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BOUNDS ON THE VON NEUMANN DIMENSION OF L^2-COHOMOLOGY AND THE GAUSS-BONNET THEOREM FOR OPEN MANIFOLDS

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0. Introduction

In this paper we continue the discussion of [6]. In §§1–3, we prove those results concerning the Von Neumann dimension of L^2-cohomology spaces (L^2 Betti numbers) whose proofs were given only for normal coverings which are profinite.[1]

The L^2-cohomology techniques turn out to be useful in other contexts as well. For example, in [7] we simplify the proof of the theorem of Gottlieb and Stallings, which states that if a $K(\pi, 1)$-space is homotopy equivalent to a finite complex and π has nontrivial center, then the Euler characteristic $\chi(K(\pi, 1))$ vanishes. In fact, we show that it suffices to assume that π has a nontrivial normal amenable subgroup.

In §4 we extend the result of [6] concerning the η-invariant to the not necessarily profinite case. For this we define a corresponding invariant $\tilde{\eta}_{(2)}$ by means of the Γ-trace.

In §5 we extend our results to certain metrics which are conformally related to those considered in §1. We also give an intrinsic criterion (Theorem 5.5) for a metric to be of this type.

As background for §§1–3 of the present paper, we now recall some material from [6]. There we considered a complete riemannian manifold M^n, whose sectional curvature, K, and volume, $\mathrm{Vol}(M)$, satisfy $|K| \leqslant 1$, $\mathrm{Vol}(M) < \infty$. Here we will be concerned exclusively with the particular case in which

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[1] A covering $\tilde{M}$ is *profinite* if there exist subgroups $\Gamma_j \subset \pi_1(M)$, of finite index, such that $\bigcap \Gamma_j = \Gamma = \pi_1(\tilde{M})$.

$\operatorname{geo}(\tilde{M}) \leqslant 1$ for some normal covering $p\colon \tilde{M} \to M$, $\tilde{M}/\Gamma = M$. The condition $\operatorname{geo}(X) \leqslant 1$ means that $|K| \leqslant 1$ and the injectivity radius, $i(X)$, satisfies $i(X) \geqslant 1$. As in [6], it actually suffices to assume $\operatorname{geo}(\tilde{U}) \leqslant 1$ for some neighborhood U of infinity. The details of this generalization are not difficult and will be omitted.

Let $\pi_{\tilde{\mathfrak{H}}_i}$ denote orthogonal projection on $\tilde{\mathfrak{H}}^i$, the space of closed and coclosed square integrable i-forms of $\tilde{M}$. Then

$$\pi_{\tilde{\mathfrak{H}}^i}(\omega) = \int_M \tilde{h}^i(x, y) \wedge *_y\, \omega(y), \tag{0.1}$$

where $\tilde{h}^i(x, y)$ is a symmetric C^∞ double form. The assumption $\operatorname{geo}(\tilde{M}) \leqslant 1$, together with the elliptic estimate for the Laplacian implies that the pointwise norm of $\tilde{h}^i(x, y)$ satisfies

$$\|h^i(x, y)\| \leqslant c(n) \tag{0.2}$$

($n = \dim M$).[2] Since $\tilde{h}^i(x, y)$ is invariant under isometries, we can regard its pointwise trace, $\operatorname{tr}(\tilde{h}^i(x, x))$, as a function on M, and put

$$\tilde{b}^i_{(2)}(M) = \int_M \operatorname{tr}(\tilde{h}^i(x, x))\, dx < \infty, \tag{0.3}$$

where the integration is with respect to the natural volume element. Observe that Γ acts on the reduced L^2-cohomology space $\overline{H}^i_{(2)}(\tilde{M}) = \ker d/\overline{\operatorname{im} d}$. The number $\tilde{b}^i_{(2)}(M)$ can be interpreted as the Von Neumann dimension (from now on we just say Γ-dimension) of the Γ-module $\overline{H}^i_{(2)}(\tilde{M})$ (see §2 and the references cited there for further discussion of Γ modules). It follows by a standard argument that $\tilde{b}^i_{(2)}(M)$ is a quasi-isometry invariant of M. A main concern here is to show that in fact, $\tilde{b}^i_{(2)}(M)$ is a *homotopy invariant* (Theorem 6.2 of [6]). To this end we show that the Γ-module $\overline{H}^i_{(2)}(\tilde{M}^n)$ is the inverse limit of the system $\overline{H}^i_{(2)}(p^{-1}(B))$, corresponding to the possibly disconnected covering spaces $p^{-1}(B)$, over all open sets with compact closure $B \subset M^n$. We can then apply the known fact that (the isomorphism class of) the Γ-module $\overline{H}^i_{(2)}(\tilde{B})$ is a homotopy invariant of B (see [13]).

The above result and others proved below are easy consequences of the assertion that M admits an exhaustion $M = \bigcup M_k$ by compact submanifolds with boundary such that

$$\operatorname{Vol}(\partial M_k) \to 0, \tag{0.4}$$

$$\|\mathrm{II}(\partial M_k)\| \leqslant c \tag{0.5}$$

[2] Throughout the paper we indicate the dependence of constants appearing in estimates on parameters, by writing, e.g., $c(n)$ for a constant which depends only on n.

for some constant c. ($\mathrm{II}(\partial M_k)$ denotes the second fundamental form of ∂M_k.) The sequence $\{M_k\}$ is called a sequence of *good choppings*. Grant for the moment that such a sequence exists. Let $\tilde{h}^i_k$ denote the kernel corresponding to projection on the harmonic i-forms for $p^{-1}(M_k) \subset \tilde{M}$. By the elliptic estimate for manifolds with (controlled geometry of the) boundary, as in (0.2),

$$\|\tilde{h}_k(x, y)\| \leqslant c(n). \tag{0.6}$$

It follows that

$$\lim_{k \to \infty} \tilde{b}^i_{(2)}(\partial M_k) = 0, \tag{0.7}$$

and in the same way

$$\lim_{k \to \infty} \tilde{b}^i_{(2)}(M \setminus M_k, \partial(M \setminus M_k)) = 0. \tag{0.8}$$

Let $\tilde{\mathbf{b}}^i_{(2)}(B)$ be defined by

$$\tilde{\mathbf{b}}^i_{(2)}(B) = \dim_\Gamma\left(\operatorname{im} H^i_{(2)}(p^{-1}(B), p^{-1}(\partial B)) \subset \overline{H}^i_{(2)}(p^{-1}(B))\right) \tag{0.9}$$

and for $A \subset B$, put

$$\underline{\tilde{b}}^i_{(2)}(A, B) = \dim_\Gamma\left(\operatorname{im} \overline{H}^i_{(2)}(p^{-1}(B)) \subset \overline{H}^i_{(2)}(p^{-1}(A))\right). \tag{0.10}$$

Then

$$\tilde{\mathbf{b}}^i_{(2)}(A) \leqslant \tilde{\mathbf{b}}^i_{(2)}(B), \tag{0.11}$$

$$\tilde{\mathbf{b}}^i(A) \leqslant \underline{\tilde{b}}^i_{(2)}(A, B) \leqslant \tilde{b}^i_{(2)}(A). \tag{0.12}$$

It follows from (0.7), (0.8) and the exact cohomology sequences for L^2-cohomology spaces which are Γ-modules (see §2) that

$$\tilde{b}^i_{(2)}(M) = \lim_{k \to \infty} \tilde{\mathbf{b}}^i_{(2)}(M_k) = \lim_{k \to \infty} \lim_{l \to \infty} \underline{\tilde{b}}^i_{(2)}(M_k, M_l). \tag{0.13}$$

By (0.11), (0.12) we then have

$$\tilde{b}^i_{(2)}(M) = \lim_{k \to \infty} \tilde{\mathbf{b}}^i_{(2)}(B_k) = \lim_{k \to \infty} \lim_{l \to \infty} \underline{\tilde{b}}^i_{(2)}(B_k, B_l) \tag{0.14}$$

for *any* exhaustion $M = \bigcup B_k$. This is easily seen to be equivalent to the statement that $\overline{H}^i_{(2)}(\tilde{M})$ is the inverse limit of $\{\overline{H}^i_{(2)}(p^{-1}(B))\}$ (where $\overline{B}$ is compact); compare [7].

However, as noted in [6], condition (0.5) in the definition of a good chopping is technically difficult to achieve. In the profinite case, we circumvented this point by passing to a sufficiently large finite covering space $p_{N(k)}\colon \tilde{M}_{N(k)} \to M$ (where $\tilde{M} \to \tilde{M}_{N(k)} \to M$) on which, say,

$$\operatorname{geo}\left(p^{-1}_{N(k)}(M_k)\right) \leqslant 2. \tag{0.15}$$

A cruder chopping result could then be applied (see [6, Theorem 2.1 (Neighborhoods of bounded geometry)]). Since the $\tilde{\mathbf{b}}^i$, $\tilde{\underline{b}}^i$ are multiplicative under finite coverings, (0.14) follows directly.

Here, rather than constructing a sequence of good choppings, we will generalize the argument just described for the profinite case. The basic observation is that in a suitable sense, *an arbitrary covering is locally profinite*. More precisely, we have the following (compare [14]). Let $|K| \leqslant 1$. There are constants $c_1(n)$, $c_2(n)$ such that if $B_R(q)$ is a metric ball of radius $R < c_1(n)$ and i: $B_R(q) \to B_{4R}(q)$ is the inclusion, then $i(\pi_1(B_R(q))) \subset B_{4R}(q)$ is a nilpotent subgroup of index $\leqslant c_2(n)$.[3] As a standard algebraic consequence, if $U \subset B_R(q)$, i: $\pi_1(U) \to \pi_1(M)$ and $\Gamma \subset \pi_1(M)$, then $i^{-1}(\Gamma) \subset \pi_1(U)$ is profinite.

The induced covering space $\tilde{U}$ corresponding to the subgroup $i^{-1}(\Gamma) \subset \pi_1(U)$ can be identified with a single component of $p^{-1}(U) \subset \tilde{M}$. It is clear from (0.3) that

$$\dim_\Gamma \overline{H}^*_{(2)}(p^{-1}(U)) = \dim_{i^{-1}(\Gamma)} \overline{H}^*_{(2)}(\tilde{U}). \tag{0.16}$$

Let $T_r(U)$ denote the tubular neighborhood of radius r and asume that $T_r(U) \subset B_R(q)$. Then applying the argument in the profinite case gives the basic local estimate

$$\tilde{\underline{b}}^i_{(2)}(U, T_r(U)) \leqslant c(n)(1 + r^{-n})\mathrm{Vol}(T_r(U)). \tag{0.17}$$

In fact, the estimate given in (0.17) holds without the hypothesis $T_r(U) \subset B_R(q)$. This can be seen by combining (0.17) with an argument based on the double complex associated to an open covering $\bigcup_\alpha U_\alpha = X \subset M$. The resulting global estimate is the main step in proving (0.14).

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1. Statement of main results

For the convenience of the reader, we begin by recalling the results of [6] concerning L^2-cohomology, whose proofs in the general case were deferred to this paper. Theorem 1.1 corresponds to Theorem 3.1(1), (3), Theorem 5.1 and Theorem 6.2 of [6]. Theorem 1.2 corresponds to Theorem 7.1 of that paper.

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[3] Actually, more is true, but the above suffices for present purposes ($c_1(n)$ is called the *Margulis constant*).

If M is a riemannian manifold with $|K| \leqslant 1$ and $\mathrm{Vol}(M) < \infty$ we put

$$\chi(M, g) = \int_M P_\chi(\Omega), \tag{1.1}$$

$$\sigma(M, g) = \int_M P_L(\Omega), \tag{1.2}$$

where $P_\chi(\Omega)$ and $P_L(\Omega)$ denote the Chern-Gauss-Bonnet form and Hirzebruch L-form, respectively.

We now define the L^2-Euler characteristic and signature by

$$\tilde{\chi}_{(2)}(M) = \sum_{i=0}^{n} (-1)^i \tilde{b}^i_{(2)}(M), \tag{1.3}$$

$$\tilde{\sigma}_{(2)}(M) = \mathrm{tr}_\Gamma(*\,\pi_{\mathfrak{H}^{2k}(\tilde{M}^{4k})}). \tag{1.4}$$

According to the L^2-index theorem of [6] (compare [1], [11], [18]).

$$\chi(M, g) = \tilde{\chi}_{(2)}(M), \tag{1.5}$$

$$\sigma(M, g) = \tilde{\sigma}_{(2)}(M). \tag{1.6}$$

Theorem 1.1.[4] *Let M be complete, $|K| \leqslant 1$, $\mathrm{Vol}(M) < \infty$ and suppose that* $\mathrm{geo}(\tilde{M}) \leqslant 1$ *for some normal covering.*

(1) *For any exhaustion $M = \bigcup M_k$,*

$$\lim_{k\to\infty} \tilde{\mathbf{b}}^i_{(2)}(M_k) = \lim_{k\to\infty} \lim_{l\to\infty} \underline{\tilde{b}}^i_{(2)}(M_k, M_l) = \tilde{b}^i_{(2)}(M). \tag{1.7}$$

In particular, the $\tilde{b}^i_{(2)}(M)$ are homotopy invariants of M.

(2) *$\chi(M, g)$ and $\sigma(M, g)$ are homotopy (respectively proper homotopy) invariants of M.*

(3) *If M has the topological type of some $M_k \subset M$, then*

$$\tilde{b}^i_{(2)}(M_k) = \tilde{b}^i_{(2)}(M), \tag{1.8}$$

$$\chi(M, g) = \chi(M_k). \tag{1.9}$$

Proof. (2) The homotopy invariance (respectively proper homotopy invariance) of $\chi(M, g)$ and $\sigma(M, g)$ is a consequence of (1) and the L^2-index theorem, (1.5), (1.6).

(3) If M has finite topological type, we can find an exhaustion such that the inclusion $M_k \to M$ is a homotopy equivalence for all k. In this case we have

$$\underline{\tilde{b}}^i_{(2)}(M_k, M_k) = \tilde{b}^i_{(2)}(M_k) \tag{1.10}$$

and the equality (1.8) follows from (1.7). Thus,

$$\tilde{\chi}_{(2)}(M_k) = \tilde{\chi}_{(2)}(M) \tag{1.11}$$

[4] We refer to [6] for earlier results concerning $\chi(M, g)$.

and (1.9) follows from (1.5) together with the corresponding L^2-index theorem for the compact space M_k (proved, for example, as in [1], [6] or [11]).

(1) The equality (1.7) is an immediate consequence of the various subcases of (3.13) of Theorem 3.3. Hence, its proof will be deferred.

The homotopy invariance of $\tilde{b}^i_{(2)}(M)$ follows from (1.7). In fact, let f: $M \to X$, g: $X \to M$ determine a homotopy equivalence and let $\tilde{M}$, $\tilde{X}$ be corresponding normal coverings. We can assume f, g are simplicial with respect to some triangulations of M, X. To each finite subcomplex, say $K^j \subset M^n$, we can associate *simplicial* L^2-cohomology spaces $H^i_{(2),c}(K^j)$, defined using square integrable cochains. By [13], the $H^i_{(2),c}(K)$ are functorial and are homotopy invariants. Moreover, we have

$$\overline{H}^i_{(2),c}(K^n) \simeq_\Gamma \overline{H}^i_{(2)}(K^n) \tag{1.12}$$

if K^n determines a compact submanifold with boundary.

Let $\bigcup_k M_k = M$ and $\bigcup_{k'} X_{k'} = X$ be exhaustions. For fixed k, let k' be chosen so large that

$$f(M_k) \subset X_{k'}. \tag{1.13}$$

Next fix l' with

$$X_{k'} \subset X_{l'}. \tag{1.14}$$

Let h_t be a homotopy of gf to the identity. Choose l so large that for each t,

$$h_t(M_k) \subset M_l, \tag{1.15}$$

$$g(X_{l'}) \subset M_l. \tag{1.16}$$

Let $\rho_{A,B}$ denote the inclusion of A into B and $\rho^*_{A,B}$ the corresponding restriction map, $\rho^*_{A,B}$: $\tilde{H}^i_{(2)}(B) \to \tilde{H}^i_{(2)}(A)$. Then we have a commutative diagram

$$\begin{tikzcd} & & X_{l'} \arrow[dr, "g"] & \\ M_k \arrow[urr, "f"] \arrow[r, "f"'] & X_{k'} \arrow[ur, "\rho_{X_{k'},X_{l'}}"] \arrow[r, "g"'] & & M_l \end{tikzcd} \tag{1.17}$$

where the composition gf is homotopic to the inclusion ρ_{M_k,M_l}. Hence

$$\begin{aligned} \underline{\tilde{b}}^i_{(2)}(M_k, M_l) &= \operatorname{rank}_\Gamma \rho^*_{M_k,M_l} = \operatorname{rank}_\Gamma (gf)^* = \operatorname{rank}_\Gamma \big(g\rho_{X_{k'},X_{\sigma'}} f\big)^* \\ &\leqslant \operatorname{rank}_\Gamma \rho^*_{X_{k'},X_{l'}} = \underline{\tilde{b}}^i_{(2)}(X_{k'}, X_{l'}). \end{aligned} \tag{1.18}$$

Then

$$\lim_{l\to\infty} \underline{\tilde{b}}^i_{(2)}(M_k, M_l) \leqslant \lim_{l'\to\infty} \underline{\tilde{b}}^i_{(2)}(X_{k'}, X_{l'}) \tag{1.19}$$

and so

$$(1.20) \qquad \lim_{k\to\infty} \lim_{l\to\infty} \underline{\tilde{b}}^i_{(2)}(M_k, M_l) \leqslant \lim_{k'\to\infty} \lim_{l'\to\infty} \underline{\tilde{b}}^i_{(2)}(X_{k'}, X_{l'}).$$

Thus, by (1.7),

$$(1.21) \qquad \tilde{b}^i_{(2)}(M) \leqslant \tilde{b}^i_{(2)}(X),$$

and, in the same way, the reverse inequality follows.

Theorem 1.2. *Let N be compact and let $\tilde{N} \to N$ be normal. If* $[0, \infty) \times N$ *admits a complete metric with* $\mathrm{Vol}([0\ \infty) \times N) < \infty$, $\mathrm{geo}([0, \infty) \times \tilde{N}) \leqslant 1$, *then for all i,*

$$(1.22) \qquad \tilde{b}^i_{(2)}(N) = 0.$$

Proof. If we apply (1.7) to the exhaustion $\bigcup [0, k] \times N$ of $[0, \infty) \times N$, it follows that

$$(1.23) \quad 0 = \tilde{\mathbf{b}}^i_{(2)}([0, k] \times N) = \underline{\tilde{b}}^i_{(2)}([0, k] \times N, [0, l] \times N) = \tilde{b}^i_{(2)}(N).$$

In summary, all the results of this section are consequences of (1.7), which is contained in Theorem 3.3.

2. Homological properties of Γ-modules

In this section we discuss some basic homological properties of Γ-modules.[5] Essentially, complexes of Γ-modules behave like complexes of finite-dimensional vector spaces, provided a certain technical condition (d is Γ-Fredholm) is satisfied. But the fact that the Γ-dimension can be an arbitrarily small real number gives a characteristic flavor to the applications (compare (0.17) which has no analog for ordinary Betti numbers).

Let Γ be a discrete group. A Γ-module A is a pre-Hilbert space on which Γ acts by isometries and which can be equivariantly isometrically imbedded as a subspace of $L^2(\Gamma) \otimes H$. Here, the action of Γ on $L^2(\Omega)$ is via the left regular representation and H is a Hilbert space on which Γ acts trivially. As in the references cited, we can attach to A a nonnegative extended real number, $0 \leqslant \dim A \leqslant \infty$, which is independent of the imbedding ϕ.

If $A \neq 0$, then $\dim A > 0$. Moreover, the Γ-dimension of a pre-Hilbert space is equal to that of its completion. As usual,

$$(2.1) \qquad \dim_\Gamma(A_1 \oplus A_2) = \dim_\Gamma A_1 + \dim_\Gamma A_2.$$

[5] We refer to [1], [6], [11], [16] for further background on Γ-modules.

The following property of Γ-dimension is crucial. If $B_{j+1} \subset B_j$, $j = 1, 2, \cdots$, are closed and $\dim B_1 < \infty$, then

$$\dim_\Gamma \cap B_j = \lim_{j \to \infty} \dim_\Gamma B_j. \tag{2.2}$$

Finally, let $\Gamma_1 \subset \Gamma_2$ and let A_1 be a Γ_1-module. If A_2 is the Γ_2-module arising from the induced representation of Γ_2, then

$$\dim_{\Gamma_1} A_1 = \dim_{\Gamma_2} A_2. \tag{2.3}$$

Example 2.1. Let $M = \tilde{M}/\Gamma_2$, $\operatorname{Vol}(M) < \infty$ and $\operatorname{geo}(\tilde{M}) \leqslant 1$. Let $p\colon \tilde{M} \to M$ and $U \subset M$ be open. Then $U = p^{-1}(U)/\Gamma_2$, but $p^{-1}(U)$ need not be connected. Consider some Γ_2-module naturally associated to $\tilde{M}$, for example the space $E^j(\lambda, M)$, corresponding to the spectral resolution of the identity for the Laplacian on i-forms of $\tilde{M}$. As in (0.2), (0.3), the elliptic estimate implies $\dim E^i(\lambda, M) < \infty$. If $i\colon U \hookrightarrow M$, then $\Gamma_2 = \pi_1(M)/\pi_1(\tilde{M})$. The isotropy group, Γ_1, of some component $\tilde{U}$ of $p^{-1}(U)$ is isomorphic to $\pi_1(U)/i^{-1}(\pi_1(\tilde{M}))$. The Γ_2-module $E^j(\lambda, p^{-1}(U))$ arises from the representation of Γ_2 induced from the Γ_1-module $E^j(\lambda, U)$, and by (2.3)

$$\dim_{\Gamma_1} E^j(\lambda, \tilde{U}) = \dim_{\Gamma_2} E^j(\lambda, p^{-1}(U)). \tag{2.4}$$

A *morphism* $i_1\colon A_1 \to A_2$ of Γ-modules is a densely defined (possibly unbounded) linear operator which commutes with the action of Γ. All maps below are assumed to be of this type.

Let

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3. \tag{2.5}$$

Since

$$\dim_\Gamma i_1(A_1) = \dim_\Gamma \overline{i_1(A_1)}, \tag{2.6}$$

we make the following convention. The sequence (2.5) is called *exact* if

$$\overline{\operatorname{im} i_1} = \ker i_2. \tag{2.7}$$

The statement that $\dim_\Gamma$ is independent of ϕ can be extended significantly. Namely, if

$$0 \to A_1 \xrightarrow{i_1} A_2 \to 0 \tag{2.8}$$

is exact, then $\dim_\Gamma A_1 = \dim_\Gamma A_2$. To see this, replace i_1 by $(i_1^* i_2)^{-1/2} i_1$, the isometric part of its polar decomposition.

A *differential* Γ-*module* is a Γ-module A, together with a morphism d: $A \to A$, satisfying $d^2 = 0$. Morphisms of differential Γ-modules are always assumed to be bounded and to commute with differentials. We define the homology spaces $\overline{H}(A)$ by

$$\overline{H}(A) = \ker \bar{d} / \overline{\operatorname{im} \bar{d}}. \tag{2.9}$$

Note that since we use $\overline{\operatorname{im} d}$ (in order to be assured of obtaining a Hilbert space), $\overline{H}(A)$ is a so-called reduced cohomology space and not a cohomology space in the usual sense. Clearly, $\overline{H}(A)$ is a Γ-module in a natural way. From now on we assume $\dim_\Gamma \overline{H}(A) < \infty$ (this assumption will have to be strengthened below). Suppose

$$0 \to \operatorname{dom} \bar{d}_1 \xrightarrow{i_1} \operatorname{dom} \bar{d}_2 \xrightarrow{i_2} \operatorname{dom} \bar{d}_3 \to 0 \tag{2.10}$$

is exact and that there exist bounded morphisms i_j^{-1}: $\operatorname{dom} \bar{d}_{j+1} \cap \operatorname{im} i_j \to \operatorname{dom} \bar{d}_j$, $j = 1, 2$. Then the reduced homology sequence

$$\begin{tikzcd} \overline{H}(A_1) \arrow[r, "i_1^*"] & \overline{H}(A_2) \arrow[dl, "i_2^*"] \\ \overline{H}(A_3) \arrow[u, "\delta"] & \end{tikzcd} \tag{2.11}$$

can be defined by a trivial modification of the standard definition for ordinary homology. However, in this degree of generality (2.11) need *not* be exact. The pathology arises as a consequence of the fact that $\bar{d}^{-1}$ need not be a bounded operator. Equivalently, $\operatorname{im} \bar{d}$ need not be a closed subspace.

Example 2.2 (*Manifolds with cylindrical ends*). Let M be a compact riemannian manifold with $\partial M \neq \varnothing$. Assume that the metric is a product near ∂M. Put $X = M \cup [0, \infty) \times \partial M$, where the union is along ∂M and the metric on $[0, \infty) \times \partial M$ is the product metric. If we view the reduced L^2-cohomology as a Γ-module with trivial Γ-action (so that $\dim_\Gamma = \dim$) we have

$$\begin{aligned} &\overline{H}^i_{(2)}(X, [0, \infty) \times \partial M) \simeq H^i(M, \partial M), \\ &\overline{H}^i_{(2)}([0, \infty) \times \partial M) = 0, \\ &\overline{H}^i_{(2)}(X) \simeq \operatorname{im} H^i(M, \partial M) \subset H^i(M). \end{aligned} \tag{2.12}$$

Thus, the reduced L^2-cohomology sequence of the pair $(X, [0, \infty) \times \partial M)$ is not exact if $\operatorname{im} H^i(M, \partial M) \neq H^i(M)$.

In order to ensure the exactness of (2.11) a condition must be placed on the operators d_j. In general, a morphism $i_1\ A_1 \to A_2$ will be called Γ-*Fredholm* if for some $\lambda_0 > 0$, the spectral projections, $E(\lambda_0)$, of the unbounded selfadjoint operators $i_1 i_1^*$ and $i_1^* i_1$ satisfy $\dim_\Gamma E(\lambda_0) < \infty$ (in our notation the spectral measure is $dE(\lambda)$).

Theorem 2.1 (*Exact homology sequence*). *Let A_j be as in* (2.10), *and assume in addition that the differentials d_j are Γ-Fredholm. Then the reduced homology sequence* (2.11) *is exact.*

Proof. We will check that $\overline{\operatorname{im} i_1^*} = \ker i_2^*$. The other cases follow by similar arguments. Since clearly $\overline{\operatorname{im} i_1^*} \subset \ker i_2^*$, if we put

$$\mathfrak{U} = \left(\overline{\operatorname{im} i_1^*}\right)^\perp \cap \ker i_2^*, \tag{2.13}$$

it suffices to show $\dim_\Gamma \mathfrak{U} = 0$. For $j = 1, 2, 3$ let

$$\mathfrak{H}_j = \{ z \in A_j | dz = d^* z = 0 \}. \tag{2.14}$$

Then $\rho\colon \mathfrak{H}_2^j \to \overline{H}(A_2^j)$ is an isometry. We identify $\mathfrak{U}$ with the corresponding subspace of $\mathfrak{H}_2$. Since $\mathfrak{U} \subset \ker i_2$,

$$i_2(\mathfrak{U}) \subset \mathfrak{H}_3^\perp \cap E_3(\infty). \tag{2.15}$$

Let π_λ denote orthogonal projection on $E_3(\lambda)$. We claim that $\ker \pi_\lambda i_2 | \mathfrak{U} = 0$ for $\lambda > 0$. For if $x \in \ker \pi_\lambda i_2$, then $i_2 x \in \operatorname{dom} d^{-1}$, and in fact,

$$\|d^{-1} i_2 x\| \leqslant \|i_2 x\| / \lambda^{1/2}. \tag{2.16}$$

Put

$$y = i_2^{-1} d^{-1} i_2(x). \tag{2.17}$$

Then

$$x - dy \in \ker i_2 = \overline{\operatorname{im} i_1}, \tag{2.18}$$

contradicting (2.13). Thus, $\pi_\lambda i_2 | \mathfrak{U}$ is an injection for all $\lambda > 0$. Since d_3 is Γ-Fredholm, by (2.2) we have

$$\lim_{\lambda \to 0} \dim_\Gamma \mathfrak{H}_3^\perp \cap E_3(\lambda) = 0. \tag{2.19}$$

So

$$\dim_\Gamma \mathfrak{U} = 0, \tag{2.20}$$

which suffices to complete the proof.

Example 2.3 (*Mayer-Vietoris*). The Mayer-Vietoris sequence, implied by Theorem 2.1, can be employed to estimate the L^2-betti numbers, $b^i(U_1^n \cup U_2^n)$, where for example, U_1^n, $U_2^n \subset M^n$ are compact submanifolds with smooth boundary such that ∂U_1^n, ∂U_2^n intersect transversally. In this case, $U_1^n \cup U_2^n$, $U_1^n \cap U_2^n$ are piecewise smooth and hence are quasi-isometric to smooth manifolds with boundary. Thus the hypothesis of Theorem 2.1 is satisfied (compare [5]) and we obtain the estimate

$$\tilde{b}^i_{(2)}(U_1 \cap U_2) \leqslant \tilde{b}^i_{(2)}(U_1) + \tilde{b}^i_{(2)}(U_2) + \tilde{b}^{i-1}_{(2)}(U_1 \cap U_2). \tag{2.21}$$

Our main estimate, (2.52), is in the spirit of (2.21) but for (possibly infinite) locally finite covers $\{U_\alpha\}$ of some $X = \bigcup_\alpha U_\alpha$, by arbitrary open sets with compact closure. Since the U_α are arbitrary, it becomes necessary to replace $\tilde{b}^i_{(2)}(U)$ by a relative invariant $\tilde{b}^i_{(2)}(r, U)$ $(0 \leqslant r \leqslant \infty)$ which is automatically finite if $r > 0$ and U_α has compact closure. In fact, keeping in mind the definition of "Γ-Fredholm", we will consider invariants $\tilde{b}^i_{(2)}(\lambda, r, U)$ $(\lambda \geqslant 0)$ defined as follows.

Let ρ: $\Lambda^i(p^{-1}(T_r(U)) \to \Lambda^i(p^{-1}(U)))$ denote the restriction map. Consider the unbounded operator $\bar{d}^{-1}\rho$ on the subspace of closed forms $\psi \in \Lambda^i(p^{-1}(T_r(U))) \cap L^2$ such that $\rho(\psi) \in \overline{\operatorname{im} d}$. Let $E(\lambda)$ denote the spectral resolution of the selfadjoint operator $(\bar{d}^{-1}\rho)^*(\bar{d}^{-1}\rho)$. Set

$$\tilde{Z}^i_{(2)}(\lambda, r, U) = \left[\ker \bar{d} | \Lambda^i\left(p^{-1}(T_r(U))\right)\right] \cap E(\lambda)^\perp. \tag{2.22}$$

In particular, for no $\phi \in \tilde{Z}^i_{(2)}(\lambda, r, U)$ does there exist η, with $\rho(\phi) = d\eta$, $\|\eta\| \leqslant \lambda^{1/2}\|\phi\|$. However, this does hold if $\phi \in E(\lambda)$.

We put

$$\underline{\tilde{b}}^i_{(2)}(\lambda, r, U) = \dim_\Gamma \tilde{Z}^i_{(2)}(\lambda, r, U), \tag{2.23}$$

and

$$\underline{\tilde{b}}^i_{(2)}(0, r, U) = \underline{\tilde{b}}^i_{(2)}(r, U) = \underline{\tilde{b}}^i_{(2)}(U, T_r(U)). \tag{2.24}$$

It follows easily that if

$$U \subset Y \subset T_{r_1}(Y) \subset T_{r_2}(U), \tag{2.25}$$

then

$$\underline{\tilde{b}}^i_{(2)}(\lambda, r_2, U) \leqslant \underline{b}^i_{(2)}(\lambda, r_1, Y). \tag{2.26}$$

In particular, if U has compact closure, taking $Y = Y^n$ to be the interior of a compact smooth submanifold with boundary, we see that

$$\underline{\tilde{b}}^i_{(2)}(\lambda, r, U) < \infty. \tag{2.27}$$

Now let $\{U_\alpha\}$ be a locally finite cover of $X = \bigcup_\alpha \subset M^n$. The *multiplicity* N_1 of $\{U_\alpha\}$ is the supremum of integers k, such that there exists

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \varnothing. \tag{2.28}$$

We will put $(\alpha_0, \cdots, \alpha_k) = (\alpha)$,

$$U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} = U_{(\alpha)}. \tag{2.29}$$

We denote by $N_2(\alpha)$ the number of sets U_β ($\beta \neq \alpha$) such that $U_\alpha \cap U_\beta \neq \varnothing$. The *norm*, N_2, of $\{U_\alpha\}$ is then defined to be

$$\sup_\alpha N_2(\alpha) = N_2. \tag{2.30}$$

Clearly, $N_1 \leqslant N_2 \leqslant \infty$.

Associated to $\{U_\alpha\}$ is the *double complex* of Γ-modules[6]

$$\bigoplus_{i,j} C^{i,j}(X) = \bigoplus_{|(\alpha)|=i} \bigoplus_j \Lambda^j\big(p^{-1}(U_{(\alpha)})\big) \cap L^2. \tag{2.31}$$

Thus, a cochain ϕ of $C^{i,j}(X)$ consists of a collection of j-forms $\phi_{(\alpha)} \in \Lambda^j(p^{-1}(U_{(\alpha)}))$ such that if π is a permutation of $\alpha_0,\cdots,\alpha_k$, then

$$\phi_{(\pi(\alpha))} = (-1)^{|\pi|}\phi_{(\alpha)}. \tag{2.32}$$

The differentials $\partial = \bigoplus \partial_{i,j}$, $d = \bigoplus d_{i,j}$,

$$\partial_{i,j}\colon C^{i,j} \to C^{i+1,j}, \tag{2.33}$$

$$d_{i,j}\colon C^{i,j} \to C^{i,j+1}, \tag{2.34}$$

are given by

$$(\partial\phi)_{(\alpha)} = \sum_k (-1)^k \phi_{(\alpha_0,\cdots,\hat{\alpha}_k,\cdots,\alpha_{j+1})}, \tag{2.35}$$

$$(d\phi)_{(\beta)} = \bar{d}(\phi)_{(\beta)}. \tag{2.36}$$

Clearly,

$$\partial d = d\partial. \tag{2.37}$$

From now on we will assume that the sets U_α have compact closure and that the multiplicity of $\{U_\alpha\}$ is finite. In this case sections of $\Lambda^j(p^{-1}(X)) \cap L^2$ can be identified with cochains $\phi \in C^{0,j}(X)$ such that

$$\phi \in \ker \partial_{0,j}. \tag{2.38}$$

$$\sum_\alpha \|\phi_\alpha\|^2 < \infty. \tag{2.39}$$

Suppose, in addition, there is a partition of unity $1 = \sum f_\alpha$ subordinate to $\{U_\alpha\}$ such that the pointwise norm $\|df\|_\alpha$ is bounded independent of α. Then for $i \geqslant 1$, an operator

$$\partial^{-1}\colon C^{i,j} \to C^{i-1,j} \tag{2.40}$$

[6] The symbol $\oplus$ denotes direct sum in the sense of Hilbert spaces.

satisfying

$$\partial^{-1}\partial + \partial\partial^{-1} = 1 \tag{2.41}$$

can be defined by the following standard formula. If $\beta = (\beta_0, \cdots, \beta_{j-1})$, we put

$$U_{(\beta\alpha)} = U_{\beta_0} \cap \cdots \cap U_{\beta_{j-1}} \cap U_\alpha \tag{2.42}$$

and

$$(\partial^{-1}\phi)_{(\beta)} = (-1)^{j-1} \sum_\alpha f_\alpha \beta_{(\beta\alpha)} | U_{(\beta)}. \tag{2.43}$$

Finally, let $\Delta = d\delta_0 + \delta_0 d$ denote the Laplacian for generalized absolute boundary conditions (see e.g. [5]). Let $\pi_{\mathfrak{H}}$ denote orthogonal projection on ker Δ and Δ^{-1} the Green's operator,

$$\Delta\Delta^{-1} = \Delta^{-1}\Delta = 1 - \pi_{\mathfrak{H}}. \tag{2.44}$$

Then as usual,

$$d^{-1} = \delta_0 \Delta^{-1}, \tag{2.45}$$

$$dd^{-1} + d^{-1}d = 1 - \pi_{\mathfrak{H}}. \tag{2.46}$$

Since there will be no danger of confusion, we also define

$$d^{-1}: C^{i,j} \to C^{i,j-1} \tag{2.47}$$

by

$$d^{-1}(\phi)_{(\alpha)} = d^{-1}(\phi_{(\alpha)}). \tag{2.48}$$

We are now in a position to state the main estimate of this section (compare [15]).

Theorem 2.2 (*Double complex estimate*). *Let $p: \tilde{M}^n \to M^n$ be normal and let $\{U_\alpha\}$ be a cover of $X = \bigcup_\alpha U_\alpha \subset M^n$ by open sets with compact closure. Assume*

(1) *$\{U_\alpha\}$ has finite norm $N_2 < \infty$ and multiplicity $N_1 \leqslant N_2$.*

(2) *There is a partition of unity $1 = \sum f_\alpha$, subordinate to $\{U_\alpha\}$, such that for all α, the pointwise norm $\|df_\alpha\|$ is uniformly bounded,*

$$\|df_\alpha\| \leqslant c. \tag{2.49}$$

For fixed j, we put

$$J = \min(j, N_1). \tag{2.50}$$

For $\lambda \geqslant 0$ (c as in (2.49)) and a suitable constant $c(N_2)$, we define μ by

$$\frac{1}{\lambda^{1/2}} = c(N_2) \sum_{k=1}^{J} (c^2/\mu)^{k/2}. \tag{2.51}$$

Then for any $0 = r_0 \leqslant r_1 < \cdots \leqslant r_{J+1}$,

$$\tilde{\underline{b}}^i_{(2)}(\lambda, r_{J+1}, X) \leqslant \sum_{k=0}^{J} \sum_{|(\alpha)|=k} \tilde{\underline{b}}^{j-k}_{(2)}\left(\mu, r_{J-k+1} - r_{J-k}, T_{r_{J-k+1}}(U_{(\alpha)})\right). \tag{2.52}$$

Remark 2.1. Of course, the implication of Theorem 2.2 is vacuous unless the sum on the right-hand side of (2.52) is finite.

Proof of Theorem 2.2. In view of (2.2), the case $\lambda = 0$ follows from the case $\lambda > 0$ by letting $\lambda \to 0$. Thus we assume $\lambda > 0$ and hence, $\mu > 0$.

Put

$$C^{i,j}(T_l(X)) = \bigoplus_{i=|(\alpha)|} \Lambda^j\left(p^{-1}\left(T_{r_l}(U_{(\alpha)})\right)\right) \cap L^2. \tag{2.53}$$

If $z \in \ker d_{i,j} \subset C^{i,j}(T_l(X))$, let $\pi_\mu(z)$ denote its projection on the subspace

$$\bigoplus_{i=|(\alpha)|} \tilde{Z}^j_{(2)}\left(\mu, r_l - r_{l-1}, T_{r_{l-1}}(U_{(\alpha)})\right).$$

We will define maps, $\psi_0 = \mathrm{Ident}|\tilde{Z}^j_{(2)}(\mu, r_{J+1}, X), \Psi_1, \cdots, \psi_J$,

$$\begin{aligned}\psi_k\colon \ker \pi_\mu \psi_{k-1} \to \operatorname{im} \partial_{k-1, j-k} \cap \ker d_{k, j-k} \\ \subset C^{k, j-k}\left(T_{r_{J-k+1}}(X)\right)\end{aligned} \tag{2.54}$$

$(1 \leqslant k \leqslant J)$ and a decreasing filtration

$$V_J \subseteq \cdots \subseteq V_0 \subseteq V_{-1} = \tilde{Z}^j_{(2)}(\lambda, r_{J+1}, X), \tag{2.55}$$

given for $k \geqslant 0$, by

$$V_k = \ker \pi_\mu \psi_k. \tag{2.56}$$

It will suffice to establish that for μ as in (2.51),

$$V_J = 0. \tag{2.57}$$

For then,

$$\underline{b}^j_{(2)}(\lambda, r_{J+1}, X) = \sum_{k=0}^{J} \dim_\Gamma (V_{k-1}/V_k), \tag{2.58}$$

and

(2.59)

$$\bigoplus_{k=0}^{J} \pi_\mu \psi_k\colon \bigoplus_{k=0}^{J} V_{k-1}/V_k \to \bigoplus_{k=0}^{J} \bigoplus_{|(\alpha)|=k} \tilde{Z}^{j-k}_{(2)}\left(\mu, r_{J-k+1} - r_{J-k}, T_{r_{J-k}}(U_{(\alpha)})\right)$$

is an injection.

For $k' > k$, let ρ_k denote the restriction map

$$\rho_k\colon \oplus C^{i,j}\left(T_{r_{k'}}(X)\right) \to C^{i,j}\left(T_{r_k}(X)\right). \tag{2.60}$$

Set

$$(2.61)\quad B_{k-1} = \partial d^{-1}\rho_{k-1}\colon \bigoplus_{i,j} C^{i,j}\big(T_{r_k}(X)\big) \to \bigoplus_{i,j} C^{i+1,j-1}\big(T_{r_{k-1}}(X)\big).$$

As in (2.38), (2.39) we identify $\tilde{Z}^j_{(2)}(\infty, 0, T_{r_{J+1}}(X))$ with the correspondiing subspace of

$$(2.62)\qquad \ker \partial_{0,j} \cap \ker d_{0,j},$$

and put

$$(2.63)\qquad \psi_0 = \mathrm{Ident}\big|\tilde{Z}^i_{(2)}(\mu, r_{J+1}, X),$$

$$(2.64)\qquad \psi_k = B_{J-k+1}\psi_{k-1}.$$

To see that $\operatorname{im}\psi_k|\ker \pi_\mu\psi_{k-1} \subset \ker d_{k,j-k}$ as claimed in (2.54), we can assume by induction that

$$(2.65)\qquad dd^{-1}\psi_{k-1}|\ker \pi_\mu\psi_{k-1} = \psi_{k-1}.$$

Then

$$(2.66)\quad d\psi_k = dB_{J-k+1}\psi_{k-1} = \partial dd^{-1}\rho_{k-1}\psi_{k-1} = \rho_{k-1}\partial\psi_{k-1} = 0.$$

Next observe that if $N_1 < j$, we have automatically

$$(2.67)\qquad \psi_{N_1+1} = 0.$$

On the other hand, if $N_1 \geqslant j$, since the only closed 0-forms are constants,

$$(2.68)\qquad \ker \pi_\mu\psi_j = \ker \psi_j$$

Thus, if we put

$$(2.69)\qquad j^* = \min(N_1, j-1)$$

on V_J we have

$$(2.70)\qquad \psi_{j^*+1} = 0.$$

To show that $V_J = 0$, we introduce

$$(2.71)\quad A = d\partial^{-1}\colon \bigoplus_{i\geqslant 1,\, j} C^{i,j}(T_l(X)) \to \bigoplus_{i\geqslant 1,\, j} C^{i-1,j+1}(T_l(X))$$

and the operator

$$(2.72)\qquad K\colon V_J \to C^{0,j-1}$$

given by

$$(2.73)\qquad K = \rho_0 \sum_{k=0}^{j^*} (-1)^k A^k d^{-1}\rho_{J-k}\psi_k$$

(where $A^0 = 1$). Since $dA = 0$, clearly

$$(2.74)\qquad dK = I_{V_J},$$

where I_{V_J} denote the identity operator on V_J. Moreover, we claim that

$$\operatorname{im}(K) \subset \ker \partial_{0, j-1}. \tag{2.75}$$

To see this note that by (2.37), (2.41), (2.66),

$$\partial A = \partial d \partial^{-1} = d\partial\partial^{-1} = d(1 - \partial^{-1}\partial) = d - A\partial, \tag{2.76}$$

from which it follows easily by induction that

$$\partial A^k = (-1)^{k-1} A^{k-1} d + (-1)^k A^k \partial. \tag{2.77}$$

Thus,

$$\begin{aligned} \partial A^k \rho_{J-k} d^{-1} \psi_k &= \left[(-1)^{k-1} A^{k-1} d + (-1)^k A^k \partial\right] d^{-1} \rho_{J-k} \psi_k \\ &= (-1)^{k-1} A^{k-1} \rho_{J-k} \psi_k + (-1)^k A^k \psi_{k+1} \\ &= (-1)^k \rho_{J-k} A^{k-1} \psi_k + (-1)^k A^k \psi_{k+1} \end{aligned} \tag{2.78}$$

By combining (2.73), (2.78) and using (2.70) we get

$$\partial K = (-1)^{j^*} \rho_0 A^{j^*} \psi_{j^*+1} = 0. \tag{2.79}$$

According to (2.75), given $x \in V_J$, we can regard

$$K(x) \in \Lambda^{j-1}(p^{-1}(X)), \tag{2.80}$$

and it suffices to show that

$$\|K(x)\| \leqslant \|x\| / \lambda^{1/2}. \tag{2.81}$$

But in view of (2.43), (2.71) (the definitions of ∂^{-1}, K) and (2.66), on V_J we have

$$\begin{aligned} \left\|\left(A^k \rho_{J-k} \psi_k(x)\right)_\beta\right\| &= \Big\| \sum_{\alpha_1 \cdots \alpha_k} df_{\alpha_1} \wedge \cdots \wedge df_{\alpha_k} d^{-1} \rho_{J-k} \psi_k(x)_{(\beta\alpha_1 \cdots \alpha_k)} \\ &\qquad + \sum_{\alpha_1 \cdots \alpha_k} f_{\alpha_k} df_{\alpha_1} \wedge \cdots \wedge df_{\alpha_{k-1}} \rho_{J-k} \psi_k(x)_{(\beta\alpha_1 \cdots \alpha_k)} \Big\| \\ &\leqslant c(N_2)\left[(c^2/\mu)^{(k-1)/2} + (c^2/\mu)^{k/2}\right] \sup_{\gamma = \alpha_1 \cdots \alpha_k, \beta} \|x_\gamma\| \end{aligned} \tag{2.82}$$

where the supremum is over α such that $U_\alpha \cap U_\beta \neq \varnothing$. For μ as in (2.51), this easily yields (2.81) which completes the proof.

Remark 2.2. Clearly the constant in (2.51) can be estimated explicitly.

Remark 2.3. There is an analog of Theorem 2.2 in the more familiar context of forms on X rather than on $p^{-1}(X)$.

Remark 2.4. In case $\tilde{b}^i_{(2)}(U_\alpha) = 0$ for $i > 0$, an easy variation of Theorem 2.2 identifies the Γ-module $\overline{H}^i_{(2)}(\tilde{X})$ with the reduced L^2-cohomology of the complex $(\{U_\alpha\}, \partial)$ with local coefficients in the system of Γ-modules $\{\overline{H}^0_{(2)}(p^{-1}(U_\alpha))\}$ (compare [13]).

3. Proof of main results

As noted in §0, the basic fact which allows us to introduce profiniteness into the general situation is the existence of constants $c_1(n)$ (the Margulis constant) and $c_2(n)$ with the following properties. If $B_R(q)$ is a metric ball of radius $R < c_1(n)$, then $i(\pi_1(B_R(q))) \subset \pi_1(B_{4R}(q))$ contains a nilpotent subgroup of index at most $c_2(n)$. As a consequence, for any $U \subset B_R(q)$ a normal covering $\tilde{U} \to U$ is profinite, provided it is induced from a covering

$$p\colon \tilde{B}_{4R}(q) \to B_{4R}(q).^7 \tag{3.1}$$

This observation together with the argument of [6] yields

Lemma 3.1. *Let $U \subset T_r(U) \subset B_R(q)$ and let p: $\tilde{B}_{4R}(q) \to B_{4R}(q)$ be a normal covering with* $\text{geo}(\tilde{T}_r(U)) \leqslant 1$. *If $R < c_1(n)$, then*

$$\underline{\tilde{b}}^i_{(2)}(\lambda, r, U) \leqslant c(N)(1 - \lambda^n + r^{-n})\text{Vol}(T_r(U)). \tag{3.2}$$

Proof. By an easy scaling argument, we can assume $\lambda \leqslant 1$, $r \geqslant 1$. Since p: $\tilde{B}_{r_1}(q) \to B_{r_1}(q)$ is profinite, there exists a finite covering p_J: $\hat{T}_r(U) \to T_r(U)$ of order j such that $\tilde{T}_r(U) \to \hat{T}_r(U) \to T(U)$. As explained in Example 2.1, we can work with $\tilde{T}_r(U)$ rather than $p^{-1}(U) \subset \tilde{M}$. By Theorem 2.2 of [6] we can find a manifold Y^n with smooth boundary such that

$$\text{Vol}(\partial Y^n) \leqslant c(n)\hat{T}_r(U), \tag{3.3}$$

$$\|\nabla^i \text{II}(\partial Y^n)\| \leqslant c(n, i), \tag{3.4}$$

$$p_j^{-1}(U) \subset Y^n \subset p_j^{-1}(T_r(U)). \tag{3.5}$$

If we apply the elliptic estimate on Y^n, together with (2.26), we obtain

$$\underline{\tilde{b}}^i_{(2)}(\lambda, r, p_j^{-1}(U)) \leqslant \underline{\tilde{b}}^i_{(2)}(\lambda, 0, Y^n) \leqslant \text{Vol}(p_j^{-1}(T_r(U))). \tag{3.6}$$

Then (3.2) follows by dividing by j.

Our first main estimate is the global version of Lemma 3.1, in which the hypothesis $T_r(U) \subseteq B_R(q)$, $R < c_1(n)$, is removed.

---

[7] For "most" small balls, $B_R(q)$ is a deformation retract of $B_{4R}(q)$. For such balls, $\pi_1(B_r(q))$ itself is nilpotent (compare [15]).

Theorem 3.2. *Let $U \subset M^n$ and p: $\tilde{T}_r(U) \to T_r(U)$ be a normal covering with* geo$(\tilde{T}_r(U)) \leqslant 1$. *Then* (3.2) *holds.*

Proof. By an easy scaling argument, we can assume $r \geqslant 1$, $\lambda \leqslant 1$. Take a maximal set of points $\{q_\alpha\}$ which is $c_1(n)/2(i+1)$ dense in U (where $\underline{i}$ is as in (2.50) and we can assume $c_1(n) \leqslant 1$). Then for $r = c_1(n)/(\underline{i}+1)$, $B_r(q_\alpha) = U_\alpha$ we have

$$U \subset X = \bigcup U_\alpha \subset T_{(\underline{i}+1)r}(X) \subset T_{r_1}(U). \tag{3.7}$$

As in [14, 2.2.A], the assumption $|K| \leqslant 1$ gives a bound

$$N_2 < c(n) \tag{3.8}$$

on the norm of the covering. Our claim now follows from (2.26), Theorem 2.2 and Lemma 3.1.

Before proceeding to our final estimate we introduce some notation. If U is an open set, we put

$$T_{-r}(U) = U \cap (T_r(\partial U))'. \tag{3.9}$$

Let τ: $\Lambda_0^i(T_{-r}(U)) \to \Lambda^i(U)$. Let $E(\lambda)$ denote the spectral resolution of $(\bar{d}^{-1}\tau)^*(\bar{d}^{-1}\tau)$, where $\phi \in \operatorname{dom} \bar{d}^{-1}\tau$ if $\phi \in \ker \bar{d}_0$ and $\tau(\phi) \in \operatorname{im} \bar{d}$. Put

$$\tilde{\mathbf{Z}}^i_{(2)}(\lambda, -r, U) = \ker \bar{d}_0 | \Lambda_0^i(T_{-r}(U)) \cap E(\lambda)^\perp. \tag{3.10}$$

$$\tilde{\mathbf{b}}^i_{(2)}(\lambda, -r, U) = \dim_\Gamma \tilde{\mathbf{Z}}_{(2)}(\lambda, -r, U). \tag{3.11}$$

Let $U_1 \subset U_2$, $0 < r_1 < s_1$, $0 < r_2 \leqslant \infty$. Then

$$T_{-s_1}(U_1) \subset T_{-r_1}(U_1) \subset U_1 \subset U_2 \subset T_{r_2}(U_2) \tag{3.12}$$

and we have

Theorem 3.3. *For all* $\varepsilon > 0$ *and* $s_1 > r_1$,

$$\begin{aligned} \tilde{\mathbf{b}}^i_{(2)}(\lambda, -r, U_1) &\leqslant \tilde{\underline{b}}^i_{(2)}(\lambda, r_2, U_2) \\ &\leqslant \mathbf{b}^i_{(2)}((1+\varepsilon)\lambda, -r, U_1) \\ &\quad + c(n)\big(1 + \lambda^n + \varepsilon^{-2n} + (s_1 - r_1)^{-n} + r_1^{-n} + r_2^{-n}\big) \\ &\quad \times \operatorname{Vol}\big(T_{r_2}(U_2) \setminus T_{-(s_1+r_2)}(U_1)\big). \end{aligned} \tag{3.13}$$

Remark 3.1. If (say for r_1 large) $T_{-r_1}(U_1) = \varnothing$, then $\tilde{\mathbf{b}}^i_{(2)}((1+\varepsilon)\lambda, -r_2, U_2) = 0$ and Theorem 3.3 reduces to Theorem 3.2.

Proof of Theorem 3.3. By scaling, we can assume $\lambda \leqslant 1$, $(s_1 - r_1)$, r_1, $r_2 \geqslant 1$. We can also assume $\varepsilon \leqslant 1$, and we have only to establish the second inequality.

Recall that for any U

$$\ker \bar{d} | p^{-1}(T_r(U)) = \bigcup_{\lambda} \tilde{Z}^i_{(2)}(\lambda, r, U). \tag{3.14}$$

We introduce the following notation for orthogonal projections.

$$\pi(\lambda, r, U) \colon \ker \bar{d} | p^{-1}(T_r(U)) \to \tilde{Z}^i_{(2)}(\lambda, r, U), \tag{3.15}$$

$$\pi(\lambda, -r, U) \colon \ker \bar{d}_0 | p^{-1}(T_{-r}(U)) \to \tilde{\mathbf{Z}}^i_{(2)}(\lambda, -r, U). \tag{3.16}$$

Also put

$$A_{-s_1} = U_2 \setminus \overline{T_{-s_1}(U_1)}\,. \tag{3.17}$$

We will show that for $212\varepsilon_1^{1/2} = \varepsilon$,

$$\begin{aligned} \underline{\tilde{b}}^i_{(2)}(\lambda, r_2, U_2) \leqslant{}& \underline{\tilde{b}}^i_{(2)}\left(\varepsilon_1^{-1}, r_2, A_{-s_1}\right) + \tilde{\mathbf{b}}^i_{(2)}((1+\varepsilon)\lambda, -r_1, U_1) \\ &+ \tilde{b}^i_{(2)}\left(\varepsilon_1^{-1}, r_1/3, T_{-r_1/3}(U_1) \setminus \overline{T_{-2r_1/3}(U_1)}\right). \end{aligned} \tag{3.18}$$

Then (3.13) follows by applying Theorem 3.2 to estimate the first and third terms of (3.18).

Let $f \colon p^{-1}(A_{-s_1}) \to [0,1]$ be a smooth function such that

$$f = \begin{cases} 1, & p^{-1}\left(A_{-r_1}\right), \\ 0, & p^{-1}\left(U_2 \setminus A_{-s_1}\right), \end{cases} \tag{3.19}$$

$$\|df\| \leqslant \frac{3}{s_1 - r_1} \leqslant 3. \tag{3.20}$$

If

$$\phi \in \ker \pi\left(\varepsilon_1^{-1}, r_2, A_{-s_1}\right) \tag{3.21}$$

there exists $\eta \in \Lambda^{i-1}(p^{-1}(A_{-s_1}))$, with

$$\eta = d^{-1}\phi, \tag{3.22}$$

$$\|\eta\| \leqslant \varepsilon_1^{1/2}\|\phi\|. \tag{3.23}$$

Set

$$B(\phi) = \phi - d(f\eta). \tag{3.24}$$

Then

$$B(\phi) | p^{-1}\left(A_{-r_1}\right) = 0, \tag{3.25}$$

$$\|B(\phi)\| \leqslant \|(1-f)\phi\| + \|df\| \cdot \|\eta\| \leqslant \left(1 + 3\varepsilon_1^{1/2}\right)\|\phi\|. \tag{3.26}$$

We have

(3.27) $\pi((1+\varepsilon)\lambda, -r, U_1)B\colon \ker \pi\left(\varepsilon_1^{-1}, r_2, A_{-s_1}\right) \to \tilde{\mathbf{Z}}^i_{(2)}((1+\varepsilon)\lambda, -r_1, U_1)$.

Suppose, also

(3.28) $$\phi \in \ker \pi((1+\varepsilon)\lambda, -r_1, U_1)B.$$

Then there exists $\gamma \in \Lambda^{i-1}(p^{-1}(U_1))$ such that

(3.29) $$\gamma = d^{-1}(B(\phi)),$$

(3.30) $$\|\gamma\| \leqslant \frac{1}{(1+\varepsilon)^{1/2}\lambda^{1/2}}\|B(\phi)\|.$$

Let

(3.31) $$g\colon p^{-1}\left(T_{-2r_1/3}(U_1) \setminus \overline{T_{-r_1/3}(U_1)}\right) \to [0,1]$$

be a smooth function such that

(3.32) $$g = \begin{cases} 1, & p^{-1}\left(T_{-7r_1/12}(U_1) \setminus \overline{T_{-2r_1/3}(U_1)}\right), \\ 0, & p^{-1}\left(T_{-5r_1/12}(U_1) \setminus \overline{T_{-7r_1/12}(U_1)}\right), \end{cases}$$

(3.33) $$\|dg\| \leqslant 12/7 \leqslant 12.$$

Suppose

(3.34) $$\gamma \in \ker \pi\left(\varepsilon_1^{-1}, r_1/3, T_{-r_1/3}(U_1) \setminus \overline{T_{-2r_1/3}(U_1)}\right).$$

Then there exists

(3.35) $$\theta \in \Lambda^{i-2}\left(p^{-1}\left(T_{-r_1/3}(U_1) \setminus \overline{T_{-2r_1/3}(U_2)}\right)\right)$$

with

(3.36) $$\theta = d^{-1}\gamma,$$

(3.37) $$\|\theta\| \leqslant \varepsilon_1^{1/2}\|\gamma\|.$$

Thus, $d(g\theta)$ extends to a form $\hat{\gamma}$ on $p^{-1}(U_2)$ and

(3.38) $$\hat{\gamma}|T_{-7r_1/12}(U_1) \setminus \overline{T_{-2r_1/3}(U_1)} = \gamma,$$

(3.39) $$d\hat{\gamma} = B(\phi),$$

(3.40) $$\|\hat{\gamma}\| \leqslant \|\gamma\| + \|dg\| \cdot \|\theta\| \leqslant \left(1 + 12\varepsilon_1^{1/2}\right)\|\gamma\|.$$

Therefore, on $p^{-1}(U_2)$

(3.41) $$\phi = d(\hat{\gamma} + f\eta),$$

(3.42) $$\|\hat{\gamma} + f\eta\| \leqslant \left[\varepsilon_1^{1/2} + \left(1 + 3\varepsilon_1^{1/2}\right)\left(1 + 12\varepsilon_1^{1/2}\right)\frac{1}{(1+\varepsilon)^{1/2}\lambda^{1/2}}\right]\|\phi\|.$$

Using $\lambda \leqslant 1$, it follows easily from the choice

$$212\varepsilon_1^{1/2} = \varepsilon, \tag{3.43}$$

that

$$\|\hat{\gamma} + f\eta\| \leqslant \lambda^{-1/2}\|\phi\|. \tag{3.44}$$

Thus (3.18) follows (see (3.21), (3.28) and (3.34)). Thus (3.13) follows as well.

To obtain our main result (1.7) from (3.13) take $U_1 = M_k$, $U_2 = M$, $r_1 = 1$, $s_1 = 2$, $r_2 = 1$ and $\lambda = 0$.

4. $\tilde{\eta}_{(2)}$-invariants and collapse with bounded covering geometry

Let N^{4l-1} be a compact oriented riemannian manifold. The η-invariant is defined by[8]

$$\eta(N^{4l-1}) = \frac{1}{\Gamma(1/2)}\int_0^\infty t^{-1/2}\operatorname{tr}\left(*de^{-\Delta_{2l-1}t}\right) dt. \tag{4.1}$$

Here $e^{-\Delta_{2l-1}t}$ is the heat kernel on $(2l-1)$-forms.

Suppose N^{4l-1} is the boundary of a riemannian manifold M^{4l} and that the metric is a product near the boundary. According to [2], we have

$$\sigma(M^{4l}) = \sigma(M^{4l}, g) + \eta(N^{4l-1}). \tag{4.2}$$

Note that $\overline{\sigma(M^{4l} g)}$, the reduction mod Z of $\sigma(M^{4l}, g)$, depends only on N^{4l-1}; it is an example of the invariants $\hat{P}(N^{4l-1})$ (where P is a polynomial in the Pontrjagin classes) which were studied in [10], [17], [9]. In the notation of those papers, $\overline{\sigma(M^{4l}, g)} = \hat{P}_L(N^{4l-1})$.

The invariants $\eta(N^{4l-1})$ does not behave multiplicatively under coverings, and hence, cannot be obtained by integrating a canonical local expression derived from the metric. However, let g_0, g_1 be a pair of metrics on N. Then, as follows easily from (4.2) (and also from (4.1)), the difference $\eta(N, g_1) - \eta(N, g_0)$ is a locally computable function of the pair (g_0, g_1). The analogous statement also holds for the invariants $\hat{P}(N)$. As a consequence, for any finite covering space $\tilde{N}$ of order d,

$$\rho(N, \tilde{N}) = \eta(N, g) - \frac{1}{d}\eta(\tilde{N}, \tilde{g}) \tag{4.3}$$

is *independent of metric*. Moreover, we can study the *nonlocal* and *topological* aspects of the invariants $\eta(N)$ and $\hat{P}(N)$ by the following approach.

[8] Everything in the present section applies equally well to the η-invariant with coefficients in a flat orthogonal bundle, but we will not mention this explicitly.

Given N^{4l-1}, possibly with some auxiliary structure, define a class of metrics (possibly degenerate) for which the invariants $\eta(N)$ and $\hat{P}(N)$ have a purely topological interpretation.

Here, isolating the relevant features of the topology is itself an element of the problem. Also, the actual geometry of those metrics which arise is of prime interest and importance. Note that we do not require uniqueness of the auxiliary structure, nor that it should always exist.

In [6], the above program was applied to the η-invariant for those manifolds N which admit a volume collapse with boundary covering geometry. A family of metrics g_ε on N is said to *volume collapse with bounded curvature* (from now on we just say volume collapse) if for all ε,

$$|K_\varepsilon| \leqslant c, \tag{4.4}$$

$$\lim_{\varepsilon \to 0} \operatorname{Vol}(N, g_\varepsilon) = 0. \tag{4.5}$$

This collapse is said to have *bounded covering geometry* if the induced metrics g_ε on the universal covering space $\tilde{N}$ satisfy

$$\operatorname{geo}(\tilde{N}, \tilde{g}_\varepsilon) \leqslant c. \tag{4.6}$$

(By scaling, we can take $c = 1$ in (4.4), (4.6).)

In [6], it was necessary to assume that the covering $\tilde{N}$ is profinite (the argument is recalled below). The purpose of this section is to remove this hypothesis by introducing an invariant $\tilde{\eta}_{(2)}(N)$ defined using the concept of Γ-trace. Most of the discussion, but not the main conclusion (Theorem 4.1), also applies to the invariants $\hat{P}(N)$.

Before proceeding further we mention that in [8], a second realization of our program is given for manifolds admitting a *volume collapse which is compatible with a polarized F-structure*. (The simplest example of a polarized F-structure is a nonvanishing Killing field; for the general definition see [8].) The discussion of [8] applies not only to the η-invariants, but to the invariants $\hat{P}(N)$ as well.

A given manifold N^{4l-1} may admit no volume collapse or it may admit (possibly infinitely many) essentially different ones. This may or may not have bounded covering geometry (the latter condition is relatively rare). If the volume collapse $(N^{4l-1}, g_\varepsilon)$ is compatible with a polarized F-strucure, then

$$\lim_{\varepsilon \to 0} \eta\left(N^{4l-1}, g_\varepsilon\right), \qquad \lim_{\varepsilon \to 0} \hat{P}\left(N^{4l-1}, g_\varepsilon\right) \tag{4.7}$$

always exist and can be evaluated explicitly as topological invariants (which are cobordism invariants of the structure; see [6] and [8]). Examples show that these limits are not independent of the choice of F-structure in general. On the other hand, we will see that the first limit in (4.7) exists for *arbitrary* collapses with bounded covering geometry. Hence, for any two such collapses the limits are equal. It would be remarkable if this were also true for the invariants $\hat{P}(N^{4l-1})$.

The main analytic point in [6] was the estimate

$$|\eta(Y^{4l-1})| \leqslant c(4l-1)\mathrm{Vol}(Y^{4l-1}), \tag{4.8}$$

provided $\mathrm{geo}(Y^{4l-1}) \leqslant 1$. This was applied as follows. Let $(N^{4l-1}, g_\varepsilon)$ be a volume collapse with bounded covering geometry, for which $\tilde{N}$ is profinite. Then for each ε, there exists a covering $\tilde{N}_\varepsilon$, of order d_ε, for which

$$\mathrm{geo}(\tilde{N}_\varepsilon, \tilde{g}_\varepsilon) \leqslant 2. \tag{4.9}$$

If we put $\tilde{\eta}_{(2)}(N, g_\varepsilon) = \eta(\tilde{N}_\varepsilon)/d_\varepsilon$, then (4.8) and (4.9) imply

$$|\tilde{\eta}_{(2)}(N, g_\varepsilon)| \leqslant c(4l-1)\mathrm{Vol}(N). \tag{4.10}$$

Since any two finite coverings have a common finite covering, it follows that

$$\lim_{\varepsilon \to 0} \eta(N, g_\varepsilon) = \lim_{\varepsilon \to 0} \rho(N, \tilde{N}_\varepsilon) \tag{4.11}$$

exists and is independent of the particular collapse with bounded covering geometry.

In order to remove the hypothesis that $\tilde{N}$ is profinite, we now define the invariant $\tilde{\eta}_{(2)}(Y^{4l-1})$ for possibly infinite normal coverings $\tilde{N}$. We will show that if $\tilde{N}_\varepsilon$ in (4.9) is replaced by $\tilde{N}$, then (4.10) remains valid. Moreover, for a pair of metrics g_0, g on Y^{4l-1},

$$\tilde{\eta}_{(2)}(Y, g_1) - \tilde{\eta}_{(2)}(Y, g_0) = \eta(Y, g_1) - \eta(Y, g_0). \tag{4.12}$$

If we put

$$\tilde{\rho}_{(2)}(Y) = \eta(X, g) - \tilde{\eta}_{(2)}(Y, g), \tag{4.13}$$

then by (4.12), $\tilde{\rho}_{(2)}(Y)$ is independent of g. Using the new (4.10), we obtain

Theorem 4.1. *Let $(N^{4l-1}, g_\varepsilon)$ be a volume collapse with bounded covering geometry. Then*

$$\lim_{\varepsilon \to 0} \eta\left(N^{4l-1}, g_\varepsilon\right) = \tilde{\rho}_{(2)}(N^{4l-1}). \tag{4.14}$$

The invariant $\tilde{\eta}_{(2)}(Y)$ is defined by

$$\tilde{\eta}_{(2)}(Y) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} \mathrm{tr}_\Gamma(e^{-\tilde{\Delta}_{2l-1}t})\, dt, \tag{4.15}$$

where $e^{-\tilde{\Delta}_{2l-1}t}$ is the heat kernel on $\tilde{Y}$. To see that this makes sense, we observe that standard arguments based on the elliptic estimate for $\tilde{\Delta}_{2l-1}$ show that $*de^{-\tilde{\Delta}_{2l-1}t}$ (initially defined by the spectral theorem) is given by a smooth kernel. In fact, since $\mathrm{geo}(\tilde{Y}) \leqslant 1$, the usual parametrix construction applies. If

$\tilde{P}(x, y, t)$ is a parametrix for $c^{-\tilde{\Delta}_{2l-1}t}$, $\square$ is the heat operator and $\tilde{Q} = \square_x \tilde{P}$, then

$$\tilde{E} = \tilde{P} - \tilde{P} * \tilde{Q} + d\tilde{P} * \tilde{Q} * \tilde{Q} \cdots \tag{4.16}$$

(see e.g. [12]). This is one way of seeing that we have the uniform pointwise estimate

$$\operatorname{tr}\left(*de^{-\Delta_{2l-1}t}\right) \sim \operatorname{tr}(*dP(t)) = O(t^{1/2}) \tag{4.17}$$

on $\tilde{Y}$.

Using the spectral theorem, a trivial modification of the argument of [6] shows that the integral in (4.15) converges. In the same way, it follows that the estimate (4.10) holds with a constant which, a priori, might depend on the higher covariant derivatives of the curvature tensor (as does Q in (4.16)). The proof in [6] that the constant can be estimated only in terms of the bound $\operatorname{geo}(\tilde{N}) \leqslant 1$, utilizes a regularization theorem ([6, Theorem 2.5]) for the metric $\tilde{g}_\varepsilon$ on $\tilde{N}_\varepsilon$. This is combined with an estimate (see Lemma 2.6 of [6]) which compares $\eta(\tilde{N}_\varepsilon, g_\varepsilon)$ with the η-invariant of the regularized metric (the covariant derivatives of whose curvature tensor are bounded). Since in [6], $\tilde{N}_\varepsilon$ is a *finite* covering, there is no need for the regularized metric to be the pullback of a metric on N (in order for its η-invariant to be defined). But in the present case, it is necessary to stay within the class of such metrics on the *infinite* covering space $\tilde{N}$, so that the expression tr_Γ in (4.15) makes sense. Equivalently, we must regularize the metric g_ε on N. Thus we cannot appeal to the hypothesis $\operatorname{geo}(\tilde{N}, \tilde{g}_\varepsilon) \leqslant 1$, which is required for Theorem 2.5 of [6]. However, according to a recent result of [3] (whose proof depends on analysis), the conclusion of Theorem 2.5 of [6] is actually valid *without assuming a lower bound on the injectivity radius*. So g_ε can be approximated by a metric, the covariant derivatives of whose curvature are bounded. The proof of (4.10) is then completed as in [6] (to which we refer for further details).

To finish the proof of Theorem 4.1 it suffices to establish (4.12). As in the case of the ordinary η-invariant this is a simple consequence of the formula for the variation of $\operatorname{tr}_\Gamma(*de^{-\tilde{\Delta}_{2l-1}t})$ under change of metric. We now indicate the derivation of this formula, following closely the arguments of [4].

Put $g_u = (1-u)g_0 + ug_1$. It is convenient to define $\tilde{\mathfrak{E}}(t)$: $\Lambda^{2l-1}(\tilde{N}) \to \Lambda^{2l-1}(\tilde{N})$ by

$$\tilde{\mathfrak{E}}(t)(\omega) = \int_{M^n} \tilde{E}(x, z, t) *_z \wedge \omega. \tag{4.18}$$

Then, in the (usual) sense of Γ-trace of linear operators, we wish to compute

$$\frac{d}{du}\operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t)) = \left(\operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t))\right)^{\cdot}. \tag{4.19}$$

Write (Duhamel's principle)

$$\begin{aligned}
&{*_0} d\tilde{\mathfrak{E}}_u(t-\varepsilon)\tilde{\mathfrak{E}}_0(\varepsilon) - {*_0} d\tilde{\mathfrak{E}}_u(\varepsilon)\tilde{\mathfrak{E}}_0(t-\varepsilon)\\
&\qquad = -\int_\varepsilon^{t-\varepsilon}\left[{*_0} d\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0(s)\right]' ds\\
&\qquad = \int_\varepsilon^{t-\varepsilon}\left[{*_0} d\tilde{\mathfrak{E}}_0'(t-s)\tilde{\mathfrak{E}}_0(s) - {*_0} d\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0'(s)\right] ds \qquad (4.20)\\
&\qquad = -\int_\varepsilon^{t-\varepsilon}\left[{*_0} d\tilde{\Delta}_u\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}(s) - {*_0} d\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0(s)\tilde{\Delta}_0\right] ds.
\end{aligned}$$

If we take the Γ-trace, the second term of the last line in (4.20) can be rewritten as

$$\begin{aligned}
&\int_\varepsilon^{t-\varepsilon}\operatorname{tr}_\Gamma\left\{\left[{*_0} d\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0(s/2)\right]\left[\tilde{\mathfrak{E}}_0(s/2)\tilde{\Delta}_0\right]\right\} ds\\
&\qquad = \int_\varepsilon^{t-\varepsilon}\operatorname{tr}_\Gamma\left\{\left[\tilde{\mathfrak{E}}_0(s/2)\tilde{\Delta}_0\right]\left[{*_0} d\tilde{\mathfrak{E}}_u(t-\varepsilon)\tilde{\mathfrak{E}}_0(s/2)\right]\right\} ds \qquad (4.21)\\
&\qquad = \int_\varepsilon^{t-\varepsilon}\operatorname{tr}_\Gamma\left({*_0} d\tilde{\Delta}_0\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0(s)\right) ds.
\end{aligned}$$

Here, note that range $\tilde{\mathfrak{E}}_u(t-s) \subset \operatorname{dom}\Delta_0$, as a consequence of the fact that the constants in the elliptic estimates for $\tilde{\Delta}_u$ are mutually bounded independent of u ($\tilde{g}_u$ is induced from g_u). Thus, (4.20) yields

$$\begin{aligned}
&\operatorname{tr}_\Gamma\left({*_0} d\tilde{\mathfrak{E}}_u(t-\varepsilon)\tilde{\mathfrak{E}}_0(\varepsilon)\right) - \operatorname{tr}_\Gamma\left({*_0} d\tilde{\mathfrak{E}}_u(\varepsilon)\tilde{\mathfrak{E}}_0(t-\varepsilon)\right)\\
&\qquad = \int_\varepsilon^{t-\varepsilon}\operatorname{tr}_\Gamma\left\{{*_0} d(\tilde{\Delta}_0-\tilde{\Delta}_u)\tilde{\mathfrak{E}}_u(t-s)\tilde{\mathfrak{E}}_0(s)\right\} ds. \qquad (4.22)
\end{aligned}$$

If we differentiate (4.22) with respect to u and set $u = 0$, the right-hand side becomes

$$\begin{aligned}
&-\int_\varepsilon^{t-\varepsilon}\operatorname{tr}_\Gamma\{* d\dot{\tilde{\Delta}}\tilde{\mathfrak{E}}(t)\} ds\\
&\qquad = -(t-2\varepsilon)\operatorname{tr}_\Gamma(* d\dot{\delta} d\tilde{\mathfrak{E}}(t)) \qquad (4.23)\\
&\qquad = (t-2\varepsilon)\operatorname{tr}_\Gamma\{*d\dot{*}d*d\tilde{\mathfrak{E}}(t) + *d*d\dot{*}d\tilde{\mathfrak{E}}^2(t)\}.
\end{aligned}$$

By permuting factors as in (4.21) this yields

$$-2(t-2\varepsilon)\operatorname{tr}_\Gamma(\dot{*}d\tilde{\Delta}\tilde{\mathfrak{E}}(t)). \qquad (4.24)$$

Taking the limit $\varepsilon \to 0$, we get

$$2t\frac{d}{dt}\operatorname{tr}_\Gamma(\dot{*}d\mathfrak{E}(t)). \qquad (4.25)$$

To make the corresponding evaluation for the left-hand side of (4.22), note that

$$\left(\operatorname{tr}_\Gamma(* d\tilde{\mathfrak{E}}(t))\right)^{\cdot} = \operatorname{tr}_\Gamma(\dot{*} d\tilde{\mathfrak{E}}(t)) + \operatorname{tr}_\Gamma(* d\dot{\tilde{\mathfrak{E}}}(t)). \qquad (4.26)$$

Also, for fixed ε,

$$(4.27)\qquad \begin{aligned}(\operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t)))^{\cdot} = \operatorname{tr}_\Gamma(\dot{*}d\tilde{\mathfrak{E}}(t)) + \operatorname{tr}_\Gamma(*d\dot{\tilde{\mathfrak{E}}}(t-\varepsilon)\tilde{\mathfrak{E}}(\varepsilon)) \\ + \operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t-\varepsilon)\dot{\tilde{\mathfrak{E}}}(\varepsilon)).\end{aligned}$$

Letting $\varepsilon \to 0$ and comparing (4.26) with (4.27) gives

$$(4.28)\qquad \lim_{\varepsilon\to 0} \operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t-\varepsilon)\dot{\mathfrak{E}}(\varepsilon)) = 0.$$

So the derivative at $u = 0$ of the left-hand side of (4.22) is

$$(4.29)\qquad \begin{aligned}\lim_{\varepsilon\to 0} \operatorname{tr}_\Gamma(*d\dot{\tilde{\mathfrak{E}}}(t-\varepsilon)\tilde{\mathfrak{E}}(\varepsilon)) - \operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t-\varepsilon)\dot{\tilde{\mathfrak{E}}}(\varepsilon)) \\ = \operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t))^{\cdot} - \operatorname{tr}_\Gamma(\dot{*}d\tilde{\mathfrak{E}}(t)),\end{aligned}$$

(the corresponding point in [4] was somewhat muddled although the correctness of the answer was unaffected). Combining (4.25) and (4.29) gives

$$(4.30)\qquad \begin{aligned}(\operatorname{tr}_\Gamma(*d\tilde{E}(t)))^{\cdot} &= \operatorname{tr}_\Gamma(*d\tilde{\mathfrak{E}}(t))^{\cdot} \\ &= \operatorname{tr}_\Gamma(\dot{*}d\tilde{\mathfrak{E}}(t)) + 2t\frac{d}{dt}\operatorname{tr}_\Gamma(\dot{*}d\tilde{\mathfrak{E}}(t)),\end{aligned}$$

which is the formula we are seeking. To obtain the analogous formula with $\tilde{\mathfrak{E}}(t)$ replaced by $\mathfrak{E}(t) = e^{-\Delta_{2l-1}t}$, replace tr_Γ by tr in (4.30).

To complete the proof of (4.12) we write

$$(4.31)\qquad \begin{aligned}&\frac{1}{\Gamma(1/2)}\int_0^\infty t^{-1/2}\operatorname{tr}_\Gamma\left(*de^{-\tilde{\Delta}_u t}\right)dt \\ &\quad = \lim_{A\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\int_\varepsilon^A t^{-1/2}\operatorname{tr}_\Gamma\left(*de^{-\tilde{\Delta}_u t}\right)dt,\end{aligned}$$

where the limits are *uniform with respect to u*. For the lower limit, this follows for example from the uniform convergence of (4.16). For the upper limit it is a direct consequence of the fact that the metrics $\tilde{g}_u$ are all uniformly quasi-isometric.

In view of (4.30) and (4.31)

$$(4.32)\qquad \begin{aligned}\frac{d}{du}\tilde{\eta}_{(2)} = \lim_{A\to\infty}\lim_{\varepsilon\to 0}\frac{1}{\Gamma(1/2)}\Bigg\{&\int_\varepsilon^A t^{-1/2}\operatorname{tr}_\Gamma\left(\dot{*}e^{-\tilde{\Delta}t}\right)dt \\ &+\int_\varepsilon^A 2t^{1/2}\frac{d}{dt}\operatorname{tr}_\Gamma\left(\dot{*}de^{-\tilde{\Delta}t}\right)dt\Bigg\},\end{aligned}$$

and so, integrating the second term by parts,

$$(4.33)\qquad \frac{d}{du}\tilde{\eta}_{(2)} = \lim_{A\to\infty} 2t^{1/2}\operatorname{tr}_\Gamma\left(\dot{*}de^{-\tilde{\Delta}t}\right) - \lim_{\varepsilon\to 0} 2t^{1/2}\operatorname{tr}_\Gamma\left(\dot{*}de^{-\tilde{\Delta}t}\right).$$

Let dE_λ denote the spectral measure for $\tilde{\Delta}$. Note that for any fixed $\lambda > 0$

$$(4.34)\qquad \begin{aligned}&\left|2A^{1/2}\operatorname{tr}_\Gamma\left(\dot{*}de^{-\tilde{\Delta}A}\right)\right| \\ &\quad\leqslant 2e^{-1}\dim_\Gamma(E_0\cap E_\lambda) + 2A^{1/2}e^{-\lambda(A-1)}\operatorname{tr}_\Gamma(e^{-\Delta}).\end{aligned}$$

Letting $A \to \infty$ in (4.34) and recalling that

$$\lim_{\lambda \to 0} \dim_\Gamma \left(E_0^\perp \cap E_\lambda \right) = 0, \tag{4.35}$$

it follows that the first term in (4.32) vanishes. In the second term, $* de^{-\tilde{\Delta} t}$ can be replaced by a parametrix. This gives

$$\frac{d}{du} \tilde{\eta}_{(2)} = \frac{d}{du} \eta, \tag{4.36}$$

which suffices to complete the proof of (4.12). Thus Theorem 4.1 follows as well.

Remark 4.1. It is tempting to try to generalize Theorem 4.1 to the invariants $\hat{P}(N^{4l-1})$ by considering η-invariants with coefficients in suitable (rational combinations of) bundles associated to the tangent bundle of N^{4l-1}. In this more general case, (4.10) is still valid but, a priori, (4.12) is not. The difficulty is well known. Namely, the kernel of Δ_u is now no longer independent of u and the possible spectral flow across 0 gives a correction to (4.12). In more concrete terms, the discussion of $\lim_{A \to \infty}$ in (4.32)–(4.35) no longer applies.

5. Conformal changes of metric

In this section we consider riemannian manifolds (M^n, g) for which

$$g = e^{-2\phi} g_0, \tag{5.1}$$

where (M, g_0) is complete and satisfies

$$|K_{g_0}| \leqslant 1, \tag{5.2}$$

$$\mathrm{Vol}_{g_0}(M) < \infty. \tag{5.3}$$

Since the Pontrjagin forms are conformal invariants, the geometric Pontrjagin numbers satisfy

$$P(M, g) = P(M, g_0). \tag{5.4}$$

In particular, we have immediately

Theorem 5.1. *If* (5-1)–(5.3) *hold and*

$$\mathrm{geo}(\tilde{M}, \tilde{g}_0) \leqslant 1, \tag{5.5}$$

then

$$\sigma(M, g) = \tilde{\sigma}_{(2)}(M, g_0), \tag{5.6}$$

where the right-hand side is a proper homotopy invariant.

The Chern-Gauss-Bonnet form, however, is not a conformal invariant. Thus, to obtain information on the geometric Euler characteristic, we put additional conditions on the function ϕ.

Let $H_{g_0}(\phi)$ denote the Hessian of ϕ with respect to g_0. By a standard calculation, for any 2-plane $\tau \in \Lambda^2(M_x)$,

$$|K_g(\tau) - e^{2\phi}K_{g_0}(\tau)| \leqslant e^{2\phi}\left\{\|d\phi\|_{g_0}^2 + \|H_{g_0}(\phi)\|_{g_0}\right\}. \tag{5.7}$$

So, if in addition to (5.1)–(5.3) we assume

$$\|d\phi\|_{g_0} \leqslant c, \qquad \|H_{g_0}(\phi)\|_{g_0} \leqslant c, \tag{5.8}$$

it follows that

$$\chi(M, g) = \int_M P_\chi(\Omega_g) < \infty. \tag{5.9}$$

Moreover, we have

Theorem 5.2. *Let (M, g_0) be complete with $|K_{g_0}| \leqslant 1$ and $\mathrm{Vol}_{g_0}(M) < \infty$. Then $g = e^{-2\phi}g_0$, $\|d\phi\|_{g_0} \leqslant c$, $\|H_{g_0}(\phi)\|_{g_0} \leqslant c$ imply*

$$\chi(M, g) = \chi(M, g_0). \tag{5.10}$$

If also $\mathrm{geo}(\tilde{M}, g_0) \leqslant 1$, *then*

$$\chi(M, g) = \chi_{(2)}(M, g_0) \tag{5.11}$$

is a homotopy invariant.

In order for Theorem 5.2 (as well as (5.4)) to be of interest there must be conditions on g which guarantee the existence of g_0, ϕ. These will be given in Theorem 5.5 below.

The proofs of Theorems 5.2 and 5.5 depend on the following lemma (the point of which is that no bound on the injectivity radius is required). If Y is a riemannian manifold, $\overline{Y}$ its completion and $x \in Y$, we put $\overline{x, \infty} = \mathrm{dist}(x, \overline{Y}\setminus Y)$.

Lemma 5.3. *Let Y be a riemannian manifold with $|K| \leqslant 1$. Let f be a function on Y such that for some $\varepsilon > 0$,*

$$|f(x) - f(y)| \leqslant c \tag{5.12}$$

if $\overline{x, y} < \varepsilon$ and $\overline{x, \infty} > \varepsilon$. Then there exists a function $f^{\#}$ such that

$$|f^{\#}(x) - f(x)| < c, \tag{5.13}$$

$$\|df\| \leqslant c \cdot c(n)\varepsilon^{-1}, \qquad \|H(f)\| \leqslant c \cdot c(n)\varepsilon^{-2}. \tag{5.14}$$

Note that (5.12) holds if f satisfies a uniform Lipschitz condition.

Proof. Let $\psi(r)$ be a nonnegative C^∞-function on $[0, 1/2]$ such that $\psi \equiv 1$ near $r = 0$ and $\psi \equiv 0$ near $r = 1/2$. Put $\psi_\varepsilon(r) = \psi(r/\varepsilon)$. Let $B_\varepsilon(O_x)$ denote the ball of radius ε about the origin in Y_x equipped with the pulled back metric, and $d\hat{y}$ the associated volume element. Set

$$(5.15)\quad f^\#(x) = \int_{B_{\varepsilon/2}(O_x)} f(\exp_x \hat{y})\psi_\varepsilon(\overline{\hat{y}, O_x})\, d\hat{y} \Big/ \int_{B_{\varepsilon/2}(O_x)} \psi_\varepsilon(\overline{\hat{y}, O_x})\, d\hat{y}.$$

Note that for $\overline{z, x} < \varepsilon/2$ and $\overline{\hat{z}, O_x} < \varepsilon/2$, we have

$$(5.16)\quad f^\#(y) = \int_{B_{\varepsilon/2}(\hat{z})} f(\exp_x \hat{y})\psi_\varepsilon(\overline{\hat{y}, \hat{z}})\, dy \Big/ \int_{B_{\varepsilon/2}(\hat{z})} \psi_\varepsilon(\overline{\hat{y}, \hat{z}})\, d\hat{y},$$

where $B_{\varepsilon/2}(\hat{z}) \subset Y_x$. Also (5.12) clearly implies

$$(5.17)\qquad |f(\exp_x \hat{y}) - f(\exp_x \hat{z})| \leqslant c$$

if $\overline{\hat{x}, \hat{z}} < \varepsilon$. Finally, since $|K| \leqslant 1$ and we may assume $\varepsilon < \pi/2$, a standard comparison argument bounds the Hessian of $\psi_\varepsilon(\overline{\hat{y}, \hat{z}})$ in terms of ε^{-2}. The claim now follows by differentiating under the integral as in the well-known Euclidean case.

Proof of Theorem 5.1. Fix $x \in M$, and apply Lemma 5.3 to the function $\psi_R(\overline{y, x})$ (where ψ_R is as in Lemma 5.3). Using (5.7), we have

$$(5.18)\qquad P(M, g) = \lim_{R\to\infty} P\left(M, e^{2\phi\psi_R^\#} g_0\right) = P(M, g_0).$$

Here, the second equality follows from the fact that $e^{2\phi\psi_R^\#}g_0$ and g_0 agree at infinity.

In order to give conditions under which g_0, ϕ can be produced from g, we introduce functions $\rho(x)$, $\beta(x)$ as follows.

If (M, g) is a riemannain manifold which is not complete and flat, set

$$(5.19)\qquad \rho(x) = \sup_{\tau\in\Lambda^2(M_x)} |K(\tau)|^{1/2}.$$

If (M, g) is complete and flat let $\rho(x)$ denote any nonnegative upper bound for the right-hand side of (5.19) such that $\rho \not\equiv 0$ and $\int_{M^n} \rho^n < \infty$.

Let $\tilde{M}$ be the universal covering space of M and $\tilde{x} \in \tilde{M}$. Define $\iota(\tilde{x})$ to be the supremum of r, such that $\exp_{\tilde{x}}$ *is defined on* $B_\gamma(0) \subset \tilde{M}_{\tilde{x}}$ and is a diffeomorphism. Put

$$(5.20)\qquad \beta(x) = \max(\rho(x), 1/\iota(x)).$$

Roughly speaking, we would like to set $g = \rho^{-2}g_0$, $g = \beta^{-2}g_0$, to produce conformally related metrics satisfying $|K_{g_0}| \leqslant 1$ (respectively geo$(\tilde{M}, g_0) \leqslant 1$) with the smallest possible volume (which might still be infinite). Then $\mathrm{Vol}_{g_0}(M) < \infty$ if and only if

$$(5.21)\qquad \int_{(M^n, g)} \rho^n < \infty, \qquad \int_{(M^n, g)} \beta^n < \infty,$$

respectively. However $\rho^{-2}g$, $\beta^2 g$ need not be complete and (5.8) does not hold for $\rho = e^{\phi}$, $\beta = e^{\phi}$, in general.

To remedy these defects, we first replace ρ, β by functions $\bar\rho$, $\bar\beta$ which do not vary too rapidly. Then $\bar\rho$, $\bar\beta$ are further regularized by a suitable local application of Lemma 5.3. If we denote the resulting functions by ρ^*, β^* and set $g_0 = (\rho^*)^2 g$, $g_0 = (\beta^*)^2 g$, then the rough argument outlined above is actually valid.

Let (M, g) be a riemannian manifold which might not be complete and let $k(x)$ be a nonnegative locally bounded function on M such that $k \not\equiv 0$. Put

$$\bar{k}(x) = \inf\left\{ \frac{1}{R} \,\middle|\, R < \overline{x, \infty}\,,\ \sup_{y \in B_R(x)} k(y) \leqslant \frac{1}{R} \right\}. \tag{5.22}$$

Lemma 5.4 (*Harnack property*). *If* $\delta \leqslant 1/2$,

$$\overline{x, y} \leqslant \delta(\bar{k}(x) + \bar{k}(y)), \tag{5.23}$$

$$\bar{k}(y) \leqslant \bar{k}(x), \tag{5.24}$$

then

$$\bar{k}(y) \leqslant \bar{k}(x) \leqslant \frac{1+\delta}{1-\delta}\bar{k}(y). \tag{5.25}$$

Proof. It follows immediately from (5.22) that if $\overline{x, y} \leqslant \bar{k}(x)$, then

$$\bar{k}(y) \geqslant \bar{k}(x) - \overline{x, y}\,. \tag{5.26}$$

This, together with (5.23) and (5.24) yields (5.25).

Theorem 5.5. *Let k denote ρ or β.*

(1) *There exists k^* such that*

$$\bar{k}/3 \leqslant k^*(x) \leqslant 3\bar{k}(x). \tag{5.27}$$

(2) *If $k^* = e^{\phi}$ and $g = e^{-2\phi} g_0$, then*

$$\|d\phi\|_{g_0} \leqslant c(n), \qquad \left\|H_{g_0}(\phi)\right\|_{g_0} \leqslant c(n). \tag{5.28}$$

(3) *(M, g_0) is complete. $|K_{g_0}| \leqslant 1$ if $k = \rho$ (respectively* $\operatorname{geo}(\tilde{M}, \tilde{g}_0) \leqslant 1$ *if $k = \beta$). Moreover,* $\operatorname{Vol}_{g_0}(M) < \infty$ *if and only if*

$$\int_{(M,g)} \bar{k}^n < \infty. \tag{5.29}$$

(4) *If (5.29) holds, then*

$$\chi(M, g) = \chi_{(2)}(M, g_0) \tag{5.30}$$

is a homotopy invariant.

Remark 5.1. It is simple to check the converse. Let $g = e^{-2\phi}g_0$, where g_0 is complete, $|K_{g_0}| \leqslant 1$ (respectively $\operatorname{geo}(\tilde{M}, \tilde{g}_0) \leqslant 1$, $\operatorname{Vol}_{g_0}(M) < \infty$, and (5.8) holds). Then (5.29) holds as well.

Proof of Theorem 5.5. Let $\{x_i\}$ be a maximal set of points such that for all i, j

$$\overline{x_i, x_j} \geqslant (\bar{k}(x_i) + \bar{k}(x_j))/7. \tag{5.31}$$

Then for all $z \in M$, there exists x_i such that

$$\overline{x_i, z} < (\bar{k}(x_i) + \bar{k}(z))/7. \tag{5.32}$$

It follows from (5.25) that

$$z \in B_{\bar{k}(x_i)/3}(x_i), \tag{5.33}$$

and hence that

$$M = \cup B_{\bar{k}(x_i)/3}(x_i). \tag{5.34}$$

The multiplicity of this covering can be bounded as follows. If $z \in B_{\bar{k}(x_i)/3}(x_i)$, then using (5.25) we have

$$B_{\bar{k}(z)/14}(x_i) \subset B_{\bar{k}(x_i)/7} \subset B_{2\bar{k}(z)/3}(z). \tag{5.35}$$

If instead of g we use the rescaled metric $\bar{k}^2(z)_g$ on $B_{2\bar{k}(z)/3}(z)$, then (5.35) becomes

$$B_{1/14}(x_i) \subset B_{\bar{k}(x_i)/7\bar{k}(z)}(x_i) \subset B_{2/3}(z). \tag{5.36}$$

By construction, the balls $B_{\bar{k}(x_i)/7}(x_i)$ in (5.35) are disjoint for different values of i. Hence, so are the balls $B_{1/14}(x_i)$ in (5.36). By (5.19), (5.20) and (5.22), on $B_{2/3}(z)$ we have

$$|K_{\bar{k}^2(z)g}| \leqslant 1. \tag{5.37}$$

So using the relative volume estimate as in [15, 2.2.A], we find that the multiplicity N of the covering in (5.34) satisfies

$$N < c(n). \tag{5.38}$$

Apply Lemma 5.3 to the function $\psi_{2/3}(\overline{x_i, y})$ on the ball $B_{2/3}(x_i)$, where the distance in each case is measured with respect to the rescaled metric $\bar{k}^2(x_i)g$. Then construct a partition of unity $\sum \alpha_i \equiv 1$ by putting

$$\alpha_i(y) = \frac{\psi^{\#}_{2/3}(\overline{x_i, y})}{\sum_j \psi^{\#}_{2/3}(\overline{x_i, y})}. \tag{5.39}$$

In view of the preceding discussion, it is clear that for all z,

$$\begin{aligned} &\|d\alpha_i(z)\|_{\bar{k}^2(z)g} \leqslant c(n), \\ &\|H_{\bar{k}^2(z)g}(\alpha_i(z))\|_{\bar{k}^2(z)g} \leqslant c(n). \end{aligned} \tag{5.40}$$

Set

$$k^*(z) = \sum_i \alpha_i(z)\bar{k}(x_i). \tag{5.41}$$

(1) It follows from (5.25) that if $\alpha_i(z) \neq 0$ and $\alpha_j(z) \neq 0$, then

$$\bar{k}(x_i) \leqslant 3\bar{k}(x_j), \qquad \bar{k}(x_j) \leqslant 3\bar{k}(x_i). \tag{5.42}$$

This gives (1).

(2) By (5.40) and (5.42), a straightforward calculation gives

$$\begin{aligned} &\|d\phi(z)\|_{\bar{k}^2(z)g} \leqslant c(n), \\ &\|H_{\bar{k}^2(z)g}(\phi)\|_{\bar{k}^2(z)g} \leqslant c(n), \end{aligned} \tag{5.43}$$

where $k^* = e^{\phi}$. But this (together with (5.27)), easily implies (5.28).

(3) Notice that each point $z \in M$ is contained in a ball of radius $1/3$ with respect to g_0, which does not meet infinity. This follows from the condition $R < \overline{x, \infty}$ in (5.22), together with (5.27). Thus M is complete. The remaining assertions are direct consequences of the definitions (5.19) and (5.20) together with (5.27).

(4) This is an immediate consequence of the fact that (5.29) implies $\mathrm{Vol}_{g_0}(M) < \infty$ and Theorem 1.1.

Example 5.1 ($|K| \leqslant c/r^2$, $\mathrm{Vol}(B_r(x)) \leqslant r^{n-\varepsilon}$). Let (M^n, g) be complete. Fix $x \in M$ and put $\overline{x, y} = r$. Suppose that for some constant c, the relation

$$\sup_{\tau \in \Lambda^2(M_y)} |K(\tau)| \leqslant \frac{c}{r^2} \tag{5.44}$$

holds outside a compact set. For example, if M^n is diffeomorphic to the interior of a compact manifold with boundary N^{n-1}, then such a metric can be chosen to be of the (exterior conical) form

$$g = dr^2 + r^2 h, \qquad r \geqslant 1, \tag{5.45}$$

where h is some metric on N^{m-1}. For g of this form,

$$\mathrm{Vol}(B_r(x)) \sim r^n. \tag{5.46}$$

Applying Theorem 5.5 yields the cylindrical metric

$$g_0 = ds^2 + h, \tag{5.47}$$

which, of course, has infinite volume.

However, if in addition to (5.44), we have

$$\int_{(M,g)} r^{-n} < \infty, \tag{5.48}$$

e.g., if

$$\mathrm{Vol}(B_r(x)) \leqslant r^{n-\varepsilon}, \tag{5.49}$$

then

$$\mathrm{Vol}_{g_0} M < \infty. \tag{5.50}$$

To obtain examples of the above type, let $\partial\overline{M}^n = N^{n-1}$ and let X be a nonvanishing Killing field on N^{n-1} for the metric h. Let q be the restriction of h to $X^\perp$. Put

$$h_{1/r} = r^{-2}\left(\frac{X^*}{\|X\|^2}\right)^2 + q. \tag{5.51}$$

Then as $r \to \infty$,

$$\mathrm{Vol}(N, h_{1/r}) = r^{-1}\mathrm{Vol}(N, h) \tag{5.52}$$

but

$$\left|K_{h_{1/r}}\right| \leqslant c, \tag{5.53}$$

(see [6], for details). If

$$g = dr^2 + r^2(h_r) = dr^2 + \left(\frac{X^*}{\|X\|^2}\right)^2 + r^2 g_0, \tag{5.54}$$

then Theorem 5.5 exhibits the conformal equivalence between g and the cusp-like metric

$$g_0 = ds^2 + h_{e^{-s}} \tag{5.55}$$

with $|K| \leqslant c$ and $\mathrm{Vol}_{g_0}(M) < \infty$, which was constructed in [6].

More generally, by starting with the metric in (5.55) and multiplying by those functions $f(s)$ for which $\log f$ is uniformly Lipschitz, we obtain examples of complete metrics and incomplete metrics for which (5.8) holds

Remark 5.2. We have only stated explicitly those applications of the present section which relate to earlier sections of this paper. However, there are analogous applications in connection with the results of [6].

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