L_2-COHOMOLOGY AND GROUP COHOMOLOGY

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§0. INTRODUCTION

LET Y be an arbitrary topological space and let Γ be a countable group which acts on Y. In this paper we study some homotopy theoretic invariants of such actions. In many respects, our treatment parallels more standard discussions of Betti numbers and the Euler characteristic. The main novelty is that our invariants are defined using the concept of Γ-dimension (Von Neumann dimension) of *singular* L_2-cohomology.

If Γ is a finite group which acts on a finite dimensional vector space V, the Γ-dimension of V is given by

$$\dim_\Gamma V = \frac{1}{\operatorname{ord}(\Gamma)} \dim V. \tag{0.1}$$

If Γ has infinite order, $\dim V$ is an extended real number, $0 \le \dim_\Gamma V \le \infty$, which is defined for certain actions of Γ on a vector space V (whose dimension in the usual sense is infinity).

In §1, seven key properties of the Γ-dimension are listed. These are the basis of the simplicial L_2-cohomology theory for free, simplicial, cocompact actions, considered in [1], [3], [4], [5], [7] and [17]. Here, we define L_2-cohomology spaces, $\bar{H}^i_{(2)}(Y:\Gamma)$, for arbitrary Y and countable Γ, by starting with the simplicial theory and taking suitable inverse limits. Thus, it is necessary to verify that the concept of Γ-dimension extends to inverse limits of Γ-modules and that the above-mentioned properties continue to hold. This is done in the Appendix.

The $\bar{H}^i_{(2)}(Y:\Gamma)$ are Γ-equivariant homotopy invariants. Moreover, they satisfy

$$\bar{H}^i_{(2)}(Y:\Gamma) = \bar{H}^i_{(2)}(Y \times K_\Gamma : \Gamma), \tag{0.2}$$

where K_Γ is a contractible complex on which Γ acts freely, and the action on $Y \times K_\Gamma$ is the diagonal action. Thus, if we set

$$b^i_{(2)}(Y:\Gamma) = \dim \bar{H}^i_{(2)}(Y:\Gamma), \tag{0.3}$$

these L_2-Betti numbers satisfy

$$b^i_{(2)}(Y_1:\Gamma) = b^i_{(2)}(Y_2:\Gamma), \tag{0.4}$$

if $Y_1 \times K_\Gamma$ is Γ-equivariantly homotopic to $Y_2 \times K_\Gamma$. From now on, we express this by saying that Y_1 is *free homotopy equivalent* to Y_2.

The $b^i_{(2)}(Y:\Gamma)$ have formal properties analogous to those of ordinary Betti numbers. They have some additional nice features as well. For example, they are sometimes finite even though $Y \times K_\Gamma / \Gamma$ has infinite topological type. Also, unlike ordinary Betti numbers, they behave multiplicatively under finite coverings. Finally, we mention that in certain circumstances, $b^i_{(2)}(Y:\Gamma) = 0$ for *all* i (including $i = 0$).

If

$$m_{(2)}(Y:\Gamma) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} b^i_{(2)}(Y:\Gamma) < \infty, \tag{0.5}$$

we define the L_2-Euler characteristic by

$$\chi_{(2)}(Y:\Gamma) = \sum (-1)^i b_{(2)}^i(Y:\Gamma). \tag{0.6}$$

If $Y = X$, an arbitrary simplicial complex on which Γ acts by simplicial automorphisms, there is a simple sufficient condition which guarantees that (0.5) holds and that $\chi_{(2)}(X:\Gamma)$ can be expressed as a sum of local terms, as in usual combinatorial formula for the Euler characteristic. When these local terms can be calculated explicitly, the resulting expression no longer directly involves L_2-cohomology, which enters only in the proof of its free homotopy invariance.

In order to describe matters in more detail, first of all, we define the *isotropy group* $\Gamma(s)$, of an open simplex $s \in X$. It consists of those γ such that $\gamma s = s$, or equivalently, those γ which fix the barycentre of s.

For Δ, a discrete group, we put

$$\bar{H}_{(2)}^i(\Delta) = \bar{H}_{(2)}^i(K_\Delta:\Delta), \tag{0.7}$$

$$b_{(2)}^i(\Delta) = b_{(2)}^i(K_\Delta:\Delta),$$

$$m_{(2)}(\Delta) = m_{(2)}(K_\Delta:\Delta).$$

If $m_{(2)}(\Delta) < \infty$, we define

$$\chi_{(2)}(\Delta) = \chi_{(2)}(K_\Delta:\Delta). \tag{0.8}$$

Clearly, for X, Γ as above, the function $m_{(2)}(\Gamma(s))$ is constant on Γ-orbits of simplices. Let S' be a set of simplices which meets each such orbit exactly once.

An action is called L_2-*finite* if

$$\sum_{s \in S'} m_{(2)}(\Gamma(s)) < \infty. \tag{0.9}$$

THEOREM 0.1.

$$m_{(2)}(X:\Gamma) \leq \sum_{s \in S'} m_{(2)}(\Gamma(s)). \tag{0.10}$$

If the right-hand side of (0.10) *is finite, then*

$$\chi_{(2)}(X:\Gamma) = \sum_{s \in S'} (-1)^{\dim s} \chi_{(2)}(\Gamma(s)). \tag{0.11}$$

Theorem 0.1 is proved by an argument which, at the formal level, is quite standard, together with a general property of Γ-dimension (reciprocity) which allows us to pass from $\dim_\Gamma$ to $\dim_{\Gamma(s)}$ (see §1 and §2 for details).

As we have indicated, Theorem 0.1 is of particular interest when the $b_{(2)}^i(\Gamma(s))$ can be computed explicitly. One case in which this is possible is that in which all $\Gamma(s)$ are *amenable*. The definition of amenability is recalled in §3. For the moment, we will simply list some important examples of amenable groups:

(a) Every finite group is amenable.
(b) Every abelian group is amenable.
(c) The union of an increasing family of amenable groups is amenable. For instance, every locally finite (that is, a union of an increasing family of finite subgroups) is amenable.
(d) Subgroups and quotient groups of amenable groups are amenable. Furthermore, if A_1 is

an amenable normal subgroup of A_1 and if the quotient group A/A_1 is amenable, then A is amenable. In particular, solvable and locally solvable groups are amenable.

(e) Let X be the infinite triadic tree (three edges at every vertex) and let G_0 be the group of automorphisms of X keeping fixed a point $\chi_0 \in X$. Grigorchuck [10] has constructed remarkable examples of finitely generated amenable subgroups $A \subset G_0$ (of subexponential growth) which cannot be obtained from finite and abelian groups by successive extensions and increasing unions.

(f) If B contains a non-abelian free subgroup, then B is non-amenable. All *known* finitely presented non-amenable groups admit such subgroups; compare [15].

THEOREM 0.2. *If A is amenable*

$$b_{(2)}^i(A) = \begin{cases} \dfrac{1}{\operatorname{ord}(A)} & i = 0 \\ 0 & i > 0 \end{cases} \tag{0.12}$$

where we consider

$$\frac{1}{\operatorname{ord}(A)} = 0, \tag{0.13}$$

if ord (A) $= \infty$.[†] *Thus,*

$$\chi_{(2)}(A) = \frac{1}{\operatorname{ord}(A)}. \tag{0.14}$$

We call an action *co-amenable* if all $\Gamma(s)$ are amenable and if

$$\sum_{s \in S'} \frac{1}{\operatorname{ord}(\Gamma(s))} < \infty. \tag{0.15}$$

A co-amenable action for which all $\Gamma(s)$ are of finite order is called *co-finite*.

Example 0.1. Let Γ be a discrete subgroup of a semisimple Lie group over a locally compact field of positive characteristic. Then Γ acts by simplicial automorphisms on the corresponding Bruhat–Tits building. As G. Prasad pointed out to us, there exist examples for which this action is co-finite and for which no subgroup of finite index acts freely.

By combining Theorems 0.1 and 0.2 we obtain:

THEOREM 0.3. (1) *If Γ acts co-amenably on X, then*

$$m_{(2)}(X : \Gamma) < \infty. \tag{0.16}$$

(2) *Moreover,*

$$\chi_{(2)}(X : \Gamma) = \sum_{s \in S'} (-1)^{\dim s} \frac{1}{\operatorname{ord}(\Gamma(s))}. \tag{0.17}$$

In particular, the expression on the right-hand side of (0.17) *is a free homotopy invariant in the class of co-amenable actions.*

(3) *If* ord $(\Gamma(s)) = \infty$ *for all* s, *then*

$$m_{(2)}(X : \Gamma) = \chi_{(2)}(X : \Gamma) = 0. \tag{0.18}$$

The vanishing of $\chi_{(2)}(X : \Gamma)$ in (3) is especially meaningful when $\chi_{(2)}(X : \Gamma)$ can be

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† This convention will be in force from now on.

expressed in terms of more standard topological invariants. Suppose X, as in Theorem 0.3, is free homotopy equivalent to a complex Z_1 on which some subgroup, Γ', of finite index, acts *freely*, with compact quotient. Define

$$\chi_{\text{virt}}(X:\Gamma) = \frac{1}{\text{ind}\,(\Gamma':\Gamma)}\,\chi(X \times K_\Gamma/\Gamma')$$

$$= \chi_{\text{virt}}(Z:\Gamma), \tag{0.19}$$

where χ denotes the usual Euler characteristic.

PROPOSITION 0.4.

$$\chi_{(2)}(X:\Gamma) = \chi_{\text{virt}}(X:\Gamma). \tag{0.20}$$

Proof. Let S be a set of simplices of Z which meets each Γ-orbit exactly once and let $S' \supset S$ be such a subset for Γ'. Then

$$\chi(X \times K/\Gamma') = \chi(Y/\Gamma') \tag{0.21}$$

$$= \sum_{s \in S'} (-1)^{\dim s}.$$

Since Γ' acts freely, distinct elements of $\Gamma(s)$ belong to distinct cosets of Γ' in Γ. It follows that each element, $s \in S'$ is equivalent to exactly

$$\frac{\text{ind}\,(\Gamma':\Gamma)}{\text{ord}\,(\Gamma(s))} \tag{0.22}$$

elements of S' under the action of Γ. Substituting this in (0.21), and using (0.17) and (0.19) gives:

$$\chi_{\text{virt}}(X:\Gamma) = \sum_{s \in S} (-1)^{\dim s}\frac{1}{\text{ord}\,(\Gamma(s))} \tag{0.23}$$

$$= \chi_{(2)}(Z:\Gamma)$$

$$= \chi_{(2)}(X:\Gamma).$$

Now, by combining Proposition 0.4 with (3) of Theorem 0.3, we get:

COROLLARY 0.5. *Let Γ act on a complex X. Suppose X is free homotopic to Y_1 on which some subgroup, Γ', of finite index, acts freely with compact quotient. If X is also free homotopic to Z_2 on which Γ acts co-amenably with all isotropy groups of infinite order, then*

$$\chi_{\text{virt}}(X:\Gamma) = 0. \tag{0.24}$$

We emphasize that in the above case, we do not require that the isotropy groups $\Gamma(s)$ be finitely generated.

If, in (0.19), Z is contractible, by definition,

$$\chi_{\text{virt}}(Z:\Gamma) = \chi_{\text{virt}}(\Gamma), \tag{0.25}$$

the virtual Euler characteristic introduced by Wall [20] (see also [5], [18]). Note that if, for example, $\Gamma_1 \subset \Gamma_2$ with Γ_2 finite, the usual Euler characteristics $\chi(\Gamma_j)$ are defined (with real

coefficients) but do not satisfy the expected multiplicative property. That is,

$$\chi(\Gamma_1) = 1 \tag{0.26}$$

$$= \chi(\Gamma_2)$$

$$\neq \mathrm{ind}\,(\Gamma_1 : \Gamma_2)\chi(\Gamma_2).$$

This stems from the fact that $K_{\Gamma_j}/\Gamma_j)$ is not finite dimensional, and hence, that $\chi(\Gamma_j)$ $= \chi(K_{\Gamma_j}/\Gamma_j)$ is not given by the usual local formula. On the other hand, $\chi_{\mathrm{virt}}(\Gamma)$ behaves multiplicatively, essentially by definition [as does $\chi_{(2)}(X:\Gamma)$].

By specializing Corollary 0.5, we now obtain one of our main results.

COROLLARY 0.6. *Let Γ contain a subgroup, Γ', of finite index such that $K_{\Gamma'}/\Gamma'$ is homotopy equivalent to a finite complex. If Γ also contains an infinite amenable normal subgroup, A, then*

$$\chi_{\mathrm{virt}}(\Gamma) = 0. \tag{0.27}$$

Proof. If Γ contains an infinite amenable normal subgroup, A, then Γ acts co-amenably on $K_{\Gamma/A}$, by composition with the quotient map $\Gamma \to \Gamma/A$. But $K_{\Gamma/A}$ is free homotopy equivalent to K_Γ, since $K_{\Gamma/A} \times K_\Gamma$ and K_Γ are Γ-equivariantly homotopy equivalent.

For *abelian* normal subgroups, the above result is due to Rosset [16], who generalized earlier theorems of Stallings [19] and Gottlieb [8]. They assumed that A is central.

We are grateful to Professor Rosset for having communicated his results to us prior to their publication. His method, like ours, is based on the notion of Von Neumann dimension.

The vanishing of $\chi(K_A/A)$ for a compact aspherical *manifold*, K_A/A, with amenable fundamental group is due to Morgan and Phillips (unpublished).

There is also a relation between the present paper and the results of [4] concerning complete manifolds of finite volume and bounded covering geometry. There, we considered isometric actions of a discrete† group Γ on a complete Riemannian manifold $\tilde{M}$, whose sectional curvature, K, and injectivity radius, $i(\tilde{M})$ satisfy $|K| \le 1$, $i(\tilde{M}) \ge 1$ [we write geo $(\tilde{M}) \le 1$]. In our present terminology, we showed that if $\mathrm{Vol}\,(\tilde{M}/\Gamma) < \infty$, then $b^i_{(2)}(\tilde{M}:\Gamma)$ can be calculated analytically,

$$b^i_{(2)}(\tilde{M}:\Gamma) = \dim_\Gamma \tilde{H}^i, \tag{0.28}$$

where $\tilde{H}^i$ is the space of L_2-harmonic forms on $\tilde{M}$. Let $P_\chi(\Omega)$ denote Chern–Gauss–Bonnet form. By the L_2-index theorem proved in [3] (see also Theorem 5.3)

$$\int_{\tilde{M}/\Gamma} P_\chi(\Omega) = \sum_i (-1)^i b_{(2)}(\tilde{M}:\Gamma). \tag{0.29}$$

Thus, in view of Corollary 0.6, we obtain:

THEOREM 0.7. *Let Γ act discretely and isometrically on a contractible manifold, $\tilde{M}$, with geo $(\tilde{M}) \le 1$, $\mathrm{Vol}\,(\tilde{M}/\Gamma) < \infty$. If Γ contains an infinite amenable normal subgroup, then*

$$\int_{\tilde{M}/\Gamma} P_\chi(\Omega) = 0. \tag{0.30}$$

Some further results in the Riemannian case will be discussed in §5. The remainder of the

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† In [4] we considered free actions, but the proofs remain valid for Γ discrete.

paper is organized as follows:

§1. Γ-Modules and simplicial L_2-cohomology
§2. Singular L_2-cohomology
§3. Vanishing in the amenable case
§4. Amalgamated products
§5. L_2-Cohomology of Riemannian manifolds
    Appendix: Inverse limits of Γ-modules.

The proofs of Theorems 0.1 and 0.2 are given in §2 and §3, respectively. As we have indicated, the other results stated in this section are direct consequences of these theorems.

§1. Γ-MODULES AND SIMPLICIAL L_2-COHOMOLOGY

Let Γ be a countable group and let $l_2(\Gamma)$ denote the Hilbert space of real valued square integrable functions on Γ. Let A be a pre-Hilbert space on which Γ acts by isometries. We call A a Γ-*module* if it is equivariantly isometric to a subspace of $l_2(\Gamma) \otimes H$, where H is some Hilbert space and Γ acts by the regular representation on $l_{(2)}(\Gamma)$ and acts trivially on H. To such an A, one can attach an extended real number, $\dim_\Gamma A \in [0, \infty]$, which [is independent of the particular identification with a subspace of $l_{(2)}(\Gamma) \otimes H$ and] enjoys the following seven properties (for further details on Γ-modules we refer the reader to [1], [3], [4], [5], [6], [12]):

(a) $\dim_\Gamma A = 0$ if and only if $A = 0$.

(b) $\dim_\Gamma \bar{A} = \dim_\Gamma A$, for the completion $\bar{A}$ of A.

(c) $\dim_\Gamma l_2(\Gamma) = 1$.

(d) If A is complete and if $\alpha : A \to B$ is a bounded Γ-invariant operator, then

$$\dim_\Gamma A = \dim_\Gamma \ker \alpha + \dim_\Gamma \operatorname{Im} \alpha. \qquad (1.1)$$

(e) (Continuity). Let $A_1 \supset A_2 \supset \dots A_i \supset \dots$ be *closed* subspaces and let $\dim_\Gamma A_1 < \infty$. Then

$$\dim_\Gamma \bigcap_{j=1}^\infty A_j = \lim_{j \to \infty} \dim_\Gamma A_j. \qquad (1.2)$$

(f) (Reciprocity). Let $\Gamma_1 \subset \Gamma_2$ and let A_2 be a Γ_2-module induced by the induced representation from a Γ_1-module A_1. Then

$$\dim_{\Gamma_1} A_1 = \dim_{\Gamma_2} A_2. \qquad (1.3)$$

(g) Let $\Gamma_1 \subset \Gamma_2$ with $\operatorname{ind}(\Gamma_1 : \Gamma_2) < \infty$ and let A be a Γ_2-module. Then A is a Γ_1-module (by restriction) and

$$\dim_{\Gamma_1} A = \dim_{\Gamma_2} A \cdot \operatorname{ind}(\Gamma_1 : \Gamma_2). \qquad (1.4)$$

Let X be a simplicial complex on which Γ acts by simplicial automorphisms. A cochain, $c \in C^i(X, R)$, is called l_2, if

$$\sum_s |c(s)|^2 < \infty, \qquad (1.5)$$

where the summation runs over all i-simplices of X. If the action of Γ on X is *free*, the space $C^i_{(2)}(X)$ of $l_{(2)}$-cochains is isometric to $l_{(2)}(\Gamma) \otimes H$, where $\dim H$ is equal to the number of i-simplices in X/Γ.

More generally, if the isotropy subgroup, $\Gamma(s)$, of each simplex s, is *finite*, then the subspace of $C^i_{(2)}(X)$ which is supported on the orbit of s, is isometric to $l_2(\Gamma/\Gamma(s))$. This space in turn is isometric to the subspace of $l_2(\Gamma)$ spanned by those functions which are constant on

cosets of $\Gamma(s)$. Since $C^i_{(2)}(X)$ is isometric to the direct sum of such subspaces, it is also a Γ-module in this case and

$$\dim_\Gamma C^i_{(2)}(X) = \sum_{s \in S'} \frac{1}{\text{ord}\,(\Gamma(s))} \tag{1.6}$$

where S' is a set of i-simplices which meets each orbit exactly once.

If, however, $\Gamma(s)$ has infinite order, functions which are constant on cosets of $\Gamma(s)$ are not in $l_2(\Gamma)$, and the above construction breaks down. This case will be dealt with in §2 by introducing the *singular* L_2-cohomology of X.

The coboundary operator,

$$d_i : C^i_{(2)}(X) \to C^{i+1}_{(2)}(X), \tag{1.7}$$

is clearly a bounded operator and $d_{i+1}\,d_i = 0$. We define

$$\bar{H}^i_{(2)}(X:\Gamma) = \ker d_i/\overline{\text{Im}\,d_{i-1}}, \tag{1.8}$$

where $\overline{\text{Im}\,d_{i-1}}$ denotes the closure of the image (compare [1], [7], [17]). Thus, if $\text{Im}\,d_{i-1}$ is not closed, $\bar{H}^i_{(2)}(X:\Gamma)$ is a so-called *reduced* cohomology space and not a cohomology space in the usual sense. When there is no danger of confusion, we will write $\bar{H}^i_{(2)}(X)$ for $\bar{H}^i_{(2)}(X:\Gamma)$.

The space, $\bar{H}^i_{(2)}(X)$ embeds isometrically into $\ker d_i \subset C^i_{(2)}(X)$ as the orthogonal complement of $\overline{\text{Im}\,d_{i-1}}$. Thus it acquires the structure of a Γ-module. The image of the embedding is, by definition, the space of *harmonic* cochains on X.

Example 1.1. If X is connected, every cochain, c, in $C^i(X)$ satisfying $d_0 c = 0$ is automatically harmonic and equal to a constant function. Hence

$$\bar{H}^0_{(2)}(X) = 0, \tag{1.9}$$

if X is connected and Γ is infinite [since in that case $c \notin C^i_{(2)}(X)$].

For the remainder of this section, we will restrict our attention to the *cofinite* case [see (0.15)].

Let E_λ denote the family of spectral projections associated to the bounded self adjoint operator $d_i^*\,d_i$. Since $\dim C^i_{(2)}(X) < \infty$, for any $\lambda_0 < \infty$,

$$\dim_\Gamma E_{\lambda_0} < \infty, \tag{1.10}$$

and it follows from (1.5) and (d) and (e) above that

$$\lim_{\lambda \to 0} \dim_\Gamma (E_\lambda \cap E_0^\perp) = 0. \tag{1.11}$$

Hence, off a subspace of arbitrarily small Γ-dimension, d_i^{-1} is a bounded operator (by definition $d_i^{-1}|\ker d_i = 0$). An operator with this property is called Γ-*Fredholm*. For such d_i, all the standard homological computations with exact sequences can be carried over in a straightforward manner [even though $\bar{H}^i_{(2)}(X)$ is a *reduced* cohomology space]; see [4], §2 for further details. In particular, if $\chi_{(2)}(X:\Gamma)$ is defined as in (0.6), by (1.6), we have

$$\chi_{(2)}(X:\Gamma) = \sum_{s \in S} (-1)^{\dim s} \frac{1}{\text{ord}\,(\Gamma(s))}, \tag{1.12}$$

which is (0.17) for the cofinite case.

Similarly, it follows that the spaces $\bar{H}^i_{(2)}$ enjoy the following properties:

(i) (Functoriality and homotopy invariance). Let $f : X \to Y$ be a Γ-equivariant simplicial map. Then the obvious homomorphism $C^i_{(2)}(Y) \to C^i_{(2)}(X)$ induces the homomorphism

$f^*: \bar{H}^i_{(2)}(Y) \to \bar{H}^i_{(2)}(X)$, which depends only on the Γ-equivariant homotopy class of f; see [7].[†]

(ii) (Exact sequence of a pair). Let $X' \subset X$ be a Γ invariant subcomplex. The relative L_2-cohomology, $\bar{H}^i_{(2)}(X, X')$, is defined in the usual way. The relative cohomology sequence

$$\bar{H}^i_{(2)}(X, X') \to \bar{H}^i_{(2)}(X) \to \bar{H}^i_{(2)}(X') \to H^{i+1}_{(2)}(X, X') \to \qquad (1.13)$$

is defined by a trivial modification of the standard procedure. It is *weakly exact* in the sense that the *closure* of the image of every map equals the kernel of the succeeding map; (see [4], §2). Since by (b) above, for any Γ-module A, $\dim_\Gamma A = \dim_\Gamma \bar{A}$, (1.13) is exact for all practical purposes.

(iii) (Excision). For all Γ-invariant subsets $V \subset X'$, whose complement $X' \backslash V$ is a closed subcomplex in X', we have

$$\bar{H}^i_{(2)}(X, X') = \bar{H}^i_{(2)}(X \backslash V, X' \backslash V). \qquad (1.14)$$

As a consequence of properties (i)–(iii), we also have the standard cohomological formalism of Mayer–Vietoris, Leray's spectral sequence, etc.; compare [4].

§2. SINGULAR L_2-COHOMOLOGY

Let Y be an arbitrary topological space on which Γ acts.

Definition 2.1. Let $\mathscr{C}(Y)$ denote the small category whose objects are pairs, (X, f), where X is a simplicial complex with free, simplicial cocompact Γ-action, X is a subcomplex of the standard simplex, $\sigma^\infty \subset R^\infty$, and $f: X \to Y$ is a Γ-equivariant continuous map.[†] The set of morphisms from (X_1, f_1) to (X_2, f_2) is empty unless X_1 is a subcomplex of X_2 and $f_2 | X_1 = f_1$. In this case it contains a single element, the inclusion map.

Corresponding to such a morphism we have the restriction map,

$$\rho_{X_1, X_2}: \bar{H}^i_{(2)}(X_2 : \Gamma) \to \bar{H}^i_{(2)}(X_1 : \Gamma), \qquad (2.1)$$

where $\bar{H}^i_{(2)}(Y : \Gamma)$ is as in §1. We define the singular L_2-cohomology, $\bar{H}^i_{(2)}(Y : \Gamma)$, to be the inverse limit (with its usual topology):

$$\bar{H}^i_{(2)}(Y : \Gamma) \overset{\text{def}}{=} \lim_{\mathscr{C}(Y)} \bar{H}^i_{(2)}(X : \Gamma)_{(X, f)}. \qquad (2.2)$$

If Y itself is a simplicial complex on which Γ acts freely with finite quotient, then the inverse limit in (2.2) is canonically isomorphic to $\bar{H}^i_{(2)}(Y : \Gamma)$ as defined in §1. Thus it inherits the structure of a Γ-module. This would be obvious if we had defined the morphisms between *any* two objects (X_1, f_1), (X_2, f_2) of $\mathscr{C}(Y)$, to be *all* Γ-equivariant maps, $g: X_1 \to X_2$, with $f_1 = f_2 g$. Then, $\mathscr{C}(Y)$ would have a final object, (Y, Ident). But it is easy to see that this second definition is actually equivalent to Definition 2.1. This is because up to homotopy, any map can be made simplicial and replaced by the inclusion into the mapping cone. Definition 2.1 is somewhat more convenient in the context of Lemma 2.3.

In general, although $\bar{H}^i_{(2)}(Y : \Gamma)$ cannot be given the structure of a Γ-module in a canonical

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[†] Actually, the argument of [7] assumes that Γ acts freely with X/Γ compact. The general case follows, for example, from the homotopy invariance of the singular theory together with Remark 2.2.

[†] If Γ is not finitely generated, then X cannot be connected. However, the Γ-module $\bar{H}^i_{(2)}(X \Gamma)$ is induced from the Γ_0-module $\bar{H}^i_{(2)}(X_0 : \Gamma)$, where Γ_0 is the (finitely generated) isotropy group of some component, X_0, of X; compare (f) of §1 and Proposition 2.5.

way, it does have a well defined Γ-dimension, for which the usual properties hold. In the cofinite case, the natural map from simplicial L_2-cohomology to singular L_2-cohomology is always an injection with dense range, and the corresponding L_2-Betti numbers coincide. This explains our use of the same notation for both theories.

To define the Γ-dimension, we first consider any system of complete, Γ-finite dimensional Γ-modules and bounded Γ-equivariant operators indexed by a category $\mathscr{C}$. We assume, as is the case for $\mathscr{C} = \mathscr{C}(Y)$, that for any pair of objects α, α':

(1) There is at most one operator,

$$p_{\alpha',\alpha} : A_\alpha \to A_{\alpha'}. \tag{2.3}$$

In this case we write $\alpha \geq \alpha'$.

(2) There exists α'' such that

$$
\begin{array}{c}
A_{\alpha''} \\
p_{\alpha,\alpha''} \swarrow \qquad \searrow p_{\alpha',\alpha''} . \\
A_\alpha \qquad\qquad A_{\alpha'}
\end{array}
\tag{2.4}
$$

For the remainder of this section, we also make the following provisional technical hypothesis.

(3) There exists a linearly ordered subset $\mathscr{L} \subset \mathscr{C}$ such that for all $\alpha \in \mathscr{C}$ there exist $\beta \in \mathscr{L}$ with $\beta \geq \alpha$.

Assumption (3) is valid, for example, for $\mathscr{C}(Y \times K_\Gamma)$, where Y is a countable simplicial complex, as follows from the fact that $Y \times K_\Gamma$ has a Γ-equivariant exhaustion, $Y = \cup Y_j$, with Y_j/Γ compact. This suffices for all applications of the present paper. However, in the Appendix we will explain how assumption (3) can be removed.

Recall that an element of $\lim_{\mathscr{C}} A_\alpha$ is, by definition, a function, $\alpha \to a_\alpha$, such that $p_{\alpha',\alpha}(a_\alpha) = a_{\alpha'}$, for all $p_{\alpha',\alpha}$. If p_β is the natural projection,

$$p_\beta : \lim_{\mathscr{C}} A_\alpha \to A_\beta, \tag{2.5}$$

and $V \subset \lim_{\mathscr{C}} A_\alpha$, we put

$$\dim_\Gamma V \overset{\mathrm{def}}{=} \sup_\beta \dim_\Gamma \operatorname{Im} p_\beta(V). \tag{2.6}$$

Properties (a)–(g) of §1 have an obvious interpretation for inverse limits, $\lim_{\mathscr{C}} A_\alpha$, and [for (d)] Γ-equivariant bounded operators. These are verified in the Appendix. In fact, for (d), we will consider only a somewhat restricted class of maps of inverse systems, see (A26), (A27). However, all maps which occur in the applications to L_2-cohomology are of this type.

The following basic Lemma 2.1 is also proved in the Appendix.

Let $(A_\alpha, \mathscr{C})$ be as above and let $B_\alpha \subset A_\alpha$ be closed. By a slight abuse of notation we denote by $\lim_{\mathscr{C}} B_\alpha$, the subspace of $\lim A_\alpha$, of elements a, such that $a_\alpha \subset B_\alpha$, for all α. Let $\bar{D}$ denote the closure of the Γ-module D.

LEMMA 2.1.

$$p_\beta\left(\lim_{\mathscr{C}} B_\alpha\right) = \overline{\bigcap_{\beta' \geq \beta} p_{\beta,\beta'}\left(\overline{\bigcap_{\beta'' \leq \beta'} p_{\beta'',\beta'}^{-1}(B_{\beta''})}\right)}. \tag{2.7}$$

The intersection in (2.7) is over all β', β'' with $B_{\beta'} \to B_\beta$, $B_{\beta'} \to B_{\beta''}$ (including $\beta' = \beta''$).

Remark 2.1. For inverse limit systems of arbitrary objects, it is easy to check that if the bars in (2.7) are removed, then the left-hand side is contained in the right. However, (2.7) itself is peculiar to inverse limits of Γ-modules. It stems from the continuity property of Γ-dimension, (e) of §1.

Lemma 2.1 has the following particular consequence which is worth noting at this point. Let X by an arbitrary countable simplicial complex with free simplicial Γ-action (if the action is not free replace X by $X \times K_\Gamma$ in what follows; see Proposition 2.2). Let $X = \cup X_j$ be an exhaustion of X by Γ-equivariant subcomplexes, with X_j/Γ compact. Then by (2.7) and the continuity property of Γ-dimension,

$$b_{(2)}^i(X:\Gamma) = \lim_{j \to \infty} \lim_{k \to \infty} \dim_\Gamma \mathrm{Im}\,(\bar{H}_{(2)}^i(X_k:\Gamma)) \subset \bar{H}_{(2)}^i(X_j:\Gamma); \qquad (2.8)$$

compare [4] and see the Appendix for further details.

The basic properties of singular L_2-cohomology are formal consequences of (a)–(g), Lemma 2.1, and the corresponding properties of the simplicial theory.

Let $g : Y \to Z$ be a Γ-equivariant map. Let $(X, f) \in \mathscr{C}(Y)$ and

$$I : \bar{H}_{(2)}^i(X:\Gamma)_{(X,f)} \simeq \bar{H}_{(2)}^i(X:\Gamma)_{(X,gf)} \qquad (2.9)$$

be the identity map. Then for $p_{(X,f)}$ as in (2.5) the family of compatible maps

$$I \circ p_{(X,f)} : \bar{H}_{(2)}^i(Z:\Gamma)_{(X,f)} \to \bar{H}_{(2)}^i(X:\Gamma)_{(X,gf)} \qquad (2.10)$$

induces

$$g^* : \bar{H}_{(2)}^i(Z:\Gamma) \to \bar{H}_{(2)}^i(Y:\Gamma). \qquad (2.11)$$

If $g_0,\, g_1 : Y \to Z$ are Γ-equivariantly homotopic, there are morphisms

$$i_j : (X, g_j f) \to (X \times I, g_t f) \quad j = 0, 1 \qquad (2.12)$$

for which the maps

$$i_j^* : \bar{H}_{(2)}^i(X) \to \bar{H}_{(2)}^i(X \times I) \quad j = 0, 1 \qquad (2.13)$$

are isomorphisms with $i_0^* = i_1^*$. Thus, the induced maps in (2.10) coincide for the choices $g = g_0$, $g = g_1$, and the same holds for (2.11). In particular, $\bar{H}_{(2)}^i(Y)$ is a Γ-equivariant homotopy invariant.†

Note that if Γ does not act freely, Y and $Y \times K_\Gamma$ need not be Γ-equivariantly homotopy equivalent. However, we have

PROPOSITION 2.2. *Let* $\pi : Y \times K_\Gamma \to Y$ *be the projection. Then*

$$\pi^* : \bar{H}_{(2)}^i(Y:\Gamma) \to \bar{H}_{(2)}^i(Y \times K_\Gamma : \Gamma) \qquad (2.14)$$

is an isomorphism.

Proof. Recall that for all complexes, X, as above, there is a Γ-equivariant map, $f : X \to K_\Gamma$, which is *unique* up to homotopy.

Let $a \in \bar{H}_{(2)}^i(Y:\Gamma)$ be non-zero. To show that $\pi^*(a) \neq 0$, we choose $f : X \to Y$ such that

$$p_{(X,f)}(a) \neq 0, \qquad (2.15)$$

---

† Here, and occasionally below, we have omitted some trivial details which are necessary to make the construction conform logically to Definition 2.1.

for the natural projection,

$$p_{(X,f)}: \bar{H}^i_{(2)}(Y:\Gamma) \to \bar{H}^i_{(2)}(X:\Gamma)_{(X,f)}. \tag{2.16}$$

Since $\pi \circ (f, \mathbf{f}) = f$, by (2.15), the image of a in $\bar{H}^*_{(2)}(X:\Gamma)_{(X,(f,\mathbf{f}))}$ is non-zero (for any choice of $\mathbf{f}$). Thus, $\pi^*(a) \neq 0$.

To see that π^* is surjective, choose $b \in H^i_{(2)}(Y \times K_\Gamma:\Gamma)$. Let $b_{(f,\mathbf{f}_j)} \in (X, (f, \mathbf{f}_j))$ be components of $b, j = 0, 1$. It suffices to show that

$$b_{(f,\mathbf{f}_0)} = b_{(f,\mathbf{f}_1)} = b_f \tag{2.17}$$

is independent of $\mathbf{f}_j$. For then, $\{b_f\}$ determines an element in $\bar{H}^i_{(2)}(Y:\Gamma)$ whose image is b. But since $\mathbf{f}_0, \mathbf{f}_1$ are Γ-equivariantly homotopic,

$$(X, (f, \mathbf{f}_j)) \xrightarrow{i_j} (X \times I, (f, \mathbf{f}_t)) \quad j = 0, 1 \tag{2.18}$$

and

$$b_{(f,\mathbf{f}_1)} = i_1^* (i_0^*)^{-1} (b_{(f,\mathbf{f}_0)}) \tag{2.19}$$

$$= b_{(f,\mathbf{f}_0)}.$$

The relative spaces, $H^i_{(2)}(Y, Y':\Gamma)$ are defined by

$$\bar{H}^i_{(2)}(Y, Y':\Gamma) = \lim_{\mathscr{C}(Y,Y')} \bar{H}^i_{(2)}(X, X':\Gamma)_{(X,X',f)}, \tag{2.20}$$

where $f: (X, X') \to (Y, Y')$ and the objects of $\mathscr{C}(Y, Y')$ are (X, X', f). To each $(X, f) \in \mathscr{C}(Y)$ we associate $(X, f^{-1}(Y'), f) \in \mathscr{C}(Y, Y')$, where $f^{-1}(Y')$ is the subcomplex of X consisting of those closed simplices s, with $f(s) \subset Y'$. The corresponding map

$$\bar{H}^i_{(2)}(Y, Y':\Gamma) \to \bar{H}^i_{(2)}(Y:\Gamma) \tag{2.21}$$

is induced by the family of compatible maps,

$$\bar{H}^i_{(2)}(Y:\Gamma) \to \bar{H}^i_{(2)}(X, f^{-1}(Y'):\Gamma)_{(X,f^{-1}(Y'),\Gamma)}$$
$$\to \bar{H}^i_{(2)}(X:\Gamma)_{(X,f)}. \tag{2.22}$$

Similarly, the coboundary map, δ, in the cohomology sequence

$$\to \bar{H}^i_{(2)}(Y, Y':\Gamma) \xrightarrow{r} \bar{H}^i_{(2)}(Y:\Gamma) \xrightarrow{t} \bar{H}^i_{(2)}(Y':\Gamma) \xrightarrow{\delta} \tag{2.23}$$

is induced by the compatible family

$$\bar{H}^i_{(2)}(Y':\Gamma) \to \bar{H}^i_{(2)}(X':\Gamma)_{(X',f)} \xrightarrow{\delta} \bar{H}^{i+1}_{(2)}(X, X', \Gamma)_{(X,X',f)} \tag{2.24}$$

where $X' \subset X$ is arbitrary.

LEMMA 2.3. *The cohomology sequence* (2.23) *is weakly exact. Thus, the Γ-dimension of the kernel of any map is equal to that of the image of the preceding one. In particular, if $\bar{H}^i_{(2)}(Y, Y':\Gamma)$, $\bar{H}^i_{(2)}(Y':\Gamma)$ have finite Γ-dimension, so does $\bar{H}^i_{(2)}(Y:\Gamma)$.*

Proof. Since by the Appendix, (d) of §1 holds in the present context, it suffices to show that (2.23) is weakly exact. This is a straightforward (but somewhat tedious) application of Lemma 2.1. So we will only give the details for the case $\overline{\text{Im}\, r} = \ker t$.

Let $(X, f) \in \mathscr{C}(Y)$, $X' \subset X$ and $f(X') \subset Y'$. Put

$$K(X, X')_{(X,f)} = \ker \rho_{X',X} \tag{2.25}$$

where

$$K(X, X')_{(X,f)} \subset \bar{H}_{(2)}^i(X : \Gamma)_{(X,f)} \tag{2.26}$$

and

$$\rho_{X', X} : \bar{H}_{(2)}^i(X : \Gamma) \to \bar{H}_{(2)}^i(X' : \Gamma) \tag{2.27}$$

is the restriction map.

If $X \subset X_1$ and $f : (X_1, X_1') \to (Y, Y')$, put

$$L(X_1, X_1', X)_{(X,f)} = \rho_{X, X_1}[K(X_1, X_1')_{(X_1,f)}]. \tag{2.28}$$

Then by (2.7), for the map t in (2.23), we have

$$\overline{p_{(X,f)}(\ker t)} = \cap \overline{L(X_1, X_1', X)_{(X,f)}} \tag{2.29}$$

$$= \underset{X \subset X_1}{\cap} \overline{L(X_1, f^{-1}(Y'), X)_{(X,f)}}$$

[where $f^{-1}(Y') \subset X_1'$ and the first intersection is over all (X_1, X_1') as above.]

On the other hand,

$$p_{(X,f)} r = \rho_{X, (X,f^{-1}(Y'))} p_{(X,f^{-1}(Y'),f)} \tag{2.30}$$

where

$$\rho_{X, (X,f^{-1}(Y'))} : \bar{H}_{(2)}^i(X, f^{-1}(Y') : \Gamma) \to \bar{H}_{(2)}^i(X : \Gamma). \tag{2.31}$$

To compute $p_{(X,f^{-1}(Y'),f)}$, let $(X, f^{-1}(Y')) \subset (X_1, X_1')$ and $f : (X_1, X_1') \to (Y, Y')$. Put

$$I(X_1, X_1', X)_{(X,f^{-1}(Y'),f)} = \text{Im } \rho_{(X,f^{-1}(Y')), (X_1, X_1')} \tag{2.32}$$

where

$$\rho_{(X,f^{-1}(Y')), (X_1, X_1')} : \bar{H}_{(2)}^i(X_1, X_1' : \Gamma) \to \bar{H}_{(2)}^i(X, f^{-1}(Y') : \Gamma). \tag{2.33}$$

Then by (2.7),

$$\overline{p_{(X,f^{-1}(Y'),f)}(\bar{H}_{(2)}^i(Y, Y' : \Gamma))} \tag{2.34}$$

$$= \cap \overline{I(X_1, X_1', X)_{(X,f^{-1}(Y'),f)}}$$

$$= \underset{X_1 \supset X}{\cap} \overline{I(X_1, f^{-1}(Y'), X)_{(X,f^{-1}(Y'),f)}}.$$

Finally,

$$\overline{\text{Im } \rho_{X, (X,f^{-1}(Y'))}} = \ker \rho_{f^{-1}(Y'), X} \tag{2.35}$$

by weak exactness for the simplicial L_2-theory. By putting together (2.29), (2.30), (2.34) and (2.35), we see that $\overline{\text{Im } r} = \ker t$ q.e.d.

Let $\bar{V} \subset \text{int}(Y') \subset Y$. Then we have the exision map

$$\bar{H}_{(2)}^i(Y, Y') \to \bar{H}_{(2)}^i(Y \backslash V, Y' \backslash V). \tag{2.36}$$

That this map is an isomorphism follows from the fact that to each $(X, X', f) \in \mathscr{C}(Y, Y')$, we can associate a pair $(Z, Z', f) \in \mathscr{C}(Y \backslash V, Y' \backslash V)$, where Z is a subcomplex of a sufficiently fine subdivision of X and the excision map

$$\bar{H}_{(2)}^i(X, X') \to \bar{H}_{(2)}^i(Z, Z') \tag{2.37}$$

is an isomorphism.

As a consequence of the results given so far, we obtain other standard cohomological machinery such as Mayer–Vietoris and spectral sequences.

The following are some further useful properties of singular L_2-cohomology.

PROPOSITION 2.4. *If Y is path connected, then*[†]

$$b_{(2)}^0 (Y:\Gamma) = \frac{1}{\text{ord}(\Gamma)}. \tag{2.38}$$

PROPOSITION 2.5. *If Γ acts transitively on the path connected components of Y and Γ_0 denotes the isotropy group of some fixed component Y_0, then*

$$b_{(2)}^i (Y:\Gamma) = b_{(2)}^i (Y_0, \Gamma_0). \tag{2.39}$$

PROPOSITION 2.6. *If $\Gamma' \subset \Gamma$ with $\text{ind}(\Gamma':\Gamma) < \infty$, then*

$$b_{(2)}^i (Y:\Gamma') = b_{(2)}^i (Y:\Gamma) \, \text{ind}(\Gamma':\Gamma). \tag{2.40}$$

In particular, taking $Y = K_\Gamma$,

$$b_{(2)}^i (\Gamma') = b_{(2)}^i (\Gamma) \, \text{ind}(\Gamma':\Gamma). \tag{2.41}$$

Proposition 2.4 is clear. Propositions 2.5 and 2.6 follow from (f) and (g) of §1, respectively.

PROPOSITION 2.7. (Kunneth formula). *If*

$$(X, \Gamma) = (Y_1 \times Y_2, \Gamma_1 \times \Gamma_2) \tag{2.42}$$

then

$$b_{(2)}^i (Y, \Gamma) = \sum_{j=0}^{i} b_{(2)}^j (Y_1 : \Gamma_1) \, b_{(2)}^{i-j} (Y_2 : \Gamma_2) \tag{2.43}$$

where we interpret

$$\infty \times 0 = 0. \tag{2.44}$$

Proof. If $f: X \to Y_1 \times Y_2$, then $f = (f_1, f_2)$, where $f_j : X \to Y_j$. Thus f can be factored as

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f_1 \times f_2} Y_1 \times Y_2 \tag{2.45}$$

where Δ denotes the inclusion of X into $X \times X$ as the diagonal. The claim now follows easily from the Kunneth formula for $X \times X$ in the simplicial theory.

COROLLARY 2.8. *Let $Y = Y_1 \times Y_2 \ldots$, an infinite direct product and $\Gamma = \bigoplus \Gamma_k$. Then if the spaces Y_k are path connected and the groups Γ_k have infinite order, then for all i,*

$$b_{(2)}^i (X : \Gamma) = 0. \tag{2.46}$$

Proof. This follows from Propositions 2.4 and 2.7.

Let $b^i(X, R)$ denote the ordinary Betti number for real coefficients.

PROPOSITION 2.9. *Let Γ_0 act by simplicial automorphisms on a finite complex X. Then*

$$b_{(2)}^i (X : \Gamma_0) = \sum_{j=0}^{i} b^j(X, R) \, b_{(2)}^{j-i} (\Gamma_0). \tag{2.47}$$

Proof. Since X is finite, there is a subgroup $\Gamma' \subset \Gamma_0$, of finite index, which acts trivially on

---

X. By Propositions 2.2 and 2.7

$$
\begin{aligned}
b^i_{(2)}(X:\Gamma') &= b^i_{(2)}(X \times K_{\Gamma'}:\Gamma) \\
&= b^i_{(2)}(X \times K_{\Gamma'}:1 \times \Gamma') \\
&= \sum_{j=0}^{i} b^j_{(2)}(X:1) b^{i-j}_{(2)}(K_{\Gamma'}:\Gamma') \\
&= \sum_{j=0}^{i} b^j(X,R) b^{i-j}_{(2)}(\Gamma').
\end{aligned}
\tag{2.48}
$$

Then (2.47) follows from (2.40), (2.41) and (2.43) q.e.d.

We note that Proposition 2.9 has an obvious generalization to the case in which X is replaced by a pair (X, X'). The case $(X, X') = (s^n, \partial s^n)$, s^n an n-simplex, also follows directly from excision and (2.47). It states

$$
\bar{H}^i_{(2)}(s^n, \partial s^n:\Gamma_0) = \begin{cases} 0 & i < n \\ b^{i-n}_{(2)}(\Gamma_0) & i \geq n. \end{cases}
\tag{2.49}
$$

We can now give the proof of Theorem 0.1.

Proof of Theorem 0.1. For X of dimension zero, the statement is immediate from Proposition 2.5. In general, we choose an exhaustion, $X = \cup X_j$, by Γ-invariant subcomplexes, such that for all j, X_{j+1} is obtained from X_j by attaching the orbit of a simplex, s^l, the orbit of whose boundary lies in X_j. Let $T_j \supset X_j$ denote a small Γ-equivariant tubular neighbourhood of X_j in X_{j+1}. Then

$$
X_{j+1} = T_j \cup \left(\bigcup_{\gamma \in \Gamma} s^l_\varepsilon \right)
\tag{2.50}
$$

where $s^l_\varepsilon \subset s^l$ is a smaller simplex. Moreover T_j is Γ-equivariantly homotopic to X_j and the isotropy group, $\Gamma(s^l_\varepsilon)$ of s^l_ε coincides with $\Gamma(s^l)$. However, if $\gamma \notin \Gamma(s^l_\varepsilon)$ then

$$
\gamma(s^l_\varepsilon) \cap s^l_\varepsilon = \varnothing
\tag{2.51}
$$

(which is not the case for s^l). Thus, by (the relative version of) Proposition 2.5 and (2.49),

$$
\begin{aligned}
& b^i_{(2)}\left(\bigcup_{\gamma \in \Gamma} \gamma(s^l_\varepsilon), \bigcup_{\gamma \in \Gamma} \gamma(\partial s^l_\varepsilon):\Gamma \right) \\
&= \begin{cases} 0 & i < l \\ b^{i-l}_{(2)}(\Gamma(s^l)) & i \geq l. \end{cases}
\end{aligned}
\tag{2.52}
$$

A standard cohomological argument based on (d) of §1, Lemma 2.3 and excision now shows that

$$
\lim_{j \to \infty} \lim_{k \to \infty} \dim_\Gamma \operatorname{Im}(\bar{H}^i_{(2)}(X_k:\Gamma)) \subset \bar{H}^i_{(2)}(X_j:\Gamma)
$$

$$
= \lim_{j \to \infty} b^i_{(2)}(X_j:\Gamma)
\tag{2.53}
$$

is finite. By Proposition 2.2 we can replace X_k, X_j in (2.53) by $X_k \times K_\Gamma$, $X_j \times K_\Gamma$. Now choose an exhaustion of $X_j \times K_\Gamma$ by Γ-equivariant subcomplexes $Z_{j,t} \subset X_j \times K_\Gamma$ such that $Z_{j,t}/\Gamma$ is compact. Then exhaust $X \times K$ by

$$
Z_{1,r_1} \subset Z_{1,t_2} \subset Z_{2,r_2} \subset Z_{2,t_2} \ldots
\tag{2.54}
$$

where

$$\lim_{j \to \infty} \dim_\Gamma \mathrm{Im}\, (\bar{H}^i_{(2)}(Z_{j,t_j}:\Gamma) \subset \bar{H}^i_{(2)}(Z_{j,r_j}:\Gamma)) \to b^i_{(2)}(X_j \times K_\Gamma:\Gamma). \qquad (2.55)$$

By combining (2.53), (2.55) and applying (2.8), the proof of Theorem 0.1 is easily completed.

Remark 2.2. An argument completely analogous to the above shows that the map from the simplicial L_2-theory to the singular L_2-theory is always an isomorphism with dense range and that the Betti numbers defined by these theories coincide.

§3. VANISHING IN THE AMENABLE CASE

It is well known and easy to see that for the standard action of Z or R,

$$b^i_{(2)}(R:Z) = b^i_{(2)}(Z) \quad i \geq 0$$
$$= 0. \qquad (3.1)$$

Thus, if A is any infinite abelian group, by Theorem 0.1 and Proposition 2.2,

$$0 = m_{(2)}(K_{A/Z}:A),$$
$$= m_{(2)}(K_{A/Z} \times K_A:A),$$
$$= m_{(2)}(A). \qquad (3.2)$$

This, together with Theorem 0.1, gives the remaining results of §1 for infinite abelian A, and in particular, Rosset's Theorem.

We now prove Theorem 0.2 [$m_{(2)}(A) = 0$ for infinite *amenable* groups, A]. A group A is called amenable if for every action of A on a compact space, there is an A-invariant Borel measure. For us, the relevant property of amenable groups is the following (see [9]). Let A act freely, simplicially and with compact quotient on a complex X. Then there exists a so-called Følner exhaustion, $X = \cup X_j$, of X with the following property: X_j is the union of N_j translates of some finite subcomplex, $D \subset X$, which is a fundamental domain D for A. Let N'_j denote the number of translates of D which intersect the topological boundary of X_j. Then

$$\lim_{j \to \infty} N'_j/N_j = 0. \qquad (3.3)$$

The proof of Theorem 0.2 is obtained by combining the following two lemmas.

LEMMA 3.1. *Let X be as above and let $\rho : \bar{H}^i_{(2)}(X:A) \to H^i(X, R)$ be the natural map. Then*

$$\ker \rho = 0. \qquad (3.4)$$

LEMMA 3.2. *For any (possibly non-amenable) group A, let $(X,f) \in \mathscr{C}(K_A)$. Then for $i > 0$,*

$$p_{(X,f)}(\bar{H}^i_{(2)}(K_A:A)) \subset \ker \rho. \qquad (3.5)$$

Proof of Lemma 3.1. We have

$$\bar{H}^i_{(2)}(X:A) \simeq \{h \in C^i_{(2)}(X) \,|\, dh = \partial h = 0\}, \qquad (3.6)$$

where $\partial h = d^* h$; see [7]. Let $\mathscr{X}^i \subset C^i_{(2)}(X)$ denote the subspace of cochains h, such that $dh = \partial h = 0$ and $[h] \in \ker \rho$. Let $\Pi_{\mathscr{X}^i}$ denote orthogonal projection onto $\mathscr{X}^i$ and Π_{X_j} denote orthogonal projection onto the space of L_2-cochains which are supported on X_j, i.e.

restriction to X_j. Let m_i denote the number of i-simplices of D. Then for any j, we have

$$\dim_\Gamma \mathscr{X}^i = \frac{1}{m_i N_j} \text{trace} \left(\Pi_{X_j} \Pi_{\mathscr{X}^i} \right)$$

$$\leq \frac{1}{m_i N_j} \text{rank} \left(\Pi_{X_j} \Pi_{\mathscr{X}^i} \right), \tag{3.7}$$

where the second inequality follows from

$$\| \Pi_{X_j} \Pi_{\mathscr{X}^i} \| \leq 1. \tag{3.8}$$

Note that in general, $\partial = \mathrm{d}^*$ does not commute with restriction. But if $\mathrm{d}g = h \in \mathscr{X}^i$ and h vanishes on all i-simplices intersecting the boundary of X_j,

$$\mathrm{d}^* \Pi_{X_j} h = 0. \tag{3.9}$$

Then, as usual, $\Pi_{X_j} h = 0$, since

$$\langle \Pi_{X_j} h, \Pi_{X_j} h \rangle = \langle \Pi_{X_j} g, \mathrm{d}^* \Pi_{X_j} h \rangle$$

$$= 0. \tag{3.10}$$

It follows that

$$\text{rank} \left(\Pi_{X_j} \Pi_{\mathscr{X}^i} \right) \leq N'_j, \tag{3.11}$$

which together with (3.3) and (3.7) gives

$$\dim_\Gamma \ker \rho = \dim_\Gamma \mathscr{X}^i$$

$$= 0. \tag{3.12}$$

Proof of Lemma 3.2. By Proposition 2.2, we may replace K_A by a point x, (with trivial A action). For any X on which A acts freely with finite quotient there is a unique A-equivariant map, $f_X : X \to x$. Thus, we may identify $\mathscr{C}(x)$ with the category of all such spaces X and all A-equivariant maps.

Let z be any cycle with support in a finite subcomplex, $|z| \subset X$. For each $\gamma \in A$, let $C(\gamma, \gamma(|z|))$ denote the cone with vertex γ, and base, $\gamma(|z|) \subset X$. The space

$$W = X \cup \left\{ \bigcup_\gamma C(\gamma, \gamma(|z|)) \right\}, \tag{3.13}$$

has an obvious free, simplicial A-action, with compact quotient, for which the inclusion, $X \hookrightarrow W$, is A-equivariant. The image of z is homologous to zero in W. Thus,

$$u(z) = 0, \tag{3.14}$$

if $u \in H^i(X, R)$ is the pullback of a class in W. Since any class in $p_{(X, f_X)}(\bar{H}^i_{(2)}(x : A))$ has this property, and z is arbitrary, the lemma follows.

§4. AMALGAMATED PRODUCTS

Let Γ_1, Γ_2 be groups with a common subgroup Δ and $\Gamma_1 *_\Delta \Gamma_2$ the free product with amalgamation along Δ. Recall that a model for $K_{\Gamma_1 *_\Delta \Gamma_2}/\Gamma_1 *_\Delta \Gamma_2$ can be obtained as follows. Let

$$f_j : K_\Delta/\Delta \to K_{\Gamma_j}/\Gamma_j \quad j = 1, 2 \tag{4.1}$$

be the natural maps (unique up to homotopy) and let

$$(K_\Delta/\Delta) \cup_{f_j} K_{\Gamma_j}/\Gamma_j \tag{4.2}$$

be the mapping cylinder of f_j. Then

$$K_{\Gamma_1 \ast_\Delta \Gamma_2}/\Gamma_1 \ast_\Delta \Gamma_2 = [K_\Delta/\Delta \cup_{f_1} K_{\Gamma_1}/\Gamma_1] \cup [K_\Delta/\Delta \cup_{f_2} K_{\Gamma_2}/\Gamma_2] \tag{4.3}$$

where the union is along K_Δ/Δ, the base of the mapping cylinders.

We can now apply the Mayer–Vietoris sequence and Proposition 2.4 to relate the L_2-cohomology of $\Gamma_1 \ast_\Delta \Gamma_2$, with that of Γ_1, Γ_2 and Δ.

Example 4.1. Let $A = A_1 \ast A_2$ be the free product of non-trivial amenable groups, A_1, A_2. Then

$$b_{(2)}^i(A) = \begin{cases} 0 & i \neq 1 \\ 1 - \dfrac{1}{\text{ord}(A_1)} - \dfrac{1}{\text{ord}(A_2)} & i = 1. \end{cases} \tag{4.4}$$

Example 4.2. Let $\Gamma_1, \Gamma_2, \ldots$ satisfy

$$b_{(2)}^i(\Gamma_k) = 0 \tag{4.5}$$

for all $i \geq 0$ and all k. Let $\Gamma(k)$ be the amalgamated product,

$$\Gamma(k) = \Gamma_{1_{F_1}} \Gamma_{2_{F_2}} \ldots \Gamma_k \tag{4.6}$$

where $F_1, \ldots$ are finite subgroups, of orders $d_1, \ldots$. Then

$$b_{(2)}^i(\Gamma(k)) = \begin{cases} 0 & i \neq 1 \\ \dfrac{1}{d_1} + \ldots \dfrac{1}{d_k} & i = 1. \end{cases} \tag{4.7}$$

Furthermore, $\Gamma = \bigcup_{k=1}^{\infty} \Gamma(k)$ satisfies

$$b_{(2)}^2(\Gamma) = \begin{cases} 0 & i \neq 1 \\ \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{d_k} & i = 1. \end{cases} \tag{4.8}$$

Since we can choose $d_1, d_2 \ldots$ such that

$$= \sum_{k=1}^{\infty} \frac{1}{d_k}, \tag{4.9}$$

for any real number β, we obtain a group Γ_β with $b_{(2)}^1(\Gamma_\beta) = \beta$ and $b_{(2)}^i(\Gamma_\beta) = 0$ for $i \neq 1$. Then, for

$$\Gamma_\beta^k = \overset{k}{\Gamma_\beta \oplus \ldots \Gamma_\beta,} \tag{4.10}$$

we have

$$b_{(2)}^k(\Gamma_\beta^k) = \begin{cases} 0 & i \neq k \\ \beta^k & i = k. \end{cases} \tag{4.11}$$

Now let β_k be any sequence of real numbers. By amalgamating the groups $\Gamma_{\beta_k^{1/k}}^k$ along any sequence of infinite cyclic subgroups, we obtain a *countable group* Γ, *with* $b_{(2)}^0(\Gamma) = 0$,

and

$$b_{(2)}^k (\Gamma) = \beta_k, \quad k \geq 1 \tag{4.12}$$

for any given sequence of real numbers, β_k.

Example 4.3. Let $\beta_1, \beta_2, \ldots$ be an arbitrary sequence of *real numbers, with* β_1 *rational.* We construct a *finitely generated* group Γ', with $b_{(2)}^0 (\Gamma') = 0$ *and*

$$b_{(2)}^i (\Gamma') = \beta_i, \quad i \geq 1 \tag{4.13}$$

as follows. By Example 4.2, there is a countable group Γ with

$$b_{(2)}^i (\Gamma) = \beta_{i+1} \quad i \geq 1. \tag{4.14}$$

By [14], Γ can be imbedded as a subgroup of some finitely generated group $\bar{\Gamma}$. We can assume $b_{(2)}^i (\bar{\Gamma}) = 0$ for $i \geq 0$; if not use $\bar{\Gamma} \oplus Z$. The group

$$\Gamma_0' = \bar{\Gamma} \underset{\Gamma}{*} \bar{\Gamma} \tag{4.15}$$

satisfies

$$b_{(2)}^i (\Gamma_0') = \begin{cases} \beta_i & i \neq 1 \\ 0 & i = 1. \end{cases} \tag{4.16}$$

Finally, take a *finitely generated* group Γ_{β_1} as in Example 4.2 with

$$b_{(2)}^i (\Gamma_{\beta_1}) = \begin{cases} 0 & i \neq 1 \\ \beta_1 & i = 1 \end{cases} \tag{4.17}$$

where β_1 is the given *rational* number. Then

$$\Gamma' = \Gamma_0' \underset{z}{*} \Gamma_{\beta_1} \tag{4.18}$$

satisfies (4.15).

Example 4.4. Let $\beta_1, \beta_2, \ldots$ be a *constructive* sequence of real numbers with β_1, β_2 *rational.* Then there is a *finitely presented* group Γ'' with

$$b_{(2)}^i (\Gamma'') = \beta_i \quad i \geq 1. \tag{4.19}$$

Recall that "constructive" means that there exists of Turing machine which computes at the N-th step, the m-th digit of β_k, for the standard numeration of pairs (m, k) by integers $1, \ldots N, \ldots$.

To see this, consider the construction of Example 4.3 and make the orders of the underlying finite groups form a recursive sequence. Then one can imbed Γ' into a finitely presented group, $\bar{\Gamma}'$ (see [13]) with $b_{(2)}^i (\bar{\Gamma}') = 0$ for all i. Now take

$$\Gamma_0'' = \bar{\Gamma}' \underset{\Gamma'}{*} \bar{\Gamma}'. \tag{4.20}$$

Then

$$b_{(2)}^i (\Gamma_0'') = \begin{cases} \beta_i & i \geq 2 \\ 0 & i = 0, 1. \end{cases} \tag{4.21}$$

Finally, take

$$\Gamma'' = \Gamma_0'' \underset{z}{*} \Gamma_{\beta_1}. \tag{4.22}$$

Remark 4.1. There is no known example of a free simplicial action with X/Γ compact for which any of the numbers $b^i_{(2)}\,(X:\Gamma)$ are irrational; compare [5].

§5. L_2-COHOMOLOBY OF RIEMANNIAN MANIFOLDS

For the most part, the results of this section are essntially restatements of those of [4], in the language of the present paper. So we will be rather brief.

Let Γ be a discrete subgroup of the isometry group of a complete Riemannian manifold Y. Then space $\mathcal{H}^i$ of L_2-harmonic forms has a natural Γ-module structure; see [3], [4]. If the quotient, Y/Γ, is a compact manifold, $\mathcal{H}^i$ is canonically isomorphic to $\bar{H}^i_{(2)}\,(Y:\Gamma)$; see [7]. Thus, for example, if Y^n is oriented, the Hodge $*$-operator gives Poincaré duality,

$$b^i_{(2)}\,(Y^n:\Gamma) = b^{n-i}_{(2)}\,(Y^n:\Gamma). \tag{5.1}$$

Remark 5.1. Actually, one can show that (5.1) holds for real homology manifolds.

Now assume that Y has bounded geometry (see §0) and that Y/Γ has finite volume.

THEOREM 5.1. *The Γ-module $\mathcal{H}^i_{(2)}$ has finite Γ-dimension and is canonically isomorphic to* $\bar{H}^i_{(2)}\,(Y:\Gamma)$. *In particular,*

$$b^i_{(2)}\,(Y:\Gamma) = \dim_\Gamma \mathcal{H}^i_2 < \infty. \tag{5.2}$$

Proof. For the case of free actions, this was proved in [4]. The proof given there applies to the case of discrete actions with only minor changes.

COROLLARY 5.2. *If* geo $(Y^n) \leq 1$, Vol $(Y^n/\Gamma) < \infty$ *and Y^n is oriented,*

$$b^i_{(2)}\,(Y^n:\Gamma) = b^{n-i}_{(2)}\,(Y^n:\Gamma). \tag{5.3}$$

In particular the dimension, n, of Y^n is a Γ-equivariant homotopy invariant of such actions, provided that for some i, $b^i_{(2)}\,(Y^n:\Gamma) \neq 0$.

Proof. The duality arises from the Hodge $*$-operator.

Example 5.1. The condition, $b^i_{(2)}\,(Y:\Gamma) \neq 0$ for some i, is actually necessary. For example, in addition to the standard action of Z on R, one has the action of Z on R^2 whose quotient is double cusp depicted below (Fig. 1).

If the metric on Y^{2n} is Kähler for some complex structure, Hodge theory puts further restrictions on $\bar{H}^*_{(2)}\,(Y^{2n}:\Gamma)$, e.g.

$$b^i_{(2)}\,(Y^{2n}:\Gamma) \leq b^{i+2}_{(2)}\,(Y^{2n}:\Gamma), \quad 0 \leq i \leq n-2. \tag{5.4}$$
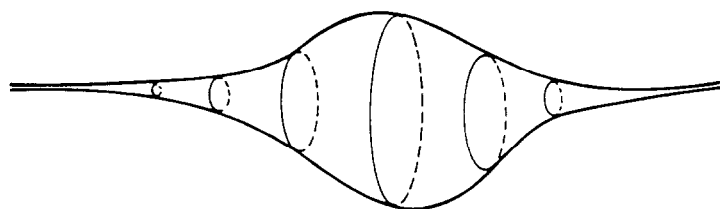

Fig. 1.

Example 5.2. Let Y^2 denote the double cusp manifold of Fig. 5.1. Then Y^2 is conformally equivalent to S^2 with two points deleted. Since $b^i_{(2)}(Y^2:Z) = 0$, $i \leq 2n$, it follows that

$$W^{2n} = Y^2 \times \ldots Y^2$$

satisfies

$$b^i_{(2)}(W^{2n}, Z^n) = 0, \quad i \leq 2n \tag{5.5}$$

and hence for the connected sum, $W^{2n} \# W^{2n}$, we have

$$b^i_{(2)}(W^{2n} \# W^{2n}: Z^n * Z^n) = \begin{cases} 0 & i \neq 1, \quad 2n-1 \\ 1 & i = 1, \quad 2n-1. \end{cases} \tag{5.6}$$

Since, for $n \geq 3$

$$b^1_{(2)}(W^{2n} \# W^{2n}, Z^n * Z^n) \geq b^3_{(2)}(W^n \# W^n, Z^n * Z^n) \tag{5.7}$$

by (5.4), $W^{2n} \# W^{2n}$ admits no complete Kähler metric of finite volume for which the universal covering space has bounded geometry. On the other hand, in view of Fig. 5.1 it obviously admits a metric of this type which is not Kähler. Finally $W^{2n} \# W^{2n}$ does admit incomplete Kähler metrics [since it can be regarded as an open subset of $CP(n)$ with a point blown up].

The specific example above can be generalized considerably. Every projective algebraic manifold, W^n, admits a Zariski open subset, V^n, which carries a complete Kähler metric of finite volume such that the universal covering, $\tilde{V}^n$, has bounded geometry. Moreover, V^n can be chosen such that $\tilde{V}^n$ is contractible and $\Gamma = \Pi_1(V^n)$ admits a chain of subgroups, $\Gamma = \Gamma_1 \supset \Gamma_2 \ldots \Gamma_{n+1} = 0$ ($n = \dim_c V^n$) with Γ_j normal in Γ_{j-1} and Γ_{j-1}/Γ_j free (see [11]). Then

$$b^i_{(2)}(\tilde{V}:\Gamma) = b^i_{(2)}(\Gamma) = \begin{cases} 0 & i \neq n \\ \sum_{j=1}^{n}(\text{rank } \Gamma_j/\Gamma_{j+1} - 1) & i = n. \end{cases} \tag{5.8}$$

For $n \geq 3$, the connected sum, $V' = V \# V$ satisfies, $b^1_{(2)}(\tilde{V}':\Gamma') = 1, b^3_{(2)}(\tilde{V}':\Gamma') = 0$. Hence there is no complete Kähler metric of finite volume on V' for which $\tilde{V}'$ has bounded geometry, even though (non-Kähler) Riemannian metrics of this type clearly exist.

Example 5.2. We continue to assume that Y is a Riemannian manifold of bounded geometry and that Vol $(Y/\Gamma) < \infty$. Let $P_\chi(\Omega)$ denote the Chern–Gauss–Bonnet form of Y.

THEOREM 5.3.

$$\int_{Y^{2n}/\Gamma} P_\chi(\Omega) = \chi_{(2)}(Y^{2n}:\Gamma). \tag{5.9}$$

Proof. As in [4] this follows from Theorem 5.1 and the L_2-index Theorem of [3] (whose proof in the discrete case requires no modification).

For an arbitrary space Y, the L_2-cohomology with compact supports, $\bar{H}^*_{(2),o}(Y:\Gamma)$ is defined as follows:

$$\bar{H}^*_{(2),o}(Y:\Gamma) \overset{\text{def}}{=\!=} \lim_{Y'} \bar{H}^i_{(2)}(Y, Y \backslash Y':\Gamma), \tag{5.10}$$

where the inverse limit in (2.45) is over all sets Y' with compact closure. Clearly, $H^*_{(2),o}(Y:\Gamma)$ is a Γ-*equivariant proper homotopy invariant.* If Y^n is an oriented pseudo-manifold and if the

action of Γ preserves the orientation, then there is a pairing,

$$\bar{H}^i_{(2),o}(Y:\Gamma) \otimes \bar{H}^{n-i}_{(2)}(Y:\Gamma) \to R. \tag{5.11}$$

For this, we define the cup product, $u \cup v \in \bar{H}^n_{(2),o}(Y:\Gamma)$, of $u \in \bar{H}^i_{(2),o}(Y:\Gamma)$ and $v \in \bar{H}^{n-i}_{(2)}(Y:\Gamma)$ by starting with complexes X, as in §2, and passing to the inverse limit. Then

$$Q(u \otimes v) = \rho(u \cup v)[Y], \tag{5.12}$$

where $[Y] \in H_{n,o}(Y,R)$, and

$$\rho : \bar{H}^n_{(2),o}(Y:\Gamma) \to \bar{H}^n_o(Y,R) \tag{5.13}$$

is the natural map. If Y is a rational homology manifold, the pairing is easily seen to be non-singular, and this implies Poincaré–Lefschetz duality,

$$b^i_{(2),o}(Y:\Gamma) = b^{n-i}_{(2)}(Y:\Gamma). \tag{5.14}$$

For $n = 4k$, if the bounded symmetric bilinear form Q has finite Γ-rank, its Γ-signature, $\sigma(Y:\Gamma)$, is defined in the usual way.

THEOREM 5.4. *Let Y^n be a complete oriented Riemannian manifold of bounded geometry on which Γ acts by orientation preserving isometries. Then for all i,*

$$\bar{H}^i_{(2),o}(Y^n:\Gamma) \simeq \bar{H}^i_{(2)}(Y^n:\Gamma). \tag{5.15}$$

If $n = 4k$, then

$$\int_{Y:\Gamma} P_L(\Omega) = \sigma(Y:\Gamma), \tag{5.16}$$

where $P_L(\Omega)$ denotes the Hirzebruch L-form.

Proof. See [4].

Remark 5.2. If Γ acts freely and the quotient is compact, Theorem 5.4 reduces to the L_2-index theorem for the signature operator of [1] and [17]:

$$\sigma(Y:\Gamma) = \sigma(Y/\Gamma). \tag{5.17}$$

Remark 5.3. The identity (5.17) generalizes to free cocompact actions on topological manifolds. In fact, the whole bounded geometry-finite volume discussion extends to a purely topological framework, including, for example, lattices in locally compact groups. This will be discussed in a future paper.

APPENDIX: INVERSE LIMITS OF Γ-MODULES

In this Appendix we verify properties (a)–(f) of §1 for inverse limits of Γ-modules, and prove Lemma 2.1.

In fact, (a), (c), (f) and (g) are trivial. To state (b), we assume that A is a subspace of some $\lim_{\mathcal{C}} A_z$, and $\bar{A}$ its closure. Then (b) is trivial as well. Since (e) is not used in the body of the paper, its proof, which is similar to that of (d), will be omitted. Before proceeding to (d) for inverse limits, we will sharpen (e) in the usual case.

Let H_z be a collection of closed submodules of some Γ-finite dimensional Γ-module, indexed by

some possibly uncountable set a. We claim that

$$\dim_\Gamma \bigcap_{x \in a} H_x = \liminf_{F \subset a} \dim_\Gamma \bigcap_{x \in \mathscr{F}} H_x \tag{A1}$$

where $\mathscr{F}$ ranges over the finite subsets of a. If a is countable this is equivalent to (e) of §1. Otherwise, use the Well Ordering Principle to write

$$a = \bigcap_\beta S_\beta \tag{A2}$$

where $\beta < \beta'$ implies $S_\beta \subset \beta_{\beta'}$ and each S_β has cardinality strictly smaller than that of a. Since $\beta < \beta'$ implies

$$\bigcap_{x \in S_{\beta'}} H_x \subset \bigcap_{x \in S_\beta} H_x, \tag{A3}$$

by a trivial extension of (e) of §1,

$$\liminf_\beta \dim_\Gamma \bigcap_{x \in S_\beta} H_x = \dim_\Gamma \bigcap_\beta \left(\bigcap_{x \in S_\beta} H_x \right)$$

$$= \dim_\Gamma \bigcap_{x \in a} H_x. \tag{A4}$$

By transfinite induction, we can assume that for each β,

$$\dim_\Gamma \bigcap_{x \in S_\beta} H_\alpha = \liminf_{\mathscr{F} \subset S_\beta} \dim_\Gamma \bigcap_{x \in \mathscr{F}} H_x \tag{A5}$$

which, together with (A4) implies (A1).

Next, we observe that the condition that H_x is closed can also be weakened. A Γ-module. B, will be called Γ-*weakly closed* if there exist closed $B_1 \subset B_2 \subset \ldots \subset B$, with $\cup B_j$ dense in B. Equivalently,

$$\lim_{j \to \infty} \dim_\Gamma B_j = \dim_\Gamma B. \tag{A6}$$

Let $\dim_\Gamma C < \infty$ and let $f : A \to C$ be bounded. It follows from (d) of §1 that if $W \subset C$ is Γ-weakly closed so is $f^{-1}(W)$. More significantly, if $V \subset A$ is Γ-weakly closed, then by the Γ-Fredholm property discussed in §1, so is $f(V)$.

Now let $B_1, B_2 \ldots$ be Γ-weakly closed, $\dim_\Gamma B_1 < \infty$. Let $C_j \subset B_j$, with C_j closed,

$$\dim_\Gamma B_j - \dim_\Gamma C_j < \varepsilon_j, \tag{A7}$$

$$\sum_{j=1}^\infty \varepsilon_j < \varepsilon. \tag{A8}$$

Then using (d) of §1, one sees that

$$\dim_\Gamma B_1 \cap \ldots \cap B_j - \dim_\Gamma C_1 \cap \ldots C_j < \varepsilon_1 + \ldots \varepsilon_j, \tag{A9}$$

which gives

$$\liminf_j \dim_\Gamma B_1 \cap \ldots B_j,$$

$$\leq \liminf_j \dim_\Gamma C_1 \cap \ldots \cap C_j + \varepsilon$$

$$= \dim_\Gamma \cap C_j + \varepsilon, \tag{A10}$$

$$\leq \dim_\Gamma \cap B_j + \varepsilon.$$

It follows that (A1) holds for countable intersections of Γ-weakly closed submodules and such intersections are again Γ-weakly closed. If we now assume that both of these properties hold for index sets of less than a fixed cardinality, it follows as above, by transfinite induction, that they hold in general.

Let $(A_\alpha, \mathscr{C})$ be as in §2 and let $B_\alpha \subset A_\alpha$ be closed for all α. We turn to the proof of Lemma 2.1 which describes the subspace $\lim_{\mathscr{C}} B_\alpha$, of elements a, such that $a_\alpha \in B_\alpha$ for all α.

Clearly, we have

$$p_{\beta'}\left(\lim_{\mathscr{C}} B_\alpha\right) \subset \bigcap_{\beta' \le \beta} p_{\beta'',\beta'}^{-1}(B_{\beta''}) \tag{A11}$$

$$\stackrel{\text{def}}{=} Q_{\beta'},$$

and $Q_{\beta'}$ is Γ-weakly closed. Moreover, if $\beta \le \beta'$, obviously

$$p_{\beta,\beta'}(Q_{\beta'}) \subset Q_\beta. \tag{A12}$$

In view of (A11),

$$p_\beta\left(\lim_{\mathscr{C}} B_\alpha\right) \subset \bigcap_{\beta' \ge \beta} p_{\beta,\beta'}(Q_{\beta'}), \tag{A13}$$

$$\stackrel{\text{def}}{=} R_\beta,$$

and R_β is Γ-weakly closed.

By (d), the continuity property, for fixed β, we can choose, $\beta_j > \beta, j = 1, \ldots, n$,

$$\dim_\Gamma \bigcap_j p_{\beta,\beta_j}(Q_{\beta_j}) - \dim_\Gamma R_\beta < \varepsilon. \tag{A14}$$

By (2.4) we can choose $\beta_0 > \beta_j, j = 1, \ldots, n$, and by (A12)

$$\dim_\Gamma p_{\beta,\beta_0}(Q_{\beta_0}) - \dim_\Gamma R_\beta < \varepsilon. \tag{A15}$$

Thus,

$$\dim_\Gamma R_\beta = \liminf_{\beta_0 \ge \beta} \dim_\Gamma p_{\beta,\beta_0}(Q_{\beta_0}). \tag{A16}$$

Let $\beta' \ge \beta$ and choose β_0 and β_0' such that (A15) holds and the corresponding relation holds for β_0' and β'. Then, if we choose $\gamma > \beta_0$, $\gamma > \beta_0'$, we have

$$\dim_\Gamma p_{\beta,\gamma}(Q_\gamma) - \dim_\Gamma R_\beta < \varepsilon \tag{A17}$$

$$\dim_\Gamma p_{\beta',\gamma}(Q_\gamma) - \dim_\Gamma R_{\beta'} < \varepsilon. \tag{A18}$$

Since

$$p_{\beta,\gamma} = p_{\beta,\beta'} \, p_{\beta',\gamma}, \tag{A19}$$

it follows that

$$\dim_\Gamma R_\beta - \dim_\Gamma p_{\beta,\beta'}(R_{\beta'}) < 2\varepsilon. \tag{A20}$$

Hence,

LEMMA A1. $p_{\beta,\beta'}(R_{\beta'})$ is weakly Γ-closed and dense in R_β.

According to the assertion of Lemma 2.1, $p_\beta\left(\lim_{\mathscr{C}} B_\alpha\right)$ is dense in R_β, provided that $\mathscr{C}$ is dominated by a linearly ordered subset $\mathscr{L}$ (for the application to L_2-cohomology, this would mean restricting attention to spaces with countable homotopy groups). As we indicated, the assumption can be removed, but some details must be modified slightly. First we will complete the discussion assuming the existence of $\mathscr{L}$.

Since the maps $p_{\beta,\beta'}$ have dense range and $\dim_\Gamma R_\beta < \infty$ for all β, we can assume that $\mathscr{L}$ is countable and in fact $\mathscr{L} = \{\beta_1 \le \beta_2 \le \ldots\}$.

Proof of Lemma 2.1 *when* $\mathscr{L}$ *exists.* By (A13), relation (2.7) follows from

$$\dim_\Gamma \lim_{\mathscr{C}} B_\alpha = \sup_\beta \dim_\Gamma R_\beta. \tag{A21}$$

By (2.4), for fixed $\{B_x\}$, we can assume that $\mathscr{L}$ has been chosen so that

$$\sup_\beta \dim_\Gamma R_\beta = \sup_j \dim_\Gamma R_{\beta_j}. \tag{A22}$$

Put

$$T_{\beta_1} = R_{\beta_1}, \tag{A23}$$

$$T_{\beta_j} = p_{\beta_{j-1},\beta_j}^{-1}(T_{\beta_{j-1}}) \cap [\ker p_{\beta_{j-1},\beta_j}]^\perp. \tag{A24}$$

Then T_{β_j} is Γ-weakly closed. It follows from (e) and Lemma A1 that $\bigcap_j p_{\beta_1,\beta_j}(T_{\beta_j})$ is dense in R_{β_1}. Moreover, clearly

$$p_{\beta_1}\left(\lim_\mathscr{C} G_x\right) \supset \bigcap_k p_{\beta_1,\beta_j}(T_{\beta_j}). \tag{A25}$$

In the same way, (A25) holds with β_1 replaced by β_i. This together with (A22) gives (A21).

We now consider maps, g, between inverse limits. We will restrict attention to those g which have a special character enjoyed by all the maps of §2 (actually it is possible to reduce the case of arbitrary bounded operators g to this one).

Let $\mathscr{D}_2 \subset \mathscr{C}_2$ be a subcategory satisfying (2.3), (2.4) and such that for all $\alpha \in \mathscr{C}_2$, there exits $\beta \in \mathscr{D}_2$ with $\beta \geq \alpha$. Let $T: \mathscr{D}_2 \to \mathscr{C}_1$ be a functor. Assume that there exist bounded operators, $g_{T(\beta)}: A_{T(\beta)}^1 \to A_\beta^2$, such that

$$\begin{array}{ccc}
A_{T(\beta)}^1 & \xrightarrow{\ g_{T(\beta)}\ } & A_\beta^2 \\
\downarrow{\scriptstyle p_{T(\beta),T(\beta')}^1} & & \downarrow{\scriptstyle p_{\beta',\beta}^2} \\
A_{T(\beta')}^1 & \xrightarrow[\ g_{T(\beta')}\ ]{} & A_{\beta'}^2
\end{array} \tag{A26}$$

commutes. The collection, $\{g_{T(\beta)}\}$, determines a unique

$$g : \lim_{\mathscr{C}_1} A_x^1 \to \lim_{\mathscr{C}_2} A_x^2$$

satisfying

$$p_\beta^2 g = g_{T(\beta)} p_{T(\beta)}^1. \tag{A27}$$

Such a map is called a *morphism* (of inverse systems).

Proof of (d) for morphisms, if $\mathscr{L}$ exists. Clearly, it suffices to assume that $\dim_\Gamma \ker g + \dim_\Gamma \operatorname{Im} g$ is finite.

Let $f: \lim_{\mathscr{C}_1} A_x^1 \to \lim_{\mathscr{C}_2} A_\beta^2$ be a morphism. An element, a, of $\lim_{\mathscr{C}_1} A_x^1$ is in $\ker g$, if and only if $a_{T(\beta)} \in \ker g_{T(\beta)}$, for all $\beta \in \mathscr{D}_2$. Let $Q_x (\ker g)$, $R_x (\ker g)$ denote the corresponding subspaces of A_x.

If follows that

$$Q_{x'}(\ker g) = \bigcap_{T(\beta) \leq x'} p_{T(\beta),x'}^{-1}(\ker g_{T(\beta)}). \tag{A28}$$

Now fix x. By (A17), there exists x' such that

$$\dim_\Gamma p_{x,x'}(Q_{x'}(\ker g)) \leq \dim_\Gamma R_x(\ker g) + \varepsilon. \tag{A29}$$

Choose $\beta_1 \ldots \beta_n$ such that

$$\dim_\Gamma \bigcap_{T(\beta_j) \leq x'} P_{T(\beta_j),x'}^{-1}(\ker g_{T(\beta_j)}) \leq \dim_\Gamma Q_{x'}(\ker g) + \varepsilon. \tag{A30}$$

Then choose $\beta_0 \geq \beta_j\, j = 1, \ldots, n$. For any x'' with $x'' \geq x', x'' \geq T(\beta_0)$, we have

$$\dim_\Gamma p_{x,x''}(p_{T(\beta_0),x''}^{-1}(\ker g_{T(\beta_0)})) \leq \dim_\Gamma R_x(\ker g) + 2\varepsilon. \tag{A31}$$

Since, by Lemma A1,

$$\dim_\Gamma \ker (p_{x,x''}) = \dim_\Gamma R_{x''} - \dim_\Gamma R_x, \tag{A32}$$

it follows that

$$\dim_\Gamma p_{T(\beta_0)x''}^{-1} (\ker g_{T(\beta_0)}) - [\dim_\Gamma R_{x''} - \dim_\Gamma R_x].$$

$$\leq \dim_\Gamma p_{x,x''} (p_{T(\beta_0)x''}^{-1} (\ker g_{T(\beta_0)})). \tag{A33}$$

Finally,

$$\dim_\Gamma p_{T(\beta_0)x''}^{-1} (\ker g_{T(\beta_0)}) = \dim_\Gamma R_{x''} - \dim_\Gamma R_{T(\beta_0)} + \dim_\Gamma \ker g_{T(\beta_0)}$$

$$= \dim_\Gamma R_{x''} - \dim_\Gamma \operatorname{Im} g_{T(\beta_0)}. \tag{A34}$$

Since ε is arbitrary, combining (A31), (A33) and (A34) gives

$$\dim_\Gamma R_\alpha \leq \dim_\Gamma R_x (\ker g) + \dim_\Gamma \operatorname{Im} g_{T(\beta_0)}. \tag{A35}$$

Using (2.4) and Lemma 2.1, we can let $\alpha \to \infty$, $T(\beta_0) \to \infty$ in such a way that

$$\lim_{\alpha \to \infty} \dim_\Gamma R_x = \dim_\Gamma \lim_{\mathscr{C}_1} A_x^1, \tag{A36}$$

$$\lim_{\beta_0 \to \infty} \dim_\Gamma R_{T(\beta_0)} (\ker g) = \dim_\Gamma \ker g. \tag{A37}$$

Then (A35) gives

$$\dim_\Gamma \lim_{\mathscr{C}_1} A_x^1 \leq \dim_\Gamma \ker g + \dim_\Gamma \operatorname{Im} g. \tag{A38}$$

In particular, if the right-hand side is finite then so is the left. Thus, as $\alpha \to \infty$,

$$\dim_\Gamma \ker p_{x,x'} \to 0, \tag{A39}$$

and in the limit, (A33) is an equality. Hence, so is (A36). This completes the proof.

Our assumption concerning the existence of $\mathscr{L}$ entered the above argument only indirectly, in our appeal to Lemma 2.1. However, we can also prove a version of Lemma 2.1 under the sole assumption that

$$\limsup_\beta \dim_\Gamma R_\beta \overset{\text{def}}{=} \overline{\dim}_\Gamma \lim_{\mathscr{C}} B_x,$$

$$< \infty. \tag{A40}$$

In this form, Lemma 2.1 asserts that $\overline{\dim}_\Gamma = \dim_\Gamma$, provided $\overline{\dim}_\Gamma < \infty$. However, if $\overline{\dim}_\Gamma = \infty$, the proof breaks down and *a priori*, one could have

$$\overline{\dim}_\Gamma \lim_{\mathscr{C}} B_x = \infty, \tag{A41}$$

$$\dim_\Gamma \lim_{\mathscr{C}} B_x = 0. \tag{A42}$$

Since $\overline{\dim}_\Gamma$ is actually an invariant of a collection $\beta_x^1 \subset A_x^1$, in order to state (d) in this context we must define $\overline{\dim}_\Gamma (\ker g)$, $\overline{\dim}_\Gamma (\operatorname{Im} g)$. For $\ker g$ we take $Q_{x'} (\ker g)$ as in (A28). Then $R_{x'} (\ker g)$, $\overline{\dim}_\Gamma (\ker g)$ are determined as usual and $\overline{\dim}_\Gamma \operatorname{Im} g$ is defined similarly.

The generalization of (d) is

(d)′ Let $g : \lim_{\mathscr{C}_1} A_x^1 \to \lim_{\mathscr{C}_2} A_x^2$ be a morphism. Then

$$\overline{\dim}_\Gamma \lim_{\mathscr{C}_1} A_x^1 = \overline{\dim}_\Gamma \ker g + \overline{\dim}_\Gamma \operatorname{Im} g. \tag{A43}$$

In particular, if the right-hand side is finite then

$$\dim_\Gamma \lim_{\mathscr{C}_1} A_x^! \dim_\Gamma \ker g + \dim_\Gamma \operatorname{Im} g. \tag{A44}$$

As we have mentioned, (d)′ follows as above from

LEMMA 2.1′. *If* $B_x \subset A_x$ *and* $\overline{\dim}_\Gamma \lim_{\mathscr{C}_1} B_x < \infty$, *then*

$$\overline{\dim}_\Gamma \lim_{\mathscr{C}_1} B_x = \dim_\Gamma \lim_{\mathscr{C}_1} B_x. \tag{A45}$$

Proof. Let $\mathscr{L} = \{\beta_1 \leq \beta_2 \dots\}$ with

$$\lim_{j \to \infty} \dim_\Gamma R_{\beta_j} = \overline{\dim}_\Gamma \lim_{\mathscr{C}_1} B_x < \infty. \tag{A46}$$

If $\mathscr{L}' = \{\beta_1' \leq \beta_2' \dots\}$ is another such system, we write $\mathscr{L}' \geq \mathscr{L}$ if for all j there exists $N(j)$ with $\beta_{N(j)}'$ $\geq \beta_j$. Let $p_{\mathscr{L},\mathscr{L}'}$ be the induced map,

$$p_{\mathscr{L},\mathscr{L}'} : \lim_{\mathscr{L}'} R_{\beta_j'} \to \lim_{\mathscr{L}} R_{\beta_j}. \tag{A47}$$

Since $\overline{\dim}_\Gamma \lim_{\mathscr{C}} B_x < \infty$, by (d) for the case in which $\mathscr{L}$ exists,

$$\ker p_{\mathscr{L},\mathscr{L}'} = 0. \tag{A48}$$

By the extension of (e)

$$\bigcap_{\mathscr{L}' \geq \mathscr{L}} p_{\beta_1} (\operatorname{Im} p_{\mathscr{L},\mathscr{L}'}) = \overline{R_{\beta_1}}. \tag{A49}$$

Thus, by (A48), every $a \in \lim_{\mathscr{L}} R_{\beta_j}$ with

$$a_{\beta_1} \in p_{\beta_1} (\operatorname{Im} p_{\mathscr{L},\mathscr{L}'}), \tag{A50}$$

determines a *unique* element, $a' \in \lim_{\mathscr{L}'} R_{\beta_j}$, for all $\mathscr{L}' > \mathscr{L}$.

Given β, using (2.4), there exists $\mathscr{L}' \geq \mathscr{L}$ with $\beta_1' = \beta$. Moreover, if $\mathscr{L}''$ is another such there exists a third $\mathscr{L}'''$ with $\mathscr{L}''' \geq \mathscr{L}'$, $\mathscr{L}''' > \mathscr{L}''$. It follows from (A48) that

$$p_{\mathscr{L}',\mathscr{L}'''}(a''') = a', \tag{A51}$$

$$p_{\mathscr{L}'',\mathscr{L}'''}(a''') = a'', \tag{A52}$$

and hence that

$$a_\beta' = a_\beta'' = a_\beta'''. \tag{A53}$$

Thus, every element a as above determines a unique element of R_β for all β, and it follows as above that these determine a unique element of $\lim_{\mathscr{C}_1} B_x$. This suffices to complete the proof.

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