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THE ℓ^2-COHOMOLOGY OF ARTIN GROUPS

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Abstract

For each Artin group, the reduced ℓ^2-cohomology of (the universal cover of) its 'Salvetti complex' is computed. This is a CW-complex which is conjectured to be a model for the classifying space of the Artin group. In the many cases when this conjecture is known to hold the calculation describes the reduced ℓ^2-cohomology of the Artin group.

1. *Introduction*

Suppose that G is a countable discrete group acting properly and cellularly on a CW-complex E with compact quotient. As in [**13**] or [**15**] one can define the reduced ℓ^2-cohomology group $\mathcal{H}^i(E)$ for each non-negative integer i. (We will give the definition in Section 3.) For each i, $\mathcal{H}^i(E)$ is a Hilbert space with an orthogonal G-action. When G is infinite, these Hilbert spaces tend to be infinite-dimensional unless they are zero. However, each $\mathcal{H}^i(E)$ is a 'Hilbert G-module', and to any Hilbert G-module V one may associate a real number, called its von Neumann dimension and denoted by $\dim_G(V)$. One can then define the ℓ^2-Betti numbers of E with respect to G by

$$\ell^2 b_i(E;G) := \dim_G \mathcal{H}^i(E).$$

If E is contractible and G acts freely on E (so that E/G is a classifying space for G), these ℓ^2-cohomology groups (and their von Neumann dimensions) are independent of the choice of E and so are invariants of G. In this case we will use the notation $\mathcal{H}^i(G)$ and $\ell^2 b_i(G)$ to stand for $\mathcal{H}^i(E)$ and $\ell^2 b_i(E;G)$ respectively.

ℓ^2-cohomology groups have proved difficult to compute directly. To date, most of the calculations have been based on two principles: (i) certain vanishing theorems, and (ii) Atiyah's formula. A typical vanishing theorem will assert that for a certain class of groups G, $\mathcal{H}^i(G)$ is zero for i within a certain range. Atiyah's formula states that in the case when G acts freely on E, the alternating sum of the ℓ^2-Betti numbers of G is equal to the ordinary Euler characteristic of E/G. In cases where $\mathcal{H}^*(E)$ is known to vanish except in one dimension, this formula provides an exact calculation of the unique non-zero ℓ^2-Betti number. In this paper we compute the ℓ^2-Betti numbers $\ell^2 b_i(G)$ when G is an Artin group. Surprisingly, there is a completely clean answer which is non-trivial in the sense that many of the $\ell^2 b_i(G)$ are non-zero.

Associated to any Coxeter matrix (m_{ij}) on a finite set I, there is a Coxeter group W, an Artin group A and a simplicial complex L with vertex set I, called the 'nerve' of the Coxeter matrix. (The definitions will be given in Section 5.) Coxeter groups are intimately connected to groups generated by reflections. For example, if

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$n = \mathrm{Card}(I)$, there is a standard representation of W as a linear reflection group on $\mathbb{R}^n$, so that there is a certain convex cone in $\mathbb{R}^n$ with W acting properly on its interior Ω.

Artin groups have a related geometric interpretation. Consider the complexification of the W-representation described above. Then $\mathbb{R}^n + i\Omega$ is a W-stable open convex subset of $\mathbb{C}^n$ on which W acts properly. Moreover, W acts freely on the hyperplane complement

$$Y = (\mathbb{R}^n + i\Omega) - \bigcup_r H_r,$$

where r ranges over all reflections in W and H_r denotes the intersection of the complex hyperplane fixed by r and $(\mathbb{R}^n + i\Omega)$. It is proved in [7] that there is a finite CW-complex X, called the 'Salvetti complex' which is homotopy equivalent to Y/W. The complex X has one vertex and, for each $k > 0$, one k-cell for each $(k-1)$-simplex of L. The 2-skeleton of X is the presentation complex associated to the standard presentation of A. It follows that $\pi_1(Y/W) = \pi_1(X) = A$. It is conjectured that the universal cover $\widetilde{X}$ of X is contractible, or equivalently that X is a classifying space BA for A. Deligne [**11**] proved this conjecture in the case when W is finite. The conjecture is also known to be true when W is 'right-angled', when $\dim L \leqslant 1$, and in many other cases (see [**6**, **7**]). Our actual calculation is of $\mathscr{H}^i(\widetilde{X})$. Thus our title is somewhat misleading: we only have a computation of the reduced ℓ^2-cohomology of A modulo the conjecture that $X = BA$.

THEOREM 1. *For any Artin group A and any i, there is an isomorphism of Hilbert A-modules:*

$$\mathscr{H}^i(\widetilde{X}) \cong \overline{H}^{i-1}(L) \otimes \ell^2(A).$$

COROLLARY 2. $\ell^2 b_i(\widetilde{X}; A) = \overline{b}_{i-1}(L)$.

Here $\overline{b}_j(L)$ denotes the ordinary 'reduced Betti number' of L, that is, $\overline{b}_j(L)$ is the dimension of the reduced homology group $\overline{H}_j(L; \mathbb{R})$.

When W is finite, the Artin group A is said to be 'spherical'. If A is spherical, then its centre always has an infinite cyclic subgroup (coming from the $\mathbb{C}^*$-action on the hyperplane complement). For such groups the reduced ℓ^2-cohomology is known to vanish (see Section 4). Hence, in the spherical case, Theorem 1 is well known. This vanishing result in the spherical case is the key ingredient for our proof in the general case.

In fact, a stronger result than Theorem 1 is true. In the following statement, the 'standard homomorphism' from an Artin group to $\mathbb{Z}$ is the homomorphism that sends each Artin generator to $1 \in \mathbb{Z}$.

THEOREM 3. *Suppose that A' is a normal subgroup of the Artin group A that is contained in the kernel of the standard homomorphism $A \to \mathbb{Z}$. Let $X' \to X$ be the corresponding covering space of X and let $G = A/A'$. Then $\mathscr{H}^i(X')$ and $\overline{H}^{i-1}(L) \otimes \ell^2(G)$ are isomorphic as Hilbert G-modules. In particular, $\ell^2 b_i(X'; G) = \overline{b}_{i-1}(L)$.*

Theorem 3 was suggested by a result of [**14**].

2. *Hilbert G-modules*

As general references for the material described in Sections 2, 3 and 4, we recommend [**13**, **18**].

Let G be a countable discrete group. The Hilbert space of square-summable real-valued functions on G is denoted by $\ell^2(G)$. There is an orthogonal (right) action of G on $\ell^2(G)$, and a diagonal action on any direct sum of copies of $\ell^2(G)$. A Hilbert space V equipped with an orthogonal (right) G-action is called a *Hilbert G-module* if it is G-equivariantly isomorphic to a G-stable closed subspace of the direct sum of finitely many copies of $\ell^2(G)$. We warn the reader that in [**18**] a more general definition of a Hilbert G-module is given, and V as above is referred to as a 'finitely generated Hilbert G-module'. A *map* $f : V \to V'$ of Hilbert G-modules is by definition a G-equivariant bounded linear operator. The kernel of $f : V \to V'$, denoted $\mathrm{Ker}(f)$, is a closed subspace of V, but the image, $\mathrm{Im}(f)$, need not be closed. The map f is *weakly surjective* if $\overline{\mathrm{Im}(f)} = V$; f is a *weak isomorphism* if it is both injective and weakly surjective.

To each Hilbert G-module V it is possible to associate a non-negative real number $\dim_G(V)$, called its *von Neumann dimension*. The three most important properties of $\dim_G(-)$ are

(1) $\dim_G(V) = 0$ if and only if $V = 0$;
(2) $\dim_G(V \oplus V') = \dim_G(V) + \dim_G(V')$;
(3) $\dim_G(\ell^2(G)) = 1$.

It follows from (2) that for any map $f : V \to V'$,

$$\dim_G(V) = \dim_G(\mathrm{Ker}(f)) + \dim_G(\overline{\mathrm{Im}(f)}).$$

Using property (1), one can then deduce the following.

LEMMA 4. *Suppose that V and V' are Hilbert G-modules and that* $\dim_G(V) = \dim_G(V')$. *The following are equivalent:*

(a) *f is injective;*
(b) *f is weakly surjective;*
(c) *f is a weak isomorphism.*

Proof. By definition, (c) is equivalent to the conjunction of (a) and (b). Thus it suffices to show that (a) and (b) are equivalent. We have $\dim_G(\overline{\mathrm{Im}(f)}) = \dim_G(V) - \dim_G(\mathrm{Ker}(f))$, and so $\dim_G(\mathrm{Ker}(f)) = 0$ if and only if $\dim_G(\overline{\mathrm{Im}(f)}) = \dim_G(V) = \dim_G(V')$. □

Next suppose that $C^* = \{(C^n, \delta_n)\}$ is a cochain complex of Hilbert G-modules, that is, the coboundary maps $\delta_i : C^i \to C^{i+1}$ are G-equivariant bounded linear operators. The cohomology group

$$H^i(C^*) := \mathrm{Ker}(\delta_i)/\,\mathrm{Im}(\delta_{i-1})$$

need not be a Hilbert space since $\mathrm{Im}(\delta_{i-1})$ is not necessarily closed in $\mathrm{Ker}(\delta_i)$. However, the *reduced cohomology groups* defined as

$$\mathscr{H}^i(C^*) := \mathrm{Ker}(\delta_i)/\overline{\mathrm{Im}(\delta_{i-1})}$$

are Hilbert G-modules. In fact, each $\mathscr{H}^i(C^*)$ is equivariantly isomorphic to the orthogonal complement of $\mathrm{Im}(\delta_{i-1})$ in $\mathrm{Ker}(\delta_i)$, and so is G-isomorphic to a closed G-invariant subspace of C^i.

LEMMA 5. *Suppose that* (C^*, δ_*) *and* (C'^*, δ'_*) *are cochain complexes of Hilbert* G-*modules and that* $f : C^* \to C'^*$ *is a weak isomorphism of cochain complexes* (*that is,* f *is a cochain map and for each* i, $f_i : C^i \to C'^i$ *is a weak isomorphism*). *Then the induced map* $\mathscr{H}(f) : \mathscr{H}^i(C^*) \to \mathscr{H}^i(C'^*)$ *is also a weak isomorphism. In particular,* $\mathscr{H}^i(C^*)$ *and* $\mathscr{H}^i(C'^*)$ *are isometric Hilbert* G-*modules.*

Proof. Let Z^i and B^i (respectively Z'^i and B'^i) denote the cocycles and coboundaries in C^* (respectively C'^*). Since f is a cochain map, $f(Z^i) \subseteq Z'^i$ and $f(B^i) \subseteq B'^i$. Moreover, since f is continuous, $f(\overline{B^i}) \subseteq \overline{B'^i}$. Since f is injective this gives

$$\dim_G(Z'^i) \geqslant \dim_G(Z^i) \quad \text{and} \quad \dim_G(B'^i) \geqslant \dim_G(B^i). \tag{1}$$

From the short exact sequences

$$0 \to Z^i \to C^i \to B^{i+1} \to 0$$
$$0 \to Z'^i \to C'^i \to B'^{i+1} \to 0$$

one obtains

$$\dim_G(C^i) = \dim_G(Z^i) + \dim_G(B^{i+1})$$

and

$$\dim_G(C'^i) = \dim_G(Z'^i) + \dim_G(B'^{i+1}). \tag{2}$$

Since f is a weak isomorphism, one has $\dim_G(C'^i) = \dim_G(C^i)$. Hence

$$\begin{aligned}\dim_G(C^i) = \dim_G(C'^i) &= \dim_G(Z'^i) + \dim_G(B'^{i+1})\\ &\geqslant \dim_G(Z^i) + \dim_G(B^{i+1}) = \dim_G(C^i),\end{aligned}$$

which together with (1) implies that

$$\dim_G(Z'^i) = \dim_G(Z^i) \quad \text{and} \quad \dim_G(B'^i) = \dim_G(B^i). \tag{3}$$

It follows from Lemma 4 that the map $f : Z^i \to Z'^i$ is a weak isomorphism. This implies that $\mathscr{H}(f) : \mathscr{H}^i(C^*) \to \mathscr{H}^i(C'^*)$ is weakly surjective.

Since $f : Z^i \to Z'^i$ is a weak isomorphism, its restriction to a map from $f^{-1}(B'^i)$ to B'^i is a weak isomorphism. Hence we see that

$$\dim_G(f^{-1}(B'^i)) = \dim_G(B'^i) = \dim_G(B^i).$$

Since B^i is contained in $f^{-1}(B'^i)$, these two subsets must coincide, and so $\mathscr{H}(f)$, the induced map on cohomology, is injective.

The final claim follows from the fact that any two weakly isomorphic Hilbert G-modules are in fact equivariantly isometric [**13**]. □

3. *Cohomology with local coefficients and ℓ^2-cohomology*

Suppose that Y is a connected CW-complex with fundamental group π and universal cover $\widetilde{Y}$. Suppose also that Y is of finite type (that is, has finite skeleta.) Let $C_*(\widetilde{Y})$ denote the chain complex of $\widetilde{Y}$, which is a chain complex of finitely generated free (left) $\mathbb{Z}\pi$-modules. By a *local coefficient system on* Y we shall mean a (left) $\mathbb{Z}\pi$-module. The cochain complex for Y with coefficients M is defined by

$$C^*(Y;M) := \mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{Y}), M).$$

Let $Y' \to Y$ be a regular covering space corresponding to a normal subgroup π' of π, and let G denote the quotient group π/π'. Then $\ell^2(G)$ may be viewed as a π-G-bimodule, where the left action of π is via the action of G. The ℓ^2-cochains on Y' are defined by

$$\begin{aligned} \ell^2 C^*(Y') &:= \mathrm{Hom}_{\mathbb{Z}G}(C_*(Y'), \ell^2(G)) \\ &= \mathrm{Hom}_{\mathbb{Z}\pi}(C_*(\widetilde{Y}), \ell^2(G)) \\ &= C^*(Y;\ell^2(G)). \end{aligned}$$

Note that the orthogonal right action of G on $\ell^2(G)$ makes $\ell^2 C^*(Y')$ a chain complex of Hilbert G-modules. The unreduced and reduced cohomology groups of $\ell^2 C^*(Y')$ will be denoted $\ell^2 H^*(Y')$ and $\mathscr{H}^*(Y')$ respectively. The ℓ^2-Betti numbers of Y' with respect to G are then defined by $\ell^2 b_i(Y';G) := \dim_G \mathscr{H}^i(Y')$.

In the case when Y' is acyclic, $C_*(Y')$ is a free resolution for $\mathbb{Z}$ over $\mathbb{Z}G$. Any two such resolutions are chain homotopy equivalent. Similarly, if Y' is only rationally acyclic, the rational chain complex $C_*(Y', \mathbb{Q})$ is a free resolution for $\mathbb{Q}$ over $\mathbb{Q}G$, and is unique up to chain homotopy equivalence. It follows that in the case when Y' is rationally acyclic, the ℓ^2-cohomology groups and ℓ^2-Betti numbers depend only on G (and not on the particular choice of Y'). In this case we shall use the notation

$$\begin{aligned} \ell^2 H^i(G) &:= \ell^2 H^i(Y'), \\ \mathscr{H}^i(G) &:= \mathscr{H}^i(Y'), \\ \ell^2 b_i(G) &:= \dim_G \mathscr{H}^i(Y'). \end{aligned}$$

(Of course, we have in mind the case when $G = \pi$ and $Y' = \widetilde{Y}$ is contractible.)

4. *A vanishing theorem*

Let G be a group and Z a central subgroup of G. Any (left) G-module M may be viewed as a G-Z-bimodule where the right action of $z \in Z$ is defined by $mz = zm$. This induces a right action of Z on $H^*(G;M)$. However one has the following.

PROPOSITION 6. *With notation as above, the action of Z on $H^*(G;M)$ is trivial.*

Proof. Let P_* be a free resolution for $\mathbb{Z}$ over $\mathbb{Z}G$, so that $H^*(G;M)$ may be computed using the cochain complex $\mathrm{Hom}_G(P_*, M)$. In terms of this cochain complex the right action of $z \in Z$ on $H^*(G;M)$ sends $f \in \mathrm{Hom}_G(P_i, M)$ to zf, defined by $zf(p) = z(f(p))$ for all $p \in P_i$. It suffices to show that this map is chain-homotopic to the identity map. Since z is central, the map $P_* \to P_*$ given by $p \mapsto zp$ is a chain map, and this chain map lifts the identity map from $\mathbb{Z}$ to $\mathbb{Z}$, so is chain-homotopic to the identity map of P_*. However for all $f \in \mathrm{Hom}_G(P_*, M)$ and all $p \in P_*$, $zf(p) = f(zp)$, and so the two actions of Z (on M and on P_*) induce the same action on $\mathrm{Hom}_G(P_*, M)$. □

COROLLARY 7. *Let π be a group admitting a finite classifying space Y, let π' be a normal subgroup of π, let $G = \pi/\pi'$, and suppose that the centre of π contains an* *element whose image in G has infinite order. Then $\mathscr{H}^*(Y') = \{0\}$, where Y' denotes the regular cover of Y corresponding π'.*

Proof. Let Z denote an infinite cyclic subgroup of the centre of π with $Z \cap \pi' = \{1\}$. The usual right action of Z on $\ell^2(G)$ and the action used in Proposition 6 agree, and hence the right action of Z on $\ell^2 H^*(Y')$ is trivial. It follows that the action of Z on $\mathscr{H}^*(Y')$ is trivial. Since the image of Z in G is infinite, no non-zero element of $\ell^2(G)$ is fixed by the right action of Z. However, each $\mathscr{H}^i(Y')$ is a Hilbert G-module, and so one obtains a contradiction unless each $\mathscr{H}^i(Y') = \{0\}$. $\square$

COROLLARY 8. *Suppose that G has a finite classifying space and that the centre of G contains an element of infinite order. Then the ℓ^2-cohomology of G vanishes.*

Proof. This is just the case $G = \pi$ of Corollary 7. $\square$

5. *Coxeter groups and Artin groups*

As a general reference for the material described in Sections 5 and 7, we recommend [**7**].

Let I be a finite set. A *Coxeter matrix* $M = (m_{ij})$ on I is an I-by-I symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that each $m_{ii} = 1$ and such that whenever $i \neq j$, $m_{ij} \geqslant 2$. Associated to M there is a *Coxeter group* denoted by W_I or W with generating set $\{s_i : i \in I\}$, with relations

$$(s_i s_j)^{m_{ij}} = 1 \qquad \text{for all } (i, j) \in I \times I.$$

There is also an *Artin group* A_I (or A) with generators $\{a_i : i \in I\}$ and with a presentation given by the relations

$$a_i a_j \ldots = a_j a_i \ldots$$

for all $i \neq j$, where there are m_{ij} terms on each side of the equation. In both the Coxeter and Artin cases, the relation is interpreted as being vacuous if $m_{ij} = \infty$.

Since each of the Artin relators contains the same number of copies of a generator on each side, there is a homomorphism from any Artin group to the infinite cyclic group $\mathbb{Z}$, defined by sending each Artin generator to 1. Call this the *standard homomorphism* from A to $\mathbb{Z}$.

Let $q : A \to W$ denote the natural homomorphism sending a_i to s_i. There is a set-theoretic section for q denoted $w \mapsto a_w$ which may be defined as follows: if $s_{i_1} \ldots s_{i_n}$ is any word of minimal length for w in terms of the s_i, then $a_w := a_{i_1} \ldots a_{i_n}$. As explained in [**6**, p. 602], it follows from Tits' solution to the word problem for Coxeter groups that $w \mapsto a_w$ is well-defined.

Given a subset J of I, let M_J denote the minor of M whose rows and columns are indexed by J, and let W_J (respectively A_J) be the corresponding Coxeter group (respectively Artin group). It is known (cf. [**2**]) that the natural map $W_J \to W_I$ (respectively $A_J \to A_I$) is injective, and hence W_J (respectively A_J) can be identified with the subgroup of W_I (respectively A_I) generated by $\{s_i : i \in J\}$ (respectively $\{a_i : i \in J\}$).

Say that the subset $J \subseteq I$ is *spherical* if W_J is finite. If this is the case, then the groups W_J and A_J are both called *spherical.* (Note that this differs from the terminology of [**6**], where spherical Artin groups were said to be 'of finite type'.) The poset of spherical subsets of I is denoted by $\mathscr{S}(I)$ or simply $\mathscr{S}$. The subposet of non-empty spherical subsets is an abstract simplicial complex which will be denoted L, and called the *nerve* of M (or the nerve of $(W, \{s_i : i \in I\})$). Thus the vertex set of L is I and a subset σ of I spans a simplex of L if and only if $\sigma \in \mathscr{S}$ if and only if W_σ is finite. (Greek letters such as σ or τ will be used to denote spherical subsets of I when viewed as simplices of L.)

EXAMPLE 9. If W is finite, then L is a simplex of dimension $\mathrm{Card}(I) - 1$.

Suppose now that $m_{ij} = \infty$ for all $i \neq j$, and let $n = \mathrm{Card}(I)$. In this case W is the free product of n copies of $\mathbb{Z}/2$, A is a free group of rank n, and L is a 0-dimensional complex consisting of n points.

A Coxeter matrix is *right-angled* if each off-diagonal entry is either 2 or ∞. The associated Coxeter group and Artin group are also said to be right-angled.

For any subset J of I and $w \in W_I$, the following two conditions are equivalent.

(1) w is the (unique) element of shortest word length in $wW_J \in W_I/W_J$.

(2) For each $j \in J$, $l(ws_j) = l(w) + 1$ (where $l(w)$ denotes the word length of w).

If w satisfies either condition, then it is called *J-reduced*. Moreover, if w is J-reduced and $v \in W_J$, then $l(wv) = l(w) + l(v)$ (cf. [**2**, Ex. 3, p. 37]). For each pair J, J' of subsets of I with $J \subseteq J'$, let $W_{J'}^J$ denote the set of J-reduced elements of $W_{J'}$.

We recall now a formula from [**20**, **10**]. Given a pair of spherical subsets τ, σ with $\tau \leqslant \sigma$, define an element $T_\sigma^\tau \in \mathbb{Z}A_\sigma$ by the formula

$$T_\sigma^\tau := \sum_{w \in W_\sigma^\tau} (-1)^{l(w)} a_w.$$

LEMMA 10. *For any three spherical subsets, $\tau, \sigma, \rho \in \mathscr{S}$ with $\tau \leqslant \sigma \leqslant \rho$, we have*

$$T_\rho^\sigma \cdot T_\sigma^\tau = T_\rho^\tau.$$

Proof. If $u \in W_\rho^\sigma$ and $v \in W_\sigma^\tau$, then $l(uv) = l(u) + l(v)$. It follows that $a_u a_v$ equals a_{uv}. Since, for each $i \in \tau$, $l(uvs_i) = l(u) + l(vs_i) = l(u) + l(v) + 1 = l(uv) + 1$, the element uv is τ-reduced. Hence, as u ranges over W_ρ^σ and v ranges over W_σ^τ, uv ranges over W_ρ^τ. Therefore,

$$\begin{aligned} T_\rho^\sigma \cdot T_\sigma^\tau &= \left(\sum_{u \in W_\rho^\sigma} (-1)^{l(u)} a_u \right) \left(\sum_{v \in W_\sigma^\tau} (-1)^{l(v)} a_v \right) \\ &= \sum_{w \in W_\rho^\tau} (-1)^{l(w)} a_w = T_\rho^\tau. \end{aligned}$$

□

6. *Some homological algebra*

Let L be a finite abstract simplicial complex. Later we shall assume that L is ordered, in the sense that there is a given linear ordering of its vertex set. The *face category* $\mathscr{F}(L)$ of L is defined to be the category whose objects are the simplices of L, with one morphism from τ to σ whenever τ is a face of σ. The *augmented face category* $\mathscr{F}^+(L)$ has the same objects and morphisms as $\mathscr{F}(L)$, but also has one extra object (the '-1-simplex', denoted $\varnothing$) equipped with one morphism to each object. If L has vertex set I, each of $\mathscr{F}(L)$ and $\mathscr{F}^+(L)$ is isomorphic to a full subcategory of the category of subsets of I. A *sheaf* on L with values in a category $\mathscr{C}$ is a covariant functor F from $\mathscr{F}^+(L)$ to $\mathscr{C}$. In the cases of interest to us, $\mathscr{C}$ will be either abelian groups or Hilbert G-modules. A homomorphism of sheaves on L is given by $\phi : F \to F'$, a natural transformation from F to F'.

If $\tau \leqslant \sigma$ are faces of L (possibly including the -1-simplex), let ι_σ^τ denote the morphism in $\mathscr{F}(L)$ from τ to σ. The morphisms of $\mathscr{F}(L)$ are generated by those ι_σ^τ such that τ is a codimension 1 face of σ, and all relations between the morphisms are consequences of the relations

$$\iota_{\tau'}^{\sigma'}\iota_\sigma^{\tau'} = \iota_\tau^{\sigma'}\iota_\sigma^\tau, \tag{4}$$

where σ' is a codimension 2 face of σ, and τ, τ' are the two codimension 1 faces of σ containing σ'. To define a sheaf F on L it suffices to specify the objects $F(\sigma)$, and morphisms $f_\sigma^\tau = F(\iota_\sigma^\tau) : F(\sigma) \to F(\sigma)$ for all codimension 1 pairs, so that relations (4) are satisfied for all codimension 2 pairs.

Now suppose that L is ordered. Then for any $n \geqslant 0$ the vertices of an n-simplex form an ordered set isomorphic to $0 < 1 < \ldots < n$. For any $0 \leqslant i \leqslant n$ and any n-simplex σ, the ith face of σ is defined to be the $(n-1)$-simplex spanned by all vertices of σ except the ith. If one writes $\partial_i = \iota_\sigma^\tau$, where τ is the ith face of σ, then the relations between the morphisms become the familiar 'cosimplicial identities' as in [**23**, 8.1].

A sheaf F of abelian groups on an ordered simplicial complex L gives rise to a cochain complex $C^*(L;F)$ defined as follows: $C^n = 0$ for $n < -1$, and for $n \geqslant -1$,

$$C^n = \bigoplus_{\tau \in L^n} F(\tau),$$

where the indexing set is the set of n-simplices of L. Under the natural isomorphism

$$\mathrm{Hom}(C^n, C^{n+1}) \cong \bigoplus_{\tau \in L^n,\ \sigma \in L^{n+1}} \mathrm{Hom}(F(\tau), F(\sigma)),$$

the coboundary map $d : C^n \to C^{n+1}$ corresponds to the matrix $(d_{\tau\sigma})$, where $d_{\tau\sigma} = 0$ unless τ is a face of σ, and is equal to $(-1)^i F(\iota_\sigma^\tau)$ if τ is the ith face of σ.

Example 11. For any abelian group N, and any simplicial complex L, there is a constant sheaf on L denoted by $\underline{N}$, which sends each object of $\mathscr{F}^+(L)$ to N and each morphism to the identity map. The cohomology of the corresponding cochain complex is isomorphic to the reduced cohomology of L with coefficients in N:

$$H^* C^*(L; \underline{N}) = \overline{H}^*(L; N).$$

Now suppose that M is a Coxeter matrix with nerve L and Artin group A, and let N be a $\mathbb{Z}A$-module. Define a sheaf $\mathscr{N}$ on L by sending each object to N, and for each $\tau < \sigma$ sending the morphism ι_σ^τ to left multiplication by T_σ^τ. (Lemma 10 implies that this does define a sheaf.)

One of the key observations of this paper is the following.

Lemma 12. *For any Coxeter matrix, M, with L, A and N as above, there is a sheaf homomorphism $\phi : \underline{N} \to \mathscr{N}$ defined by*

$$\phi(\sigma) : n \mapsto T_\sigma^\varnothing \cdot n.$$

Proof. The proof is immediate from Lemma 10. □

COROLLARY 13. *Let N be a $\mathbb{Z}A$-module upon which $T_\sigma^\varnothing$ acts isomorphically for each $\sigma \in L$. Then*

$$H^*C^*(L;\mathcal{N}) \cong \overline{H}^*(L;N).$$

Proof. In this case the map described in Lemma 12 is an isomorphism of sheaves, and so induces an isomorphism of cochain complexes $C^*(L;\underline{N}) \to C^*(L;\mathcal{N})$. □

7. *The Salvetti complex*

Let M be a spherical Coxeter matrix on a set I of cardinality n. By definition the associated Coxeter group W is finite, and so admits a standard representation as an orthogonal reflection group on $\mathbb{R}^n$. In this case, Salvetti [**20**, **21**] defined a regular CW-complex X' which is equivariantly homotopy equivalent to Y, the complexification of $\mathbb{R}^n$ minus the hyperplanes fixed by reflections in W. Each closed cell of X' is naturally identified with a face of a certain convex polytope associated to W (namely the convex hull of a generic W-orbit in $\mathbb{R}^n$). Such a polytope is called a 'Coxeter cell', and denoted C_W. The group W acts freely on X'. The quotient space, denoted by X_I or X, is called the *Salvetti complex*, and has fundamental group A.

When W is infinite it still has a representation as a group generated by linear reflections on $\mathbb{R}^n$; however, the action is not proper (since the stabilizer of the origin is infinite). W does act properly on the interior of a certain W-stable convex cone (see [**22**]). Denote the interior of this cone by Ω. Rather than considering the W-action on $\mathbb{C}^n$, consider instead the proper W-action on $\mathbb{R}^n + i\Omega \subset \mathbb{C}^n$. Let Y denote the complement in $\mathbb{R}^n + i\Omega$ of the hyperplanes fixed by the reflections in W.

As mentioned in the introduction, the definitions of X' and X were extended to arbitrary Coxeter matrices in [**7**]. The definition of X' goes as follows. First one defines an open covering of Y by a certain collection of convex subsets. These convex subsets are indexed by the poset $W \times \mathcal{S}$, where the partial order is given by $(w,\tau) < (v,\sigma)$ if and only if $\tau < \sigma$ and $v^{-1}w \in W_\sigma^\tau$. Moreover, a collection of convex sets in the cover has non-empty intersection if and only if the corresponding elements of $W \times \mathcal{S}$ form a chain. That is to say, the nerve of this covering of Y is the derived complex of $W \times \mathcal{S}$. X' is defined to be this nerve. Since each element of $W \times \mathcal{S}$ corresponds to a contractible subset of Y, X' and Y are homotopy equivalent. It is also proved in [**7**, p. 118] that the subposet $(W \times \mathcal{S})_{\leqslant(w,\sigma)}$ consisting of all elements which are $\leqslant (w,\sigma)$ is isomorphic to the poset of faces of the Coxeter cell C_{W_σ} associated to W_σ. Thus, by assembling into a single cell all simplices of X' with vertices less than some given element, we can regard X' as a cell complex whose poset of cells is isomorphic to $W \times \mathcal{S}$. The dimension of the cell corresponding to (v,σ) is $\mathrm{Card}(\sigma)$; its vertex set corresponds to the set $\{(vu,\varnothing) : u \in W_\sigma^\varnothing\}$. Henceforth we identify a cell with the corresponding element of $W \times \mathcal{S}$.

The group W acts freely on X' and the quotient space $X = X'/W$ is the *Salvetti complex*. For J a subset of I, let W_J denote the Coxeter subgroup of W generated by $\{s_i : i \in J\}$, and let X_J (respectively X'_J) denote the Salvetti complex for W_J (respectively the analogue of X'). It may be checked that the induced W-complex

$W \times_{W_J} X'_J$ is a W-subcomplex of X'. The following important properties of X are easily verified.

(1) X has one open cell e_σ for each spherical subset σ of I. Moreover the closure of e_σ in X is the subcomplex X_σ corresponding to the spherical Artin group A_σ.

(2) The 2-skeleton of X is the presentation complex for A and hence $\pi_1(X) = A$.

(3) For $J \subseteq I$, X_J is a subcomplex of X, and X is equal to the union of the subcomplexes X_σ, where σ runs over the spherical subsets of I.

Example 14. Let $\mathbb{T}^n$ denote the usual cubical structure on the n-torus, with one k-dimensional cell for each subset of $I = \{1, \ldots, n\}$ of cardinality k. In the case when $W = (\mathbb{Z}/2)^n$, then $A = \mathbb{Z}^n$, and $X = \mathbb{T}^n$.

More generally, if A is right-angled with generating set I of cardinality n, then X is a subcomplex of $\mathbb{T}^n$, consisting of those cells of $\mathbb{T}^n$ that correspond to spherical subsets of I.

An edge or 1-cell of X' has the form $(w, \{i\})$ for some $w \in W$ and $i \in I$. Orient this edge by declaring $(w, \emptyset)$ to be its initial vertex and $(ws_i, \emptyset)$ to be its terminal vertex. Since the W-action preserves the edge orientations, this gives rise to an orientation of the edges of $X = X'/W$. Let $\{e_i : i \in I\}$ denote the set of edges of X with their induced orientation. Since X has only one vertex, the closure of each e_i gives rise to an oriented loop. By definition the generator a_i of the Artin group $A = \pi_1(X)$ is the homotopy class of the oriented loop around e_i.

A vertex x of a cell C in X' is called a *top vertex* (respectively *bottom vertex*) of C if each edge of C that contains x points away from x (respectively towards x). Each cell (v, σ) of X' has a unique top vertex $(v, \emptyset)$ and a unique bottom vertex $(vu_\sigma, \emptyset)$, where u_σ denotes the element of longest length in the finite Coxeter group W_σ.

Let $\widetilde{X}$ denote the universal cover of X (or equivalently of X'). The orientations of the edges of X' lift to orientations on the edges of $\widetilde{X}$. Each cell of $\widetilde{X}$ then also has a unique top vertex and a unique bottom vertex. The 1-skeleton of $\widetilde{X}$ is an oriented graph and each edge is labelled by an element of $\{a_i : i \in I\}$ so that if e is an oriented edge labelled by a_i whose initial vertex is x, then the terminal vertex of e is $x \cdot a_i$.

Now fix a lift $\widetilde{x} \in \widetilde{X}$ of the vertex $(1, \emptyset)$ of X'. This gives an identification of the vertex set of $\widetilde{X}$ with A and of the 1-skeleton of $\widetilde{X}$ with the Cayley graph of the standard presentation of A.

For each $\sigma \in \mathscr{S}$, let $\widetilde{e}_\sigma$ be the lift of the cell $(1, \sigma)$ whose top vertex is $\widetilde{x}$. Using the A-action we get an identification of the poset of cells in $\widetilde{X}$ with $A \times \mathscr{S}$. The partial order on $A \times \mathscr{S}$ corresponding to 'is a face of' is defined by $(b, \tau) < (a, \sigma)$ if and only if $\tau < \sigma$ and $b = aa_u$ for some $u \in W_\sigma^\tau$. Thus for a given cell (a, σ) of $\widetilde{X}$, its set of faces of type τ is $\{(aa_u, \tau) : u \in W_\sigma^\tau\}$.

Once we have fixed an ordering on I there is an induced orientation for each cell of $\widetilde{X}$: a cell C with top vertex x is oriented so that the edges at x in their natural order give an oriented basis for the tangent space of C at x (recall that C may be viewed as a convex polytope in $\mathbb{R}^n$ for some n). Suppose that C is of type σ, $\sigma = \{i_1 < \ldots < i_k\}$ and that D is a codimension 1 face of type $\tau = \sigma - \{i_j\}$. Suppose also that D has the same top vertex as C. In this case the natural orientation of D and its orientation induced by the orientation on C (by taking an outward pointing normal) differ by the sign $\epsilon(\sigma, \tau)$ defined by $\epsilon(\sigma, \tau) = (-1)^{j-1}$.

If (v, σ) is a cell of X', then the (right) action of W_σ on the cell (induced by the action on its vertex set) is same as the action of W_σ on this cell viewed as a convex cell in $\mathbb{R}^{\mathrm{Card}(\sigma)}$ under the reflection group action. In particular, each $s \in \sigma$ acts as a reflection on the cell. It follows that the action of $u \in W_\sigma$ changes the orientation of the cell by $(-1)^{l(u)}$. Hence, we have a formula for the boundary maps in $C_*(\widetilde{X})$:

$$\begin{aligned} \partial(a\widetilde{e}_\sigma) = \partial(a, \sigma) &= \sum_\tau \epsilon(\sigma, \tau) \sum_{u \in W_\sigma^\tau} (-1)^{l(u)}(aa_u, \tau) \\ &= a \sum_\tau \epsilon(\sigma, \tau) T_\sigma^\tau \widetilde{e}_\tau \end{aligned}$$

where the summation is over all codimension 1 faces τ of σ.

Remark 15. Note the similarity with the formula occurring in Lemma 10. An alternative proof of Lemma 10 can be given by considering the oriented incidence relation between cells in $\widetilde{X}$ of types τ, σ and ρ, where $\tau \subseteq \sigma \subseteq \rho$.

The cells $\widetilde{e}_\sigma$ for $\sigma \in \mathscr{S}$ form a $\mathbb{Z}A$-basis for the cellular chain groups of $\widetilde{X}$. In matrix terms, the ith cellular chain group of $\widetilde{X}$ may be viewed as consisting of row vectors over $\mathbb{Z}A$, with the standard basis elements corresponding to the cells $\widetilde{e}_\sigma$, where σ ranges over the subsets of $\mathscr{S}$ of size i. The $\mathbb{Z}A$-action is the standard left action, and the boundary map in the chain complex is given by right multiplication by a matrix whose entry in the (σ, τ)-position is $\epsilon(\sigma, \tau) T_\sigma^\tau$ if $\tau \subseteq \sigma$ and is zero otherwise.

Now suppose that N is any $\mathbb{Z}A$-module. After identifying $C_i(\widetilde{X})$ with a space of row vectors with entries in $\mathbb{Z}A$ as above, it is natural to identify $\mathrm{Hom}_A(C_i(\widetilde{X}), N)$ with a space of column vectors (of the same length) with entries in N, so that the evaluation map corresponds to multiplying a row vector and a column vector to produce a single element of N. In these terms, the coboundary map in the cochain complex $\mathrm{Hom}_A(C_*(\widetilde{X}), N)$ is described as *left* multiplication by the same matrix over $\mathbb{Z}A$ as used previously to describe the boundary map in $C_*(\widetilde{X})$. This gives the following theorem.

Theorem 16. *For any A-module N, there is an isomorphism between $C^*(X; N) = \mathrm{Hom}_A(C_*(\widetilde{X}), N)$ as defined at the beginning of Section* 3 *and $C^{*-1}(L; \mathscr{N})$ as defined in Section* 6.

In the spherical case, a statement equivalent to the above theorem is contained in [**21**, **10**], and it is used to compute some cohomology.

Remark 17. There is a simple description of the CW-complex X. Firstly, for each $\sigma \in \mathscr{S}$ we describe a CW-complex X_σ. Start with a Coxeter cell C_σ of type W_σ. For each $\tau < \sigma$ identify C_τ with the face of C_σ of type τ which has the same top vertex as C_σ. Now for each $\tau < \sigma$ and each $u \in W_\sigma^\tau$ glue together C_τ and uC_τ via the homeomorphism induced by u. The result is X_σ. To construct X, start with the disjoint union of the X_σ for $\sigma \in \mathscr{S}$, and then use the natural maps to identify X_τ with a subcomplex of X_σ whenever $\tau < \sigma$.

Remark 18. In [**6**, Section 3] it is shown how to associate a 'simple complex of groups' $\mathscr{A}$ to a Coxeter matrix. The underlying poset is $\mathscr{S}$. To each $\sigma \in \mathscr{S}$ one associates the Artin group A_σ, and for each $\tau < \sigma$, the associated homomorphism $A_\tau \to A_\sigma$ is the natural inclusion. There is a natural projection from (the barycentric subdivision of) X to the geometric realization of $\mathscr{S}$ such that the inverse image of the vertex corresponding to σ is a copy of X_σ. Since it follows from [**11**] that $X_\sigma \simeq BA_\sigma$, it follows that X is an aspherical realization of the complex of groups $\mathscr{A}$ in the sense of [**16**]. Thus X is homotopy equivalent to the classifying space $B\mathscr{A}$ of $\mathscr{A}$ (which is defined as a homotopy colimit of BA_σ over the category $\mathscr{S}$). The main conjecture of [**6**] is that $B\mathscr{A}$ (or equivalently X) is aspherical. This has been proved in many cases; for example, it holds if the simplicial complex L is a flag complex (for example, if the Coxeter matrix is either spherical or right-angled) and it holds whenever the dimension of L is at most 1.

8. *Cohomology with generic coefficients*

This section represents a slight digression, although the methods used are similar to those that will be applied to ℓ^2-cohomology in later sections.

Let A be an Artin group with Artin generators $\{a_i : i \in I\}$, let G be the abelianization of A, and let α_i be the image of a_i in G. In the case when m_{ij} is even, the abelianization of the Artin relator between a_i and a_j is trivial, and in the case when m_{ij} is odd the abelianization of the Artin relator is $\alpha_i = \alpha_j$. Hence one sees that G is free abelian, of rank equal to the number of components of the graph with vertex set I and an edge joining i to j if and only if m_{ij} is odd. In particular, G is never trivial. Let m denote the rank of G, and (after changing the ordering on I), suppose that $\alpha_1, \dots, \alpha_m$ freely generate G.

Now suppose that k is a field of characteristic zero, and that N is a kA-module whose underlying k-vector space is 1-dimensional. The action of A on N factors through G, a free abelian group of rank m, and so N is classified up to isomorphism by an m-tuple

$$\Lambda = (\lambda_1, \dots, \lambda_m) \in (k^*)^m,$$

where $k^* = k - \{0\}$ and for $1 \leqslant i \leqslant m$, the generator α_i acts on N as multiplication by λ_i. As an illustration of the methods we shall apply to ℓ^2-cohomology, we shall compute the cohomology of the Salvetti complex X with local coefficients N for 'generic N', that is, for N corresponding to elements of a dense Zariski open subset of $(k^*)^m$.

THEOREM 19. *For a generic* 1*-dimensional kA-module N, there is an isomorphism*

$$\overline{H}^{*-1}(L;k) \cong H^*(X;N).$$

Proof. Each side of the supposed isomorphism is isomorphic to the cohomology of L with coefficients in a certain sheaf. The left-hand side is

$$\overline{H}^*(L;k) \cong \overline{H}^*(L;N) \cong H^*C^*(L;\underline{N})$$

and by Theorem 16, the cochain complex for the right-hand side is

$$C^*(X;N) \cong C^{*-1}(L;\mathscr{N}).$$

It therefore suffices to show that for generic N, the sheaf homomorphism $\phi : \underline{N} \to \mathscr{N}$ described in Lemma 12 is an isomorphism. Equivalently, it suffices to show that for each simplex σ of L, multiplication by $T_\sigma^\varnothing$ is an isomorphism for generic N.

As in the introduction to this section, let G denote the abelianization of A, and let $\alpha_1,\ldots,\alpha_m$ be free generators for G which are images of the Artin generators for A. For $\Lambda=(\lambda_1,\ldots,\lambda_m)\in(k^*)^m$, let $\psi'_\Lambda:\mathbb{Z}G\to k$ be defined by $\psi'_\Lambda(\alpha_i)=\lambda_i$, and let $\psi_\Lambda:\mathbb{Z}A\to k$ be the composite of the projection $\mathbb{Z}A\to\mathbb{Z}G$ and ψ'. For each $\sigma\in L$, the set of Λ such that $\psi_\Lambda(T_\sigma^{\varnothing})=0$ is a closed subset of $(k^*)^m$. The union over all σ of these closed sets is not the whole of $(k^*)^m$, since in the case when $\Lambda=(-1,-1,\ldots,-1)$, $\psi_\Lambda(T_\sigma^{\varnothing})=\mathrm{Card}(W_\sigma)$. Since $(k^*)^m$ is connected, it follows that the set of Λ for which each $\psi_\Lambda(T_\sigma^{\varnothing})\neq 0$ is open and dense in $(k^*)^m$ as required. □

REMARK 20. The theorem does not hold for arbitrary 1-dimensional coefficients, as can be seen by comparing the case of 'generic' 1-dimensional coefficients N with the trivial 1-dimensional module k. As a first example, consider H^1. For any A, $H^1(A;k)$ is naturally isomorphic to $\mathrm{Hom}(A,k)$, which has dimension equal to the rank of the abelianization of A. In particular it is non-zero for every Artin group, whereas for A of spherical type and generic N, we have seen that $H^*(A;N)=\{0\}$. (Note that in the case when A is of spherical type, $H^*(A;k)$ is computed in [**4**].)

As a further example, consider the case when A is right-angled. In this case it may be shown that $H^*(A;k)$ is isomorphic to the 'exterior face ring of L', defined by taking the exterior algebra over k with generators $\{x_i : i\in I\}$ with each x_i in degree 1, and adding in the relation $x_{i_1}\ldots x_{i_m}=0$ whenever $i_1,\ldots,i_m$ do not form the vertex set of a simplex of L [**12**, **17**]. In particular, the rank of $H^i(X;k)=H^i(A;k)$ is equal to the number of $(i-1)$-simplices in L for each $i>0$.

9. *Reduced ℓ^2-cohomology*

Each Artin group of spherical type has a non-trivial centre. If A is a spherical Artin group and w is the longest element of the corresponding Coxeter group, then the element $(a_w)^2$ is in the centre of A (cf. [**11**]). Note that the expression for this element contains only positive powers of the Artin generators, and so in particular its image under the standard homomorphism $A\to\mathbb{Z}$ is non-zero.

LEMMA 21. *Suppose that A is an Artin group of spherical type, and that A' is a normal subgroup of A contained in the kernel of the standard homomorphism. Let $X'\to X$ be the corresponding covering space of X and let $G=A/A'$. For each i, the Hilbert G-module $\mathscr{H}^i(X')$ is the zero module.*

Proof. In view of the remarks preceding the statement of the lemma, this is a special case of Corollary 7. □

PROPOSITION 22. *Let $H\leqslant G$, and let $\xi\in\mathbb{Z}H$. Then left multiplication by ξ induces a weak isomorphism of $\ell^2(H)$ if and only if it induces a weak isomorphism of $\ell^2(G)$.*

Proof. Consider $\mathscr{H}$, defined to be the set of all functions on G that are square-summable on each right coset Hg of H. As a left $\mathbb{Z}H$-module, $\mathscr{H}$ is isomorphic to a direct product of copies of $\ell^2(H)$ indexed by the coset space $H\backslash G$. One has $\ell^2(H)\leqslant\ell^2(G)\leqslant\mathscr{H}$, and so multiplication by ξ is injective as a self-map of $\ell^2(H)$ if and only if it is injective as a self-map of $\ell^2(G)$. The corresponding assertion with 'weak isomorphism' in place of 'injective' follows from Lemma 4. □

$$
\begin{tikzcd}
\arrow[r] & C^{n-1}(L;\underline{N}) \arrow[r, "\delta'"] \arrow[d, "\phi_{n-1}"] & C^n(L;\underline{N}) \arrow[d, "\phi_n"] \\
\arrow[r] & C^{n-1}(L;\mathcal{N}) \arrow[r, "\delta"] & C^n(L;\mathcal{N})
\end{tikzcd}
$$

FIGURE 1.

$$
\begin{tikzcd}
\arrow[r] & \bigoplus_\tau \ell^2(G) \arrow[r, "\delta'"] \arrow[d, "\bigoplus_\tau \phi_\tau"] & \ell^2(G) \arrow[d, "\phi_\sigma"] \\
\arrow[r] & \bigoplus_\tau \ell^2(G) \arrow[r, "\delta"] & \ell^2(G)
\end{tikzcd}
$$

FIGURE 2.

LEMMA 23. *Suppose that $A = A_\sigma$ is an Artin group of spherical type, and that A' is a normal subgroup of A contained in the kernel of the standard homomorphism $A \to \mathbb{Z}$. Let $G = G_\sigma = A/A'$, and let $\phi_\sigma : \ell^2(G) \to \ell^2(G)$ denote left multiplication by the element $T_\sigma^\varnothing \in \mathbb{Z}A_\sigma$. Then ϕ_σ is a weak isomorphism.*

Proof. Let π denote the quotient map from A_σ to G_σ. For any $\tau < \sigma$, let i_τ denote the inclusion of A_τ in A_σ, and define a subgroup $G_\tau \leqslant G_\sigma$ by $G_\tau = \pi \circ i_\tau(A_\tau)$. The composite of i_τ and the standard homomorphism for A_σ is equal to the standard homomorphism for A_τ, and so the pair (A_τ, G_τ) satisfy the hypotheses of the lemma.

The lemma is trivially satisfied if $\mathrm{Card}(\sigma) = 0$, since $T_\varnothing^\varnothing = 1$. Hence by induction on $\mathrm{Card}(\sigma)$, we may assume that for any $\tau < \sigma$, the map $\phi_\tau : \ell^2(G_\tau) \to \ell^2(G_\tau)$ is a weak isomorphism. Here ϕ_τ is defined to be left multiplication by the element $T_\tau^\varnothing \in \mathbb{Z}A_\tau$. By Proposition 22, we may assume that $\phi_\tau : \ell^2(G_\sigma) \to \ell^2(G_\sigma)$ is also a weak isomorphism.

Now let L be the simplex with vertex set σ, where $n = \mathrm{Card}(\sigma)$. Let N denote $\ell^2(G_\sigma) = \ell^2(G)$, and consider the sheaves of Hilbert G-modules on L that were denoted by $\underline{N}$ and $\mathcal{N}$ in Section 6. Note that in this case, $C^*(L;\underline{N})$ and $C^*(L;\mathcal{N})$ are cochain complexes of Hilbert G-modules. Since the reduced cohomology of L is trivial, the cochain complex $C^*(L;\underline{N})$ is exact. By Theorem 16 the cochain complex $C^*(L;\mathcal{N})$ (with a shift of degree) may be used to compute the cohomology of A with coefficients in $\ell^2(G)$, and so by Lemma 21, it follows that $C^*(L;\mathcal{N})$ is weakly exact.

Now consider the sheaf homomorphism $\phi : \underline{N} \to \mathcal{N}$ defined in Lemma 12, and the induced map

$$\phi^* : C^*(L;\underline{N}) \to C^*(L;\mathcal{N}).$$

In degree i, each of $C^i(L;\underline{N})$ and $C^i(L;\mathcal{N})$ is isomorphic to a direct sum of copies of $\ell^2(G)$ indexed by the subsets $\tau \leqslant \sigma$ with $\mathrm{Card}(\tau) = i$. With respect to these bases, the map ϕ^i is given by a diagonal matrix, whose (τ,τ)-entry is the map ϕ_τ. By induction, this map is a weak isomorphism in each degree except possibly n, the top degree. Consider the commutative diagram in Figure 1, consisting of the right-hand ends of the two cochain complexes. This can be rewritten as shown in Figure 2, where τ runs over the subsets of σ of size $n-1$. The coboundary map δ' is surjective, the coboundary map δ is weakly surjective, and by induction the left-hand vertical map is a weak isomorphism. It follows that the right-hand vertical map, $\phi_\sigma : \ell^2(G) \to \ell^2(G)$ is weakly surjective. The claim now follows from Lemma 4. $\square$

COROLLARY 24. *Let A be any Artin group, let A' be a normal subgroup of A contained in the kernel of the standard homomorphism $A \to \mathbb{Z}$, let $G = A/A'$, and let $N = \ell^2(G)$. Then the map $\phi : \underline{N} \to \mathcal{N}$ defined in Lemma 12 is a weak isomorphism of cosheaves of Hilbert G-modules on the simplicial complex L.*

Proof. On each object $\sigma \in \mathcal{S}$, the map $\phi(\sigma) : \underline{N}(\sigma) \to \mathcal{N}(\sigma)$ is equal to left multiplication by $T_\sigma^\varnothing$. By Lemma 23 and Proposition 22, each of these maps is a weak isomorphism. □

The next corollary is immediate from the above.

COROLLARY 25. *With notation as in Corollary 24, the induced map of cochain complexes*

$$\phi^* : C^*(L; \underline{N}) \to C^*(L; \mathcal{N})$$

is a weak isomorphism.

The proofs of Theorems 1 and 3 are now completed by combining the description of $\mathcal{H}^*(X')$ as the reduced cohomology of $C^*(L; \mathcal{N})$, given in Theorem 16, with Corollary 25 and Lemma 5.

10. *Closing remarks*

In this section we make some remarks concerning ℓ^p-cohomology for $p \neq 2$, and describe an alternative method for proving Theorems 1 and 3, using a spectral sequence argument. In a subsequent paper with T. Januszkiewicz, we plan to use this technique to calculate the ℓ^2-cohomology of the universal cover of the Salvetti complex associated to an arbitrary finite real hyperplane arrangement. To simplify the notation, we shall discuss this proof for Theorem 1 but not for Theorem 3.

10.1. *A spectral sequence argument*

Recall from Section 7 that for a general Artin group A, the Salvetti complex X may be expressed as a union of subcomplexes X_σ, where σ ranges over the spherical subsets of the Artin generating set, and X_σ is a copy of the Salvetti complex for A_σ. Let Y_σ be the subcomplex of $\widetilde{X}$, the universal cover of X, consisting of lifts of cells of X_σ. Each Y_σ is a free A-CW-complex. By Corollary 8 and Proposition 22, the reduced ℓ^2-cohomology of Y_σ vanishes for $\sigma \neq \varnothing$. In contrast, $Y_\varnothing$ is just the 0-skeleton of $\widetilde{X}$, which consists of a single free G-orbit of cells. Hence the reduced ℓ^2-cohomology of $Y_\varnothing$ is a copy of $\ell^2(G)$ in degree zero.

One may construct a 'Mayer–Vietoris double chain complex' $C_{*,*}$ for $\widetilde{X}$ expressed as the union of the Y_σ. Let $\mathcal{L}$ denote the set of maximal simplices in L, define $Y(\varnothing) = \widetilde{X}$, and for $\varnothing \neq S \subseteq \mathcal{L}$, define $Y(S) = \bigcap_{\sigma \in S} Y_\sigma$. Define $C_{*,0}$ to be the cellular chain complex for $\widetilde{X}$, and for $j > 0$ define $C_{*,j}$ to be the direct sum over the subsets $S \subseteq \mathcal{L}$ of size j of the cellular chain complex for $Y(S)$. The boundary maps in $C_{*,*}$ of degree $(-1, 0)$ are the boundary maps in the chain complexes $C_*(Y(S))$, and the boundary maps of degree $(0, -1)$ are given by the matrices whose (S, T)-entry is $\epsilon(S, T)$ times the map induced by the inclusion of $Y(S)$ in $Y(T)$, where $\epsilon(S, T) = (-1)^i$ if T is obtained from S by omitting the ith element of S (for some fixed ordering of $\mathscr{L}$). By construction, this double complex has trivial homology, since the boundary map of degree $(0, -1)$ is exact. Since each $C_{i,j}$ is free, it follows that for each i, the chain complex $C_{i,*}$ is split exact.

Now suppose that N is a $\mathbb{Z}A$-module such that for each $\sigma \in L$, $H^*(A_\sigma; N) = \{0\}$. Define a double cochain complex by

$$E_0^{*,*} = \mathrm{Hom}_A(C_{*,*}, N),$$

and let $E_*^{*,*}$ denote the spectral sequence in which the differential d_0 is induced by the boundary map of degree $(-1, 0)$ on $C_{*,*}$. The boundary map on $E_0^{*,*}$ of degree $(0, 1)$ is exact, and so the homology of the total complex for $E_0^{*,*}$ is zero. It follows that the E_∞-page of the spectral sequence is identically zero. The E_1-page has $E_1^{i,0} \cong H^i(X; N)$, $E_1^{i,j} = \{0\}$ if both $i > 0$ and $j > 0$, and $E_1^{0,j}$ is isomorphic to a direct sum of copies of N, indexed by those j-element subsets of $\mathscr{L}$ such that the intersection of the corresponding simplices of L is empty.

The structure of the E_1-page implies that $E_2^{i,j} = E_1^{i,j}$ for $i > 0$, and it may be shown that $E_2^{0,j} \cong \overline{H}^j(L; N)$. To see this, note that the cochain complex $E_1^{0,*}$ embeds as a subcomplex in an exact complex C^*, where C^j is isomorphic to a direct sum of copies of N indexed by *all* j-element subsets of $\mathscr{L}$, and the quotient complex $C^*/E_1^{0,*}$ is isomorphic to the augmented cochain complex for the nerve of the covering of L by the elements of $\mathscr{L}$, shifted in degree by 1. The long exact sequence in cohomology coming from this short exact sequence of cochain complexes gives the claimed isomorphism.

Since the E_∞-page is identically zero, and the only non-zero groups on the E_2-page are those $E_2^{i,j}$ for which either $i = 0$ or $j = 0$, it follows that the differential d_r must be an isomorphism from $E_r^{0,r-1} = E_2^{0,r-1}$ to $E_r^{r,0} = E_1^{r,0}$. Thus we obtain the following.

THEOREM 26. *For any $\mathbb{Z}A$-module N such that $H^*(A_\sigma; N) = \{0\}$ for all non-trivial spherical subgroups $A_\sigma \leqslant A$, there is an isomorphism*

$$H^*(X; N) \cong \overline{H}^{*-1}(L; N).$$

One corollary of this is a version of Theorem 19.

COROLLARY 27. *Let k be a field of characteristic zero, and let N be a 1-dimensional kA-module, such that for each non-trivial spherical subgroup A_σ, the centre of A_σ acts non-trivially on N. Then $H^*(X; N) \cong \overline{H}^{*-1}(L; N)$.*

Proof. By Proposition 6, the action of the centre Z of A_σ on $H^*(A_\sigma : N)$ is trivial. However, as a kZ-module $H^i(A_\sigma; N)$ is isomorphic to a submodule of a direct sum of copies of N. The only possibility is that $H^*(A_\sigma; N) = \{0\}$. Hence Theorem 26 may be applied. □

Unfortunately, Theorem 26 does not apply directly to the case when $N = \ell^2(A)$, since only the *reduced* ℓ^2-cohomology groups $\mathscr{H}^*(A_\sigma)$ are equal to $\{0\}$. One way around this difficulty is to use W. Lück's linear equivalence of categories (for any discrete group G) between the category of Hilbert G-modules and the category of finitely generated projective modules for the group von Neumann algebra $\mathscr{N}(G)$, as explained in [**18**, Chapter 6]. The von Neumann dimension is defined for projective $\mathscr{N}(G)$-modules in such a way that it is preserved by the above equivalence of categories. Furthermore, von Neumann dimension extends to a dimension function on all $\mathcal{N}(G)$-modules so that it is additive on extensions, and such that $\dim_G(M) = 0$ if and only if $\mathrm{Hom}_G(M, \mathcal{N}(G)) = \{0\}$. For any CW-complex Y with fundamental group G, the von Neumann dimension of the ordinary homology group $H_i(Y; \mathcal{N}(G))$ is equal to the von Neumann dimension of $\mathscr{H}^i(Y)$. Hence the following statement is a version of Theorem 1.

THEOREM 28. *For any Artin group A and any i, the von Neumann dimensions of $H_i(X; \mathcal{N}(A))$ and $\overline{H}_{i-1}(L; \mathcal{N}(A)) \cong \mathcal{N}(A) \otimes \overline{H}_{i-1}(L)$ are equal.*

Proof. Use the homology version of the Mayer–Vietoris spectral sequence described above. In more detail, define a double chain complex

$$E^0_{*,*} = \mathcal{N}(A) \otimes_{\mathbb{Z}A} C_{*,*}.$$

The corresponding spectral sequence has the following properties:

(1) $E^\infty_{*,*} = \{0\}$.
(2) $E^2_{0,j} \cong \overline{H}_j(L; \mathcal{N}(A))$.
(3) $E^1_{i,0} \cong H_i(A; \mathcal{N}(A))$.
(4) For $i, j > 0$, $\dim_G(E^1_{i,j}) = 0$.

(Property (4) follows from Corollary 7.) It may be shown that for $1 < s \leqslant r$,

$$\dim_G \left(E^s_{0,r-1}\right) = \dim_G \left(E^2_{0,r-1}\right) \quad \text{and} \quad \dim_G \left(E^s_{r,0}\right) = \dim_G \left(E^1_{r,0}\right).$$

Since the E^∞-page is identically zero, the map $d^r : E^r_{r,0} \to E^r_{0,r-1}$ is an isomorphism, and hence $\dim_G(E^r_{r,0}) = \dim_G(E^r_{0,r-1})$. □

REMARK 29. In the case when the Artin group A contains no finite conjugacy classes, the von Neumann dimension of a Hilbert A-module determines the module up to isomorphism. Hence in this case, Theorem 28 is equivalent to Theorem 1. In fact, as was pointed out to us by the referee, the general version of Theorem 1 may also be proved in this way. For general G, the isomorphism type of a Hilbert G-module is determined not by its von Neumann dimension, but instead by the centre-valued von Neumann dimension (which takes values in the centre of $\mathcal{N}(G)$ instead of in $\mathbb{R}$). Lück's theory can be extended to the centre-valued von Neumann dimension, and this gives the general version of Theorem 1.

10.2. *ℓ^p-cohomology*

For $1 \leqslant p < \infty$, let $\ell^p(G)$ denote the space of real-valued functions f on G for which $\sum_{g \in G} |f(g)|^p$ is finite. Each $\ell^p(G)$ is a Banach space with respect to the obvious norm. If Y' is a free G-CW-complex with only finitely many orbits of cells of each dimension, one may define reduced ℓ^p-cohomology groups $\mathscr{H}^*_{(p)}(Y')$ in the same way as the reduced ℓ^2-cohomology groups were defined in Section 3. For $1 < p < \infty$, Gromov has proved vanishing theorems that imply the ℓ^p-analogues of Corollaries 7 and 8 in [**15**].

The main problem with attempting to generalize Theorems 1 and 3 to ℓ^p-cohomology is that there is no analogue of von Neumann dimension in this generality, and we do not know whether the analogue of Lemma 5 holds. Nevertheless, we conjecture that Theorems 1 and 3 remain valid for reduced ℓ^p-cohomology, for $1 < p < \infty$.

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References

1. M. Bestvina and N. Brady, 'Morse theory and finiteness properties of groups', *Invent. Math.* 129 (1997) 445–470.
2. N. Bourbaki, *Groupes et algèbres de Lie* (Masson, Paris, 1981) Chapitres 4–6.
3. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature* (Springer, Berlin, 1999).
4. E. Brieskorn, *Sur les groupes de tresses. Séminaire Bourbaki* (1971/72), Lecture Notes in Mathematics 317 (Springer, Berlin, 1973) 21–44.
5. K. S. Brown, *Cohomology of groups* (Springer, Berlin, 1982).
6. R. Charney and M. W. Davis, 'The $K(\pi,1)$-problem for hyperplane complements associated to infinite reflection groups', *J. Amer. Math. Soc.* 8 (1995) 597–627.
7. R. Charney and M. W. Davis, 'Finite $K(\pi,1)$ s for Artin groups', *Prospects in Topology*, Annals of Mathematics Studies 138 (ed. F. Quinn, Princeton University Press, Princeton, NJ, 2000) 110–124.
8. M. W. Davis, 'Nonpositive curvature and reflection groups', *Handbook of geometric topology* (ed. R. Daverman and R. Sher, Elsevier, Amsterdam, 2002) 373–422.
9. M. W. Davis and B. Okun, 'Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups', *Geometry and Topology* 5 (2001) 7–74.
10. C. De Concini and M. Salvetti, 'Cohomology of Artin groups', *Math. Res. Lett.* 3 (1996) 293–297.
11. P. Deligne, 'Les immeubles des groupes de tresses généralisés', *Invent. Math.* 17 (1972) 273–302.
12. W. Dicks and I. J. Leary, 'Presentations for subgroups of Artin groups', *Proc. Amer. Math. Soc.* 127 (1999) 343–348.
13. B. Eckmann, 'Introduction to ℓ_2-methods in topology: reduced ℓ_2-homology, harmonic chains, ℓ_2-Betti numbers', *Israel J. Math.* 117 (2000) 183–219.
14. M. Farber, 'Von Neumann Betti numbers and Novikov type inequalities', *Proc. Amer. Math. Soc.* 128 (2000) 2819–2827.
15. M. Gromov, 'Asymptotic invariants of infinite groups', *Geometric group theory* 2, London Mathematical Society Lecture Note Series 182 (ed. G. Niblo and M. Roller, Cambridge University Press, Cambridge, 1993).
16. A. Haefliger, 'Extensions of complexes of groups', *Ann. Inst. Fourier (Grenoble)* 42 (1992) 275–311.
17. K. H. Kim and F. W. Roush, 'Homology of certain algebras defined by graphs', *J. Pure Appl. Algebra* 17 (1980) 179–186.
18. W. Lück, L^2-invariants and K-theory (Springer, Berlin, 2002).
19. D. Quillen, *Higher algebraic K-theory* I. *Battelle Institute Conference*, 1972, Lecture Notes in Mathematics 341 (Springer, Berlin, 1973) 77–139.
20. M. Salvetti, 'Topology of the complement of real hyperplanes in $\mathbb{C}^n$', *Invent. Math.* 88 (1987) 603–618.
21. M. Salvetti, 'The homotopy type of Artin groups', *Math. Res. Lett.* 1 (1994) 565–577.
22. E. B. Vinberg, 'Discrete linear groups generated by reflections', *Math. USSR Izvestija* 5 (1971) 1083–1119.
23. C. A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).

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