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Geometry of growth: approximation theorems for L^2 invariants

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0. Introduction

0.1. In 1994 Wolfgang Lück [L] proved a beautiful theorem stating that von Neumann Betti numbers of the universal covering of a finite polyhedron can be found as the limits of the normalized Betti numbers of finitely sheeted normal coverings. Before Lück it was only known that there is an inequality (called Kazhdan's inequality [Ka], cf. also Gromov [Gr], pages 13 and 153).

One of the goals of the present paper is to generalize the Lück's theorem in two directions. First, instead of finitely sheeted normal coverings we consider flat vector bundles of finite dimension. Secondly, instead of L^2-Betti numbers we study the von Neumann dimensions of the homology of infinite dimensional flat bundles determined by unitary representations in a von Neumann category with a trace.

The other main purpose of this paper is to investigate the situations, when the statement of the Lück's theorem in its original form is incorrect. We show that the correcting additional term has a very interesting meaning (*the torsion dimension*); it can be understood in the framework of the formalism of extended cohomology and von Neumann categories. As examples in the paper show, vanishing of this correcting term happens in fact rarely, under very special arithmetic assumptions.

0.2. In order to illustrate our results, we formulate here three approximation theorems, dealing with the towers of coverings and the L^2-Betti numbers, which are

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corollaries of the main Theorem 9.2 below. The first (Theorem 0.3) generalizes the Lück's theorem by admitting towers of non-normal finitely sheeted coverings. The second (Theorem 0.4) generalizes Theorem 0.3 by allowing twisted coefficients; here we impose some important restrictions coming from the algebraic number theory.

0.3. Theorem. *Let π be an infinite discrete group and let $\pi \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ be a sequence of subgroups of finite index. For any $k = 1, 2, \dots$ denote by n_k the total number of the subgroups of π, conjugate to Γ_k; given $g \in \pi$, we denote by $n_k(g)$ the number of subgroups conjugate to Γ_k, containing g. Suppose, that for $g \neq 1$ holds*

$$\lim_{k\to\infty} \frac{n_k(g)}{n_k} = 0. \tag{0-1}$$

For any finite polyhedron X with $\pi_1(X) = \pi$, consider $[\pi : \Gamma_k]$-sheeted coverings $\tilde{X}_k \to X$ corresponding to the subgroups $\Gamma_k \subset \pi$, where $k = 1, 2, \dots$. Then the sequence of the normalized Betti numbers

$$\lim_{k\to\infty} \frac{\dim H_i(\tilde{X}_k)}{[\pi : \Gamma_k]} = b_i^{(2)}(X) \tag{0-2}$$

converges to the L^2-Betti number $b_i^{(2)}(X)$.

Proof of Theorem 0.3 will be given in Sect. 9.

Note that the condition (0-1) of Theorem 0.3 implies that π is residually finite: for fixed k denote by $P_k \subset \pi$ the intersection of all the subgroups of π, conjugate to Γ_k; then because of (0-1) we have $\cap P_k = \{1\}$.

Assuming that all subgroups $\Gamma_k \subset \pi$ are normal, Theorem 0.3 reduces to the theorem of Lück.

Here is another generalization of the theorem of Lück:

0.4. Theorem. *Let π be an infinite discrete group and let $\pi \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ be a sequence of normal subgroups of finite index such that the intersection $\cap\Gamma_k = \{1\}$ is trivial. Let*

$$\rho : \pi \to \mathrm{Mat}(m \times m, \mathfrak{o}) \tag{0-3}$$

be a unitary representation, where $\mathfrak{o}$ denotes the ring of algebraic integers in an algebraic number field $\mathscr{F} \subset \mathbb{C}$. We assume that $\mathscr{F}$ comes imbedded into $\mathbb{C}$ such that it is invariant under the complex conjugation and we consider the induced involution on $\mathscr{F}$ and on $\mathfrak{o}$. For any finite polyhedron X with $\pi_1(X) = \pi$, consider the normal covering $\tilde{X}_k \to X$, where $k = 1, 2, \dots$, corresponding to the subgroup Γ_k and denote by V^k the flat vector bundle over $\tilde{X}_k$, determined by the representation ρ, restricted onto Γ_k. Then the sequence

$$\lim_{k\to\infty} \frac{\dim H_i(\tilde{X}_k, V^k)}{\dim V^k \cdot [\pi : \Gamma_k]} = b_i^{(2)}(X) \tag{0-4}$$

converges to the L^2-Betti number $b_i^{(2)}(X)$ of X.

The proof of Theorem 0.4 will be given in 9.3; it follows from a more general Theorem 9.2, dealing with sequences of flat bundles, satisfying some arithmeticity conditions, and such that their normalized characters converge to the character of a unitary representation in a von Neumann category. The properties of arithmetic approximation include an important condition on the sequence of Galois groups acting on the characters of the approximating sequence of representations; we show in Sect. 10 that the theorem becames false, if this condition is violated.

It is interesting to emphasize that under the conditions of arithmetic approximation the dimensions of the flat bundles, approximating a von Neumann flat bundle, have to tend to infinity.

Here is another corollary of Theorem 9.2, which we prove in Sect. 9:

0.5. Theorem. *Let X be a finite polyhedron, and let $\rho : \pi_1(X) \to \mathrm{Mat}(m \times m, \mathfrak{o}_{\mathscr{F}})$ be a unitary representation, where $\mathscr{F} \subset \mathbb{C}$ denotes a cyclotomic field and $\mathfrak{o}_{\mathscr{F}} \subset \mathscr{F}$ denotes its ring of algebraic integers. Suppose that ρ is injective and its image has trivial intersection with the the center of the matrix algebra* $\mathrm{Mat}(m \times m, \mathfrak{o}_{\mathscr{F}})$*. Let $\mathscr{E} \to X$ denote the flat vector bundle of rank m determined by the representation ρ. Then*

$$\lim_{k\to\infty} \frac{\dim H_i(X, \mathscr{E}^{\otimes k})}{(\dim \mathscr{E})^k} = b_i^{(2)}(X). \tag{0-5}$$

A more general statements of this type can be found in Sect. 9, cf. Theorem 9.6.

0.6. In this paper we use the language of von Neumann categories, which provides a natural environment for developing the L^2-homology theory, cf. [F]. We review this material briefly in Sect. 1. Traces on von Neumann categories play an important role; the traces allow to assign dimensions to objects of the von Neumann category, which generalize the von Neumann dimension.

Given a polyhedron X and a representation of the fundamental group π of X on an object of a von Neumann category with a trace, they determine *a character* on π. It is a class function $\chi : \pi \to \mathbb{C}$, which satisfies certain positivity condition, cf. Sect. 7. We show here that knowing this character as the only information about the representation allows to find the von Neumann Betti numbers and the spectral density function of the extended L^2-homology. Conversely, we show that one may construct von Neumann categories with traces starting from class functions on the fundamental group π.

The problem of describing the behavior of the L^2-invariants under deformations of the von Neumann representation, seems to be of central importance. For example, one wants to approximate von Neumann representations by finite dimensional ones (as in the Lück's theorem). Since the character of a von Neumann representation determines completely the most important L^2-homological invariants, we study situations, when we have a sequence of finite dimensional representations with the property that their normalized characters converge (pointwise, i.e. as functions on the group π) to the character of the given infinite dimensional representation. Our aim is to find the homological (spectral) invariants corresponding to the infinite dimensional representation in terms of the approximating finite dimensional family; this seems to be a natural generalization of the situation studied by Lück [L].

0.7. It turns out that any approximating sequence of finite dimensional representations can also be treated as a single representation is a finite von Neumann category. Moreover, this von Neumann category admits *a Dixmier type* (i.e. not normal) *trace*; the construction of this trace uses universal summation machines of von Neumann [vN]. Note that Dixmier type traces play a very important role in the noncommutative geometry of A. Connes [C]. We show in Sect. 2, that not normal traces allow to define a dimension type function for the torsion objects of the extended category. We call this function *the torsion dimension*. Its main property is that it determines a non-trivial homomorphism on the Grothendieck group of the torsion subcategory.

This von Neumann category allows to study *the growth processes* - families of finite dimensional chain complexes. A sequence of flat bundles over a finite polyhedron (more precisely, the corresponding sequence of the chain complexes) is an instance of a growth process. Any growth process defines its asymptotic invariants: the projective dimension, the torsion dimension, and the spectral density function. As another geometrically interesting example of growth processes we may mention the sequence of choppings (exhaustion) of a non-compact Riemannian manifold.

0.8. In the most general approximation theorems established in Sect. 8 (cf. Theorems 8.2, 8.3), we find that the torsion dimension of the extended homology appears as the additional correcting term. In many cases one may expect the torsion dimension to be independent of the choice of the summation machine ω, which is a part of the Dixmier type trace. We show that such independence happens in the analytic situation (Theorem 8.4). We also analyse examples showing that sometimes one may realize a sequence of approximating Betti numbers by an arbitrary sequence consisting of 0's and 2's, cf. 6.3.

However, if we want to guarantee vanishing of the torsion dimension in the general approximation theorem 8.2, we have to impose some assumptions from algebraic number theory. The idea of *integrality* is also very important in the original Lück's theorem. We develop this idea further, by allowing representations over the algebraic integers of algebraic number fields; this adds flexibility and makes possible many interesting applications.

0.9. Finally, I want to mention an approximation theorem of a different type; it is Theorem 11.1. Here we assume that the fundamental group admits a chain of normal subgroups with index being a power of p, where p is a fixed prime number. We show that the Betti numbers over the finite field $\mathbb{F}_p$ behave *in a monotone fashion*; this produces an inequality

$$b_i^{(2)}(X) \leq \dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p) \tag{0-6}$$

between the L^2-Betti numbers and the usual $\mathbb{F}_p$-homology, cf. Corollary 11.2.

0.10. This paper was written while the author was visiting IHES in Bures-sur-Yvette (France); I am very thankful to the IHES for hospitality.

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1. A review of extended L^2-homology, von Neumann categories, and traces

Intuitively, the extended homology provides a rigorous formalism to study a homology theory based on the (usual) infinite L^2-cycles together with the "cycles" of the form

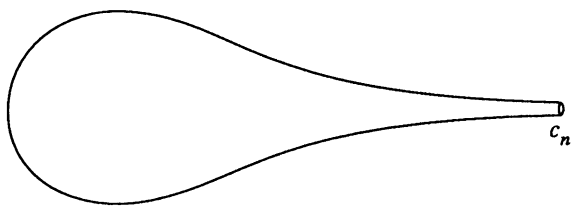

Fig. 1

More precisely, we study geometry of non-compact manifolds or flat infinite dimensional bundles over compact manifolds; the cycle on the Fig. 1 above represents in fact a sequence of cycles c_n, where $n = 1, 2, \dots$ such that each c_n is a boundary, but the *size* of a minimal chain, spanned by c_n is much greater (*asymptotically*) than the volume of c_n.

A precise definition of the extended L^2 homology uses a generalization of the notion of Hilbert space – the functor of extended homology assigns to a manifold such generalized Hilbert space. It turns out that the familiar category of Hilbert spaces is not good enough; we complete it by adding *"torsion Hilbert spaces"*, such that the obtained category becomes an abelian category. In order to obtain a good category and to include some interesting applications, it is reasonable to study this construction of abelian extension starting from a von Neumann category.

In this section we will give a brief review of the notion of von Neumann category, the extended abelian categories, and traces, which will be used in the rest of the paper. In full detail all this material is described in [F].

1.1. von Neumann categories. Let $\mathscr{A}$ be an algebra over $\mathbb{C}$ having an involution which will be denoted by the star $*$. *A Hilbert representation of $\mathscr{A}$ (or a Hilbert module)* is a Hilbert space $\mathscr{H}$ supplied with a left action of $\mathscr{A}$ on $\mathscr{H}$ by bounded linear maps such that for any $a \in \mathscr{A}$ holds

$$\langle ax, y \rangle = \langle x, a^* y \rangle \tag{1-1}$$

for all $x, y \in \mathscr{H}$. *A morphism* between Hilbert representations $\phi : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear map commuting with the action of the algebra $\mathscr{A}$. We obtain the additive category of all Hilbert representations of a given $*$-algebra $\mathscr{A}$.

Assume that $\mathscr{C}_{\mathscr{A}}$ is an additive subcategory of the category of all Hilbert representations of $\mathscr{A}$. We say that $\mathscr{C}_{\mathscr{A}}$ is *a von Neumann category* if the following properties are verified:

(i) *The kernel of any morphism* $\phi : \mathscr{H}_1 \to \mathscr{H}_2$ *in* $\mathscr{C}_{\mathscr{A}}$ *and the natural inclusion* $\ker\phi \to \mathscr{H}_1$ *belong to* $\mathscr{C}_{\mathscr{A}}$.
(ii) For any morphism $\phi : \mathscr{H}_1 \to \mathscr{H}_2$ *of* $\mathscr{C}_{\mathscr{A}}$ *the adjoint operator* $\phi^* : \mathscr{H}_2 \to \mathscr{H}_1$ *is also a morphism of* $\mathscr{C}_{\mathscr{A}}$.
(iii) *for any pair of representations* $\mathscr{H}_1, \mathscr{H}_2 \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$, *the corresponding set of morphisms* $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}_1, \mathscr{H}_2)$ *is a weakly closed subspace in the space of all bounded linear operators between* $\mathscr{H}_1$ *and* $\mathscr{H}_2$.

Note, that for any object $\mathscr{H} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ of a von Neumann category the set of endomorphisms $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$ is a von Neumann algebra.

1.2. Finite objects. We will say that an object $\mathscr{H} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ of a von Neumann category *is finite* if any closed $\mathscr{C}_{\mathscr{A}}$-submodule $\mathscr{H}_1 \subset \mathscr{H}$ which is isomorphic to $\mathscr{H}$ in $\mathscr{C}_{\mathscr{A}}$, coincides with $\mathscr{H}$.

This property is equivalent to the requirement that the von Neumann algebra $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$ of endomorphisms of $\mathscr{H}$ is finite. Cf. [Di], part III, chapter 8, Sect. 1.

A von Neumann category $\mathscr{C}_{\mathscr{A}}$ is called *finite* if all its objects are finite.

1.3. Trace and dimension. Let $\mathscr{C}_{\mathscr{A}}$ be a von Neumann category.

Definition. *A trace on category* $\mathscr{C}_{\mathscr{A}}$ *is a function, denoted* tr, *which assigns to each object* $\mathscr{H} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ *a finite, non-negative trace*

$$\mathrm{tr}_{\mathscr{H}} : \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H}) \to \mathbb{C} \tag{1-2}$$

on the von Neumann algebra $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$*; in other words* $\mathrm{tr}_{\mathscr{H}}$ *assumes (finite) values in* $\mathbb{C}$, $\mathrm{tr}_{\mathscr{H}}(a)$ *is non-negative on positive elements* a *of* $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$, *and* $\mathrm{tr}_{\mathscr{H}}$ *is traceful, i.e.* $\mathrm{tr}_{\mathscr{H}}(ab) = \mathrm{tr}_{\mathscr{H}}(ba)$, *for* $a, b \in \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$. *It is also assumed that for any pair of representations* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *the corresponding traces* $\mathrm{tr}_{\mathscr{H}_1}$, $\mathrm{tr}_{\mathscr{H}_2}$ *and* $\mathrm{tr}_{\mathscr{H}_1\oplus\mathscr{H}_2}$ *are related as follows: if* $f \in \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}_1 \oplus \mathscr{H}_2, \mathscr{H}_1 \oplus \mathscr{H}_2)$ *is given by a* 2×2 *matrix*(f_{ij}), *where* $f_{ij} : \mathscr{H}_i \to \mathscr{H}_j$, $i, j = 1, 2$, *then*

$$\mathrm{tr}_{\mathscr{H}_1\oplus\mathscr{H}_2}(f) \ = \ \mathrm{tr}_{\mathscr{H}_1}(f_{11}) + \mathrm{tr}_{\mathscr{H}_2}(f_{22}). \tag{1-3}$$

For the notion of positive elements of the von Neumann algebra $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$ we refer to [T], page 24.

We will say that a trace tr on a von Neumann category is *normal* iff for each non-zero $\mathscr{H} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ the trace $\mathrm{tr}_{\mathscr{H}}$ on the von Neumann algebra $\mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H}, \mathscr{H})$ is normal. Recall that this means that

$$\sup_i\{\mathrm{tr}_{\mathscr{H}}(a_i)\} = \mathrm{tr}_{\mathscr{H}}(\sup_i\{a_i\}) \tag{1-4}$$

for any bounded increasing net $a_i \in \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{H},\mathscr{H})$ consisting of positive operators; cf. [T], page 309.

Given a trace tr on a category $\mathscr{C}_{\mathscr{A}}$, one can define the following *dimension function*:

$$\dim \mathscr{H} = \dim_{\mathrm{tr}} \mathscr{H} = \mathrm{tr}_{\mathscr{H}}(\mathrm{id}_{\mathscr{H}})\ . \tag{1-5}$$

The real number $\dim_{\mathrm{tr}} \mathscr{H}$ is called *the von Neumann dimension (or the projective dimension) of* $\mathscr{H}$ *with respect to the trace* tr.

1.4. The abelian extension. Given a von Neumann category $\mathscr{C}_{\mathscr{A}}$, there exists a bigger category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$, which is abelian and which contains $\mathscr{C}_{\mathscr{A}}$ as a full subcategory. The construction of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ was suggested in [F1], [F] using ideas of P. Freyd [Fr].

An object of the category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ is defined as a morphism $(\alpha : A' \to A)$ in the category $\mathscr{C}_{\mathscr{A}}$. Given a pair of objects $\mathscr{X} = (\alpha : A' \to A)$ and $\mathscr{Y} = (\beta : B' \to B)$ of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$, a *morphism* $\mathscr{X} \to \mathscr{Y}$ in the category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ is an equivalence class of morphisms $f : A \to B$ of category $\mathscr{C}_{\mathscr{A}}$ such that $f \circ \alpha = \beta \circ g$ for some morphism $g : A' \to B'$ in $\mathscr{C}_{\mathscr{A}}$. Two morphisms $f : A \to B$ and $f' : A \to B$ of $\mathscr{C}_{\mathscr{A}}$ represent *identical morphisms* $\mathscr{X} \to \mathscr{Y}$ *of* $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ iff $f - f' = \beta \circ F$ for some morphism $F : A \to B'$ of category $\mathscr{C}_{\mathscr{A}}$. This defines an equivalence relation. The morphism $\mathscr{X} \to \mathscr{Y}$, represented by $f : A \to B$, is denoted by

$$[f] : (\alpha : A' \to A) \ \to\ (\beta : B' \to B) \quad \text{or by} \quad [f] : \mathscr{X} \to \mathscr{Y}. \tag{1-6}$$

The *composition* of morphisms is defined as the composition of the corresponding morphisms f in the category $\mathscr{C}_{\mathscr{A}}$.

1.5. Embedding of $\mathscr{C}_{\mathscr{A}}$ into $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$. Given an object $A \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ one defines the following object $(0 \to A) \in \mathrm{ob}(\mathscr{E}(\mathscr{C}_{\mathscr{A}}))$ of the extended category. Since any morphism $f : A \to B$ determines a morphism $[f] : (0 \to A) \to (0 \to B)$ in the extended category, we obtain a full embedding $\mathscr{C}_{\mathscr{A}} \to \mathscr{E}(\mathscr{C}_{\mathscr{A}})$.

It is possible to characterize the objects of the extended category which are isomorphic in $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ to objects coming from $\mathscr{C}_{\mathscr{A}}$ in intrinsic terms. Namely, *an object* $\mathscr{X} \in \mathrm{ob}(\mathscr{E}(\mathscr{C}_{\mathscr{A}}))$ *is projective if and only if it is isomorphic in* $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ *to an object of the form* $(0 \to A)$*, where* $A \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$

1.6. The torsion subcategory. An object $\mathscr{X} = (\alpha : A' \to A)$ of the extended category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ is called *torsion* iff the image of α is dense in A.

We will denote by $\mathscr{T}(\mathscr{C}_{\mathscr{A}})$ the full subcategory of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ generated by all torsion objects. $\mathscr{T}(\mathscr{C}_{\mathscr{A}})$ is called *the torsion subcategory of* $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$. If $\mathscr{C}_{\mathscr{A}}$ is a finite von Neumann category, then the torsion subcategory $\mathscr{T}(\mathscr{C}_{\mathscr{A}})$ is an abelian subcategory of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$.

Given an arbitrary object $\mathscr{X} = (\alpha : A' \to A)$ of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ one considers the following torsion object $T(\mathscr{X}) = (\alpha : A' \to \mathrm{cl}(\mathrm{im}(\alpha)))$ which is called *the* *torsion part of* $\mathscr{X}$. There is an obvious monomorphism $T(\mathscr{X}) \to \mathscr{X}$. The factor $P(\mathscr{X}) = \mathscr{X}/T(\mathscr{X})$ is projective, called *the projective part of* $\mathscr{X}$. We have $\mathscr{X} = T(\mathscr{X}) \oplus P(\mathscr{X})$. Thus, *the isomorphism type of an object of the extended category* $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ *is determined by the isomorphism types of its projective and torsion parts.*

1.7. Novikov-Shubin invariants. Given a trace on a von Neumann category $\mathscr{C}_{\mathscr{A}}$, one obtains the numerical invariant $\mathfrak{ns}(\mathscr{X})$ of torsion objects, called the Novikov - Shubin invariant. We refer to [F], Sect. 3.9, where it is described. There exist also other invariants of torsion objects, independent of the Novikov - Shubin invariant, cf. [F1].

In the next section we will define new numerical invariant of torsion objects, which is sometimes more convenient.

1.8. Extended homology. The functor of extended homology is constructed as follows, cf. [F], [F1]. Suppose that X is a finite polyhedron with fundamental group π. Let $\mathscr{C}_{\mathscr{A}}$ be a von Neumann category, and let $\rho : \pi \to \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$ be a representation, where $\mathscr{M} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$. Consider the chain complex $C_*(\tilde{X})$ (the cellular chain complex of the universal covering $\tilde{X}$). Then

$$\mathscr{M} \otimes_{\pi} C_*(\tilde{X})$$

is a chain complex in category $\mathscr{C}_{\mathscr{A}}$. Thus, it lies in the abelian category $\mathscr{E}(\mathscr{C}_{\mathscr{A}}))$ and its homology (calculated in $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$, called *extended* L^2 *homology of* X *with coefficients in* $\mathscr{M}$) is denoted by $\mathscr{H}_*(X, \mathscr{M})$. Being an object of $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$, it is a direct sum of its projective and torsion parts. The projective part of the extended homology coincides with the reduced L^2 homology, cf. [A] (defined by dividing the space of infinite L^2 chains by the closure of L^2 boundaries). The torsion part of the extended homology is responsible for the “almost cycles” or “asymptotic cycles” as the one shown on Fig. 1.

2. Torsion dimension

In this section we define a new numerical invariant of torsion objects, which we call torsion dimension. It behaves in better way, than the known invariants (such as the Novikov-Shubin invariant and the minimal number of generators, introduced in [F1]). We will use the torsion dimension in the next section to study the Grothendieck group of torsion objects. Also, we will use the torsion dimension in approximation theorems for L^2 topological invariants, cf. Theorems 8.2 and 8.4, where it produces a correcting additional term.

Everywhere in this section $\mathscr{C}_{\mathscr{A}}$ will denote a finite von Neumann category. We will assume that we have a fixed trace tr on $\mathscr{C}_{\mathscr{A}}$, cf. Subsect. 1.3. We will *not* assume that the trace tr is normal, since in the case of normal traces the torsion dimension is always zero. Also, the trace tr is not supposed to be faithful.

Not normal traces are usually called *Dixmier type traces*, cf. [C], since J. Dixmier [D] was the first who constructed such traces. Dixmier type traces play very important role in the non-commutative geometry of A. Connes [C].

2.1. First we will show that any non-normal trace determines *a dimension function* of torsion objects. We will see that it behaves sub-additively under extensions.

Let $\mathscr{X} = (\alpha : A' \to A)$ be a torsion object of the extended category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$ and let $F(\lambda)$ be its spectral density function with respect to the trace tr, cf. [F], formula (3-12).

Definition. We will define *the torsion dimension of $\mathscr{X}$ (with respect to the trace* tr*)* as the following real number

$$\mathfrak{tordim}\,\mathscr{X} = \mathfrak{tordim}_{\mathrm{tr}}\,\mathscr{X} = \lim_{\lambda \to +0} F(\lambda). \tag{2-1}$$

Note that $F(\lambda)$ is increasing and so the limit exists.

We will also define the *reduced spectral density function* by

$$\tilde{F}(\lambda) = F(\lambda) - \mathfrak{tordim}\,\mathscr{X}. \tag{2-2}$$

Note, that if the trace tr is normal, then the torsion dimension $\mathfrak{tordim}\,\mathscr{X}$ is always zero.

2.2. Proposition. *The torsion dimension $\mathfrak{tordim}\,\mathscr{X}$ depends only on the isomorphism type of $\mathscr{X}$ as an object of the extended category. The reduced spectral density functions corresponding to isomorphic torsion objects $\mathscr{X}$ and $\mathscr{Y}$ are dilatationally equivalent.*

Proof. The proof of Proposition 3.8 in [F] does not use the assumption of normality of the trace. It shows that if $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic torsion objects of the extended category then the corresponding spectral density functions are dilatationally equivalent. This implies our statement. □

Now we will establish the following internal characterization of the torsion dimension. Let us recall that any trace tr on von Neumann category $\mathscr{C}_{\mathscr{A}}$ determines a dimension function on $\mathscr{C}_{\mathscr{A}}$, cf. 1.3.

2.3. Proposition. *Given a torsion object $\mathscr{X}$ of the extended category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$, its torsion dimension $\mathfrak{tordim}\,\mathscr{X}$, equals to the infimum of the von Neumann dimensions* $\dim P$ *(with respect to the trace* tr*) of projective objects P of $\mathscr{C}_{\mathscr{A}}$ such that there exists an epimorphism $P \to \mathscr{X}$.*

Proof. Suppose that $\mathscr{X} = (\alpha : A' \to A)$ and α is injective. Recall that the spectral density function $F(\lambda)$ is defined as follows. We consider the positive square root T of the equation $T^2 = \alpha^*\alpha$ and the spectral decomposition $T = \int_0^\infty \lambda dE_\lambda$. Then $F(\lambda)$ is the von Neumann dimension of the subspace $E_\lambda A'$.

Thus, for any $\lambda > 0$ the spectral projection E_λ determines a projective object $P = E_\lambda A'$ which has von Neumann dimension $F(\lambda)$ and which maps epimorphically onto $\mathscr{X}$. Indeed, the torsion object $\mathscr{X}_\lambda = (\alpha : E_\lambda A' \to \alpha(E_\lambda A'))$ is isomorphic to $\mathscr{X}$ and $E_\lambda A'$ and $\alpha(E_\lambda A')$ are isomorphic (by Lemma 2.3 of [F]. Therefore $\dim \mathscr{X} \geq \inf P$.

On the other hand, if P is projective and maps epimorphically onto $\mathscr{X}$ then $\mathscr{X}$ admits a representation of the form $(\gamma : P' \to P)$ with some P' and γ and thus we obtain (using Proposition 2.2) that $\mathfrak{tordim}\mathscr{X} \leq \inf P$. □

Now we will show that the torsion dimension is sub-additive for extensions.

2.4. Proposition. *For any short exact sequence*

$$0 \to \mathscr{X}' \to \mathscr{X} \to \mathscr{X}'' \to 0, \tag{2-3}$$

consisting of torsion objects of the extended abelian category $\mathscr{E}(\mathscr{C}_{\mathscr{A}})$*, holds*

$$\max\{\mathfrak{tordim}(\mathscr{X}'), \mathfrak{tordim}(\mathscr{X}'')\} \leq \mathfrak{tordim}(\mathscr{X}) \leq \mathfrak{tordim}(\mathscr{X}') + \mathfrak{tordim}(\mathscr{X}''). \tag{2-4}$$

Moreover, if the sequence (2-3) splits, then

$$\mathfrak{tordim}(\mathscr{X}) = \mathfrak{tordim}(\mathscr{X}') + \mathfrak{tordim}(\mathscr{X}''). \tag{2-5}$$

Proof. We will use the internal characterization of the torsion dimension given by Proposition 2.3. It is clear that if P' can be mapped epimorphically onto $\mathscr{X}'$ and P'' can be mapped epimorphically onto $\mathscr{X}''$, then their direct sum $P' \oplus P''$ can be mapped epimorphically onto $\mathscr{X}$. Thus we obtain the right side of inequality (2-4).

From Proposition 2.3 clearly follows that $\mathfrak{tordim}(\mathscr{X}) \geq \mathfrak{tordim}(\mathscr{X}'')$. Suppose now that $P \to \mathscr{X}$ is an epimorphism with P being a projective object of $\mathscr{C}_{\mathscr{A}}$. Let $P' \to P$ be the kernel of the composite $P \to \mathscr{X} \to \mathscr{X}''$. Then we have an epimorphism $P' \to \mathscr{X}'$. Observe that P' is isomorphic to P in $\mathscr{C}_{\mathscr{A}}$ by Lemma 2.3 of [F]; therefore $\dim P = \dim P'$. This proves that $\mathfrak{tordim}(\mathscr{X}) \geq \mathfrak{tordim}(\mathscr{X}')$.

The equality (2-5) obviously follows from the definitions. □

3. Grothendieck group of torsion objects

Note that equality (2-5) represents a very important distinction between the properties of two functions on isomorphism types of torsion objects - the torsion dimension, which we introduced above in Sect. 2, and the well known Novikov - Shubin invariant. Recall that the Novikov - Shubin invariant of a direct sum equals to the minimum of the Novikov - Shubin invariants of the summands:

$$\mathfrak{ns}(\mathscr{X}' \oplus \mathscr{X}'') = \min\{\mathfrak{ns}(\mathscr{X}'), \mathfrak{ns}(\mathscr{X}'')\}. \tag{3-1}$$

The advantage of (2-5) is that it implies that *the torsion dimension determines a homomorphism with values in* $\mathbb{R}$ *from the Grothendieck group constructed out* *of abelian category $\mathscr{T}(\mathscr{C}_{\mathscr{A}})$ of torsion objects in $\mathscr{C}_{\mathscr{A}}$.* Thus existence of a non-normal trace on $\mathscr{C}_{\mathscr{A}}$ implies non-triviality of the Grothendieck group. We will make all this precise in the following subsection.

3.1. Grothendieck group of torsion objects of $\mathscr{C}_{\mathscr{A}}$. We will denote by $K(\mathscr{T}(\mathscr{C}_{\mathscr{A}}))$ the Grothendieck group of the abelian category $\mathscr{T}(\mathscr{C}_{\mathscr{A}})$, cf. [K], page 53. Recall that $K(\mathscr{T}(\mathscr{C}_{\mathscr{A}}))$ is an abelian group generated by the symbols $[\mathscr{X}]$, one for each isomorphism type of torsion objects $\mathscr{X}$ in $\mathscr{C}_{\mathscr{A}}$, with the addition given by $[\mathscr{X}]+[\mathscr{Y}] = [\mathscr{X} \oplus \mathscr{Y}]$. The torsion dimension gives a well-defined homomorphism

$$\mathfrak{tordim} : K(\mathscr{T}(\mathscr{C}_{\mathscr{A}})) \to \mathbb{R} \tag{3-2}$$

(by Propositions 2.2 and 2.4).

3.2. Theorem. *If the given trace* tr *on the category $\mathscr{C}_{\mathscr{A}}$ is non-normal, then homomorphism (3-2) is non-trivial and thus the Grothendieck group $K(\mathscr{T}(\mathscr{C}_{\mathscr{A}}))$ is non-zero.*

Proof. If the trace is non-normal then we may find a sequence

$$\mathscr{H} = \mathscr{H}_1 \supset \mathscr{H}_2 \supset \mathscr{H}_3 \supset \dots,$$

where $\mathscr{H}$ is an object of $\mathscr{C}_{\mathscr{A}}$ and $\mathscr{H}_n$'s are its closed subobjects, such that $\cap \mathscr{H}_n = 0$ and $\lim \dim(\mathscr{H}_n) = c > 0$. Define the following projector valued function E_λ for $\lambda \in [0,1]$, by setting E_λ = the projection onto $\mathscr{H}_n$, for $(n+1)^{-1} < \lambda \le n^{-1}$. Then we consider the morphism

$$\alpha : \mathscr{H} \to \mathscr{H}, \quad \text{where} \quad \alpha = \int_0^1 \lambda dE_\lambda.$$

Then $\mathscr{X} = (\alpha : \mathscr{H} \to \mathscr{H})$ is a torsion object, and clearly $\mathfrak{tordim}(\mathscr{X}) = c > 0$. □

4. An example of von Neumann category with Dixmier type trace

Our purpose now is to describe the simplest example of a finite von Neumann category with a Dixmier type trace. This category will be important for our applications to the problem of approximation of L^2 invariants; we will see in Subsect. 4.8 that this category allows to describe geometry and topology of growth processes.

4.1. Fix a sequence of non-negative real numbers $\mu = (\mu^n)$, $\mu^n > 0$, $\mu^n \in \mathbb{R}$. We will call μ *the growth rate*. Normally, we will have in our applications μ^n tending to 0, or being constant. The von Neumann category, we are going to construct will depend on this choice; we will denote it $\mathscr{C}(\mu)$.

Objects of the category $\mathscr{C}(\mu)$ are sequences $\mathscr{V} = (V^n)$ of finite dimensional Euclidean spaces, where n runs over non-negative integers, such that the growth rate of the dimension of V^n is bounded above by the given sequence μ:

$$\dim V^n = O((\mu^n)^{-1}). \tag{4-1}$$

In other words, we assume that the product $\mu^n \cdot \dim V^n$ is bounded. Note that each V^n *is Euclidean*, i.e. it is supplied with a scalar product.

Each object $\mathscr{V}$ of $\mathscr{C}(\mu)$ determines *a Hilbert space* $\mathscr{H}_{\mathscr{V}}$, where

$$\mathscr{H}_{\mathscr{V}} = \{v = (v^n), v^n \in V^n; \sum ||v^n||^2 < \infty\}. \tag{4-2}$$

Here the norm $||v^n||$ denotes the norm of the space V^n.

A morphism $f : \mathscr{V} \to \mathscr{W}$ in $\mathscr{C}(\mu)$, where $\mathscr{V} = (V^n)$ and $\mathscr{W} = (W^n)$, is a sequence $f = (f^n)$, where $f^n : V^n \to W^n$ is a linear map, such that there exists a common upper bound

$$||f^n|| \leq M \tag{4-3}$$

(M is independent of n). Any morphism $f : \mathscr{V} \to \mathscr{W}$ of $\mathscr{C}(\mu)$ clearly induces a bounded linear map of the corresponding Hilbert spaces, which we denote $f : \mathscr{H}_{\mathscr{V}} \to \mathscr{H}_{\mathscr{W}}$. Now one checks easily, that all properties from the definition of von Neumann category (cf. 1.1) are satisfied. The algebra $\mathscr{A}$ in this case is $\mathscr{A} = \mathbb{C}$.

The category $\mathscr{C}(\mu)$ is clearly *finite*, cf. 1.2.

4.2. Now we will describe a Dixmier type trace on $\mathscr{C}(\mu)$. First, we recall from [D] and [C], page 305, that there exists a linear form Lim_ω (invented by J. von Neumann [vN]) on the space $\ell^\infty(\mathbb{N})$ of bounded sequences of complex numbers, that satisfies the following conditions:

$$\begin{array}{ll} (\alpha) & \mathrm{Lim}_\omega(\alpha_n) \geq 0 \quad \text{if} \quad \alpha_n \geq 0, \\ (\beta) & \mathrm{Lim}_\omega(\alpha_n) = \mathrm{Lim}\,\alpha_n \quad \text{if} \quad \alpha_n \quad \text{is convergent}, \\ (\gamma) & \mathrm{Lim}_\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \dots) = \mathrm{Lim}_\omega(\alpha_n). \end{array}$$

Note that the form Lim_ω is not unique; it depends on the choice of the "rule" ω, which is sometimes called *the summation machine*.

Now for any object $\mathscr{V}$ of $\mathscr{C}(\mu)$ and for any endomorphism $f : \mathscr{V} \to \mathscr{V}$ in $\mathscr{C}(\mu)$ define its trace $\mathrm{tr}_\omega(f)$ by

$$\mathrm{tr}_\omega(f) = \mathrm{Lim}_\omega(\mu^n \cdot \mathrm{Tr}(f^n)). \tag{4-4}$$

Here on the right hand side of (4-4) $\mathrm{Tr}(f^n)$ denotes the usual finite dimensional trace of the linear map $f^n : V^n \to V^n$. Note that because of condition (4-3) we have $|\mathrm{Tr}(f^n)| \leq M \dim V^n$ and using (4-1) we see that the sequence $\mu^n \,\mathrm{Tr}(f^n)$ is bounded, and therefore the definition is correct.

It is easy to see that (4-4) defines a trace on the category $\mathscr{C}(\mu)$ (in the sense of [F], Definition 2.7) which is non-negative and traceful. We will see later that

tr_ω is not normal and not faithful. Note that the constructed trace tr_ω is not unique – it depends on the choice of the functional Lim_ω (i.e. on the "rule" ω).

According to philosophy of A. Connes [C], in problems, having geometric origin, the answer will be often independent of ω; such problems A. Connes calls *measurable*. Cf. for example Proposition 5 in [C], chapter IV, Sect. 2.β concerning the Wodzicki residue.

We will also see examples of measurable problems (Theorems 8.4, 9.2 and 11.1) and not measurable problems (example 6.3) later in this paper.

4.3. The projective dimension. We know that any trace on a von Neumann category determines a dimension function, cf. 1.3 above. The trace tr_ω on $\mathscr{C}(\mu)$ defined by (4-4) determines the following dimension function

$$\mathfrak{projdim}_\omega \mathscr{V} = \mathrm{Lim}_\omega(\mu^n \cdot \dim V^n), \tag{4-5}$$

which we will call *the projective dimension* of V.

Note that the projective dimension $\mathfrak{projdim}_\omega \mathscr{V}$ depends only on the asymptotic behavior of the numbers $\dim V^n$ for large n and does not depend on any finite number of $\dim V^n$. In particular, the projective dimension $\mathfrak{projdim}_\omega \mathscr{V}$ vanishes if V^n is non-zero only for finitely many n. This shows that the projective dimension $\mathfrak{projdim}_\omega$ (or, more precisely, the trace (4-4)) is not faithful – nontrivial object may have trivial dimension.

Also, given an object $\mathscr{V}$ of $\mathscr{C}(\mu)$ with $\mathfrak{projdim}_\omega \mathscr{V} \neq 0$, consider the following sequence $\mathscr{V}(m), \quad m = 1, 2, 3, \dots$ of truncated objects of $\mathscr{C}(\mu)$, where $\mathscr{V}(m)^n$ equals to V^n for $n \leq m$ and $V(m)^n = 0$ for $n > m$. We see that $\mathscr{V}(m) \subset \mathscr{V}$ and

$$\sup_m \mathscr{V}(m) = \mathscr{V}.$$

However $\mathfrak{projdim}_\omega \mathscr{V}(m) = 0 \neq \mathfrak{projdim}_\omega \mathscr{V}$ for all m. Therefore *the trace (4-4) is not normal*.

4.4. The torsion dimension. Any torsion object of the extended abelian category $\mathscr{E}(\mathscr{C}(\mu))$, constructed out of $\mathscr{C}(\mu)$, cf. 1.4 and also [F], Sect. 1, is represented by a morphism of $\mathscr{C}(\mu)$ $\mathscr{X} = (\alpha : \mathscr{V} \to \mathscr{V})$,. Recall that $\alpha = (\alpha^n)$, where $\alpha^n : V^n \to V^n$ is a linear map. We want to translate the general definition of the spectral density function cf. [F], Subsect. 3.7, to the present situation. Given a positive $\lambda > 0$, denote by $F^n(\lambda)$ the maximal dimension of a linear subspace contained in the following cone

$$\{v \in V^n; (\alpha^n(v), \alpha^n(v)) \leq \lambda^2(v, v)\}. \tag{4-6}$$

It is also equal to the number of eigenvalues of $(\alpha^n)^*\alpha^n : V^n \to V^n$ which are less than λ^2. Then the spectral density function of $\mathscr{X}$ is given by

$$F(\lambda) = \mathrm{Lim}_\omega(\mu^n F^n(\lambda)). \tag{4-7}$$

The *torsion dimension* (defined in Sect. 1 above) of $\mathscr{X}$ is by definition

$$\mathfrak{tordim}_\omega \mathscr{X}^\cdot = \lim_{\lambda \to +0} F^n(\lambda). \tag{4-8}$$

Roughly, the torsion dimension in this situation can be characterized as *the density near zero of eigenvalues of* $(\alpha^n)^*\alpha^n$ *with respect to the chosen scale* $\mu = (\mu^n)$.

4.5. An example. Fix an arbitrary sequence a^n of positive real numbers with tends to 0. Consider an arbitrary object $\mathscr{V}^\cdot = (V^n)$ of $\mathscr{C}(\mu)$ and let the morphism $\alpha : \mathscr{V}^\cdot \to \mathscr{V}^\cdot$ be given as follows: $\alpha^n : V^n \to V^n$ is multiplication by a^n. Then we obtain a torsion object $\mathscr{X}^\cdot = (\alpha : \mathscr{V}^\cdot \to \mathscr{V}^\cdot)$ of $\mathscr{C}(\mu)$. For this $\mathscr{X}^\cdot$ we have

$$F^n(\lambda) = \begin{cases} 0, & \text{if} \quad a^n > \lambda, \\ \dim V^n, & \text{if} \quad a^n \le \lambda. \end{cases}$$

Thus we obtain that the spectral density function of $\mathscr{X}^\cdot$ $F(\lambda) = \mathrm{Lim}_\omega\, \mu^n F^n(\lambda)$ *is constant and equals the Dixmier dimension of* $\mathscr{V}^\cdot$, $F(\lambda) = \mathfrak{projdim}_\omega(\mathscr{V}^\cdot)$. Therefore the torsion dimension of $\mathscr{X}^\cdot$ (defined in Sect. 2) equals to $\mathfrak{projdim}_\omega(\mathscr{V}^\cdot)$.

This is example shows that the torsion dimension may assume arbitrary non-negative real numbers.

4.6. Extended homology in $\mathscr{C}(\mu)$. Consider a chain complex in $\mathscr{C}(\mu)$ of length m. Any such chain complex C is just a sequence $C = (C^n, d^n)$, where $n = 1, 2, \dots$ of finite-dimensional complexes

$$C^n = (0 \to C^n_m \xrightarrow{\ d^n\ } C^n_{m-1} \xrightarrow{\ d^n\ } \dots C^n_0 \to 0) \tag{4-9}$$

such that

(1) each chain space C^n_i, *where* $i = 0, 1, \dots, m$, *has a fixed Euclidean structure;*

(2) the dimension growth rate satisfies $\dim C^n_i = O((\mu^n)^{-1})$*;*

(3) the norm of the differentials d^n *has a common upper bound* $||d^n|| \le M$.

Given such chain complex C, it determines the extended homology, having the projective and torsion parts, and we want to understand the Dixmier dimension of the projective part and also the torsion dimension of the torsion part. Let Z^n_i denote the space of cycles $\ker[d^n : C^n_i \to C^n_{i-1}]$; then $Z_i = (Z^n_i)$ and $C_i = (C^n_i)$ are objects of $\mathscr{C}(\mu)$. We clearly have for the extended homology of C:

$$\mathscr{H}_i(C) = (d : C_{i+1} \to Z_i). \tag{4-10}$$

The projective part of $\mathscr{H}_i(C)$ is just $(H_i(C^n))$, i.e. it is given by the sequence consisting of the usual homology of the complexes C^n. Therefore, the Dixmier dimension of the projective part of the extended homology is given by

$$\mathfrak{projdim}_\omega(P(\mathscr{H}_i(C))) = \mathrm{Lim}_\omega(\mu^n \cdot \dim H_i(C^n)). \tag{4-11}$$

Let B^n_i be the subspace of boundaries $B^n_i = \mathrm{im}[d^n : C^n_{i+1} \to C^n_i]$ and let $B_i = (B^n_i) \in \mathrm{ob}(\mathscr{C}(\mu))$. Then the torsion part of the extended homology is given by

$$T(\mathscr{H}_i(C)) = (d : C_{i+1}/Z_{i+1} \to B_i) \tag{4-12}$$

We summarize now the above discussion as follows:

4.7. Proposition. *Suppose that a chain complex (4-9) in $\mathscr{C}(\mu)$ is given. For any pair of integers n and i denote by h_i^n the number of zero eigenvalues of the operator*

$$(d^n)^* d^n : C_{i+1}^n \to C_{i+1}^n \tag{4-13}$$

(the "Half-Laplacian") and for any $\lambda > 0$ denote by $G_i^n(\lambda)$ the number of eigenvalues of (4-13) lying in the interval $(0, \lambda^2)$. Then the Dixmier dimension of the projective part of the extended i-dimensional homology $\mathscr{H}_i(C)$ equals $\mathrm{Lim}_\omega\, \mu^n h_i^n$ *and the spectral density function of the torsion part of $\mathscr{H}_i(C)$ is* $G_i(\lambda) = \mathrm{Lim}_\omega\, \mu^n G_i^n(\lambda)$*. In particular, the torsion dimension of the torsion part of $\mathscr{H}_i(C)$ is*

$$\mathfrak{tordim}_\omega T(\mathscr{H}_i(C)) = \lim_{\lambda \to +0} (\mathrm{Lim}_\omega\, \mu^n G_i^n(\lambda)) \tag{4-14}$$

4.8. Asymptotic invariants of a growth process. A typical geometric situation, when the above numerical invariants of chain complexes in $\mathscr{C}(\mu)$ (the projective and torsion dimension and the Novikov - Shubin invariants) can be applied consists in the following.

Suppose that $K = (K^n)$ is *growth process*, i.e. a sequence of finite simplicial complexes, such that for any integer i the number of i-dimensional simplices in K^n is $O((\mu^n)^{-1})$.

As a concrete example (which will be studied in detail later in this paper) we may assume that the complexes $K = (K^n)$ form a tower of finitely sheeted coverings over a fixed finite polyhedron.

Another source of examples of growth processes is the following. Suppose that we have an infinite polyhedron and the finite polyhedra K^n with $K^n \subset K^{n+1}$ form its exhaustion.

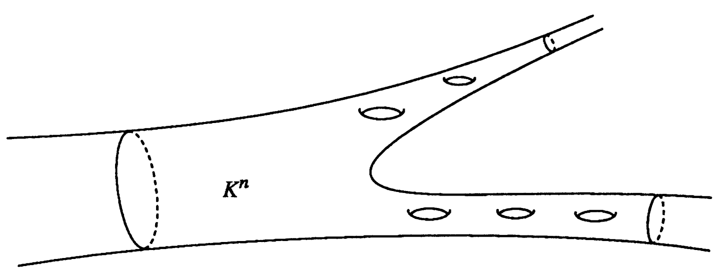

Fig. 2

Growth process of this type was considered in a recent preprint [DM] of J. Dodziuk and V. Mathai.

Another example of a growth process provides a sequence of smaller and smaller polyhedral approximations to a given compact Riemannian manifold.

Let us return now to the general situation. Given a growth process (K^n), we obtain the chain complexes $C^n = C_*(K^n)$ corresponding to the given simplicial structures on the complexes K^n. We may introduce the euclidean structure on $C_*(K^n)$, such that the simplices of K^n form an orthonormal base. (Note, that in fact there may be different geometrically interesting ways of choosing the scalar product on the chain space $C_*(K^n)$.)

We also have to verify condition (3) in Subsect. 4.6. Note that this condition will be automatically satisfied if the growth process (K^n) has *bounded geometry:*

there is a constant M (independent of n) such that each i-dimensional simplex of K^n is adjacent in K^n to at most M simplices of dimension $(i+1)$*.*

The sequence $C = (C^n)$ of chain complexes is now a single chain complex in the abelian category $\mathscr{C}(\mu)$ considered above (with an appropriately chosen growth rate μ), so we may apply the construction of extended homology and study the projective dimension, the torsion dimension, and the Novikov - Shubin invariants. We will call these invariants *the asymptotic invariants* of the sequence K^n. Note that the asymptotic invariants really depend only on the geometry of K^n for large $n \to \infty$.

In order to construct the chain complex $C_*(K^n)$, one has to choose orientations for all simplices of K^n. But it is easy to see that different choices of orientations do not influence the spectrum of the "Half - Laplacians" (4-13) and so the obtained invariants do not depend on these orientations.

Note also that the asymptotic invariants are in general *geometric and not topological*, i.e. they will depend on the simplicial decomposition of K^n's and not on the topology of K^n.

For future references, let us make the following simple observation.

4.9. Proposition. *Given a growth process* (K^n) *as above, its asymptotic invariants in dimension i depend only on the growth process consisting of the skeletons of K^n of dimension* $(i+1)$*. In particular, the asymptotic invariants in dimension zero depend only on the 1-skeletons of K^n.* $\square$

5. Spectrum of towers: theorem of Lück

In this section we will reformulate the theorem of Lück [L].

Lück considers a sequence of normal subgroups

$$\cdots \subset \Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset \Gamma_1 \subset \pi$$

such that the index $[\pi : \Gamma_k]$ is finite for all k and the intersection $\cap \Gamma_k$ is the trivial group. Let X be a finite polyhedron with fundamental group π. For each k we have the finite sheeted covering $X^k \to X$ corresponding to the subgroup Γ_k, and therefore we have a growth process (X^k) (in the terminology of Sect. 4) determined by this tower of covering. The theorem of Lück [L] computes the asymptotic invariants of this growth process.

5.1. Theorem (Luc̆k [L]). *Choose for the growth rate* $\mu = (\mu^k)$ *the numbers*

$$\mu^k = |\pi : \Gamma_k|^{-1} \tag{5-1}$$

(the inverses of the orders of the quotients π/Γ_k). Then

(i) The projective dimension of the growth process (X^k) equals to the L^2 Betti number of the universal covering of X in the corresponding dimension.

(ii) The torsion dimension of the extended homology vanishes.

We may conclude that the towers of coverings represent a very special class of growth processes.

6. Growing flat bundles

Here we will consider an example of a growth process, which is a generalization of the construction of tower of coverings, considered in the previous section. We will fix a polyhedron X and study a sequence of flat bundles over X of growing dimension. Our aim is to understand the asymptotic invariants in this situation.

This section contains only a general discussion of the problem; the results are given by Theorems 8.2, 8.3, 8.4, 9.2 and 11.1.

6.1. Let X be a fixed finite simplicial polyhedron and let $\mathscr{E}^k$ be a sequence of finite dimensional flat bundles over X. We will assume that each bundle $\mathscr{E}^k$ is supplied with a flat metric.

Define the growth rate $\mu = (\mu^k)$ as

$$\mu^k = (\operatorname{rank} \mathscr{E}^k)^{-1}. \tag{6-1}$$

For each integer k we have the chain complex $C^k = C_*(X, \mathscr{E}^k)$ over $\mathbb{C}$. The basis of this chain complex is formed by the flat sections of $\mathscr{E}^k$ defined over the oriented simplices of X. The boundary homomorphism is given by restricting a flat section over a simplex σ on all the faces of σ, multiplied by the sign, expressing compatibility of the orientations of the simplex σ with the orientation of the face.

We want to view this sequence of complexes C^k, where $k = 1, 2, 3, \ldots$, as a single complex in the category $\mathscr{C}(\mu)$. To meet all the requirements of Sect. 4.6, we need to introduce a scalar product in $C_*(X, \mathscr{E}^k)$. We will do it as follows: the scalar product of two flat sections s_1 and s_2, which are defined over two different simplices of X is zero; if s_1 and s_2 are defined over the same simplex σ of X, then the scalar product (s_1, s_2) equals to the scalar product of $s_1(v)$ and $s_2(v)$ in the fiber of the bundle $\mathscr{E}^k$ over v, where v is any vertex of the simplex σ - the result is independent on the choice of v, since the metric on $\mathscr{E}^k$ is supposed to be flat.

We obtain a chain complex $C = (C^k)$ in the abelian category $\mathscr{C}(\mu)$ and we want to understand its asymptotic invariants.

6.2. Note, that the construction of the tower of coverings (cf. Sect. 5) is a special case of this construction. In fact, in the situation of Sect. 5 for any k we have the action of π on the group ring of the finite quotient $V^k = \mathbb{C}[\pi/\Gamma_k]$. More precisely, we consider the action of π from the left on the group algebra $\mathbb{C}[\pi/\Gamma_k]$ and the corresponding flat bundle $\mathscr{E}^k$ over X. Note that this bundle has a flat metric, which comes from the metric of $\mathbb{C}[\pi/\Gamma_k]$ in which the elements of π/Γ_k form an orthonormal base. The homology of the flat bundle $\mathscr{E}^k$ over X coincides with the homology of the normal covering $X^k \to X$, corresponding to Γ_k.

Example 6.3. Here we consider an example, which behaves unlikely the situation with the towers of coverings.

Let X be the closed 3-manifold obtained from the trefoil knot

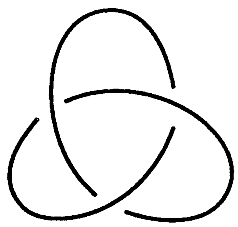

Fig. 3

by 0-framed surgery. We have the canonical epimorphism $\phi : \pi_1(X) \to Z$ (the abelinization), and therefore for any complex number ξ with $|\xi| = 1$, there is a unique flat Hermitian line bundle $\mathscr{E}_\xi$ with monodromy given by $g \mapsto \chi_\xi(g) = \xi^{\phi(g)}$ for $g \in \pi$. The dimension of homology $H_1(X, \mathscr{E}_\xi)$ is zero for all ξ with $\xi^2 - \xi + 1 \neq 0$. Here $\Delta(\xi) = \xi^2 - \xi + 1$ is the Alexander polynomial of the trefoil. If ξ is one of the roots of the Alexander polynomial, i.e. if $\xi = \xi_\pm = e^{\pm\pi i/3}$, then the dimension of the homology $H_1(X, \mathscr{E}_\xi)$ is 2.

Now, choose a sequence of complex numbers ξ_k with $|\xi_k| = 1$, such that $\xi_k \to \xi_+$. Then we have a sequence of flat bundles $\mathscr{E}_{\xi_k}$, such that *the sequence of dimensions* $\dim H_1(X, \mathscr{E}_{\xi_k})$ *may be an arbitrary sequence consisting of 0 and 2*: we obtain 0 if $\xi_k \neq \xi_+$ and we obtain 2 if $\xi_k = \xi_+$.

Therefore, *the projective dimension in this situation* $\mathrm{Lim}_\omega \dim_{\mathbb{C}} H_1(X, \mathscr{E}_\xi)$ *may actually depend on the choice of the summation machine* ω.

Note also that in this example the corresponding characters χ_{ξ_k} converge and their limit is the character χ_{ξ_+} at the root of the Alexander polynomial.

Suppose now that in the situation described above ξ_k tends to ξ_+, but $\xi_k \neq \xi_+$. Then we see that the projective dimension of the growth process is zero (independently of ω). However, we will have the torsion dimension equal to 2.

7. Characters of representations and extended L^2-homology

In this section we will show that the extended homology of a finite polyhedron X with coefficients in a representation $\mathscr{M}$ in a von Neumann category with a trace, depends mainly on the character $\chi_{\mathscr{M}} : \pi \to \mathbb{C}$ of the fundamental group of X determined by $\mathscr{M}$. We also show that any positive self-adjoint class function on the fundamental group can be realized as the character of a unitary representation in a von Neumann category.

7.1. Suppose that $\mathscr{C}_{\mathscr{A}}$ is a von Neumann category with a fixed trace tr.

Let π be a discrete group. We will consider representations of π on objects of $\mathscr{C}_{\mathscr{A}}$. More precisely, let $\mathscr{M}$ be an object of $\mathscr{C}_{\mathscr{A}}$; then a representation of π is a ring homomorphism $\rho : \mathbb{C}[\pi] \to \hom_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$. Such representation will be called *unitary* if ρ is a $*$-homomorphism, i.e. if it preserves the involutions. Here we assume that the group ring is supplied with the standard involution $g \mapsto g^{-1}$ for $g \in \pi$.

Any representation $\rho : \mathbb{C}[\pi] \to \hom_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$ as above determines *the character*

$$\chi_{\mathscr{M}} : \pi \to \mathbb{C}, \quad g \mapsto \mathrm{tr}(\rho(g)), \quad g \in \pi. \tag{7-1}$$

The character $\chi_{\mathscr{M}}$ is clearly constant on the conjugacy classes of π. Also, if the representation is unitary, then the character $\chi_{\mathscr{M}}$ has the property

$$\chi_{\mathscr{M}}(g^{-1}) = \overline{\chi_{\mathscr{M}}(g)} \tag{7-2}$$

for any $g \in \pi$. Class functions with this property are called *self-adjoint*. Another important property of characters is *positivity*: for any element $a \in \mathbb{C}[\pi]$ of the group algebra $\mathbb{C}[\pi]$ holds

$$\chi_{\mathscr{M}}(a^* a) \geq 0. \tag{7-3}$$

It is not true in general that the character determines the representation up to the natural equivalence.

Using the construction of [F], we know that to any finite CW space X with fundamental group $\pi_1(X) = \pi$ we may assign extended homology $\mathscr{H}_*(X, \mathscr{M})$ with coefficients in $\mathscr{M}$.

Our observation here is that (assuming that the trace tr on $\mathscr{C}_{\mathscr{A}}$ is normal) the most important invariants of the extended homology can be computed using only the character $\chi_{\mathscr{M}}$ of the representation $\mathscr{M}$:

7.2. Theorem. *Suppose that the chosen trace* tr *on the von Neumann category $\mathscr{C}_{\mathscr{A}}$ is normal. Let X be a finite polyhedron with fundamental group π. Then for any unitary representation $\rho : \pi \to \hom_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$, one can find the spectral density function $F_i(\lambda)$ of the extended homology $\mathscr{H}_i(X, \mathscr{M})$ with coefficients in $\mathscr{M}$ using the character $\chi_{\mathscr{M}}$ of $\mathscr{M}$ as the only information on the representation $\mathscr{M}$. In particular, the von Neumann dimension and the Novikov - Shubin invariants of $\mathscr{H}_i(X, \mathscr{M})$ depend only on the character $\chi_{\mathscr{M}}$ (and on X, of course).*

The proof of Theorem 7.2 is given in Sect. 12.

We will see later in 10.1 that Theorem 7.2 is false without assuming normality of the trace tr.

Also, the Theorem is not true if the representation ρ is not unitary. Although Theorem 7.2 can be generalized to non-unitary representations, but the conclusion then is different; we will consider this generalization elsewhere.

As a simple example, consider a finite dimensional unitary representation V of π and form the tensor product $\mathscr{M} = V \otimes_{\mathbb{C}} \ell^2(\pi)$. The character $\chi_{\mathscr{M}}$ of this representation equals to the character of $\mathscr{M}' = \ell^2(\pi) \oplus \cdots \oplus \ell^2(\pi)$ (dim V

times). Then we obtain from Theorem 7.2 that the spectral density functions of $\mathscr{H}_i(X, \mathscr{M})$ and $\mathscr{H}_i(X, \mathscr{M}')$ coincide.

7.3. Constructing representations with given characters. Here we will consider the following problem: given a class function $\chi : \pi \to \mathbb{C}$, which is self-adjoint (7-2) and non-negative (7-3), we want to construct a unitary representation $\rho : \pi \to \hom_{\mathscr{C}_\mathscr{A}}(\mathscr{M}, \mathscr{M})$ in certain von Neumann category $\mathscr{C}_\mathscr{A}$ with a normal trace tr such that the character $\chi_\mathscr{M}$ of this representation is the given function χ. We will see that there is a canonical construction for this purpose. This construction is very similar to the classical constructions (cf. [N], Sect. 30, and also [G]); therefore we will be very brief.

First, we will associate a Hilbert space $\mathscr{H}_\chi$ with a given self-adjoint non-negative class function $\chi : \pi \to \mathbb{C}$. We will denote by J_χ the following two-sided ideal of $\mathbb{C}[\pi]$:

$$J_\chi = \{a \in \mathbb{C}[\pi]; \chi(ab) = 0 \quad \text{for any} \quad b \in \mathbb{C}[\pi]\}. \tag{7-4}$$

Then we define the Hilbert space $\mathscr{H}_\chi$ as the completion of the factor-ring $\mathbb{C}[\pi]/J_\chi$ with respect to the following scalar product

$$(a, b) = \chi(ab^*), \quad a, b \in \mathbb{C}[\pi]. \tag{7-5}$$

It is easy to check that the obvious left and right actions of π on the factor-ring $\mathbb{C}[\pi]/J_\chi$ are continuous with respect to the norm determined by the scalar product (7-5), and thus these actions extend to the left and right actions of π on $\mathscr{H}_\chi$. Both these actions are in fact unitary.

Note that the previous construction applied to the case when χ is the delta-function at the unit element of the group π, gives the standard Hilbert space $\ell^2(\pi)$ which is usually associated with the group π.

Now we will construct a von Neumann algebra $\mathscr{N}(\chi)$ acting on $\mathscr{H}_\chi$. We will denote by $\mathscr{N}(\chi)$ the space of all bounded linear maps $A : \mathscr{H}_\chi \to \mathscr{H}_\chi$, commuting with the action of π from the left. We obtain that

$$\mathbb{C}[\pi]/J_\chi \subset \mathscr{N}(\chi) \tag{7-6}$$

(where $\mathbb{C}[\pi]$ acts from the right on the Hilbert space $\mathscr{H}_\chi$).

Now we will define the following function (the trace)

$$\tau : \mathscr{N}(\chi) \to \mathbb{C}. \tag{7-7}$$

For $A \in \mathscr{N}(\chi)$ set

$$\tau(A) = (A \cdot 1, 1) \tag{7-8}$$

where $1 \in \mathbb{C}[\pi]/J_\chi \subset \mathscr{N}(\chi)$ denotes the unit element and the brackets (,) denote the scalar product (7-5). One easily check that:

(1) *τ is a trace on the von Neumann algebra $\mathscr{N}(\chi)$;*
(2) τ is normal;
(3) τ is faithful;

(4) on the subring $\mathbb{C}[\pi]/J_\chi \subset \mathcal{N}(\chi)$ *the trace* τ *coincides with* χ.

As shown in Sect. 2.6 (example 3) of [F], the von Neumann algebra $\mathcal{N}(\chi)$ acting on $\mathcal{H}_\chi$ generates a finite von Neumann category $\mathcal{C}_{\mathcal{A}}$, where $\mathcal{A}$ is the group algebra $\mathbb{C}[\pi]$. The trace τ on the algebra $\mathcal{N}(\chi)$ determines a trace tr on the category $\mathcal{C}_{\mathcal{A}}$. This trace on $\mathcal{C}_{\mathcal{A}}$ is clearly normal (since τ is normal).

Now, we have a unitary action of π on $\mathcal{M} = \mathcal{H}_\chi \in \mathrm{ob}(\mathcal{C}_{\mathcal{A}})$ and the corresponding character $\chi_{\mathcal{M}}$ equals χ.

8. Approximating characters

Here we study the general problem about the relation between the von Neumann Betti numbers and the dimensions of the homology of a sequence of finitely dimensional representations, assuming that their characters converge to the character of the von Neumann representation. We find a relation, involving an additional term, the torsion dimension, which was studied in Sect. 2.

In the next section we consider the situation (which we call arithmetic approximation) when this additional term vanishes.

8.1. In this section we will study the following generalization of the situation considered by W. Lück [L].

Suppose that π is a discrete group and we are given a sequence of finite dimensional unitary representations $\rho_k : \pi \to \mathrm{End}(V^k)$, where $k = 1, 2, \dots$. We will denote by $\chi_k : \pi \to \mathbb{C}$ the corresponding characters. The dimensions of these representations $\dim V^k = \chi_k(1)$ are not supposed to be constant. We will denote by $\mu^k = (\chi_k(1))^{-1}$ the inverse numbers; the numerical sequence $\mu = (\mu^k)$ describes the growth rate of the dimensions. We will consider also the *normalized characters* $\tilde{\chi}_k = \mu^k \chi_k : \pi \to \mathbb{C}$, where $k = 1, 2, \dots$.

Using the von Neumann category $\mathcal{C}(\mu)$ of Sect. 4.1 (where $\mu = (\mu^k)$ is the growth rate specified in the previous paragraph), we can view the given sequence of representations $\rho_k : \pi \to \mathrm{End}(V^k)$, where $k = 1, 2, \dots$ as a single representation $\rho_0 : \pi \to \mathrm{hom}_{\mathcal{C}(\mu)}(\mathcal{V}, \mathcal{V})$. Here $\mathcal{V} = (V^k)$ is the object of $\mathcal{C}(\mu)$ determined by the given sequence V^k of finite dimensional Hilbert spaces.

Using the construction of 1.8, for any finite polyhedron X with fundamental group π we have the extended homology $\mathcal{H}_*(X, \mathcal{V})$ with coefficients in $\mathcal{V}$. If we choose the Dixmier type trace (4-4) on the category $\mathcal{C}(\mu)$, we obtain the numerical invariants of the extended homology - the projective dimension, the torsion dimension, and the Novikov - Shubin invariants, cf. Sect. 1. We will denote by $\mathfrak{projdim}_\omega P(\mathcal{H}_i(X, \mathcal{V}))$ the projective dimension and by $\mathfrak{tordim}_\omega T(\mathcal{H}_i(X, \mathcal{V}))$ the torsion dimension. Recall that the projective dimension is just

$$\mathfrak{projdim}_\omega P(\mathcal{H}_i(X, \mathcal{V})) = \mathrm{Lim}_\omega\left[\frac{\dim_{\mathbb{C}} H_i(X, V^k)}{\dim_{\mathbb{C}} V^k}\right]. \tag{8-1}$$

Note also that these invariants depend in general on the choice of the summation procedure ω.

8.2. Theorem. *In the situation described in Subsect. 8.1, suppose that we are given another von Neumann category $\mathscr{C}_{\mathscr{A}}$ with a normal trace* tr *and a unitary representation $\rho : \pi \to \hom_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$. Suppose that the normalized characters of the finite dimensional representations $\tilde{\chi}_k$ converge pointwise (as functions on the group π) to the character $\chi_{\mathscr{M}} : \pi \to \mathbb{C}$ of $\mathscr{M}$, when $k \to \infty$. In other words, we assume that for any $g \in \pi$ holds $\lim_{k\to\infty} \tilde{\chi}(g) = \chi_{\mathscr{M}}(g)$. Then the following formula holds, which expresses the von Neumann dimension of the extended homology $\mathscr{H}_i(X, \mathscr{M})$ with coefficients in $\mathscr{M}$ (with respect to the trace* tr*) by means of the dimensions of the homology of the approximating finite dimensional representations:*

$$\dim_{\mathrm{tr}} P(\mathscr{H}_i(X, \mathscr{M})) = \mathfrak{projdim}_\omega P(\mathscr{H}_i(X, \mathscr{V}))+\mathfrak{tordim}_\omega T(\mathscr{H}_i(X, \mathscr{V})). \quad (8\text{-}2)$$

Note that the RHS of (8-2) contains only information obtained from the finite dimensional flat bundles. The LHS of (8-2) is the L^2-Betti number with respect to a normal trace; it is independent of the choice of the summation process ω. Thus, (8-2) implies that the sum of the projective dimension and the torsion dimension is independent of ω. Note, however, that the choice of ω may influence each of the numbers in the RHS of (8-2), as example 6.3 shows.

Compared with Lück's theorem [L], formula (8-2) contains an additional summand (the torsion dimension). We will discuss in Sect. 9 the conditions under which this torsion dimension vanishes.

The proof of Theorem 8.2 is given in Sect. 12.

The following statement shows that one may recapture the entire spectral density function of the extended homology $\mathscr{H}_i(X, \mathscr{M})$ in terms of the finite dimensional approximations.

8.3. Theorem (Approximation of the spectral density function). *Under the condition of Theorem 8.2 the spectral density function $F_i(\lambda)$ of extended homology $\mathscr{H}_i(X, \mathscr{M})$ and the spectral density function $G_i(\lambda)$ of extended homology $\mathscr{H}_i(X, \mathscr{V})$ are related as follows*

$$F_i(\lambda) = \lim_{\epsilon\to+0} G_i(\lambda+\epsilon) \quad \textit{for all} \quad \lambda \geq 0. \quad (8\text{-}3)$$

In other words, F_i coincides with G_i, made right continuous.

The proof of Theorem 8.3 is also postponed until Sect. 12.

In the next theorem we point out conditions under which the projective dimension and the torsion dimension in the RHS of (8-2) are both independent of the summation procedure ω.

8.4. Theorem (Analytic curve of representations). *Let π be a discrete group and let $\mathscr{R}_N(\pi)$ denote the real analytic variety of all representations of π into the N-dimensional unitary group $\mathscr{U}(N)$. Suppose, that $\rho_n \in \mathscr{R}_N(\pi)$, where $n = 1, 2, \dots$ is an infinite sequence of representations such that there exists a real analytic curve $\rho : [0, \epsilon) \to \mathscr{R}_N(\pi)$ and a sequence $t_n \in [0, \epsilon)$, such that $\rho_n = \rho(t_n)$, $t_n \to 0$, $t_n \neq 0$. Let $\rho_0 = \rho(0)$ denote the limit representation. Then for any finite* *polyhedron X and for any homomorphism $\phi : \pi_1(X) \to \pi$ we obtain the sequence of unitary representations*

$$\psi_n : \pi_1(X) \xrightarrow{\phi} \pi \xrightarrow{\rho_n} \mathscr{U}(N). \tag{8-4}$$

Each ψ_n produces a flat N-dimensional unitary bundle over X, which we will denote by V^n. Then the dimension of the homology $H_i(X, V^n)$ is constant for large n. Therefore, the torsion dimension of the sequence $\mathscr{V} = (V^n)$ of these flat bundles equals to the jump in the Betti number

$$\mathfrak{tor}\mathfrak{Dim}_\omega H_i(X, \mathscr{V}) = \dim H_i(X, V^0) - \dim H_i(X, V^n), \tag{8-5}$$

where n is sufficiently large. In particular, we see that the torsion dimension is independent of the choice of ω.

Proof of Theorem 8.4. We only have to show that the dimension $\dim H_i(X, V^n)$ stabilizes for large n; the rest follows from Theorem 8.2.

We use the well known property of upper semi continuity of the dimension, cf. [H], Ch.3, Sect. 12. For any $t \in [0, \epsilon)$ denote by V^t the flat bundle over X with monodromy $\phi \circ \rho(t)$. Then there exists a non-constant real analytic function $f(t)$ such that the dimension $\dim H_i(X, V^t)$ assumes the constant value, say D, for all t with $f(t) \neq 0$; moreover, $\dim H_i(X, V^t) \geq D$ for all t. Suppose that we have a sequence of points t_n with $t_n \to 0$, $t_n \neq 0$. If $f(t_n)$ is zero only for finitely many n, then we obtain that the dimension $\dim H_i(X, V^n)$ is constant for large n. However, if $f(t_n) = 0$ for infinitely many n, then the function $f(t)$ must be identically zero – a contradiction. □

9. Arithmetic approximation

It turns our that one may impose some arithmetic conditions on the approximating sequence of flat bundles, which would imply vanishing of the additional correcting term (the torsion dimension), appearing in Theorems 8.2 and 8.4. Roughly, the arithmeticity condition requires that each approximating finite dimensional representation be definable over the ring of algebraic integers of an algebraic number field, and the degrees of these number fields must have a common upper bound.

This result implies the theorem of Lück. Namely, Lück [L] considers the tower of finitely sheeted regular coverings, which is equivalent to studying the homology of X with coefficients in the representations of π on $\mathbb{C}[\pi/\Gamma_k]$ (cf. 6.2); these representations are clearly defined over the integers $\mathbb{Z}$.

The main theorem of this section contains also a statement that the torsion part of the extended homology $\mathscr{H}_i(X, \mathscr{M})$ is of determinant class assuming that the character $\chi_{\mathscr{M}}$ of $\mathscr{M}$ admits an arithmetic approximation . This result generalizes a theorem proven by D. Burghelea, L. Friedlander and T. Kappeler [BFK] for the case $\mathscr{M} = \ell^{(}\pi)$. The proof presented in this paper (cf. Sect. 12), is quite similar to the proof suggested in [BFK].

9.1. Definition (Arithmetic approximation). Suppose that $\chi : \pi \to \mathbb{C}$ is a positive self-adjoint class function $\chi : \pi \to \mathbb{C}$, cf. 7.1. We will assume that χ is normalized so that $\chi(1) = 1$. We will say that χ *admits an arithmetic approximation*, if there exists a sequence of finite dimensional unitary representations $\rho_k : \pi \to \mathrm{End}_{\mathbb{C}}(V^k)$, where $k = 1, 2, \dots$, such that the following conditions are verified:

A. Let $\tilde{\chi}_k : \pi \to \mathbb{C}$ denote the normalized character of ρ_k, i.e. $\tilde{\chi}_k = \chi_k(1)^{-1}\chi_k$, where χ_k is the character of ρ_k. Then the sequence $\tilde{\chi}_k(g)$ converges to $\chi(g)$ for any $g \in \pi$.

B. For each k there is given an algebraic number field $\mathscr{F}_k \supset \mathbb{Q}$, imbedded into the field of complex numbers $\mathbb{C}$ such that its image $\mathscr{F}_k \subset \mathbb{C}$ is invariant under the complex conjugation. We will consider $\mathscr{F}_k$ together with the involution induced from $\mathbb{C}$. We suppose that *the representation ρ_k can be defined over $\mathscr{F}_k$*. In other words, there exists a representation $\tilde{\rho}_k : \pi \to \mathrm{End}_{\mathscr{F}_k}(W_k, W_k)$, preserving a positively defined Hermitian form $\langle\ ,\ \rangle_k : W_k \times W_k \to \mathscr{F}_k \subset \mathbb{C}$, which produces ρ_k by extension of the scalars from $\mathscr{F}_k$ to $\mathbb{C}$.

C. Denote by $\mathfrak{o}_k$ the ring of algebraic integers of $\mathscr{F}_k$. We suppose that for each k there exists an $\mathfrak{o}_k$-lattice $\mathscr{L}_k \subset W_k$ (i.e. a finitely generated $\mathfrak{o}_k$-submodule generating W_k over $\mathscr{F}_k$), which is invariant under the action of π and such that the form $\langle\ ,\ \rangle_k$ restricted on $\mathscr{L}_k$ assumes values in $\mathfrak{o}_k$.

D. Denote by $\mathscr{L}_k^D$ the dual lattice

$$\mathscr{L}_k^D = \{w \in W_k; \langle w, x\rangle \in \mathfrak{o}_k \quad \text{for any} \quad x \in \mathscr{L}_k\},$$

cf. [FT], page 122. Then $\mathscr{L}_k \subset \mathscr{L}_k^D$ and the factor $\mathscr{L}_k^D/\mathscr{L}_k$ is a finite group. We suppose that one can choose the lattices $\mathscr{L}_k$ such that that there is an integer $M > 0$ (independent of k) with $M \cdot \mathscr{L}_k^D/\mathscr{L}_k = 0$.

E. Let h_k denote the degree of the number field $\mathscr{F}_k$ (cf. above) over the rationals. We assume that there exists a common upper bound

$$h_k \le h \tag{9-1}$$

for all k.

F. There exists a function $N : \pi \to \mathbb{R}_+$, having the property

$$N(gg') \le N(g)N(g') \quad \text{for any} \quad g, g' \in \pi, \tag{9-2}$$

and such that for any group element $g \in \pi$ the following inequality holds

$$|\sigma_j(\tilde{\chi}_k(g))| \le N(g) \tag{9-3}$$

for all $k = 1, 2, \ldots$ and for all the embeddings $\sigma_j : \mathscr{F}_k \to \mathbb{C}$, $j = 1, \dots h_k$ of the algebraic number field $\mathscr{F}_k$.

Note, that conditions **E** and **F** imply that, if the dimensions of the representations $\dim V^k$ are bounded, then for any $g \in \pi$ the sequence $\tilde{\chi}_k(g)$ stabilizes for large k. This statement easily follows applying Lemma A (cf. Subsect. 12.3). However, this is not true if the dimensions $\dim V^k$ grow; for example this stabilization does not happen in Theorem 0.5, although all the conditions above hold.

The number h (appearing in property **E**) will be called *the degree* of the arithmetic approximation. The number M (appearing in **D**) will be called *the denominator* of the arithmetic approximation.

Now we show that the conditions of arithmetic approximation imply vanishing of the torsion dimension, which appears in Theorem 8.2.

9.2. Theorem. *Suppose that $\mathscr{C}_{\mathscr{A}}$ is a finite von Neumann category with a fixed normal trace, $\mathscr{M} \in \mathrm{ob}(\mathscr{C}_{\mathscr{A}})$ is a fixed object with $\dim_{\mathrm{tr}}(\mathscr{M}) = 1$, and $\rho : \pi \to \mathrm{End}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M})$ is a unitary representation, having character $\chi_{\mathscr{M}} : \pi \to \mathbb{C}$. Suppose that we are given a sequence of finite dimensional representations $\rho_k : \pi \to \mathrm{End}_{\mathbb{C}}(V^k)$ which provide an arithmetic approximation of the character $\chi_{\mathscr{M}}$, cf. Sect. 9.1. Then for any finite polyhedron X with fundamental group π holds:*

(i) the sequence of the normalized Betti numbers

$$\frac{\dim_{\mathbb{C}} H_i(X, V^k)}{\dim_{\mathbb{C}} V^k}$$

converges and its limit coincides with the von Neumann dimension of the projective part of the extended homology

$$\lim_{k\to\infty} \frac{\dim_{\mathbb{C}} H_i(X, V^k)}{\dim_{\mathbb{C}} V^k} = \dim_{\mathrm{tr}} \mathscr{H}_i(X, \mathscr{M}). \tag{9-4}$$

(ii) Let $F_i(\lambda)$ denote the spectral density function of extended homology $\mathscr{H}_i(X, \mathscr{M})$. Then the following inequality holds

$$F_i(\lambda) - F_i(0) \leq \frac{c}{-\log(\lambda)} \tag{9-5}$$

for small $\lambda > 0$, where $c > 0$ is a constant.

(iii) For any i the torsion part of the extended homology $\mathscr{H}_i(X, \mathscr{M})$ is of determinant class.

Lück's theorem [L] follows from this by taking $V^k = \mathbb{C}[\pi/\Gamma_k]$, cf. 6.2. This flat bundle can be defined over the integers.

Intuitively, the integrality condition in Theorem 9.2 allows to conclude at some point of the proof, that certain nonzero quantity cannot be too small.

The proof of Theorem 9.2 is given in Sect. 12.

For the definition of the notion *determinant class* (which appears in the statement (iii) of Theorem 9.2) we refer to [BFKM]. Cf. also [CFM], Sect. 3.8, where it is explained why the condition of being of determinant class depends only on the torsion part of the extended homology.

It is natural to ask for which groups π the character χ of the natural representation of π on $\ell^2(\pi)$ (which is the delta-function at the unit $1 \in \pi$) admits an arithmetic approximation. It is easy to see that *it happens if and only if π is residually finite*. In fact, if π is residually finite, we may construct the arithmetic approximation as in Lück's theorem: if $\pi \supset \Gamma_1 \supset \Gamma_2 \dots$ is a sequence of normal subgroups with trivial intersection, then we may take $V^k = \mathbb{C}[\pi/\Gamma_k]$, which can be realized over integers. Conversely, if we are given an arithmetic approximation (cf. 9.1) with $\tilde{\chi}_k$ converging to χ, then for any $g \in \pi$, where $g \neq 1$, we have $\tilde{\chi}_k(g) \to 0$, and so there exists k with $\tilde{\chi}_k(g) < 1$. This implies that the image of g under the k-th representations ρ_k is nontrivial, $\rho_k(g) \neq 1$. Since the automorphism of the lattice $\mathrm{GL}_{\mathfrak{o}_k}(\mathscr{L}_k)$ is residually finite, we obtain that π must be residually finite.

Now we will give proofs of Theorems 0.3, 0.4 and 0.5 (cf. Sect. 0), deduced from Theorem 9.2.

9.3. Proof of Theorem 0.3. We will use the notations introduced in Theorem 0.3.

With the sequence of subgroups $\pi \supset \Gamma_1 \supset \Gamma_2 \ldots$ we associate the sequence of the unitary representations $\rho_k : \pi \to \mathrm{End}(\mathbb{Z}[\pi/\Gamma_k])$ defined over Z. Here we assume that π acts on the group ring $\mathbb{Z}[\pi/\Gamma_k]$ as the left regular representation. It is clear that if V^k denotes the flat bundle over X determined by this representation then $H_i(X, V^k) \simeq H_i(\tilde{X}_k)$.

All conditions of arithmetic approximation (of Sect. 6.1) are obviously satisfied. We only need to compute the normalized character $\tilde{\chi}_k$ of ρ_k. An elementary calculation shows that for $g \in \pi$

$$\tilde{\chi}_k(g) = \frac{n_k(g)}{n_k},$$

where n_k is the total number of different subgroups of π conjugate to Γ_k, and $n_k(g)$ is the number of them, containing g. Therefore, our assumption (0-1) implies that the normalized characters $\tilde{\chi}_k$ converge pointwise to the character of the standard representation of π on $\ell^2(\pi)$. Applying Theorem 9.2, we complete the proof. □

9.4. Proof of Theorem 0.4. Consider the representation

$$\nu_k : \pi \to \mathrm{End}_{\mathfrak{o}}(\mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\Gamma_k]} \mathfrak{o}^m), \tag{9-6}$$

induced from the restriction onto Γ_k of the given unitary representation

$$\rho : \pi \to \mathrm{End}_{\mathfrak{o}}(\mathfrak{o}^m). \tag{9-7}$$

We want to apply Theorem 9.2 to the obtained sequence of representations ν_k. It is clear that if X is any polyhedron with $\pi_1(X) = \pi$, then the homology of X with coefficients twisted by ν_k is the same as the homology of the covering space $\tilde{X}_k$ with coefficients twisted by ρ_k (here $\tilde{X}_k \to X$ denotes covering corresponding to Γ_k).

From the general properties of induced representations (cf. [CR], Sect. 10) we see that the conditions **B, C, D, E** of Sect. 9.1 are satisfied. We only need to check that the sequence of normalized characters of ν_k converge to $\chi_0 : \pi \to \mathbb{C}$, where $\chi_0(g) = 0$ for $g \in \pi, g \neq 1$ and $\chi_o(1) = 1$. Also, we need to check condition **F** in 9.1.

If we denote by $\chi : \Gamma \to \mathbb{C}$ the character of ρ and by $\eta_k : \pi \to \mathbb{C}$ the character of ν_k, then $\eta_k(g) = \chi(g)$ for $g \in \Gamma_k$ and $\eta_k(g) = 0$ for $g \notin \Gamma_k$. Therefore we see that $\eta_k \to \chi_0$. The property **F** of Sect. 9.1 holds with the function $N(g)$ given by

$$N(g) = m \cdot \max_j \{||\sigma_j(\rho(g))||\}.$$

Here j runs over $1, 2, \dots, h$ (where h denotes the degree of $\mathscr{F}$), $\sigma_j : \mathscr{F} \to \mathbb{C}$ are the embeddings of $\mathscr{F}$ into $\mathbb{C}$, and $\sigma_j(\rho(g))$ denotes the complex matrix obtained by applying the embedding σ_j to the matrix $\rho(g)$ with entries in $\mathscr{F}$. Applying Theorem 9.2 finishes the proof. □

9.5. Proof of Theorem 0.5. Let $\chi : \pi \to \mathfrak{o}_{\mathscr{F}}$ denote the character of the given representation $\rho : \pi \to \mathrm{Mat}(m \times m, \mathfrak{o}_{\mathscr{F}})$. Then the character of the tensor power $\rho^{\otimes k}$ is $g \mapsto \chi(g)^k$. We claim that for $g \neq 1$ holds $|\chi(g)| < m$ and therefore the normalized character of $\rho^{\otimes k}$ tends to 0:

$$\frac{|\chi(g)|^k}{m^k} \to 0.$$

In fact, $\rho(g)$ viewed as a complex $m \times m$ matrix, can be diagonalized, and on the diagonal we will obtain m numbers with norm 1. Therefore $|\chi(g)| \leq m$ and the equality holds if and only if $\rho(g)$ belongs to the center.

Thus we obtain condition **A** of 9.1. Conditions **B, C, D, E** are obvious. Condition **F** follows from the assumption that the field $\mathscr{F}$ is cyclotomic: then all the Galois transformations preserve the complex norm. □

Next we will formulate a corollary of Theorem 9.2 which may be useful.

9.6. Theorem. *Let $\mathscr{F} \subset \mathbb{C}$ be an algebraic number field invariant under the complex conjugation, and let $\mathscr{F}' \subset \mathbb{C}$ be a cyclotomic field. We will denote by $\mathfrak{o}_{\mathscr{F}}$ and $\mathfrak{o}_{\mathscr{F}'}$ the corresponding rings of algebraic integers. Let π be a discrete group and let $\rho : \pi \to \mathrm{Mat}(n \times n, \mathfrak{o}_{\mathscr{F}})$ and $\rho_k : \pi \to \mathrm{Mat}(n_k \times n_k, \mathfrak{o}_{\mathscr{F}'})$, where $k = 1, 2, \dots$, be unitary representations, such that for any $g \in \pi$ the limit*

$$\lim_{k\to\infty} \frac{\chi_k(g)}{n_k} = \chi_0(g) \tag{9-8}$$

exists; here $\chi_k : \pi \to \mathbb{C}$ denotes the character of ρ_k. Let X be a compact polyhedron with $\pi_1(X) = \pi$ and let $\mathscr{E}$ and $\mathscr{E}_k$ (for $k = 1, 2, \dots$) denote the complex flat vector bundles over X determined by ρ and ρ_k correspondingly. Then the sequence of the normalized Betti numbers

$$\frac{\dim H_i(X, \mathscr{E} \otimes \mathscr{E}_k)}{n \cdot n_k}, \quad \textit{where} \quad k = 1, 2, \dots \tag{9-9}$$

converges and its limit can be found as follows. Let $\mathscr{C}_{\mathscr{A}}$ be a finite von Neumann category with a normal trace tr *and let $\mathscr{M}$ be an object of $\mathscr{C}_{\mathscr{A}}$ supplied with a unitary action of π having the character*

$$g \mapsto \chi_{\mathscr{M}}(g) \ = \ \frac{\chi_0(g)\chi(g)}{n}, \quad g \in \pi, \tag{9-10}$$

where χ denotes the character of ρ (we know from Sect. 7 that such von Neumann representation $\mathscr{M}$ exists). Then the limit of the sequence (9-9) equals to the dimension (with respect to the trace tr*) of the extended homology $\mathscr{H}_i(X, \mathscr{M})$:*

$$\lim_{k\to\infty} \frac{\dim H_i(X, \mathscr{E} \otimes \mathscr{E}_k)}{\dim \mathscr{E} \cdot \dim \mathscr{E}_k} = \dim_{\mathrm{tr}} \mathscr{H}_i(X, \mathscr{M}). \tag{9-11}$$

Proof. Theorem 9.6 follows by applying Theorem 9.2 similarly to the arguments given in 9.3, 9.4, 9.5. We will only point out here how one constructs the function $N : \pi \to \mathbb{R}_+$, which appears in **F**. For $g \in \pi$ we define

$$N(g) = n^{-1} \cdot \sup_j \{||\sigma_j(\chi(g))||\}, \tag{9-12}$$

where $\sigma_j : \mathscr{F} \to \mathbb{C}$ runs over all the embeddings of $\mathscr{F}$. We consider the representations $\rho \otimes \rho_k$, where $k = 1, 2, \dots,$ as defined over the ring of algebraic integers of the compositum $\mathscr{F}\mathscr{F}'$ of the fields $\mathscr{F}$ and $\mathscr{F}'$; any embedding of $\mathscr{F}\mathscr{F}'$ into $\mathbb{C}$ determines embeddings of $\mathscr{F}$ and $\mathscr{F}'$, and (9-12) is clearly enough to establish property **F** (cf. 6.1), since $\mathscr{F}'$ is assumed to be cyclotomic and so its Galois transformations are unitary. □

10. Examples

10.1. Here we show that Theorem 7.2 is false if the trace tr on von Neumann category $\mathscr{C}_{\mathscr{A}}$ is not normal.

We will construct two finite von Neumann categories with traces (one normal and one not normal) and two unitary representations of the fundamental group of a polyhedron X on objects of these categories, such that the characters of this representations are equal but the projective dimensions of the corresponding homology are distinct.

As the first von Neumann category we will take the category $\mathscr{C}_1$ of finite dimensional Euclidean vector spaces with the usual trace. As the second category $\mathscr{C}_2$ we will take the category $\mathscr{C}(\mu)$ (cf. Sect. 4), where μ is the constant sequence $\mu^k = 1$. We will consider the Dixmier type trace tr_ω in $\mathscr{C}_2$, cf. (4-4).

Now we will return to the example described in Subsect. 6.3. The space X was obtained by 0-framed surgery on the trefoil knot, and for any $\xi \in S^1$ we had a unitary flat bundle $\mathscr{E}_\xi$ over X. We will suppose that the sequence of points ξ_k on the unit circle is chosen so that $\xi_k \to \xi_+$ and $\xi_k \neq \xi_+$, where $\xi_+ = e^{\pi i/3}$ is a root of the Alexander polynomial, cf. 6.3. Then the sequence of flat bundles $\mathscr{E}_{\xi_k}$, viewed as a single flat bundle $\mathscr{V}$ with fiber an object of $\mathscr{C}_2$, has character $g \mapsto \xi_+^{\phi(g)}$, where $g \in \pi_1(X)$ and $\phi : \pi_1(X) \to \mathbb{Z}$ is the abelinization. We see that the same character has the line bundle $\mathscr{E}_{\xi_+}$ (viewed as bundle in $\mathscr{C}_1$). Then we have $\dim_{\mathrm{tr}_\omega} \mathscr{H}_1(X, \mathscr{V}) = 0$, however $\dim H_1(X, \mathscr{E}_{\xi_+}) = 2$.

10.2. Here we will show that Theorem 9.2 is false without assumption (9-1) that the degrees of the number fields h_k are bounded.

We will again use the example 6.3. We choose points ξ_k on the unit circle such that ξ_k converges to ξ_+ and ξ_k for any k is a root of 1. We will assume that $J = \{k; \xi_k = \xi_+\}$ is a subsequence; we may actually choose the sequence ξ_k such that J is arbitrary. Note that the corresponding sequence of flat line bundles $\mathscr{E}_{\xi_k}$ satisfies all the conditions of arithmetic approximation, cf. 9.1, besides (9-1). We see that the sequence of dimensions $\dim H_1(X, \mathscr{E}_{\xi_k})$ is the following: we have 2, for $k \in J$ and we have 0, for $k \notin J$. Thus sequence (9-4) is not convergent.

10.3. Algebraic integers on the unit circle. Here (preparing tools for the next example) we observe that *there exist algebraic integers on the unit circle, which are not roots of unity.* The simplest example is as follows. Consider the roots of the equation

$$\begin{aligned} & z^4 - z^3 - z^2 - z + 1 = \\ & \quad = (z^2 - \frac{1+\sqrt{13}}{2} \cdot z + 1) \cdot (z^2 - \frac{1-\sqrt{13}}{2} \cdot z + 1) = 0 \, . \end{aligned} \tag{10-1}$$

Two of its roots are complex, lying on the unit circle, they are roots of the second factor in (10-5). We will denote them $e^{i\alpha}$ and $e^{-i\alpha}$. Here $\alpha \simeq 130.6463$ degrees.

Two other roots are real, we will denote them by r and r^{-1}, where $r \simeq 0.5807$.

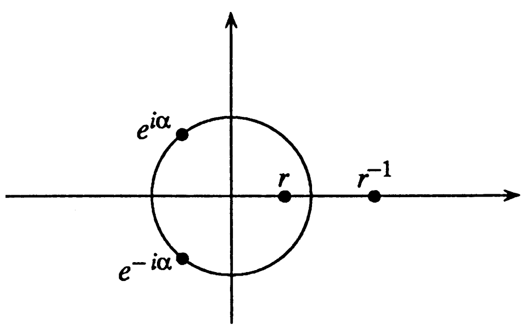

Fig. 4

The numbers $e^{i\alpha}$, $e^{-i\alpha}$, r and r^{-1} are algebraic integers, which are all conjugate to each other. We conclude that $e^{i\alpha}$ is not root of 1 since otherwise all its conjugates would be roots of 1, and so they would be points of the unit circle.

Also, the numbers $e^{i\alpha}$, $e^{-i\alpha}$, r and r^{-1} are in fact units of the corresponding ring of algebraic integers.

We observe that the powers of $e^{i\alpha}$ are dense on the unit circle. The powers of r tend to 0.

10.4. Here we will show that Theorem 9.2 is false without condition **F** in 9.1.

Let X be the 3-manifold obtained by 0-framed surgery from the trefoil, as in example 6.3. We will use the notations introduced in 6.3 and in 10.1 and 10.2.

Let $e^{i\alpha}$ denote the algebraic integer on the unit circle, constructed in 10.3. The powers $e^{in\alpha}$, where $n \in \mathbb{Z}$, are dense on the circle, and thus we can find a subsequence n_k such that $e^{in_k\alpha}$ converges to ξ_+ (recall that ξ_+ denotes the root of the Alexander polynomial of the trefoil). We will denote by $\mathscr{E}_k$ the unitary flat line bundle over X corresponding to the point $e^{in_k\alpha}$ (as in 10.1). Then we obtain, that the sequence of Betti numbers $\dim H_1(X, \mathscr{E}_k)$ consists of zeros, and the corresponding characters converge to the character of $\mathscr{E}_{\xi_+}$, but $\dim H_1(X, \mathscr{E}_{\xi_+}) = 2$.

Note that in this example all the conditions of arithmetic approximation of 9.1 except **F** are satisfied. Our field $\mathscr{F}$ in this example has 4 embeddings $\sigma_j : \mathscr{F} \to \mathbb{C}$, $j = 1, 2, 3, 4$. The embedding, which sends the number $e^{i\alpha}$ to r^{-1} (cf. notations of 10.3) sends $e^{in_k\alpha}$ to r^{-n_k}, which tends to ∞, violating **F**.

11. Approximation in characteristic *p*

11.1. Theorem. *Let p be a prime number. Suppose that*

$$\pi \supset \Gamma_1 \supset \Gamma_2 \supset \dots, \quad \textit{where} \quad \cap \Gamma_j = \{1\}, \tag{11-1}$$

is a chain of normal subgroups such that for each j the index $[\pi : \Gamma_j]$ is a power of p. Let X be a finite CW complex with fundamental group π and let $\tilde{X}_j \to X$ be the normal covering corresponding to Γ_j. Then for any i the sequence

$$\frac{\dim_{\mathbb{F}_p} H_i(\tilde{X}_j, \mathbb{F}_p)}{[\pi : \Gamma_j]}, \quad j = 1, 2, \dots \tag{11-2}$$

decreases and so the limit

$$\lim_{j\to\infty} \frac{\dim_{\mathbb{F}_p} H_i(\tilde{X}_j, \mathbb{F}_p)}{[\pi : \Gamma_j]} \tag{11-3}$$

exists.

11.2. Corollary. *If for some prime p the fundamental group π of a finite CW complex X admits a chain of normal subgroups (11-1) such that all the factors π/Γ_j are p-groups, then the following inequality holds*

$$b_i^{(2)}(X) \leq \dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p), \tag{11-4}$$

for the L^2-Betti number $b_i^{(2)}(X)$.

Corollary 11.2 follows immediately from Theorem 11.1 using the Theorem of Lück [L] and the inequality

$$\dim_{\mathbb{C}} H_i(\tilde{X}_j, \mathbb{C}) \leq \dim_{\mathbb{F}_p} H_i(\tilde{X}_j, \mathbb{F}_p).$$

Proof of Theorem 11.1. Using Corollary on page 25 of [La], we may assume that in the given chain of normal subgroups (11-1) all the factors π/Γ_j are cyclic of order p. Thus, to prove Theorem 11.1 it is enough to show that

$$\dim_{\mathbb{F}_p} H_i(\tilde{X}_{j+1}, \mathbb{F}_p) \leq p \cdot \dim_{\mathbb{F}_p} H_i(\tilde{X}_j, \mathbb{F}_p) \tag{11-5}$$

for any j.

Fix a tringulation of X and consider the induced triangulations on all the coverings $\tilde{X}_j$.

From this moment we will assume that j is fixed. We will consider the p-sheeted covering $\tilde{X}_{j+1} \to \tilde{X}_j$. Let C denote the chain complex of simplicial chains of $\tilde{X}_{j+1}$ with coefficients in the finite field $\mathbb{F}_p$. C is a free finitely generated chain complex over the ring $\Lambda = \mathbb{F}_p[\mathbb{Z}/p]$. Note that Λ has a unique maximal ideal $\mathfrak{m} = (t-1)\Lambda$, where t denotes the generator of $\mathbb{Z}/p$. We have the following filtration on Λ:

$$\Lambda \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^{p-1} \supset 0.$$

Therefore, we obtain the filtration

$$C \supset \mathfrak{m}C \supset \mathfrak{m}^2 C \supset \cdots \supset \mathfrak{m}^{p-1} C \supset 0$$

and all the factor-complexes $\mathfrak{m}^r C/\mathfrak{m}^{r+1} C$, where $r = 0, 1, \ldots p-1$, are isomorphic to the chain complex of $\tilde{X}_j$ with coefficients in $\mathbb{F}_p$. We obtain that there is a spectral sequence, starting from

$$\bigoplus_{p \text{ times}} H_i(\tilde{X}_j, \mathbb{F}_p)$$

and converging to $H_i(\tilde{X}_{j+1}, \mathbb{F}_p)$. This proves (11-5). $\square$

11.3. Questions. Can limit (11-3) be greater than the L^2-Betti number $b_i^{(2)}(X)$?

Does sequence (11-3) always stabilize after a finitely many steps?

12. Proofs of Theorems 7.2, 8.2, 8.3, 9.2

Here we finally present proofs of the main theorems of this paper. These proofs are related to each other. Therefore we use the same notations and terminology. In fact, we assume that the reader will read the proofs in the proper order (7.2, 8.2, 8.3 and then 9.2 - lexicographical ordering!). Also, we very much use arguments of Lück's paper [L], and sometimes we do not repeat them, but instead refer to [L]. Thus, it will be very helpful for the reader to have a copy of [L] at hand while reading this section.

12.1. Proof of Theorem 7.2. Suppose that X has a fixed tringulation. Consider the chain complex $C_*(\tilde{X})$ of the simplicial chains in the universal covering $\tilde{X}$. It is a complex of free finitely generated $\mathbb{Z}[\pi]$-modules; its basis is formed by the lifts of the oriented simplices of X. Note that each chain module $C_i(\tilde{X})$ is naturally supplied with a non-degenerate $\mathbb{Z}[\pi]$-valued Hermitian scalar product which is defined using the basis formed by the lifts of the cells as the orthonormal basis. The boundary homomorphism $d : C_{i+1}(\tilde{X}) \to C_i(\tilde{X})$ is given by the matrix with entries in $\mathbb{Z}[\pi]$. Consider the "adjoint" homomorphism $d^* : C_i(\tilde{X}) \to C_{i+1}(\tilde{X})$ which is defined using the above mentioned $\mathbb{Z}[\pi]$-valued Hermitian scalar product. Then we have the following self-adjoint homomorphism

$$d^*d : C_{i+1}(\tilde{X}) \to C_{i+1}(\tilde{X}), \quad d^*d \in \mathbb{Z}[\pi] \otimes \mathrm{Mat}(a \times a, \mathbb{Z}), \tag{12-1}$$

(where a denotes the number of $(i+1)$-dimensional simplices in X). If $p(z)$ is any polynomial with real coefficients then we may form $p(d^*d)$ and the result will be a self-adjoint matrix with entries in $\mathbb{R}[\pi]$. Now, applying the character $\chi_{\mathscr{M}}$ to this matrix produces a matrix $\chi_{\mathscr{M}}(p(d^*d))$ with entries in $\mathbb{C}$, which is Hermitian. We will consider then the trace (in the usual sense) of this Hermitian matrix $\mathrm{Tr}(\chi_{\mathscr{M}}(p(d^*d)))$. Note that the same answer will be obtained if we will first map the matrix d^*d via the representation $\rho : \mathbb{C}[\pi] \to \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}, \mathscr{M})$, then applying the polynomial $p(z)$ to get

$$p(\rho(d^*d)) \in \mathrm{Hom}_{\mathscr{C}_{\mathscr{A}}}(\mathscr{M}^a, \mathscr{M}^a), \tag{12-2}$$

and finally computing the trace $\mathrm{tr}_{\mathscr{M}^a}$ of (12-2):

$$\mathrm{Tr}(\chi_{\mathscr{M}}(p(d^*d))) = \mathrm{tr}_{\mathscr{M}^a}(p(\rho(d^*d))). \tag{12-3}$$

This follows from the definition of the trace on a category (cf. 1.3) and the definition of the character.

We would like to be able to compute (using only the character) more general expressions of the form $\mathrm{Tr}(\chi_{\mathscr{M}}(f(d^*d)))$, where $f(z)$ is a real valued function. The most important for us is the case, when the function $f(z)$ above is the characteristic function of an interval $[0, \lambda^2]$, which we will denote by $f_\lambda(z)$.

According to W. Lück [L], this can be done as follows. Choose a sequence of real polynomials $p_n(z)$ such that

$$p_n(z) \to f_\lambda(z) \quad \text{and} \quad |p_n(z)| \le L, \tag{12-4}$$

where both properties (12-4) hold for any $z \in [0, N]$. Here N is a fixed apriori large number such that

$$|\rho(d^*d)| \le N \quad \text{for any unitary representation} \quad \rho. \tag{12-5}$$

We will take N to be a times the sum of all coefficients, which appear in the matrix elements of d^*d (this claim is similar to Lemma 2.5 in [L]). To be more precise, we know, that $d^*d = (b_{ij})$, where the entries b_{ij} of this $a \times a$-matrix belong to the group ring $\mathbb{C}[\pi]$, $b_{ij} = \sum \beta_{ij}(g) \cdot g$, where the sum is taken over $g \in \pi$ (only finitely many terms are nonzero). We define N as

$$N = \sum_{i,j,g} |\beta_{ij}(g)|.$$

Using the Lebesgue theorem on Majorized convergence and the assumption that the trace tr is normal and, therefore it is continuous with respect to the ultraweak topology on $\hom_{\mathscr{C}_\mathscr{A}}(\mathscr{M}^a, \mathscr{M}^a)$ (cf. [Di], Part I, chapter 6, Sect. 1), we obtain that the operator $p_n(\rho(d^*d))$ converges ultraweakly to

$$f_\lambda(\rho(d^*d)) = \int_0^{\lambda^2} dE_\lambda, \tag{12-6}$$

in the von Neumann algebra $\hom_{\mathscr{C}_\mathscr{A}}(\mathscr{M}^a, \mathscr{M}^a)$, where E_λ is the right continuous spectral decomposition of $\rho(d^*d)$. Thus using (12-3), we find

$$\mathrm{Tr}(\chi_\mathscr{M}(p_n(d^*d))) = \mathrm{tr}_{\mathscr{M}^a}(p_n(\rho(d^*d))) \to \mathrm{tr}_{\mathscr{M}^a}(f_\lambda(\rho(d^*d))). \tag{12-7}$$

We obtain finally the following formula for the spectral density function $F_i(\lambda)$ of the extended homology $\mathscr{H}_i(X, \mathscr{M})$:

$$F_i(\lambda) = \lim_{n\to\infty} \mathrm{Tr}(\chi_\mathscr{M}(p_n(d^*d))), \quad \lambda > 0. \tag{12-8}$$

The last formula involves only the character $\chi_\mathscr{M}$. Since $F_i(\lambda)$ is right continuous, we find also (using only the character $\chi_\mathscr{M}$) the von Neumann dimension of the extended homology $\dim_{\mathrm{tr}} P(\mathscr{H}_i(X, \mathscr{M}))$ as the limit $\lim_{\lambda\to+0} F_i(\lambda)$. This completes the proof of Theorem 7.2. □

12.2. Proof of Theorems 8.2 and 8.3. The proof uses the methods of Lück [L] with certain adjustments. We will use the notations introduced in the proof of Theorem 7.2, cf. 12.1. In particular we will use formula (12-8). As in the proof of 7.2 we will denote by $F_i(\lambda)$ the spectral density function of the extended homology $\mathscr{H}_i(X, \mathscr{M})$. Since the trace tr is assumed to be normal, we will assume that $F_i(\lambda)$ is right continuous. The von Neumann dimension $\dim_{\mathrm{tr}} P(\mathscr{H}_i(X, \mathscr{M})$ is by the definition $F_i(0)$.

For $k = 1, 2, \dots$ denote by $F_i^k(\lambda^2)$ the spectral density function of the finite dimensional operator $\rho_k(d^*d)$, where $\rho_k : \pi \to \mathrm{End}(V^k)$ is the k-th representation. As in the proof of 7.2, we regard here d^*d as the matrix with entries in the group ring $\mathbb{Z}[\pi]$, i.e. $d^*d \in \mathbb{Z}[\pi] \otimes \mathrm{Mat}(a \times a, \mathbb{Z})$, where a denotes the number of $(i+1)$-dimensional cells in X. Therefore, $\rho_k(d^*d) \in \mathrm{End}(V^k) \otimes \mathrm{Mat}(a \times a, \mathbb{Z})$. Similarly to (12-8) we have for $\lambda > 0$

$$F_i^k(\lambda) = \lim_{n\to\infty} \mathrm{Tr}(\chi_k(p_n(d^*d))), \tag{12-9}$$

where $p_n(z)$ is any sequence of polynomials constructed as in the proof of Theorem 7.2. Let us introduce also the functions

$$G_i(\lambda) = \mathrm{Lim}_\omega[\mu^k F_i^k(\lambda)] \text{ and} \tag{12-10}$$

$$G_i^+(\lambda) = \lim_{\epsilon\to+0} G_i(\lambda + \epsilon), \tag{12-11}$$

defined for $\lambda \geq 0$. According to our definitions, we have

$$G_i(0) = \mathfrak{projdim}_\omega P(\mathscr{H}_i(X, \mathscr{V})) \text{ and} \tag{12-12}$$

$$G_i^+(0) = \mathfrak{projdim}_\omega P(\mathscr{H}_i(X, \mathscr{V})) + \mathfrak{tordim}_\omega T(\mathscr{H}_i(X, \mathscr{V})) . \tag{12-13}$$

Therefore to prove Theorem 8.2 we have to show that $F_i(0) = G_i^+(0)$.

Note that $G_i(\lambda)$ is the spectral density function of extended homology $\mathscr{H}_i(X, \mathscr{V})$ as defined in [F], Sect. 3.7. We will see now that the second function $G_i^+(\lambda)$ is in fact more important.

We will choose the polynomials $p_n(z)$ as follows. Denote by $g_n : \mathbb{R} \to \mathbb{R}$ the function

$$g_n(z) = \begin{cases} 1 + 1/n \quad \text{for} \quad z \leq \lambda^2, \\ 1 + 1/n - n(z - \lambda^2) \quad \text{for} \quad \lambda^2 \leq z \leq \lambda^2 + 1/n, \\ 1/n \quad \text{for} \quad \lambda^2 + 1/n \leq z \end{cases} \tag{12-14}$$

and construct the polynomials $p_n(z)$ such that

$$g_n(z) \leq p_n(z) \leq 2 \quad \text{and} \quad \lim_{n\to\infty} p_n(z) = f_\lambda(z)$$

for all $z \in [0, N]$. Here $f_\lambda(z)$ denotes the characteristic function of the interval $[0, \lambda^2]$ and N is the large number constructed in the proof of Theorem 7.2, cf. (12-5).

With this choice of the polynomials $p_n(z)$ we may show that for any n, k and $\lambda > 0$ holds

$$\mu^k F_i^k(\lambda) \leq \mathrm{Tr}(\tilde{\chi}_k(p_n(d^*d))) \leq (1 + 1/n)\mu^k F_i^k(\lambda + 1/n) + a/n, \tag{12-15}$$

where a denotes the number of $(i+1)$-dimensional cells in X. To prove this one denotes by $E_k(\lambda)$ the ordered set of eigenvalues z of $\rho_k(d^*d)$ satisfying $z \leq \lambda$ listed with multiplicities. Then

$$\mathrm{Tr}(\tilde{\chi}_k(p_n(d^*d))) = \mu^k \cdot \sum_{z\in E_k(\lambda^2)} p_n(z),$$

and now to obtain (12-15) one just repeats the arguments on page 469 of [L].

Taking in (12-15) for fixed n the limit Lim_ω with respect to k and using the assumption that $\tilde{\chi}_k \to \chi_{\mathscr{M}}$ we obtain

$$G_i(\lambda) \leq \mathrm{Tr}(\chi_{\mathscr{M}}(p_n(d^*d))) \leq (1 + 1/n)G_i(\lambda + 1/n) + a/n. \tag{12-16}$$

Therefore, taking the limit in (12-16) when $n \to \infty$ and using (12-8) we get

$$G_i(\lambda) \leq F_i(\lambda) \leq G_i^+(\lambda). \tag{12-17}$$

From the last inequality we obtain for $\epsilon > 0$

$$F_i(\lambda) \leq G_i^+(\lambda) \leq G_i(\lambda + \epsilon) \leq F_i(\lambda + \epsilon), \tag{12-18}$$

and since we know that $F_i(\lambda)$ is right continuous, this shows (by passing to the limit when $\epsilon \to 0$) that

$$F_i(\lambda) = G_i^+(\lambda). \tag{12-19}$$

This is precisely the statement of Theorem 8.3.

Since both functions in the last equality are right continuous, we obtain $F_i(0) = G_i^+(0)$, which completes the proof of Theorem 8.2 (cf. 12-13)). □

12.3. Proof of Theorem 9.2.(i) and 9.2.(ii). We will use the following Lemma from algebraic number theory:

Lemma A. *Let $\mathscr{F}_k \subset \mathbb{C}$ be a number field of degree $h_k \leq h$ and let $\mathfrak{o}_k$ be the ring of algebraic integers of $\mathscr{F}_k$. Let $\sigma_1, \sigma_2, \dots, \sigma_{h_k} : \mathscr{F}_k \to \mathbb{C}$ denote all the distinct embeddings of $\mathscr{F}_k$ into the complex numbers. Then for any element $\alpha \in \mathfrak{o}_k$ with $\alpha \neq 0$, the condition*

$$|\sigma_i(\alpha)| \leq R \quad \textit{for all} \quad i = 1, 2, \dots, h_k \tag{12-20}$$

implies

$$|\sigma_j(\alpha)| \geq R^{1-h_k} \tag{12-21}$$

for any $j = 1, 2, \dots, h_k$.

This Lemma is well known, however we will give a simple independent proof. Similar argument is used in [Sh], in the proof of Theorem 11 in chapter 1.

Proof of Lemma A. The product

$$\prod_{i=1}^{h_k} \sigma_i(\alpha)$$

(the norm of α) is a nonzero integer. Therefore we obtain

$$|\sigma_j(\alpha)| \geq \prod_{i=1, i\neq j}^{h_k} |\sigma_i(\alpha)|^{-1} \geq R^{1-h_k}.$$

This completes the proof of Lemma A.

Here is another lemma, which we will need:

Lemma B. *Let $A = (a_{ij})$ be a $k \times k$-matrix with complex entries. Suppose that for some $C > 0$ and $K \geq 1$ holds*

$$|\operatorname{Tr}(A^r)| \leq C \cdot K^r \tag{12-22}$$

for all $r = 1, 2, \dots k$. Then we have the following estimate for the coefficients $s_r = s_r(A)$ of the characteristic polynomial $\det(\lambda - A) = \sum_{r=0}^{k} (-1)^{k-r} s_{k-r}(A)\lambda^r$ *of A:*

$$|s_r(A)| \leq \frac{C(C+1)\dots(C+r-1)}{r!} \cdot K^r. \tag{12-23}$$

Proof of Lemma B. For $r = 1, 2, \dots k$ denote

$$p_r = \mathrm{Tr}(A^r) = \sum_{i=1}^{k} \lambda_i^r,$$

where $\lambda_1, \lambda_2, \dots \lambda_k$ denote the eigenvalues of A. We have

$$s_r = s_r(A) = \sum_{i_1 < \dots < i_r} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r}. \tag{12-24}$$

We will prove (12-23) by induction on r using the following Newton's identity

$$(-1)^r \cdot r \cdot s_r = s_1 p_{r-1} - s_2 p_{r-2} + \dots + (-1)^r s_{r-1} p_1 - p_r, \tag{12-25}$$

cf. [CR], page 314.

Since $s_1 = p_1$, the inequality (12-23) holds for $r = 1$; suppose that it has been established for all values of r which are less than the given one. Then from (12-25) we obtain

$$r \cdot |s_r| \le C \cdot K^r \cdot \{\sum_{j=1}^{r-1} \binom{C+j-1}{j} + 1\},$$

and now the desired inequality (12-23) follows from the identity

$$\frac{C}{r} \cdot \{\sum_{j=1}^{r-1} \binom{C+j-1}{j} + 1\} = \binom{C+r-1}{r}, \tag{12-26}$$

which can be easily checked by induction. This completes the proof of Lemma B.

Now we will prove statement (i) of Theorem 9.2. We will use the notations introduced in the proofs of Theorems 7.2 and 8.2. Also we will use the notations introduced in 9.1.

Let us fix some k. We know that the matrix $\rho_k(d^*d) \in \mathrm{End}(V^k) \otimes \mathrm{Mat}(a \times a, \mathbb{Z})$ is congruent over $\mathbb{C}$ to the matrix $\tilde{\rho}_k(d^*d) \in \mathrm{End}_{\mathscr{F}_k}(W^k) \otimes \mathrm{Mat}(a \times a, \mathbb{Z})$ (by condition **B** in 9.1. The latter matrix has entries in the field $\mathscr{F}_k$ and therefore the characteristic polynomial

$$q_k(t) = \det(t - \rho_k(d^*d)) \tag{12-27}$$

(of $\rho_k(d^*d)$ or equivalently of $\tilde{\rho}_k(d^*d)$) has coefficients in $\mathscr{F}_k$. Write $q_k(t) = t^\nu \overline{q}_k(t)$, where $\overline{q}_k(0) \neq 0$ and $\overline{q}_k(0) \in \mathscr{F}_k$.

We claim now that

$$M^{a \dim V^k} \cdot \overline{q}_k(0) \quad \text{belongs to} \quad \mathfrak{o}_k, \tag{12-28}$$

where M is the denominator of the arithmetic approximation, cf. 9.1. To show this we note, that from our assumptions **C** and **D** in 9.1 it follows that M times the dual (over the field $\mathscr{F}_k$) of the $\mathfrak{o}_k$-homomorphism

$$d : \mathscr{L}_k \otimes_\pi C_{i+1}(\tilde{X}) \to \mathscr{L}_k \otimes_\pi C_i(\tilde{X})$$

is well defined as a homomorphism

$$\mathscr{L}_k \otimes_\pi C_i(\tilde{X}) \to \mathscr{L}_k \otimes_\pi C_{i+1}(\tilde{X}).$$

In other words,

$$M \cdot \tilde{\rho}_k(d^*d) : W_k \otimes_\pi C_{i+1}(\tilde{X}) \to W_k \otimes_\pi C_{i+1}(\tilde{X})$$

preserves the $\mathfrak{o}_k$-lattice $\mathscr{L}_k \otimes_\pi C_{i+1}(\tilde{X})$. If we assume that the last lattice is free over $\mathfrak{o}_k$, then we may take its basis as the basis of $W_k \otimes_\pi C_{i+1}(\tilde{X})$ and compute the characteristic polynomial with respect to this basis. We obtain that the polynomial $M^{a \dim V^k} \cdot q_k(tM^{-1})$ has all coefficients in $\mathfrak{o}_k$. Therefore all coefficients of $M^{a \dim V^k - \nu} \cdot \overline{q}_k(tM^{-1})$ belong to $\mathfrak{o}_k$ and therefore $M^{a \dim V^k - \nu} \cdot \overline{q}_k(0) \in \mathfrak{o}_k$, which implies our statement (12-28).

In order to prove (12-28) in the general case (not assuming that the lattice $\mathscr{L}_k \otimes_\pi C_{i+1}(\tilde{X})$ is free), we proceed as follows. For any prime ideal $\mathfrak{p} \subset \mathfrak{o}_k$ we consider the localization of $\mathscr{L}_k \otimes_\pi C_{i+1}(\tilde{X})$ with respect to the complement of $\mathfrak{p}$ which is now a free $(\mathfrak{o}_k)_\mathfrak{p}$-module (since $(\mathfrak{o}_k)_\mathfrak{p}$ is a principal ideal ring, cf. [FT], page 59). The arguments of the previous paragraph show that the valuation

$$v_\mathfrak{p}(M^{a \dim V^k} \cdot \overline{q}_k(0)) \geq 0$$

is non-negative. Since this is true for any prime ideal $\mathfrak{p}$ of $\mathfrak{o}_k$, we obtain that the fractional ideal generated by $M^{a \dim V^k} \cdot \overline{q}_k(0)$ is contained in $\mathfrak{o}_k$, which proves (12-28).

We know that $\overline{q}_k(0)$ is the product of all the nonzero eigenvalues of the matrix $\rho_k(d^*d)$ and from (12-5) we know that any of these eigenvalues is less or equal than N, where the number $N \geq 1$ was constructed in the proof of Theorem 7.2. Note that N is determined only by the matrix d^*d (i.e. by the polyhedron X) and does not depend on k. Therefore we obtain that

$$|\overline{q}_k(0))| \leq N^{a \dim V^k}, \quad i = 1, 2, \ldots, h_k. \tag{12-29}$$

We claim that a similar estimate

$$|\sigma_j(\overline{q}_k(0))| \leq N_1^{a \dim V^k}, \quad i = 1, 2, \ldots, h_k. \tag{12-30}$$

holds for any embedding $\sigma_j : \mathscr{F}_k \to \mathbb{C}$ of the number field $\mathscr{F}_k$ into $\mathbb{C}$, where the constant N_1 is independent on k and of $j = 1, 2, \ldots h_k$. To show this, we will denote by $g_1, g_2, \ldots g_s$ all elements of π, which appear with nonzero coefficients in the matrix d^*d. We set

$$N_1 = 4 \cdot N \cdot L, \quad \text{where} \quad L = \max\{N(g_1), N(g_2), \ldots, N(g_s), 1\}; \tag{12-31}$$

the numbers $N(g_i)$ are given by property **F** in 9.1. In order to prove inequality (12-30) we will apply Lemma B to the following $(a \dim V^k \times a \dim V^k)$-matrix

$$A = \sigma_j(\rho_k(d^*d)) \tag{12-32}$$

for fixed k and j; here we view $\rho_k(d^*d)$ as the matrix with entries in the field $\mathscr{F}_k$ and we obtain a complex matrix applying the embedding $\sigma_j : \mathscr{F}_k \to \mathbb{C}$. An obvious argument using our assumption (9-3) gives the estimate

$$|\operatorname{tr}(A)| \leq a \dim V^k \cdot N \cdot L. \tag{12-33}$$

Similarly, using our assumption (9-2) about the behavior of the function $N(g)$, gives

$$|\operatorname{tr}(A^i)| \leq a \dim V^k \cdot (NL)^i \tag{12-34}$$

for all i. Now, by Lemma B, we may conclude (using the notation introduced in Lemma B) that

$$|s_r(A)| \leq \binom{a \dim V^k + r - 1}{r} \cdot (NL)^r \tag{12-35}$$

for $r = 0, 1, 2, \ldots, a \dim V^k$. Clearly, $\sigma_j(\overline{q}_k(0)) = s_r(A)$ for the largest $r \leq a \dim V^k$ with $s_r(A) \neq 0$, and since we have the following obvious estimate for the binomial coefficient

$$\binom{a \dim V^k + r - 1}{r} \ \leq\ 2^{2a \dim V^k - 1} \ <\ 4^{a \dim V^k}, \tag{12-36}$$

combining (12-35) and (12-36), we obtain (12-30).

Using inequality (12-21) of Lemma above and also (12-23), we obtain

$$|M^{a \dim V^k} \overline{q}_k(0)| \geq (MN_1)^{a \dim V^k (1-h_k)}, \tag{12-37}$$

which implies

$$|\overline{q}_k(0)| \geq M^{-ha \dim V^k} N_1^{(1-h)a \dim V^k}. \tag{12-38}$$

Now we use Lemma 2.8 of Lück [L]. We apply it to the operator $\rho_k(d^*d)$ and estimate (12-38). We obtain (using the notations introduced in the proofs of Theorems 7.2 and 8.2 and after some elementary transformations)

$$\mu^k[F_i^k(\lambda) - F_i^k(0)] \leq \frac{c}{-\ln(\lambda)}, \tag{12-39}$$

where the constant c is $c = h \log M + (h-1) \log N_1 + \log N$. Taking $\operatorname{Lim}_\omega$ with respect to $k \to \infty$ in (12-27) we obtain

$$G_i(\lambda) - G_i(0) \leq \frac{c}{-\ln(\lambda)}. \tag{12-40}$$

By the definition $\mathfrak{tordim}_\omega T(\mathscr{H}_i(X, \mathscr{V}^{\cdot})) = G_i^+(0) - G_i(0)$ (cf. (12-12) and (12-13)). Comparing this with (12-40) we see that the torsion dimension vanishes.

Now to finish the proof we only have to show that the sequence $\mu^k \dim H_i(X, V^k) = \mu^k F_i^k(0)$ converges. To do so, we will introduce the following notations:

$$\underline{G}_i(\lambda) = \lim\inf_{k\to\infty} \mu^k F_i^k(\lambda),$$

$$\overline{G}_i(\lambda) = \lim\sup_{k\to\infty} \mu^k F_i^k(\lambda),$$

$$\underline{G}_i^+(\lambda) = \lim_{\epsilon\to+0} \underline{G}_i(\lambda+\epsilon),$$

$$\overline{G}_i^+(\lambda) = \lim_{\epsilon\to+0} \overline{G}_i(\lambda+\epsilon).$$

From (12-15) we obtain (by passing to the limits with respect to k)

$$\overline{G}_i(\lambda) \leq \mathrm{Tr}(\chi_{\mathscr{M}}(p_n(d^*d)))) \leq (1+1/n)\underline{G}_i(\lambda+1/n) + a/n$$

and the limit $n\to\infty$ gives

$$\overline{G}_i(\lambda) \leq F_i(\lambda) \leq \underline{G}_i^+(\lambda).$$

Now we see that for $\epsilon > 0$ we have

$$F_i(\lambda) \leq \underline{G}_i^+(\lambda) \leq \underline{G}_i^+(\lambda+\epsilon) \leq \overline{G}_i(\lambda+\epsilon) \leq F_i(\lambda+\epsilon)$$

and thus when $\epsilon\to 0$ we get

$$F_i(\lambda) = \underline{G}_i^+(\lambda) = \overline{G}_i^+(\lambda). \tag{12-41}$$

In particular, we obtain

$$F_i(0) = \underline{G}_i^+(0) = \overline{G}_i^+(0). \tag{12-42}$$

On the other hand, using inequality (12-27) we obtain

$$\underline{G}_i(\lambda) \leq \underline{G}_i(0) + \frac{c}{-\ln(\lambda)},$$

$$\overline{G}_i(\lambda) \leq \overline{G}_i(0) + \frac{c}{-\ln(\lambda)},$$

which give for $\lambda\to 0$

$$\underline{G}_i^+(0) = \underline{G}_i(0), \qquad \text{and} \qquad \overline{G}_i^+(0) = \overline{G}_i(0).$$

Comparing the last equalities with (12-42) gives $\underline{G}_i(0) = \overline{G}_i(0)$. This proves the convergence of $\mu^k \dim H_i(X, V^k)$ and completes the proof of (i).

Statement (ii) was proven above by (12-40).

12.4. Proof of Theorem 9.2.(iii). The arguments here are similar to those used by D.Burghelea, L.Friedlander and T.Kappeler, cf. Appendix of [BFK].

We will use the notations introduced in the proof of Theorem 8.2 and in Theorem 7.2. We want to show that

$$\log\ \det{}'(\rho(d^*d)) \overset{def}{=} \int_{+0}^{\infty} \ln(\lambda) dF_i(\lambda) > -\infty. \tag{12-43}$$

If N is the large number constructed in Proof of Theorem 7.2, (cf. (12-5)), then $F_i(\lambda)$ is constant for $\lambda \geq N$ and so we may write (integrating by parts)

$$\begin{aligned}\log\ \det{}'(\rho(d^*d)) =&(F_i(N) - F_i(0))\cdot \ln N \ + \\ &+ \lim_{\epsilon\to+0}\{[F_i(\epsilon)-F_i(0)](-\ln\epsilon) - \int_\epsilon^N \frac{F_i(\lambda)-F_i(0)}{\lambda} d\lambda\}.\end{aligned} \tag{12-44}$$

From the last formula we obtain the inequality

$$\log\,\det{}'(\rho(d^*d)) \geq (F_i(N) - F_i(0)) \cdot \ln N \;\; - \int_{+0}^{N} \frac{F_i(\lambda) - F_i(0)}{\lambda} d\lambda. \qquad (12\text{-}45)$$

Note that in the similar formula for the finite dimensional operator $\rho_k(d^*d)$ we have the equality

$$\log\,\det{}'(\rho_k(d^*d)) = (F_i^k(N) - F_i^k(0)) \cdot \ln N \;\; - \int_{0}^{N} \frac{F_i^k(\lambda) - F_i^k(0)}{\lambda} d\lambda\}, \qquad (12\text{-}46)$$

since the spectral density function $F_i^k(\lambda)$ is constant for small $\lambda \neq 0$. Now using inequality (12-38) we find

$$\log\,\det{}'(\rho_k(d^*d)) = \log|\overline{q}_k(0)| \geq -ha\log(MN_1) \cdot \dim V^k, \qquad (12\text{-}47)$$

where a denotes the number of $(i+1)$-dimensional simplices in X and N_1 is given by (12-31). Multiplying by $\mu^k = (\dim V^k)^{-1}$, we get

$$\mu^k \cdot \log\,\det{}'(\rho_k(d^*d)) \geq -ha\log(MN_1). \qquad (12\text{-}48)$$

Similarly to statement 1 of Lemma 3.3 of Lück [L], we obtain the inequality

$$\int_0^N \frac{F_i(\lambda) - F_i(0)}{\lambda} d\lambda \leq \lim\inf_{k\to\infty}\{\int_0^N \frac{\mu^k F_i^k(\lambda) - \mu^k F_i^k(0)}{\lambda} d\lambda\}. \qquad (12\text{-}49)$$

Now, we multiply equality (12-34) by μ^k and pass to the limit infimum, when k tends to infinity. Since $\mu^k F_i(N) \to F_i(N)$ (by Theorem 8.3) and $\mu^k F_i(0) \to F_i(0)$ (by Theorem 9.2.(i)), we obtain finally (combining inequalities (12-45), (12-48) and (12-49))

$$\log\,\det{}'(\rho(d^*d)) \geq -ha\log(MN_1). \qquad (12\text{-}50)$$

This completes the proof. □

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