operatorname{im} \phi)^\perp$ coincides with $\ker(\phi\phi^*)$. Since the image of $\phi\phi^*$ is closed we conclude that zero is an isolated point in the spectrum of $\phi\phi^*$ and hence we may write

$$Q = \frac{1}{2\pi i} \int_\Gamma (z - \phi\phi^*)^{-1} dz,$$

where Γ is a small circle round zero. Therefore, *in the basis* x_{ijk} *the projector* Q *is given an* (8×8)*-matrix with entries in* $C^*_{\mathbf{R}}(\pi)$.

The projective $C^*_{\mathbf{R}}(\pi)$-module determined by Q is stably free; we know that adding a free one-dimensional module (generated by y) makes it free. Therefore we may consider the bouquet $X_1 = X \vee S^2$ so that $\mathcal{H}_2(X_1; \ell^2(\pi)) = \mathcal{H}_2(X; \ell^2(\pi)) \oplus \ell^2(\pi)$ and $\pi_2(X_1) = \pi_2(X) \oplus \mathbf{Z}[\pi]$. Thus, the exact sequence (2.17) for X_1

$$0 \to \mathcal{H}_3(\pi; \ell^2(\pi)) \xrightarrow{\psi} \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \to 0 \tag{2.18}$$

will have the following property: *the orthogonal projection*

$$Q_1 : \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X_1) \to \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X_1)$$

onto $(\operatorname{im} \psi)^\perp$ *is given by a* (9×9)*-matrix with entries in* $C^*_{\mathbf{R}}(\pi)$ *which determines a free* $C^*_{\mathbf{R}}(\pi)$*-module of rank* 8.

We may reformulate the last statement as follows: there exists a $\mathbf{Z}[\pi]$-homomorphism

$$\gamma : (\mathbf{Z}[\pi])^8 \to C^*_{\mathbf{R}}(\pi) \tilde{\otimes}_\pi \pi_2(X_1) \tag{2.19}$$

such that the following composite

$$\ell^2(\pi) \tilde{\otimes}_\pi (\mathbf{Z}[\pi])^8 \xrightarrow{1 \otimes \gamma} \ell^2(\pi) \tilde{\otimes}_\pi \pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1; \ell^2(\pi)) \tag{2.20}$$

is an isomorphism. Now we will use the fact that the rational group ring $\mathbf{Q}[\pi]$ is dense in $C^*_{\mathbf{R}}(\pi)$ with respect to the operator norm topology. Hence we may approximate γ by a $\mathbf{Z}[\pi]$-homomorphism

$$\gamma_1 : (\mathbf{Z}[\pi])^8 \to \mathbf{Q}[\pi] \otimes_\pi \pi_2(X_1)$$

so that the similar composition (2.20) is an isomorphism. Finally, we may multiply γ_1 by a large integer N to obtain a $\mathbf{Z}[\pi]$-homomorphism

$$\gamma_2 : (\mathbf{Z}[\pi])^8 \to \mathbf{Z}[\pi] \otimes_\pi \pi_2(X_1) = \pi_2(X_1)$$

such that the composition

$$(\ell^2(\pi))^8 = \ell^2(\pi)\tilde{\otimes}_\pi(\mathbf{Z}[\pi])^8 \overset{1\otimes\gamma_2}{\to} \ell^2(\pi)\tilde{\otimes}_\pi\pi_2(X_1) \xrightarrow{h} \mathcal{H}_2(X_1;\ell^2(\pi)) \tag{2.21}$$

is an isomorphism.

Let $z_1, \dots, z_8 \in \pi_2(X_1)$ be images of a free basis of $(\mathbf{Z}[\pi])^8$ under γ_2. Realize each z_j by a continuous map $f_j : S^2 \to X_1$, where $j = 1, \dots, 8$, and let

$$Y = X_1 \cup e_1^3 \cup \dots \cup e_8^3$$

be obtained from X_1 by glueing eight three-dimensional cells to X_1 along $f_1, \dots, f_8$. We claim that

$$\mathcal{H}_j(Y;\ell^2(\pi)) = 0 \quad \text{for all} \quad j = 0, 1, \dots \ . \tag{2.22}$$

In order to show this, we note that $\mathcal{H}_j(Y, X;\ell^2(\pi))$ vanishes for all $j \neq 3$ and the 3-dimensional extended L^2-homology $\mathcal{H}_3(Y, X;\ell^2(\pi))$ equals $(\ell^2(\pi))^8$. The boundary homomorphism $\partial : \mathcal{H}_3(Y, X;\ell^2(\pi)) \to \mathcal{H}_2(X;\ell^2(\pi))$ is an isomorphism since it coincides with (2.21). Hence (2.22) follows from the homological exact sequence of the pair (Y, X). This completes the proof of Theorem 3.

G. Now we may complete the proof of Theorem 2, having constructed above a finite 3-dimensional polyhedron Y. For any $n \geq 6$ we may embed Y into $\mathbf{R}^{n+1}$ as a subpolyhedron. Let $N \subset \mathbf{R}^{n+1}$ be a regular neighborhood of $Y \subset \mathbf{R}^{n+1}$. We will define M as the boundary of N, i.e. $M = \partial N$.

First note that the inclusion $M \to N$ induces an isomorphism of the fundamental groups and thus $\pi_1(M) = \pi = F \times F \times F$, where F is a free group in two generators. We want to show that

$$\mathcal{H}_j(M;\ell^2(\pi)) = 0, \quad \text{for all} \quad j = 0, 1, \dots \ . \tag{2.23}$$

In the exact homological sequence

$$\dots \to \mathcal{H}_j(M;\ell^2(\pi)) \to \mathcal{H}_j(N;\ell^2(\pi)) \to \mathcal{H}_j(N, M;\ell^2(\pi)) \to \dots$$

we have $\mathcal{H}_j(N;\ell^2(\pi)) = 0$. Also, $\mathcal{H}_j(N, M;\ell^2(\pi)) \simeq \mathcal{H}^{n+1-j}(N;\ell^2(\pi))$ by the Poincaré duality (cf. [3]) and $\mathcal{H}^{n+1-j}(N;\ell^2(\pi)) = 0$ because of (2.22), by the Universal Coefficient Theorem (cf. [3]). Hence, (2.23) follows. $\square$

H. Theorem 4 follows from Theorem 3 combined with a theorem of Kan and Thurston [9], as refined by Baumslag, Dyer and Heller [1]; see also Maunder [14]. Indeed, the "asphericalization" procedure of [9] and [1], applied to the complex Y of Theorem 3 gives a finite 3-dimensional aspherical polyhedron Z and a continuous map $t : Z \to Y$, which induces an epimorphism of the fundamental groups and such that for every local coefficient system A on Y, the induced map is an isomorphism $t_* : H_*(Z; t^*A) \simeq H_*(Y; A)$. We apply the above isomorphism in the case $A = \ell^2(\pi_1(Y))$. Since vanishing of unreduced L^2-homology $H_*(Y; \ell^2(\pi_1(Y))$ is equivalent to vanishing of the extended L^2-homology $\mathcal{H}_*(Y; \ell^2(\pi_1(Y))$, we obtain that $\mathcal{H}_p(Z; \ell^2(\pi/H)) = 0$ for all p, where $\pi = \pi_1(Z)$ and H denotes the kernel of $t_* : \pi_1(Z) \to \pi_1(Y)$. $\square$

3. Appendix: The Künneth theorem for extended L^2-cohomology

1. A *Hilbert category* $\mathcal{C}$ is defined as an additive subcategory of the category of Hilbert spaces and bounded linear maps, such that for any morphism $f : H \to H'$ of $\mathcal{C}$ the inclusion $\ker(f) \subset H$ belongs to $\mathcal{C}$ and also the adjoint map $f^* : H' \to H$ belongs to $\mathcal{C}$; cf. [4]. It is shown in [4] that any Hilbert category can be canonically embedded into an abelian category $\mathcal{E}(\mathcal{C})$, called *the extended abelian category.*

Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ be three Hilbert categories and let

$$\hat{\otimes} : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}'' \tag{3.1}$$

be a covariant functor of two variables (*the "tensor product"*) such that:

(a) For $H \in \mathrm{Ob}(\mathcal{C})$ and $H' \in \mathrm{Ob}(\mathcal{C}')$ the image $H \hat{\otimes} H'$ has as the underlying Hilbert space the tensor product of Hilbert spaces H and H';

(b) If $f : H \to H_1$ is a morphism of $\mathcal{C}$ and $f' : H' \to H'_1$ is a morphism of $\mathcal{C}'$ then $f \hat{\otimes} f' : H \hat{\otimes} H' \to H_1 \hat{\otimes} H'_1$ is the tensor product of bounded linear maps f and f'.

Recall that the tensor product if Hilbert spaces $H \hat{\otimes} H'$ is defined as the Hilbert space completion of the algebraic tensor product $H \otimes H'$ with respect to the following scalar product $\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \cdot \langle w, w' \rangle$.

Suppose that (C, d) and (C', d) are chain complexes in $\mathcal{C}$ and $\mathcal{C}'$ correspondingly. We assume that all chain complexes are graded by nonnegative integers and have a finite length. Their tensor product $(C, d) \hat{\otimes} (C', d)$ (defined in the usual way) is a chain complex in $\mathcal{C}''$. Also, $(C, d) \hat{\otimes} (C', d)$ is a projective chain complex in the abelian category $\mathcal{E}(\mathcal{C}'')$ and its extended homology $\mathcal{H}_*(C \hat{\otimes} C')$ is an object of the extended category $\mathcal{E}(\mathcal{C}'')$. Our purpose is to express the extended homology of $(C, d) \hat{\otimes} (C', d)$ in terms of the extended homology $\mathcal{H}_*(C)$ of (C, d) and $\mathcal{H}_*(C')$ of (C', d).

2. *Example.* Suppose that G and H are discrete groups. Let $\mathcal{C}_G$ denote the category of Hilbert representations of G. Recall, that an object of $\mathcal{C}_G$ is a Hilbert space with a unitary G-action which can be continuously and G-equivariantly embedded into a finite direct sum $\ell^2(G) \oplus \ldots \oplus \ell^2(G)$; morphisms of $\mathcal{C}_G$ are bounded linear maps commuting with the G-action. Then we have the tensor product functor

$$\hat{\otimes} : \mathcal{C}_G \times \mathcal{C}_H \to \mathcal{C}_{G\times H} \tag{3.2}$$

which is of primary interest to us.

3. *Tensor and periodic products.* A tensor product (3.1) defines two bifunctors $\mathcal{E}(\mathcal{C}) \times \mathcal{E}(\mathcal{C}') \to \mathcal{E}(\mathcal{C}'')$, which we now describe. Let $\mathcal{X} = (\alpha : A' \to A) \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$ and $\mathcal{Y} = (\beta : B' \to B) \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}'))$ be two objects with α and β injective. Consider the following chain complex in $\mathcal{C}''$:

$$0 \to A'\hat{\otimes}B' \overset{\begin{pmatrix} -1\hat{\otimes}\beta \\ \alpha\hat{\otimes}1 \end{pmatrix}}{\longrightarrow} (A'\hat{\otimes}B) \oplus (A\hat{\otimes}B') \overset{(\alpha\hat{\otimes}1,\, 1\hat{\otimes}\beta)}{\longrightarrow} A \otimes B \to 0. \tag{3.3}$$

In other words, we view the objects $\mathcal{X}$ and $\mathcal{Y}$ as chain complexes of length 1 and then (3.3) is the tensor product of these chain complexes. The extended homology of (3.3) in dimension 0 will be called *the tensor product of $\mathcal{X}$ and $\mathcal{Y}$*:

$$\mathcal{X}\hat{\otimes}\mathcal{Y} = ((\alpha\hat{\otimes}1, 1\hat{\otimes}\beta) : (A'\hat{\otimes}B) \oplus (A\hat{\otimes}B') \to A\hat{\otimes}B). \tag{3.4}$$

The extended homology of (3.3) in dimension 1 will be called *the periodic product of $\mathcal{X}$ and $\mathcal{Y}$*:

$$\mathcal{X} * \mathcal{Y} = \left(\begin{pmatrix} -1\hat{\otimes}\beta \\ \alpha\hat{\otimes}1 \end{pmatrix} : A'\hat{\otimes}B' \to Z\right), \tag{3.5}$$

where

$$Z = \ker\left[\begin{pmatrix} -1\hat{\otimes}\beta \\ \alpha\hat{\otimes}1 \end{pmatrix} : (A'\hat{\otimes}B) \oplus (A\hat{\otimes}B') \to A\hat{\otimes}B)\right]. \tag{3.6}$$

It is easy to see that $\mathcal{X}\hat{\otimes}\mathcal{Y}$ and $\mathcal{X}*\mathcal{Y}$ are covariant functors of two variables.

Proposition 5. *Let $\hat{\otimes} : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be a tensor product functor* (3.1). *Let $\mathcal{X} \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}))$ and $\mathcal{Y} \in \mathrm{Ob}(\mathcal{E}(\mathcal{C}'))$. Then*

(a) *$\mathcal{X}\hat{\otimes}\mathcal{Y}$ is projective if both $\mathcal{X}$ and $\mathcal{Y}$ are projective*;

(b) *$\mathcal{X} * \mathcal{Y} = 0$ if $\mathcal{X}$ or $\mathcal{Y}$ is projective*;

(c) *$\mathcal{X}\hat{\otimes}\mathcal{Y}$ is torsion if $\mathcal{X}$ or $\mathcal{Y}$ is torsion*;

(d) *If $\mathcal{C}''$ is a finite von Neumann category then $\mathcal{X} * \mathcal{Y}$ is torsion for any $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. Statements (a) and (b) follow directly from the definitions.

We prove (c) assuming that $\mathcal{X} = (\alpha : A' \to A)$ is torsion, i.e. $\operatorname{im} \alpha \subset A$ is dense. From the definition of the tensor product $\hat{\otimes}$ it follows then that the image of $\alpha \hat{\otimes} 1 : A' \hat{\otimes} B \to A \hat{\otimes} B$ is dense and hence from (3.4) we see that $\mathcal{X} \hat{\otimes} \mathcal{Y}$ is torsion.

It is enough to prove (d), assuming that both $\mathcal{X}$ and $\mathcal{Y}$ are torsion. Let $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ with α and β injective and with dense images. Then A' is isomorphic to A and B' is isomorphic to B (cf. [4, §2]). Therefore (d) will follow if we can show that Z (given by (3.6)) is isomorphic to $A \hat{\otimes} B$. The projection of Z on the first coordinate gives a morphism $Z \to A' \hat{\otimes} B$ which is injective (obviously) and has a dense image (this follows from the proposition in §2 of [4]). Hence we obtain (using the lemma in §2 of [4]) that Z is isomorphic to $A' \hat{\otimes} B \simeq A \hat{\otimes} B$. $\square$

THEOREM 6 (The Künneth formula). *Extended homology $\mathcal{H}_*(C \hat{\otimes} C')$ of a tensor product, where (C, d) is a chain complex in $\mathcal{C}$ and (C', d) is a chain complex in $\mathcal{C}'$, gives the equality*

$$\mathcal{H}_n(C \hat{\otimes} C') = \bigoplus_{i+j=n} \mathcal{H}_i(C) \hat{\otimes} \mathcal{H}_j(C') \oplus \bigoplus_{i+j=n-1} \mathcal{H}_i(C) * \mathcal{H}_j(C'). \tag{3.7}$$

Proof. Let $Z_i \subset C_i$ and $Z'_i \subset C'_i$ denote the subspaces of cycles. We have the decomposition $C_i = Z_i \oplus Z_i^\perp$; the boundary homomorphism vanishes on Z_i and maps $Z_i^\perp$ into Z_{i-1}. We denote by D_i the short chain complex $D_i = (d : Z_{i+1}^\perp \to Z_i)$, where Z_i stands in degree i and $Z_{i+1}^\perp$ stands in degree $i + 1$. Then $C \simeq \bigoplus_{i=0}^\infty D_i$; i.e., C is isomorphic to the direct sum of the chain complexes D_i.

Similarly, we define chain complexes $D'_j = (d : {Z'_{j+1}}^\perp \to Z'_j)$ and $C' \simeq \bigoplus_{j=0}^\infty D'_j$. Hence we obtain

$$C \hat{\otimes} C' \simeq \bigoplus_{i,j} (D_i \hat{\otimes} D'_j), \quad \mathcal{H}_n(C \hat{\otimes} C') = \bigoplus_{i,j} \mathcal{H}_n(D_i \hat{\otimes} D'_j). \tag{3.8}$$

Now we observe that D_i has nontrivial homology only in dimension i and $\mathcal{H}_i(D_i) = \mathcal{H}_i(C)$; similarly, D'_j has nontrivial homology only in dimension j and $\mathcal{H}_j(D'_j) = \mathcal{H}_j(C')$. Therefore $D_i \hat{\otimes} D'_j$ has nontrivial homology only in dimensions $i + j$ and $i + j + 1$, and

$$\mathcal{H}_{i+j}(D_i \hat{\otimes} D'_j) = \mathcal{H}_i(C) \hat{\otimes} \mathcal{H}_j(C'), \quad \mathcal{H}_{i+j+1}(D_i \hat{\otimes} D'_j) = \mathcal{H}_i(C) * \mathcal{H}_j(C')$$

according to our definition of the tensor and periodic products. Formula (3.7) now follows. $\square$

Theorem 7 (The Künneth formula for extended L^2-homology). *Let X, X' be finite cell complexes with $\pi = \pi_1(X)$, $\pi' = \pi_1(X')$. Then*

$$\begin{aligned}\mathcal{H}_n(X \times X'; \ell^2(\pi \times \pi')) \\ &\simeq \bigoplus_{i+j=n} \mathcal{H}_i(X; \ell^2(\pi)) \hat{\otimes} \mathcal{H}_j(X'; \ell^2(\pi')) \\ &\oplus \bigoplus_{i+j=n-1} \mathcal{H}_i(X; \ell^2(\pi)) * \mathcal{H}_j(X'; \ell^2(\pi')),\end{aligned} \tag{3.9}$$

where the tensor and periodic products are understood with respect to functor (3.2).

Proof. Let $C_*(\tilde{X})$ and $C_*(\tilde{X}')$ be the cell chain complexes of the universal coverings $\tilde{X}$ and $\tilde{X}'$. We apply the previous theorem to chain complexes $C = \ell^2(\pi)\tilde{\otimes}_\pi C_*(\tilde{X})$ and $C' = \ell^2(\pi')\tilde{\otimes}_{\pi'} C_*(\tilde{X}')$. Note that C is a chain complex in category $\mathcal{C}_\pi$ (cf. example above) and $\mathcal{H}_n(C) = \mathcal{H}_n(X; \ell^2(\pi))$. Similarly C' is a chain complex in $\mathcal{C}_{\pi'}$ and $\mathcal{H}_n(C') = \mathcal{H}_n(X'; \ell^2(\pi'))$. Formula (3.9) follows from (3.7) using the isomorphism $\ell^2(\pi)\hat{\otimes}\ell^2(\pi') = \ell^2(\pi \times \pi')$ and the fact that the chain complex $C_*(\tilde{X}) \otimes_{\mathbf{Z}} C_*(\tilde{X}')$ over $\mathbf{Z}[\pi \times \pi']$ is isomorphic to $C_*(\widetilde{X \times X'})$, where we consider the obvious product cell structure on $X \times X'$. $\square$

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