CHEEGER CONSTANTS AND L^2-BETTI NUMBERS

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Abstract
We prove the existence of positive lower bounds on the Cheeger constants of manifolds of the form X/Γ *where* X *is a contractible Riemannian manifold and* $\Gamma < \mathrm{Isom}(X)$ *is a discrete subgroup, typically with infinite covolume. The existence depends on the* L^2*-Betti numbers of* Γ*, its subgroups, and a uniform lattice of* $\mathrm{Isom}(X)$*. As an application, we show the existence of a uniform positive lower bound on the Cheeger constant of any manifold of the form* $\mathbb{H}^4/\Gamma$ *where* $\mathbb{H}^4$ *is real hyperbolic* 4*-space and* $\Gamma < \mathrm{Isom}(\mathbb{H}^4)$ *is discrete and isomorphic to a subgroup of the fundamental group of a complete finite-volume hyperbolic* 3*-manifold. Via Patterson–Sullivan theory, this implies the existence of a uniform positive upper bound on the Hausdorff dimension of the conical limit set of such a* Γ *when* Γ *is geometrically finite. Another application shows the existence of a uniform positive lower bound on the zeroth eigenvalue of the Laplacian of* $\mathbb{H}^n/\Gamma$ *over all discrete free groups* $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ *whenever* $n \geq 4$ *is even. (The bound depends on* n*.) This extends results of Phillips–Sarnak and Doyle, who obtained such bounds for* $n \geq 3$ *when* Γ *is a finitely generated Schottky group.*

1. Introduction

The Cheeger constant of a smooth Riemannian manifold M is defined by

$$h(M) := \inf \frac{\mathrm{area}(\partial M_0)}{\mathrm{vol}(M_0)},$$

where the infimum is over all smooth compact submanifolds $M_0 \subset M$ with $\mathrm{vol}(M_0) \leq \mathrm{vol}(M)/2$. For most of the paper we will be applying the Cheeger constant to infinite-volume manifolds, in which case the infimum in the formula above is over all smooth compact submanifolds. In this case, we could call $h(M)$ the *Følner constant* instead of the Cheeger constant. For example, if M has infinite volume, then $h(M) = 0$ if and only if M is amenable. This paper is motivated by the following general problem.

Problem 1.1
Given a contractible smooth Riemannian manifold X and a family $\mathcal{F}$ of abstract

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groups let $I(X|\mathcal{F}) = \inf_\Gamma h(X/\Gamma)$ where the infimum is over all $\Gamma < \mathrm{Isom}(X)$ such that

- Γ acts freely and properly discontinuously on X;
- Γ is isomorphic to a group in $\mathcal{F}$.

Compute $I(X \mid \mathcal{F})$ for interesting special cases (e.g., when X is real hyperbolic n-space $\mathbb{H}^n$ and $\mathcal{F}$ is the class of free groups). We are especially interested in knowing whether $I(X \mid \mathcal{F}) = 0$.

For example, let Free denote the class of free groups. For every $\epsilon > 0$ there is a free group $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ such that the compact core of $\mathbb{H}^2/\Gamma$ is a pair of pants with geodesic boundary components each of length ϵ. The compact core has area 2π but $\mathbb{H}^2/\Gamma$ has infinite area. It follows that $h(\mathbb{H}^2/\Gamma) \leq 3\epsilon/(2\pi)$. Since ϵ is arbitrary, $I(\mathbb{H}^2 \mid \text{Free}) = 0$. Likewise, $\mathrm{Isom}(\mathbb{H}^3)$ admits a nonuniform lattice isomorphic to the fundamental group of a fiber bundle over the circle so that the fundamental group of the fiber surface is a rank 2 free subgroup Λ of $\mathrm{Isom}(\mathbb{H}^3)$ with $h(\mathbb{H}^3/\Lambda) = 0$. So $I(\mathbb{H}^3 \mid \text{Free}) = 0$. The exact value of $I(\mathbb{H}^n \mid \text{Free})$ is unknown for $n > 3$. It is not even known whether $I(\mathbb{H}^n \mid \text{Free})$ is monotone in n.

To explain our main result it is convenient to introduce the following definitions.

Definition 1

Given a Riemannian manifold X and $\Gamma < \mathrm{Isom}(X)$, we say that Γ is *geometric* if the action of Γ on X is free and properly discontinuous. This ensures that X/Γ is a manifold and the quotient map $X \to X/\Gamma$ is a cover.

Definition 2

Let us say that a residually finite countable group Γ has *asymptotically vanishing lower dth Betti number* if

$$\liminf_N \frac{b_d(N)}{[\Gamma : N]} = 0,$$

where the lim inf is with respect to the net of finite-index normal subgroups $N \lhd \Gamma$ ordered by reverse inclusion. Equivalently, this holds if, for every $\epsilon > 0$, for every finite-index normal subgroup $N \lhd \Gamma$ there exists a subgroup $N' < N$ such that N' is normal and has finite index in Γ and $b_d(N')/([\Gamma : N']) < \epsilon$. For example, if Γ has a finite classifying space, then Γ has asymptotically vanishing lower dth Betti number if and only if $b_d^{(2)}(\Gamma) = 0$ by Lück's approximation theorem [25, Theorem 0.1] (where $b_d^{(2)}(\Gamma)$ denotes the d-dimensional L^2-Betti number of Γ).

Our main result is the following.

THEOREM 1.2
Let X be a smooth contractible complete Riemannian manifold. Let $\mathscr{G}_d$ be the class of all residually finite countable groups Γ such that every finitely generated subgroup $\Gamma' < \Gamma$ has asymptotically vanishing lower dth Betti number. Suppose there is a residually finite geometric subgroup $\Lambda < \mathrm{Isom}(X)$ such that X/Λ is compact and $b_d^{(2)}(\Lambda) > 0$. Then $I(X \mid \mathscr{G}_d) > 0$.

This is derived from a more general result (Theorem 7.1) concerning metric-measure spaces (mm-spaces) in place of manifolds. In general, it appears to be a difficult problem to determine whether a given group is in $\mathscr{G}_d$. However, we show in Proposition 8.2 below that if Γ is the fundamental group of a complete finite-volume hyperbolic 3-manifold, then $\Gamma \in \mathscr{G}_d$ for all $d > 1$ (and the same holds for every subgroup of Γ). Using this we obtain the following.

COROLLARY 1.3
If $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is geometric and isomorphic with a subgroup of the fundamental group of a complete finite-volume hyperbolic 3-manifold and $n \geq 4$ is an even integer, then $h(\mathbb{H}^n/\Gamma) \geq I(\mathbb{H}^n \mid \mathscr{G}_{n/2}) > 0$. In particular, $I(\mathbb{H}^n \mid \mathrm{Free}) > 0$ for every even integer $n \geq 4$.

Observe that we do not require the group Γ to be finitely generated in the result above.

Instead of the Cheeger constant, one might be interested in the bottom of the spectrum of the Laplace operator of a smooth Riemannian manifold M, which we denote by $\lambda_0(M)$. By [9],

$$h(M)^2/4 \leq \lambda_0(M). \tag{1}$$

More precisely, Cheeger proved (1) when M is compact, but it is well known that it generalizes to the noncompact case.

In order to compare Corollary 1.3 with previous results, recall that a *classical Schottky group* is a subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by elements $g_1, \dots, g_m$ such that there exist pairwise disjoint conformally round balls $B_1, B_2, \dots, B_m$ and $B_1', B_2', \dots, B_m'$ in the sphere at infinity $S^{n-1} = \partial\mathbb{H}^n$ such that $g_i(S^{n-1}/B_i') = \mathrm{int}(B_i)$ for every i. Phillips and Sarnak [30, Theorem 5.4] showed that for every $n \geq 4$ there is a constant $f_n > 0$ such that if Γ is any classical Schottky subgroup of $\mathrm{Isom}(\mathbb{H}^n)$, then $\lambda_0(\mathbb{H}^n/\Gamma) \geq f_n$ where λ_0 denotes the bottom of the spectrum of the Laplace operator. This result was extended by Doyle [12] to $n = 3$. No such bound exists for $n = 2$.

Classical Schottky groups are free groups so it makes sense to ask whether these results hold for free groups more generally. Corollary 1.3 and (1) show that indeed $\lambda_0(\mathbb{H}^n/\Gamma) \geq I(\mathbb{H}^n \mid \mathcal{G}_{n/2})^2/4 > 0$ whenever $n \geq 4$ is even and Γ is a free group.

Instead of the Cheeger constant or λ_0, one might be interested in the Hausdorff dimension of the limit set. The limit set $L\Gamma$ of a subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is the intersection of the sphere at infinity $S^{n-1} = \partial\mathbb{H}^n$ with the closure of Γx for any $x \in \mathbb{H}^n$. Let $\mathrm{HD}(L\Gamma)$ denote the Hausdorff dimension of $L\Gamma$. In [33, Theorem 2.21], Sullivan shows that if $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ is geometrically finite without cusps and $\mathrm{HD}(L\Gamma) \geq (n-1)/2$, then

$$\lambda_0(\mathbb{H}^n/\Gamma) = \big(n - 1 - \mathrm{HD}(L\Gamma)\big)\,\mathrm{HD}(L\Gamma). \tag{2}$$

(Our definition of λ_0 differs from the definition in [33] by a sign.) If Γ is merely geometrically finite, then this result holds with the limit set replaced by the conical limit set by [5, Corollary 2.6] and [33, Theorem 2.17]. These results have been generalized to other rank 1 symmetric spaces in [11]. From Corollary 1.3 and (1) we now obtain the following.

COROLLARY 1.4
For every integer $n \geq 2$, there exists a number $d_{2n} < 2n-1$ such that if $\Lambda < \mathrm{Isom}(\mathbb{H}^{2n})$ is a geometrically finite discrete group isomorphic to a subgroup of the fundamental group of a finite-volume complete hyperbolic 3-manifold, then the Hausdorff dimension of the conical limit set of Λ is at most d_{2n}.

In the opposite direction, it is shown in [20] and [21] that any Kleinian group whose limit set has sufficiently small Hausdorff dimension is a classical Schottky group.

The corollary above partially solves [23, Problem 10.27, p. 530]. Let us mention in passing that the survey [23] is a rich source for examples and open problems about Kleinian groups in higher dimensions.

It is a long-standing open problem to determine whether there exists a closed real hyperbolic 4-manifold M that fibers over a surface with fiber a surface. Recently U. Hamenstädt [18] has proven that no such manifold exists if both base and fiber are closed. However, the other cases remain open. If there is such a manifold, then the universal cover $\widetilde{M}$ is naturally identifiable with hyperbolic space $\mathbb{H}^4$ and therefore the fundamental group $\pi_1(M)$ can be represented as a lattice in $\mathrm{Isom}(\mathbb{H}^4)$. Moreover, the fundamental group of a fiber surface can be represented as a discrete group $\Lambda < \mathrm{Isom}(\mathbb{H}^4)$. This group is isomorphic to the fundamental group of a surface. So Corollary 1.3 implies that $h(\mathbb{H}^4/\Lambda) > 0$. Because Λ is a normal subgroup of a lattice, its limit set is the entire 3-sphere boundary of $\mathbb{H}^4$. However, it is not geometrically finite. It might seem reasonable, by analogy with the 3-dimensional case, to suspect that by deforming Λ slightly (or by passing to a subgroup), it should be possible to find, for every $\epsilon > 0$, a geometrically finite discrete group $\Lambda' < SO(4,1)$ such that the Hausdorff dimension of the conical limit set of Λ' is at least $3 - \epsilon$ and Λ' is isomorphic to the fundamental group of a compact surface. Corollary 1.4 implies this intuition is incorrect.

Question 1

Let Surface denote the class of fundamental groups of closed surfaces of genus $g \geq 2$. Is $I(\mathbb{H}^4 \mid \text{Surface})$ realized? In other words, does there exist a geometric surface group $\Gamma < \text{Isom}(\mathbb{H}^4)$ such that $h(\mathbb{H}^4/\Gamma) = I(\mathbb{H}^4 \mid \text{Surface})$? Suppose that there exists a closed hyperbolic 4-manifold which fibers over a surface and Γ is the fundamental group of the fiber surface. Then is it true that $h(\mathbb{H}^4/\Gamma) = I(\mathbb{H}^4 \mid \text{Surface})$? This question admits natural variations by replacing the Cheeger constant with λ_0 or $\text{HD}(L\Gamma)$ for example.

Question 2

Is $I(\mathbb{H}^n \mid \text{Free}) > 0$ when n is odd? Does the limit $\lim_{n\to\infty} I(\mathbb{H}^n \mid \text{Free})$ exist? If so, is it positive?

Remark 1

Corollaries 1.3 and 1.4 can be generalized to complex-hyperbolic space (by using [26, Theorem 5.12] and Proposition 8.2 below). In fact, complex-hyperbolic manifolds are always even-dimensional and it is known that the L^2-Betti number of a lattice acting complex-hyperbolic space does not vanish in the middle dimension. Therefore, we obtain $I(\mathbb{CH}^n \mid \mathcal{G}_n) > 0$ for all $n \geq 2$.

Remark 2

There are stronger results for quaternionic hyperbolic space and the octonionic hyperbolic plane because the isometry groups of these spaces have property (T) (see [10], [11]). In fact it is known that if Γ is a geometrically finite subgroup of the isometry group of one of these spaces but Γ is not a lattice, then there is a nontrivial lower bound on the codimension of the Hausdorff dimension of the limit set of Γ which does not depend on Γ. The analogous statement for real or complex hyperbolic n-space is false (see [23]) essentially because there exist lattices which surject onto infinite amenable groups.

Remark 3

It is an open question whether Theorem 1.2 holds without the residual finiteness assumptions (either on Λ or $\mathcal{G}_d$). However, in many interesting cases $\mathrm{Isom}(X)$ is linear, and therefore, by Malćev's theorem, every finitely generated subgroup of $\mathrm{Isom}(X)$ is residually finite.

Remark 4

It might be possible to obtain an explicit bound on $I(\mathbb{H}^{2n} \mid \mathcal{G}_n)$ from the proof of Theorem 1.2 and the results of [13], which show that Betti numbers are testable.

1.1. Outline

We begin by explaining the Benjamini–Schramm convergence of simplicial complexes in Section 2. The highlight of this section is G. Elek's result: if $\{K_i\}_{i=1}^{\infty}$ is a Benjamini–Schramm-convergent (BS-convergent) sequence of finite connected simplicial complexes, then the normalized Betti numbers of $\{K_i\}_{i=1}^{\infty}$ converge as $i \to \infty$. This result is the key to the whole proof. In Section 3 we review mm-spaces, deferring the proofs to the appendices. We generalize G. Elek's result in Section 4 to sequences of mm-spaces following an outline provided by G. Elek [14] in the closed Riemannian manifold case. Section 5 reviews L^2-Betti numbers and Section 6 provides some tools for proving Benjamini–Schramm convergence.

The proof of Theorem 1.2 is in Section 7. Here is a brief and rough outline. It suffices to prove the contrapositive: if $\{\Gamma_i\}_{i=1}^{\infty}$ is a sequence of residually finite geometric subgroups of $\mathrm{Isom}(X)$ and $\lim_{i\to\infty} h(X/\Gamma_i) \to 0$, then for all but finitely many i there exist subgroups $\Gamma_i' < \Gamma_i$ such that

$$\widehat{b}_d^{(2)}(\Gamma_i') := \liminf_N \frac{b_d(N)}{[\Gamma' : N]} > 0,$$

where the limit is over the net of finite-index normal subgroups of Γ ordered by reverse inclusion.

We are assuming the existence of a residually finite uniform lattice $\Lambda < \mathrm{Isom}(X)$ with $b_d^{(2)}(\Lambda) > 0$. We use a lemma due to Buser (see Lemma 7.3 below) to find compact smooth submanifolds $M_i \subset X/\Gamma_i$ such that for every $r > 0$ the ratio $\mathrm{vol}(N_r(\partial M_i))/(\mathrm{vol}(M_i))$ tends to zero as $i \to \infty$ where $N_r(\partial M_i)$ denotes the radius r neighborhood of the boundary of M_i. After passing to a subgroup of Γ_i if necessary, we can also require that M_i has no short homotopically nontrivial loops (meaning that every homotopically nontrivial loop in M_i has length at least r_i where $\lim_{i\to\infty} r_i = +\infty$). From these results we conclude that $\{M_i\}_{i=1}^{\infty}$ Benjamini–Schramm converges to X. So our generalization of Elek's result implies that $\lim_{i\to\infty} b_d(M_i)/(\mathrm{vol}(M_i)) = b_d^{(2)}(\Lambda)/(\mathrm{vol}(X/\Lambda))$ where $b_d(M_i)$ denotes the ordinary dth Betti number of M_i.

The Mayer–Vietoris sequence is employed to show (roughly speaking) that the normalized Betti numbers $b_d(M_i)/(\mathrm{vol}(M_i))$ are asymptotically bounded by the normalized Betti numbers of Γ_i. Lück approximation and residual finiteness allow us to replace ordinary Betti numbers with L^2-Betti numbers and to compare these limits with the L^2-Betti numbers of the lattice Λ, proving Theorem 1.2. In Section 8, we use treeability, almost treeability, and well-known results about L^2-Betti numbers of hyperbolic lattices to obtain Corollaries 1.4 and 1.3 from Theorem 1.2.

2. Benjamini–Schramm convergence of simplicial complexes

A *rooted simplicial complex* is a pair (K, v) where K is a simplicial complex and v is a vertex of K. We say (K_1, v_1) and (K_2, v_2) are *root-isomorphic* if there is an isomorphism from K_1 to K_2 which takes v_1 to v_2. We let $[K, v]$ denote the root-isomorphism class of (K, v).

Let RSC denote the set of all root-isomorphism classes of connected rooted locally finite simplicial complexes.

We define a topology on RSC as follows. Given a finite rooted simplicial complex (L, w) and an integer $r > 0$, let $U_r(L, w) \subset \mathrm{RSC}$ be the set of all $[K, v] \in \mathrm{RSC}$ such that the closed ball of radius r centered at v in K is root-isomorphic to (L, w). Here we are employing a standard convention: the closed ball of radius r is the subcomplex consisting of all simplices σ in K with the property that every vertex v' of σ is of distance at most r from v with respect to the path metric on the 1-skeleton of K.

We give RSC a topology by declaring each $U_r(L, w)$ to be a closed set. For $\Delta > 0$ let $\mathrm{RSC}(\Delta) \subset \mathrm{RSC}$ denote the set of all root-isomorphism classes of connected rooted simplicial complexes $[K, v]$ so that every vertex of K has degree at most Δ. With the subspace topology, $\mathrm{RSC}(\Delta)$ is compact and metrizable. Moreover, each $U_r(L, w) \cap \mathrm{RSC}(\Delta)$ is a clopen subset of $\mathrm{RSC}(\Delta)$.

Definition 3

In general, if X is a topological space, then we let $\mathcal{M}(X)$ denote the space of all Borel measures on X with the weak* topology. Therefore a sequence $\{\lambda_i\}_{i=1}^{\infty} \subset \mathcal{M}(X)$ converges to an element $\lambda_\infty \in \mathcal{M}(X)$ if and only if, for every compactly supported continuous function $f \in C(X)$, $\int f d\lambda_i$ converges to $\int f d\lambda_\infty$ as $i \to \infty$. Also let $\mathcal{M}_1(X)$ denote the subspace of Borel probability measures on X.

Given a finite connected simplicial complex K, let $\mu_K \in \mathcal{M}_1(\mathrm{RSC})$ denote

$$\mu_K = \frac{1}{|V(K)|} \sum_{v \in V(K)} \delta_{[K,v]},$$

where $V(K)$ denotes the set of vertices of K and $\delta_{[K,v]}$ denotes the Dirac probability measure concentrated on $[K, v] \in \mathrm{RSC}$.

A sequence of finite connected simplicial complexes $\{K_i\}_{i=1}^{\infty}$ is *BS-convergent* if the sequence $\{\mu_{K_i}\}_{i=1}^{\infty}$ converges in the weak* topology on $\mathcal{M}_1(\mathrm{RSC})$. In the special case in which there is a uniform degree bound Δ on the K_i's, this means that, for every finite $(L, w) \in \mathrm{RSC}$ and every $r > 0$, $\lim_{i\to\infty} \mu_{K_i}(U_r(L, w))$ exists. The graph-theoretic version of this notion was introduced in [4]. The next lemma is crucial to our entire approach.

LEMMA 2.1
Let $\Delta > 0$, and let $\{K_i\}_{i=1}^{\infty}$ be a sequence of finite connected simplicial complexes such that every vertex of every K_i has degree at most Δ. If $\{K_i\}_{i=1}^{\infty}$ is BS-convergent, then $\lim_{i\to\infty} b_d(K_i)/|V(K_i)|$ exists for any $d \geq 0$ where $b_d(K_i)$ denotes the ordinary dth Betti number of K_i.

Proof
This is [13, Lemma 6.1]. □

The next lemma is a generalization of the above to convex sums of finite connected simplicial complexes.

LEMMA 2.2
Let $\{\eta_i\}_{i=1}^{\infty} \in \mathcal{M}_1(\mathrm{RSC}(\Delta))$ be a convergent sequence in the weak topology. In addition, assume that for each i there exist finite connected simplicial complexes $K_{i,1}, \ldots, K_{i,m_i}$ and positive real numbers $t_{i,1}, \ldots, t_{i,m_i}$ such that*

$$\eta_i = \sum_{j=1}^{m_i} t_{i,j} \mu_{K_{i,j}}.$$

Suppose as well that there exist natural numbers N_i such that $|V(K_{i,j})| \geq N_i$ for all i, j and $\lim_{i\to\infty} N_i = +\infty$. Then for any $d \geq 1$,

$$\lim_{i\to\infty} \frac{\sum_{j=1}^{m_i} t_{i,j} b_d(K_{i,j})}{\sum_{j=1}^{m_i} t_{i,j} |V(K_{i,j})|}$$

exists.

Proof
By approximating the coefficients $t_{i,j}$ by rational numbers, we see that it suffices to prove the special case in which each $t_{i,j}$ is a rational number, which we now assume. Let $D_i > 0$ be a natural number such that $D_i t_{i,j} \in \mathbb{N}$ for all i, j.

Let $K_{i,j}^{(1)}, \ldots, K_{i,j}^{(D_i t_{i,j})}$ be disjoint complexes, each of which is isomorphic to $K_{i,j}$. Let $v_{i,j}^k$ be a vertex of $K_{i,j}^{(k)}$. Let L_i be the disjoint union of $K_{i,j}^{(k)}$ over all $1 \leq k \leq D_i t_{i,j}$ and $1 \leq j \leq m_i$. Let L_i' be the smallest complex containing L_i such that there is an edge in L_i' from $v_{i,j}^k$ to $v_{i,j}^{k+1}$ for all $1 \le k < D_i t_{i,j}$, $1 \le j \le m_i$, and an edge from $v_{i,j}^{D_i t_{i,j}}$ to $v_{i,j+1}^1$ for all $1 \le j < m_i$. Then L_i' is a connected complex with vertex degree bound $\Delta + 2$. Moreover,

$$b_d(L_i') = \sum_{j=1}^{m_i} D_i t_{i,j} b_d(K_{i,j}), \qquad \left|V(L_i')\right| = \sum_{j=1}^{m_i} D_i t_{i,j} \left|V(K_{i,j})\right|,$$

which implies that

$$b_d(L_i')/\left|V(L_i')\right| = \frac{\sum_{j=1}^{m_i} t_{i,j} b_d(K_{i,j})}{\sum_{j=1}^{m_i} t_{i,j} |V(K_{i,j})|}.$$

So it suffices to show that $\lim_{i\to\infty} b_d(L_i')/|V(L_i')|$ exists. By Lemma 2.1, it suffices to show that $\{L_i'\}_{i=1}^\infty$ is BS-convergent.

Let (A, a) be a finite rooted simplicial complex, let $r \in \mathbb{N}$, and as above, let $U_r(A, a) \subset \mathrm{RSC}$ be the set of all $[K, v] \in \mathrm{RSC}$ such that the closed ball of radius r centered at v in K is root-isomorphic to (A, a). It suffices to show that $\mu_{L_i'}(U_r(A, a))$ converges as $i \to \infty$. However, we observe that $|\mu_{L_i'}(U_r(A, a)) - \eta_i(U_r(A, a))| \le 2|X_i|/|V(L_i')|$ where $X_i \subset V(L_i')$ is the set of vertices at distance less than or equal to r from the set $\{v_{i,j}^k\}_{j,k} \subset V(L_i')$. Since $\{\eta_i\}_{i=1}^\infty$ is convergent by hypothesis, it suffices to show that $\lim_{i\to\infty} |X_i|/|V(L_i')| = 0$.

Because the vertex degrees of L_i' are bounded by $\Delta + 2$, it follows that

$$|X_i| \le (\Delta + 2)^r \left|\{v_{i,j}^k\}_{j,k}\right| \le (\Delta + 2)^r \sum_{j=1}^{m_i} D_i t_{i,j}.$$

On the other hand,

$$\left|V(L_i')\right| = \sum_{j=1}^{m_i} D_i t_{i,j} \left|V(K_{i,j})\right| \ge N_i \sum_{j=1}^{m_i} D_i t_{i,j}.$$

So

$$\frac{|X_i|}{|V(L_i')|} \le (\Delta + 2)^r / N_i,$$

which implies that $\lim_{i\to\infty} |X_i|/|V(L_i')| = 0$ as required. □

3. mm-spaces

In Section 4 we generalize Elek's theorem (Lemma 2.1 above) by replacing the space of rooted simplicial complexes with the space of pointed mm-spaces. In this section, we present the basic definitions and results we will need. The standard reference for this subject is [17]. Our definition of mm^n-spaces, given below, and the topology on $\mathbb{M}^n$ appear to be nonstandard. (At least, we did not find them in the literature.) We should also mention that the Benjamini–Schramm convergence of random-length spaces first appeared in [1]. Our notion is similar, although not exactly the same.

Definition 4
An *mm-space* is a triple $(M, \mathrm{dist}_M, \mathrm{vol}_M)$ where (M, dist_M) is a complete separable proper metric space and vol_M is a Radon measure on M. We will usually denote such a space by M leaving dist_M and vol_M implicit. A *pointed mm-space* is a quadruple $(M, p, \mathrm{dist}_M, \mathrm{vol}_M)$ where $(M, \mathrm{dist}_M, \mathrm{vol}_M)$ is an mm-space and $p \in M$. More generally, a *pointed mm^n-space* is an $(n+3)$-tuple $(M, p, \mathrm{dist}_M, \mathrm{vol}_M^{(1)}, \dots, \mathrm{vol}_M^{(n)})$ where (M, dist_M) is a complete separable proper metric space, $p \in M$, and $\mathrm{vol}_M^{(i)}$ is a Radon measure on M for every i. We will often denote a pointed mm^n-space by (M, p) leaving the rest of the data implicit. Two pointed mm^n-spaces $(M, p), (M', p')$ are *isomorphic* if there is an isometry from M to M' that takes p to p' and $\mathrm{vol}_M^{(i)}$ to $\mathrm{vol}_{M'}^{(i)}$ for $i = 1, \dots, n$. We let $[M, p]$ denote the isomorphism class of (M, p). Let $\mathbb{M}^n$ denote the set of all isomorphism classes of pointed mm^n-spaces. Let $\mathbb{M} = \mathbb{M}^1$.

Definition 5 (A topology on $\mathbb{M}^n$)
We define a topology on $\mathbb{M}^n$ by declaring that a sequence $\{[M_i, p_i]\}_{i=1}^\infty$ converges to $[M_\infty, p_\infty]$ in $\mathbb{M}^n$ if and only if there exist a complete proper separable metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that

- $$\lim_{i\to\infty} \big(\varphi_i(M_i), \varphi_i(p_i)\big) = \big(\varphi_\infty(M_\infty), \varphi_\infty(p_\infty)\big)$$

  in the pointed Hausdorff topology (see Definition 28 in Appendix A for the definition of this topology);
- $\lim_{i\to\infty} (\varphi_i)_* \mathrm{vol}_{M_i}^{(k)} = (\varphi_\infty)_* \mathrm{vol}_{M_\infty}^{(k)}$ (in the weak* topology on $\mathcal{M}(Z)$) for all k.

THEOREM 3.1
With the topology above, $\mathbb{M}^n$ is separable and metrizable.

The proof of this theorem is in Appendix B.

Definition 6
Every nonnull finite-volume mm-space M is associated with a measure $\mu_M \in \mathcal{M}_1(\mathbb{M})$ obtained by pushing forward the probability measure $\mathrm{vol}_M/(\mathrm{vol}_M(M))$ under the map from M to $\mathbb{M}$ given by $p \mapsto [M, p]$. A sequence $\{M_i\}_{i=1}^\infty$ of nonnull finite-

volume mm-spaces *Benjamini–Schramm converges* if $\{\mu_{M_i}\}_{i=1}^{\infty}$ converges in the weak* topology on $\mathcal{M}_1(\mathbb{M})$.

4. A variant of Elek's theorem

The purpose of this section is to prove a version of Lemma 2.1 for mm-spaces. We first need some definitions to state the result properly.

Definition 7 (Special mm-spaces)

Let M be an mm-space. We say M is *special* if

- vol_M is nonatomic (i.e., $\mathrm{vol}_M(\{x\}) = 0$ for every $x \in M$);
- vol_M is fully supported (i.e., $\mathrm{vol}_M(O) > 0$ for every nonempty open set $O \subset M$);
- spheres have measure zero (i.e., for all $p \in M$, $\epsilon > 0$, $\mathrm{vol}_M(\{q \in M : \mathrm{dist}_M(p,q) = \epsilon\}) = 0$);
- M is pathwise connected.

Let $\mathbb{M}_{sp} \subset \mathbb{M}$ denote the subspace of isomorphism classes of pointed special mm-spaces.

Definition 8

Let M be a metric space. We say that m is a *midpoint* of x, y (for $m, x, y \in M$) if $\mathrm{dist}_M(x,m) = \mathrm{dist}_M(m,y) = (1/2)\,\mathrm{dist}_M(x,y)$. We say a subset $X \subset M$ is *strongly convex* if every pair $x, y \in X$ has a unique midpoint $m \in X$.

Definition 9

Let M be a metric space, and let $\epsilon > 0$. A set $S \subset M$ is ϵ*-separated* if $\mathrm{dist}_M(s,s') > \epsilon$ for every $s, s' \in S$ with $s \neq s'$. If $Q \subset M$, then S ϵ*-covers* Q if for every $q \in Q$ there is an $s \in S$ such that $\mathrm{dist}_M(q,s) < \epsilon$.

Definition 10

Given a metric space M, $p \in M$, and $R > 0$, let $B_M(p,R)$ denote the closed ball of radius R centered at p. Let $B^o_M(p,R)$ denote the open ball of radius R centered at p.

The main result of this section is the following.

THEOREM 4.1

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of finite-volume special mm-spaces. Suppose that $\lim_{i\to\infty} \mu_{M_i} = \mu_\infty \in \mathcal{M}_1(\mathbb{M}_{sp})$ exists. We assume there are constants ϵ, v_0, v_1 such that for every $p \in M_i$ (and every $i = 1, 2, \ldots$)

- $v_1 > \mathrm{vol}_{M_i}(B^o_{M_i}(p, 20\epsilon)) \geq \mathrm{vol}_{M_i}(B^o_{M_i}(p, \epsilon/2)) > v_0 > 0,$

- *$B^o_{M_i}(p,r)$ is strongly convex for every $r \le 10\epsilon$.*

Then $\lim_{i\to\infty} b_d(M_i)/(\mathrm{vol}(M_i))$ exists for every $d \ge 1$ where $b_d(M_i)$ denotes the dth ordinary Betti number of M_i.

The main ideas for the proof of Theorem 4.1 are due to G. Elek [14].

4.1. A brief outline

First we show how to construct for every rooted special mm-space (M,p) a random discrete subset $S \subset M$ which is ϵ-separated and 3ϵ-covering. The main difficulty is showing that this construction can be made to depend continuously on $[M,p]$. Second we let $\rho^S : S \to [5\epsilon, 6\epsilon]$ be a random map and we consider the nerve complex K of the open covering $B^o_M(s,\rho^S(s))$. To be precise, the vertex set of K is S and a subset $S' \subset S$ spans a simplex in K if $\bigcap_{s\in S'} B^o_M(s,\rho^S(s)) \neq \emptyset$. Considering the case $M = M_i$ with M_i as in Theorem 4.1, we see that its random complex K_i has degree bound Δ. Moreover, we show that $\{K_i\}_{i=1}^\infty$ is BS-convergent and K_i is homotopic to M_i. (This uses a variant of Borsuk's nerve theorem.) So we can use Lemma 2.1 to finish the argument.

4.2. Pointed mm-spaces with a weighted discrete set

We will use the following definitions as technical tools for proving Theorem 4.1.

Definition 11

A *pointed mm-space with a weighted discrete set* is a quadruple (M,p,S,f) where (M,p) is a pointed mm-space, $S \subset M$ is a locally finite set, and $f : S \to [0,1]$ is a function. By *locally finite* we mean that $B_M(p,R) \cap S$ is finite for every $R > 0$. Two such spaces $(M,p,S,f), (M',p',S',f')$ are *isomorphic* if there is an isomorphism from (M,p) to (M',p') which takes S to S' and f to f'. Let $\mathbb{MSF}$ denote the set of all isomorphism classes of pointed mm-spaces with a weighted discrete set. We let $[M,p,S,f] \in \mathbb{MSF}$ denote the isomorphism class of (M,p,S,f).

Definition 12 (A topology on $\mathbb{MSF}$)

Given $[M,p,S,f] \in \mathbb{MSF}$, define $\mathrm{vol}^{(2)}_M$ on M to be the counting measure on S, and define $\mathrm{vol}^{(3)}_M$ on M to be the atomic measure corresponding to f. So

$$\mathrm{vol}^{(2)}_M(E) = |E \cap S|, \qquad \mathrm{vol}^{(3)}_M(E) = \sum_{s\in E\cap S} f(s)$$

for any $E \subset M$. This defines an embedding of $\mathbb{MSF}$ into $\mathbb{M}^3$. We give $\mathbb{MSF}$ the induced topology.

Definition 13 (Pointed mm-spaces with a discrete set)
A *pointed mm-space with a discrete set* is a triple (M, p, S) where (M, p) is a pointed mm-space and $S \subset M$ is locally finite. Two such spaces $(M, p, S), (M', p', S')$ are *isomorphic* if there is an isomorphism from (M, p) to (M', p') (as elements of $\mathbb{M}$) which maps S bijectively to S'. Let $\mathbb{MS}$ denote the set of all pointed mm-spaces with a discrete set up to isomorphism. We let $[M, p, S] \in \mathbb{MS}$ be the isomorphism class of (M, p, S). There is an obvious projection map $\mathbb{MSF} \to \mathbb{MS}$. We endow $\mathbb{MS}$ with the quotient topology. Alternatively, $\mathbb{MS}$ can be embedded into $\mathbb{M}^2$ by $[M, p, S] \mapsto [M, p, \mathrm{dist}_M, \mathrm{vol}_M, \mathrm{vol}_M^{(2)}]$ where $\mathrm{vol}_M^{(2)}$ is the measure $\mathrm{vol}_M^{(2)}(E) = |E \cap S|$.

4.3. Random discrete subsets of mm-spaces
The first step in the proof of Theorem 4.1 is to associate to an mm-space a random discrete subset in a natural way. First we need a few more definitions.

Notation 1
Given a random variable X, let $\mathrm{Law}(X)$ denote the law of X. So $\mathrm{Law}(X)$ is a probability measure on the space of all values of X.

Definition 14
If (Y, λ) is a purely nonatomic finite measure space and $k \geq 1$ is an integer, then (Y^k, λ^k) denotes the direct product of k-copies of (Y, λ) and $(\binom{Y}{k}, \binom{\lambda}{k})$ denotes the projection of (Y^k, λ^k) onto the space of all *unordered* subsets of Y of cardinality k. Because λ is purely nonatomic, this is well defined: the large diagonal in Y^k has measure zero with respect to λ^k. A *uniformly random subset* $S \subset Y$ *of cardinality* k is a random subset with law equal to $\binom{\lambda}{k}/|\binom{\lambda}{k}|$ where $|\binom{\lambda}{k}|$ denotes the total mass of $\binom{\lambda}{k}$.

LEMMA 4.2
Let $\epsilon > 0$. *There exists a continuous map* $\mathcal{F} : \mathbb{M}_{sp} \to \mathcal{M}_1(\mathbb{MS})$ *such that, for any* $[M, p] \in \mathbb{M}_{sp}$, *if* $[M', p', S'] \in \mathbb{MS}$ *is random with* $\mathrm{Law}([M', p', S']) = \mathcal{F}([M, p])$, *then* $[M', p'] = [M, p]$ *and* S' *is* ϵ-*separated and* 3ϵ-*covers* M *almost surely. Moreover,* $\mathcal{F}$ *does not depend on the point* p *in the following sense. If* $[M, p], [M, q] \in \mathbb{M}_{sp}$ *and* $[M, p, S], [M, q, T] \in \mathbb{MS}$ *are random with* $\mathrm{Law}([M, p, S]) = \mathcal{F}([M, p])$, $\mathrm{Law}([M, q, T]) = \mathcal{F}([M, q])$, *then* $\mathrm{Law}(S) = \mathrm{Law}(T)$.

Proof
Fix (M, p) to be a pointed special mm-space. For $j \in \mathbb{N}$, let S_j^M be a Poisson point process on M of intensity 1. To be precise S_j^M is a random subset of M characterized by the following properties.

1. If $Q \subset M$ has finite volume, then $S_j^M \cap Q$ is uniformly random with cardinality $\eta_{j,Q}$ where $\eta_{j,Q}$ is a discrete Poisson random variable with parameter $\lambda = \mathrm{vol}_M(Q)$. So $\mathrm{Prob}(\eta_{j,Q} = n) = [\mathrm{vol}_M(Q)^n \exp(-\mathrm{vol}_M(Q))]/(n!)$ for $n = 0, 1, 2, \ldots$.
2. If $\{Q_i\}_{i=1}^\infty$ are pairwise disjoint Borel subsets of M of finite volume, then the random variables $\{S_j^M \cap Q_i\}_{i=1}^\infty$ are jointly independent.

Also let $f_j^M : S_j^M \to [0,1]$ be a random function with law $\mathrm{Leb}^{S_j^M}$ where Leb denotes the Lebesgue measure on the interval $[0,1]$. We require that $\{S_j^M\}_{j=1}^\infty$ and $\{f_j^M\}_{i,j=1}^\infty$ are jointly independent.

CLAIM 1

The map $[M,p] \in \mathbb{M}_{sp} \mapsto \mathrm{Law}([M,p,S_j^M,f_j^M]) \in \mathcal{M}_1(\mathbb{MSF})$ *is continuous for each* j.

Proof

Let $\{[M_i,p_i]\}_{i=1}^\infty \subset \mathbb{M}_{sp}$ be a sequence with $\lim_{i\to\infty}[M_i,p_i] = [M_\infty,p_\infty] \in \mathbb{M}_{sp}$. So there are a complete separable proper metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ (for $1 \le i \le \infty$) such that

$$\lim_{i\to\infty} \varphi_i(M_i,p_i) = \varphi_\infty(M_\infty,p_\infty), \qquad \lim_{i\to\infty} (\varphi_i)_* \mathrm{vol}_{M_i} = (\varphi_\infty)_* \mathrm{vol}_{M_\infty}.$$

The first limit above is in the pointed Hausdorff topology, and the second is in the weak* topology. These limits imply that the Poisson point process with intensity 1 with respect to the measure $(\varphi_i)_* \mathrm{vol}_{M_i}$ converges in law to the Poisson point process with intensity 1 with respect to the measure $(\varphi_\infty)_* \mathrm{vol}_{M_\infty}$. Similarly, if f'_{ij} is defined on $\varphi_i(S_j^{M_i})$ by $f'_{ij}(\varphi_i(s)) = f_j^{M_i}(s)$, then $\mathrm{Law}(\varphi_i(S_j^{M_i}), f'_{ij})$ converges to $\mathrm{Law}(\varphi_\infty(S_j^{M_\infty}), f'_{\infty j})$, which implies that $\mathrm{Law}([M_i,p,S_j^{M_i},f_j^{M_i}])$ converges to $\mathrm{Law}([M_\infty,p,S_j^{M_\infty},f_j^{M_\infty}])$ as $i \to \infty$. $\square$

The idea behind the proof is to construct a random subset $S^M \subset \bigcup_{j\in\mathbb{N}} S_j^M$ such that the map $[M,p] \mapsto \mathrm{Law}([M,p,S^M])$ satisfies the conclusions of the lemma. We build S^M in stages. In the first stage, we identify a random subset $T_1^M \subset S_1^M$ such that $U_1^M := S_1^M \setminus T_1^M$ is ϵ-separated. In the nth stage we identify a random subset $T_n^M \subset S_n^M$ such that if $U_n^M := S_n^M \setminus T_n^M$, then $\bigcup_{j<n} U_j^M$ is ϵ-separated. Finally we let $S^M = \bigcup_{j=1}^\infty U_j^M$. Randomness is used in the construction of each T_j^M in order to ensure continuity of the map $[M,p] \mapsto \mathrm{Law}([M,p,S^M])$. Next we present the details.

Let $\phi : [0,\infty) \to [0,1]$ be a continuous function satisfying the following:

- $\phi(t) = 1$ if $t \le \epsilon$,
- $\phi(t) = 0$ if $t \ge 2\epsilon$.

For each pair $s, t \in \bigcup_{j=1}^{\infty} S_j^M$, let $X(s,t) \in [0,1]$ be a random variable with Lebesgue distribution. We require that the $X(s,t)$'s are jointly independent. Let T_1^M consist of every $s \in S_1^M$ such that there is some $t \in S_1^M$ with $f_1^M(s) \le f_1^M(t)$ and $\phi(\mathrm{dist}_M(s,t)) \ge X(s,t)$. Let $U_1^M = S_1^M \setminus T_1^M$. Note that U_1^M is ϵ-separated almost surely.

CLAIM 2
The map $[M, p] \in \mathbb{M} \mapsto \mathrm{Law}([M, p, U_1^M]) \in \mathcal{M}_1(\mathbb{MS})$ *is continuous.*

Proof
Let $\{[M_i, p_i, S_1^{M_i}, f_1^{M_i}]\}_{i=1}^{\infty} \subset \mathbb{MSF}$ be a (deterministic) sequence with $\lim_{i\to\infty}[M_i, p_i, S_1^{M_i}, f_1^{M_i}] = [M_\infty, p_\infty, S_1^{M_\infty}, f_1^{M_\infty}] \in \mathbb{MSF}$ and such that $f_1^{M_\infty}$ is injective. So there are a complete separable metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that

$$\lim_{i\to\infty} \varphi_i(M_i, p_i) = \varphi_\infty(M_\infty, p_\infty), \qquad \lim_{i\to\infty} (\varphi_i)_* \mathrm{vol}_{M_i}^{(k)} = (\varphi_\infty)_* \mathrm{vol}_{M_\infty}^{(k)}$$

for each $k = 1, 2, 3$ where $\mathrm{vol}_{M_i}^{(2)}, \mathrm{vol}_{M_i}^{(3)}$ are as in Definition 12. By Claim 1, it suffices to show that $\mathrm{Law}(\varphi_i(T_1^{M_i}))$ converges to $\mathrm{Law}(\varphi_\infty(T_1^{M_\infty}))$.

Suppose that $x_i \in S_1^{M_i}$ and

$$\lim_{i\to\infty} \varphi_i(x_i) = \varphi_\infty(x_\infty)$$

for some $x_\infty \in S_1^{M_\infty}$. Then $f_1^{M_i}(x_i)$ converges to $f_1^{M_\infty}(x_\infty)$.

Let $W(x_i)$ be the set of all $s \in S_1^{M_i} \cap B_{M_i}(x_i, 2\epsilon)$ such that $f_1^{M_i}(x_i) \le f_1^{M_i}(s)$. The probability that $x_i \in T_1^{M_i}$ is the probability that $\phi(\mathrm{dist}_{M_i}(x_i, s)) \ge X(x_i, s)$ for some $s \in W(x_i)$. Note that $W(x_i)$ is finite and $\varphi_i(W(x_i))$ converges to $\varphi_\infty(W(x_\infty))$ as $i \to \infty$ in the Hausdorff topology because $f_1^{M_\infty}$ is injective. Also the values of the functions $f_1^{M_i}$ converge in the sense that if $y_i \in W(x_i)$ and $\lim_{i\to\infty} \varphi_i(y_i) = \varphi_\infty(y_\infty)$, then $f_1^{M_i}(y_i)$ converges to $f_1^{M_\infty}(y_\infty)$. Since ϕ is continuous, the probability that $x_i \in T_1^{M_i}$ converges to the probability that $x_\infty \in T_1^{M_\infty}$ as $i \to \infty$. Because $\{x_i\}_{i=1}^{\infty}$ is arbitrary, this implies the claim. □

For $(M, p) \in \mathbb{M}$, we inductively define T_n^M, U_n^M (for $n \ge 2$) by: T_n^M consists of every $x \in S_n^M$ such that there exists $y \in S_n^M \cup \bigcup_{j<n} U_j^M$ with $f_n^M(x) \le f_n^M(y)$ and $\phi(\mathrm{dist}_M(x, y)) \ge X(x, y)$. Let $U_n^M = S_n^M \setminus T_n^M$. Note $\bigcup_{j\le n} U_j^M$ is ϵ-separated almost surely.

CLAIM 3
The map $[M, p] \mapsto \mathrm{Law}([M, p, U_n^M]) \in \mathcal{M}_1(\mathbb{MS})$ *is continuous for every* n.

The proof of this is similar to the proof of Claim 2 so we will skip it. Let $S^M = \bigcup_{j=1}^{\infty} U_j^M$. Note that S^M is ϵ-separated almost surely. We claim that S^M 3ϵ-covers M if M is special. To see this, let $q \in M$. Let $n > 0$ be an integer, and consider the event that $\bigcup_{j<n} U_j^M$ has trivial intersection with $B_M^o(q, 3\epsilon)$. Conditioned on this event, the probability that U_n^M has nontrivial intersection with $B_M^o(q, 3\epsilon)$ is bounded below by the probability that $S_n^M \cap B_M^o(q, 3\epsilon)$ consists of a single point contained in $B_M(q, \epsilon)$. In particular, there is a positive lower bound on this probability (depending on q) which is independent of n. This uses the hypothesis that vol_M is fully supported because M is special. By the law of large numbers then, with probability 1, $S^M \cap B_M^o(q, 3\epsilon) \neq \emptyset$. This proves that S^M 3ϵ-covers M as claimed. To finish the lemma, define $\mathcal{F}([M, p]) := \mathrm{Law}([M, p, S^M])$. The continuity of $\mathcal{F}$ follows from Claim 3. $\square$

Proof of Theorem 4.1

Let $\mathbb{MS}'$ be the set of all $[M, p, S] \in \mathbb{MS}$ such that there is a unique $s \in S$ with $\mathrm{dist}_M(p, s) \leq \mathrm{dist}_M(p, s')$ for all $s' \in S$. Given $[M, p, S] \in \mathbb{MS}'$, let $\rho^S : S \to [5\epsilon, 6\epsilon]$ be a random function defined by the following:

- for each $t \in S$, $\mathrm{Law}(\rho^S(t))$ is the normalized Lebesgue measure on the interval $[5\epsilon, 6\epsilon]$;
- the family $\{\rho^S(t) : t \in S\}$ is jointly independent.

In other words, the law of ρ^S is the product measure $(\mathrm{Leb}_{[5\epsilon,6\epsilon]})^S$ where $\mathrm{Leb}_{[5\epsilon,6\epsilon]}$ denotes the Lebesgue measure on the interval $[5\epsilon, 6\epsilon]$ normalized to have total mass 1. Let $\Sigma(M, S, \rho^S)$ be the nerve complex of $\{B_M^o(s, \rho^S(s)) : s \in S\}$. To be precise, the vertex set of $\Sigma(M, S, \rho^S)$ is S and for every $S' \subset S$ there is a simplex in $\Sigma(M, S, \rho^S)$ spanning S' if and only if $\bigcap_{s \in S'} B_M^o(s, \rho^S(s)) \neq \emptyset$. Let $v \in S$ be the unique element closest to p, let $\Sigma(M, S, \rho^S)_v$ be the connected component of $\Sigma(M, S, \rho^S)$ containing v, and let $\nu_{M,p,S} = \mathrm{Law}(\Sigma(M, S, \rho^S)_v, v) \in \mathcal{M}(\mathrm{RSC})$.

Let (K, v) be a finite rooted simplicial complex, let $r > 0$ be an integer, and let $U_r(K, v)$ be the set of all $[K', v'] \in \mathrm{RSC}$ such that the ball of radius r centered at v' is isomorphic to (K, v) as rooted simplicial complexes.

CLAIM 1

The map $[M, p, S] \in \mathbb{MS}' \mapsto \nu_{M,p,S}(U_r(K, v))$ is continuous for every (K, v), $r > 0$.

Note that the reason why we choose the radii ρ^S randomly rather than deterministically is to make this claim true.

Proof of Claim 1

Let $W_r(K, v)$ be the union of all sets of the form $U_r(K', v')$ where $[K', v'] \in \mathrm{RSC}$ is such that there is a simplicial embedding $\phi : K \to K'$ which maps v to v' and is bijective on the 0-skeleton. With the use of inclusion-exclusion, it is possible to express $\nu_{M,p,S}(U_r(K,v))$ as a finite linear combination of numbers of the form $\nu_{M,p,S}(W_r(K',v'))$. So it suffices to show that the map $(M,p,S) \mapsto \nu_{M,p,S}(W_r(K, v))$ is continuous.

So let $\{[M_i,p_i,S_i]\}_{i=1}^{\infty} \subset \mathbb{MS}'$ be a sequence with $\lim_{i\to\infty}[M_i,p_i,S_i] = [M_\infty, p_\infty, S_\infty] \in \mathbb{MS}'$. Without loss of generality, we may assume that there is a complete proper separable metric space Z containing M_i for $1 \le i \le \infty$ such that

- dist_{M_i} is the restriction of dist_Z to M_i (for all i);
- (M_i, p_i) converges to (M_∞, p_∞) in the pointed Hausdorff topology;
- (S_i, p_i) converges to (S_∞, p_∞) in the pointed Hausdorff topology.

Let $R = 100\epsilon r$. Since each S_i is locally finite, there is an integer $n > 0$ and $s_{i,1}, \dots, s_{i,n} \in S_i$ such that

- $\lim_{i\to\infty} s_{i,j} = s_{\infty,j}$ for each j,
- $B_Z(p_i, R) \cap S_i \subset \{s_{i,1}, \dots, s_{i,n}\}$ for all i.

Let E_i be the set of all $t = (t_1, \dots, t_n) \in [5\epsilon, 6\epsilon]^n$ such that if $\rho : S_i \to [5\epsilon, 6\epsilon]$ is any function with $\rho(s_{i,j}) = t_j$ for all j, then $(\Sigma(M_i, S_i, \rho)_{v_i}, v_i) \in W_r(K, v)$ where $v_i \in S_i$ is the unique closest point to p_i. By definition, $\nu_{M_i,p_i,S_i}(W_r(K,v)) = \mathrm{Leb}^n_{[5\epsilon,6\epsilon]}(E_i)$.

Note that E_i is open (because the nerve complexes are defined in terms of open sets). Also, the definition of $W_r(K, v)$ implies that E_i has the following monotone property: if $t \in E_i$ and $t' \in [5\epsilon, 6\epsilon]^n$ satisfies $t'_j \ge t_j$ for all j, then $t' \in E_i$. In order to estimate the volume of E_i, let $f_i : [5\epsilon, 6\epsilon]^{n-1} \to [5\epsilon, 6\epsilon]$ be the function $f_i(t_1, \dots, t_{n-1}) = t_n$ where t_n is the largest number in $[5\epsilon, 6\epsilon]$ such that $(t_1, \dots, t_n) \notin E_i$ if such a number exists. Otherwise, set $f_i(t_1, \dots, t_{n-1}) = 5\epsilon$. Then the complement of E_i is the region below the graph of f_i. So

$$\begin{aligned}\nu_{M_i,p_i,S_i}\big(W_r(K,v)\big) &= \mathrm{Leb}^n_{[5\epsilon,6\epsilon]}(E_i)\\ &= 1 - \int f_i(t_1,\dots,t_{n-1})\, d\,\mathrm{Leb}^{n-1}_{[5\epsilon,6\epsilon]}(t_1,\dots,t_{n-1}).\end{aligned}$$

Because $\lim_{i\to\infty} s_{i,j} = s_{\infty,j}$ for each j and (M_i, p) converges to (M_∞, p_∞), it follows that $\{f_i\}_{i=1}^{\infty}$ converges pointwise to f_∞. The bounded convergence theorem now implies that $\nu_{M_i,p_i,S_i}(W_r(K,v))$ converges to $\nu_{M_\infty,p_\infty,S_\infty}(W_r(K,v))$ as $i \to \infty$. $\square$

Given a special mm-space M, $s \in M$, and $r > 0$, let $\kappa(s,r) \ge 0$ be the smallest radius such that $\mathrm{vol}_M(B_M(s, \kappa(s,r))) = r$ if such a number exists. Let $\kappa(s,r) = +\infty$ if no such number exists. Let $\mathbb{MS}(r)$ be the set of all $(M', p', S') \in \mathbb{MS}$ such that $\mathrm{dist}_{M'}(p',s) \le \kappa(s,r)$ for some $s \in S'$. Similarly, let $\mathbb{MS}^o(r)$ be the set of all $(M',p',S') \in \mathbb{MS}$ such that $\mathrm{dist}_{M'}(p',s) < \kappa(s,r)$ for some $s \in S'$.

Let M_i be as in the statement of Theorem 4.1, let $p_i \in M_i$ be uniformly random, let $S_i \subset M_i$ be such that $\mathrm{Law}([M_i,p_i,S_i]) = \mathcal{F}([M_i,p_i])$ as in Lemma 4.2, and let $\lambda_i = \mathrm{Law}([M_i,p_i,S_i])$ for $1 \le i < \infty$. By the hypotheses of Theorem 4.1 and Lemma 4.2, λ_i converges as $i \to \infty$ to a measure $\lambda_\infty \in \mathcal{M}_1(\mathbb{M}_{sp})$. Let $[M_\infty, p_\infty, S_\infty] \in \mathbb{MS}$ be random with law λ_∞. By hypothesis, M_∞ is a special mm-space almost surely.

CLAIM 2
(a) $\lambda_\infty(\partial\mathbb{MS}(v_0/2)) = 0$ *where* $\partial\mathbb{MS}(v_0/2) = \overline{\mathbb{MS}(v_0/2)} \cap \overline{\mathbb{MS} \setminus \mathbb{MS}(v_0/2)}$;
(b) $\lim_{i\to\infty} \lambda_i(\mathbb{MS}(v_0/2)) = \lambda_\infty(\mathbb{MS}(v_0/2)) \ge v_0/(2v_1)$.

Proof of Claim 2
Note that, for every $s \in M_i$,

$$\kappa(s, v_0/2) < \kappa(s, v_0) < \epsilon/2$$

because $\mathrm{vol}_{M_i}(B^o_{M_i}(s,\epsilon/2)) > v_0 > 0$ and because M_i is special, so spheres in M_i have measure zero. Because λ_i converges to λ_∞ and $\mathbb{MS}(r)$ is closed in $\mathbb{MS}$, the portmanteau theorem implies that

$$\limsup_{i\to\infty} \lambda_i\big(\mathbb{MS}(r)\big) \le \lambda_\infty\big(\mathbb{MS}(r)\big) \quad \forall r > 0. \tag{3}$$

Because $\mathbb{MS}^o(r)$ is open in $\mathbb{MS}$,

$$\liminf_{i\to\infty} \lambda_i\big(\mathbb{MS}^o(r)\big) \ge \lambda_\infty\big(\mathbb{MS}^o(r)\big) \quad \forall r > 0. \tag{4}$$

Now observe that

$$\lambda_i\big(\mathbb{MS}^o(r)\big) = \lambda_i\big(\mathbb{MS}(r)\big) = \frac{|S_i| r}{\mathrm{vol}(M_i)}$$

if $r \le v_0$ because spheres in M_i have measure zero, $\kappa(s,v_0) < \epsilon/2$, and S_i is ϵ-separated. In particular, if $0 < r_1, r_2 < v_0$, then

$$\frac{\lambda_\infty(\mathbb{MS}^o(r_1))}{\lambda_\infty(\mathbb{MS}(r_2))} \le \frac{\liminf_{i\to\infty} \lambda_i(\mathbb{MS}^o(r_1))}{\limsup_{i\to\infty} \lambda_i(\mathbb{MS}(r_2))} \le \liminf_{i\to\infty} \frac{\lambda_i(\mathbb{MS}^o(r_1))}{\lambda_i(\mathbb{MS}(r_2))} = \frac{r_1}{r_2}.$$

Similarly,

$$\frac{\lambda_\infty(\mathbb{MS}(r_1))}{\lambda_\infty(\mathbb{MS}^o(r_2))} \ge \frac{\limsup_{i\to\infty} \lambda_i(\mathbb{MS}(r_1))}{\liminf_{i\to\infty} \lambda_i(\mathbb{MS}^o(r_2))} \ge \limsup_{i\to\infty} \frac{\lambda_i(\mathbb{MS}(r_1))}{\lambda_i(\mathbb{MS}^o(r_2))} = \frac{r_1}{r_2}.$$

So for any sufficiently small $\delta > 0$,

$$\frac{r_1-\delta}{r_2+\delta} \le \frac{\lambda_\infty(\mathbb{MS}(r_1-\delta))}{\lambda_\infty(\mathbb{MS}^o(r_2+\delta))} \le \frac{\lambda_\infty(\mathbb{MS}^o(r_1))}{\lambda_\infty(\mathbb{MS}(r_2))}$$
$$\le \frac{\lambda_\infty(\mathbb{MS}(r_1))}{\lambda_\infty(\mathbb{MS}^o(r_2))} \le \frac{\lambda_\infty(\mathbb{MS}^o(r_1+\delta))}{\lambda_\infty(\mathbb{MS}(r_2-\delta))} \le \frac{r_1+\delta}{r_2-\delta}.$$

By sending $\delta \searrow 0$ we see that

$$\frac{r_1}{r_2} = \frac{\lambda_\infty(\mathbb{MS}(r_1))}{\lambda_\infty(\mathbb{MS}^o(r_2))} = \frac{\lambda_\infty(\mathbb{MS}^o(r_1))}{\lambda_\infty(\mathbb{MS}(r_2))}.$$

In particular, $\lambda_\infty(\mathbb{MS}(v_0/2)) = \lambda_\infty(\mathbb{MS}^o(v_0/2))$, which implies that $\lambda_\infty(\partial\mathbb{MS}(v_0/2)) = 0$. By (3) and (4),

$$\lim_{i\to\infty} \lambda_i\big(\mathbb{MS}(v_0/2)\big) = \lambda_\infty\big(\mathbb{MS}(v_0/2)\big).$$

Because $B_{M_i}(q, 3\epsilon) < v_1$ (for any $q \in M_i$) and S_i is 3ϵ-covering, it follows that

$$v_1|S_i| \ge \mathrm{vol}_{M_i}(M_i).$$

Because $\kappa(s, v_0/2) \le \epsilon/2$ and S_i is ϵ-separated it follows that the collection of balls of radii $\kappa(s, v_0/2)$ centered at $s \in S_i$ is pairwise disjoint. Therefore

$$\lambda_i\big(\mathbb{MS}(v_0/2)\big) = \frac{|S_i|v_0/2}{\mathrm{vol}_{M_i}(M_i)} \ge \frac{v_0}{2v_1} > 0.$$

So $\lambda_\infty(\mathbb{MS}(v_0/2)) \ge v_0/(2v_1) > 0$. $\square$

By Claim 2 and the portmanteau theorem, λ_i' converges to λ_∞' in the weak* topology as $i \to \infty$ where λ_i' denotes the normalized restriction of λ_i to $\mathbb{MS}(v_0/2)$. More precisely,

$$\lambda_i'(E) := \frac{\lambda_i(E \cap \mathbb{MS}(v_0/2))}{\lambda_i(\mathbb{MS}(v_0/2))}$$

for every Borel $E \subset \mathbb{MS}$.

If $T \subset B_{M_i}(s, 20\epsilon)$ is any ϵ-separated subset, then because

$$v_1 > \mathrm{vol}_{M_i}\big(B^o_{M_i}(q, 20\epsilon)\big) \ge \mathrm{vol}_{M_i}\big(B^o_{M_i}(q, \epsilon/2)\big) > v_0 > 0$$

for every $q \in M_i$, we must have $v_0|T| \le v_1$. So $|T| \le v_1/v_0$. So setting $\Delta := v_1/v_0$, we see that the degree of any vertex in $\Sigma(M_i, S_i, \rho^{S_i})$ is at most Δ. So if $\nu_i' := \int \nu_{M_i,p_i,S_i}\, d\lambda_i'(M_i, p_i, S_i)$, then $\nu_i' \in \mathcal{M}_1(\mathrm{RSC}(\Delta))$. By Claim 1 and the fact that

$\mathbb{MS}(v_0/2) \subset \mathbb{MS}'$, $\lim_{i\to\infty} \nu_i'(U_r(K,v)) = \nu_\infty'(U_r(K,v))$ for every finite $(K,v) \in \mathrm{RSC}(\Delta)$ and $r > 0$. Because each $\nu_i' \in \mathcal{M}_1(\mathrm{RSC}(\Delta))$ and the sets $U_r(K,v)$ generate the Borel sigma-algebra of $\mathrm{RSC}(\Delta)$, it follows that ν_i' converges to ν_∞' in the weak* topology as $i \to \infty$.

Let $[K_i, w_i] \in \mathrm{RSC}$ be random with law ν_i'. We claim that the law of w_i given K_i is uniform over the vertex set of K_i (for $1 \le i < \infty$). Indeed, the set of vertices of K_i is S_i and $w_i \in S_i$ is the nearest point to p_i when $p_i \in M_i$ is chosen uniformly at random subject to the condition that $\mathrm{dist}_{M_i}(p_i, w_i) \le \kappa(w_i, v_0/2)$. The element w_i is uniquely determined by p_i because S_i is ϵ-separated with $\epsilon/2 \ge \kappa(w_i, v_0/2)$. So the balls $B_{M_i}(s, \kappa(s, v_0/2))$ are pairwise disjoint for $s \in S_i$ and each has the same volume, namely, $v_0/2$. Therefore w_i is uniformly distributed over S_i as required.

Because each M_i is special, each is pathwise connected. This implies that K_i is connected. It now follows from Lemma 2.2 that

$$\lim_{i\to\infty} \frac{\mathbb{E}[b_k(K_i)]}{\mathbb{E}[|V(K_i)|]} \tag{5}$$

exists, where $\mathbb{E}[\cdot]$ denotes the expected value.

Because $B^o_{M_i}(s,r)$ is strongly convex for every $r \le 10\epsilon$, for any subset $S' \subset S_i$, either $\bigcap_{s\in S'} B^o_{M_i}(s, \rho^{S_i}(s))$ is empty or it is strongly convex. In the latter case, it is contractible by [31]. This implies that K_i is homotopy equivalent to M_i by [19, Corollary 4G.3]. (This is a slightly stronger version of Borsuk's nerve theorem [6].) So $\mathbb{E}[b_k(K_i)] = b_k(M_i)$. Because of (5) it now suffices to prove that

$$\lim_{i\to\infty} \frac{\mathbb{E}[|V(K_i)|]}{\mathrm{vol}(M_i)}$$

exists.

Note that $|V(K_i)| = |S_i| = \mathrm{vol}(M_i(v_0/2))(v_0/2)^{-1}$ where $M_i(v_0/2)$ is the set of all $q \in M_i$ such that $\mathrm{dist}_{M_i}(q,s) \le \kappa(s, v_0/2)$ for some $s \in S_i$. So

$$\begin{aligned}\lim_{i\to\infty} \frac{\mathbb{E}[|V(K_i)|]}{\mathrm{vol}(M_i)} &= (v_0/2)^{-1} \lim_{i\to\infty} \frac{\mathbb{E}[\mathrm{vol}(M_i(v_0/2))]}{\mathrm{vol}(M_i)} \\ &= (v_0/2)^{-1} \lim_{i\to\infty} \lambda_i\big(\mathbb{MS}(v_0/2)\big) = (v_0/2)^{-1} \lambda_\infty\big(\mathbb{MS}(v_0/2)\big). \quad \square\end{aligned}$$

The next result is not needed in the remainder of the paper. However, it seems worth recording for the sake of future research. This result was first obtained by G. Elek [14].

Definition 15

We consider any Riemannian manifold X as an mm-space with distance dist_X equal to the Riemannian distance and measure vol_X equal to the Riemannian volume form.

COROLLARY 4.3

Let $\{M_i\}_{i=1}^{\infty}$ be a sequence of connected closed smooth Riemannian n-manifolds. Suppose that $\{M_i\}_{i=1}^{\infty}$ Benjamini–Schramm converges in the sense of Definition 6. Suppose also that there are constants δ, κ such that, for each M_i, all sectional curvatures are bounded from above by κ and all Ricci curvatures are bounded from below by δ. Suppose also that the injectivity radius of M_i tends to infinity as $i \to \infty$. Then the normalized limit

$$\lim_{i\to\infty} \frac{b_d(M_i)}{\operatorname{vol}(M_i)}$$

exists for every $d \geq 1$.

Proof

It suffices to check that the conditions of Theorem 4.1 are met. The volume bounds on balls follow from [8, Theorems 3.7 and 3.9]. The strong convexity of small balls follows from [8, Theorem 7.9]. The other conditions are trivial to verify. □

5. L^2-Betti numbers

In this section, we quickly review facts about L^2-invariants used in the proof of Theorem 1.2. We refer the reader to [26] and [27] for background.

Given a topological space X with a continuous Γ-action (where Γ is a countable discrete group), we may define the L^2-Betti numbers $b_k^{(2)}(X; \mathcal{N}(\Gamma))$ (for $k \in \mathbb{N}$) (where $\mathcal{N}(\Gamma)$ denotes the von Neumann algebra of Γ). For simplicity, we let $b_k^{(2)}(X)$ denote $b_k^{(2)}(\tilde{X}; \mathcal{N}(\pi_1(X)))$ where $\tilde{X}$ is the universal cover of X and $\pi_1(X)$ acts on $\tilde{X}$ in the usual way. These numbers are known to be homotopy invariants. Hence we may define the L^2-Betti numbers of a countable discrete group Γ by $b_k^{(2)}(\Gamma) := b_k^{(2)}(B\Gamma)$ where $B\Gamma$ is any classifying space for Γ (i.e., $B\Gamma$ is a connected CW-complex with $\pi_1(B\Gamma)$ isomorphic to Γ and $\pi_n(B\Gamma) = 0$ for all $n \geq 2$).

THEOREM 5.1

Let M be a finite connected CW-complex. Suppose there is a decreasing sequence $\{N_i\}_{i=1}^{\infty}$ of finite-index normal subgroups $N_i \lhd \pi_1(M)$ such that $\bigcap_{i=1}^{\infty} N_i = \{e\}$. Let $M_i \to M$ be the finite cover associated to N_i. Then for any integer $k \geq 0$,

$$\lim_{i\to\infty} \frac{b_k(M_i)}{[\pi_1(M) : N_i]} = b_k^{(2)}(M),$$

where $b_k^{(2)}(M)$ is the kth L^2-Betti number of M and $b_k(M_i)$ is the ordinary kth Betti number of M_i (with real coefficients).

Proof

This is [25, Theorem 0.1]. □

6. Unimodular measures

Measures of the form μ_M (where M is a nonnull finite-volume mm-space) have a special property called *unimodularity*, which is a kind of statistical homogeneity. We will use this property to prove the convergence of certain sequences in $\mathcal{M}_1(\mathbb{M})$. To begin we need a few definitions.

Definition 16

A *doubly pointed mm-space* is a quintuple $(M, p, q, \mathrm{dist}_M, \mathrm{vol}_M)$ where $(M, \mathrm{dist}_M, \mathrm{vol}_M)$ is an mm-space and $p, q \in M$. We will usually denote such a space by (M, p, q), leaving the rest implicit. We say (M, p, q) and (M', p', q') are *doubly pointed-isomorphic* if there is an isometry from M to M' which takes p to p', q to q', and vol_M to $\mathrm{vol}_{M'}$. Let $\mathbb{DM}$ denote the set of all isomorphism classes of doubly pointed mm-spaces. We let $[M, p, q] \in \mathbb{DM}$ denote the isomorphism class of (M, p, q). We can embed $\mathbb{DM}$ into $\mathbb{M}^2$ by $[M, p, q, \mathrm{dist}_M, \mathrm{vol}_M] \mapsto [M, p, \mathrm{dist}_M, \mathrm{vol}_M, \delta_q]$ where δ_q is the Dirac probability measure concentrated on $\{q\}$. We give $\mathbb{DM}$ the induced topology.

Definition 17

Let $\lambda \in \mathcal{M}_1(\mathbb{M})$. Define measures λ_l, λ_r on $\mathbb{DM}$ by

$$d\lambda_l\big([M, p, q]\big) = d\,\mathrm{vol}_M(q)\, d\lambda\big([M, p]\big),$$
$$d\lambda_r\big([M, p, q]\big) = d\,\mathrm{vol}_M(p)\, d\lambda\big([M, q]\big).$$

For example, this means that if f is a positive Borel function on $\mathbb{DM}$, then

$$\int f\big([M, p, q]\big)\, d\lambda_l\big([M, p, q]\big) = \int f\big([M, p, q]\big)\, d\,\mathrm{vol}_M(q)\, d\lambda\big([M, p]\big).$$

We say that λ is *unimodular* if $\lambda_l = \lambda_r$. This term originally appeared in percolation theory (see, e.g., [3] and the references therein).

Example 1

Let M be a nonnull finite-volume mm-space, and let $p \in M$ be a uniformly random point. Then $\mathrm{Law}([M, p]) = \mu_M \in \mathcal{M}_1(\mathbb{M})$ is unimodular. Assume that M is connected, let $\widetilde{M}$ be the universal cover of M, and let $\tilde{p} \in \widetilde{M}$ be an inverse image of p. The pointed-isometry class of $(\widetilde{M}, \tilde{p})$ does not depend on the choice of $\tilde{p}$. Also $\mathrm{Law}([\widetilde{M}, \tilde{p}])$ is unimodular.

LEMMA 6.1
The space of unimodular measures in $\mathcal{M}_1(\mathbb{M})$ is closed in $\mathcal{M}_1(\mathbb{M})$.

Proof
Let $\pi : \mathcal{M}_1(\mathbb{M}) \to \mathcal{M}(\mathbb{DM}) \times \mathcal{M}(\mathbb{DM})$ be the map $\pi(\lambda) = (\lambda_l, \lambda_r)$. This is a continuous map. Since the space of unimodular measures is $\pi^{-1}(\{(\lambda_1, \lambda_2) : \lambda_1 = \lambda_2\})$, it must be closed in $\mathcal{M}_1(\mathbb{M})$. $\square$

Remark 5
Let $\mathcal{F} \subset \mathcal{M}_1(\mathbb{M})$ be the space of all measures of the form μ_M where M is a finite-volume mm-space and $\mu_M = \mathrm{Law}([M, p])$ where $p \in M$ is uniformly random. The relative closure $\overline{\mathcal{F}} \cap \mathcal{M}_1(\mathbb{M}) \subset \mathcal{M}_1(\mathbb{M})$ is the space of *sofic measures*. Are all unimodular measures sofic? This question is a generalization of the well-known problem: are all groups sofic? It is also a generalization of the problem: are all unimodular networks sofic? This was first asked in [3].

Definition 18
If X is an mm-space, then $\mathrm{Isom}(X)$ denotes the group of all measure-preserving isometries $\phi : X \to X$. To be precise, we require $\phi_* \operatorname{vol}_X = \operatorname{vol}_X$. A subgroup $\Lambda < \mathrm{Isom}(X)$ is a *lattice* if there exists a measurable subset $\Delta \subset X$ of positive finite-volume such that $\{\gamma\Delta : \gamma \in \Lambda\}$ is a partition of X. Such a set is called a *fundamental domain* for Λ.

LEMMA 6.2
Let X be an mm-space. Suppose there is a lattice $\Lambda < \mathrm{Isom}(X)$. Then there is a unique unimodular measure $\mu \in \mathcal{M}_1(\mathbb{M})$ such that μ-almost every $[M, p] \in \mathbb{M}$ is such that $(M, \operatorname{dist}_M, \operatorname{vol}_M)$ is isomorphic with $(X, \operatorname{dist}_X, \operatorname{vol}_X)$.

Proof
Let $\Delta \subset X$ be a measurable fundamental domain for Λ. Let $\pi : X \to \mathbb{M}$ be the map $\pi(p) = [X, p]$. Let $\nu = \pi_*[(\operatorname{vol}_X)|_\Delta/(\operatorname{vol}_X(\Delta))]$ be the pushforward of the normalized volume on X restricted to Δ. It is easy to check that ν is a unimodular measure on $\mathbb{M}$. This shows existence.

Now suppose that μ is as in the statement of the lemma. To be precise, $\mu \in \mathcal{M}_1(\mathbb{M})$ is a unimodular measure such that μ-almost every $[M, p] \in \mathbb{M}$ is such that $(M, \operatorname{dist}_M, \operatorname{vol}_M)$ is isomorphic with $(X, \operatorname{dist}_X, \operatorname{vol}_X)$. It suffices to show that $\mu = \nu$. Let $A \subset \mathbb{M}$ be measurable. Suppose that $\nu(A) = 0$. We will show that $\mu(A) = 0$. Note that $\operatorname{vol}_X(\pi^{-1}(A) \cap \Delta) = 0$. Since Δ is a fundamental domain of a lattice, this implies that $\operatorname{vol}_X(\pi^{-1}(A)) = 0$. Define a function f on $\mathbb{DM}$ by $f([M, p, q]) = 1$ if there is a doubly pointed-isomorphism from (M,p,q) to (X,p',q') and $p' \in \pi^{-1}(A) \cap \Delta$, $q' \in \Delta$. Let $f([M,p,q]) = 0$ otherwise. Because μ is unimodular,

$$\begin{aligned}\mathrm{vol}_X(\Delta)\mu(A) &\le \iint f\big([M,p,q]\big)\, d\,\mathrm{vol}_M(q)\, d\mu\big([M,p]\big)\\ &= \iint f\big([M,p,q]\big)\, d\,\mathrm{vol}_M(p)\, d\mu\big([M,q]\big) = 0.\end{aligned}$$

So $\mu(A) = 0$. Because A is arbitrary, μ is absolutely continuous to ν. So there exists a nonnegative measurable function r' such that $d\mu = r'\, d\nu$. By pulling back under π we see that there is a nonnegative measurable function r on Δ such that

$$d\mu\big([X,p]\big) = r(p)\, d\Big(\frac{\pi_* \mathrm{vol}_X|_\Delta}{\mathrm{vol}_X(\Delta)}\Big)(p).$$

Because μ is unimodular $d\,\mathrm{vol}_X(q)\, d\mu([X,p]) = d\,\mathrm{vol}_X(p)\, d\mu([X,q])$. Therefore

$$r(p)\, d\,\mathrm{vol}_X(q)\, d\,\mathrm{vol}_X(p) = r(q)\, d\,\mathrm{vol}_X(p)\, d\,\mathrm{vol}_X(q).$$

In particular, $r(p) = r(q)$ for almost every $p,q \in \Delta$. This implies that $\mu = \nu$ as required. $\square$

Next, we determine conditions under which a sequence of mm-spaces Benjamini–Schramm-converges to the unique unimodular measure concentrated on pointed-isomorphism classes of mm-spaces that are isomorphic with X.

Definition 19
Given a metric space M and a subset $M' \subset M$, let $\partial M' = \overline{M'} \cap \overline{M \setminus M'}$. For $r > 0$, let $N_r(M')$ be the closed radius-r neighborhood of M' in M.

Definition 20
If M is a path-connected metric space and $M' \subset M$, then $\mathrm{covrad}(M' \mid M)$ is the supremum over all $r > 0$ such that if $\pi : \widetilde{M} \to M$ is the universal cover and $p \in \pi^{-1}(M') \subset \widetilde{M}$, then π restricted to $B_{\widetilde{M}}(p,r)$ is an isometry onto its image.

LEMMA 6.3
Let X be a pathwise-connected mm-space with a cocompact subgroup $\Lambda < \mathrm{Isom}(X)$. Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a sequence of geometric subgroups of $\mathrm{Isom}(X)$*, and let $M_i \subset X/\Gamma_i$ be a finite-volume closed subspace. Suppose that*

- $\lim_{i\to\infty} \mathrm{covrad}(M_i \mid X/\Gamma_i) = +\infty$ *and*
- $\lim_{i\to\infty} (\mathrm{vol}(N_r(\partial M_i)))/(\mathrm{vol}(M_i)) = 0$ *for every $r > 0$.*

Then $\lim_{i\to\infty} \mu_{M_i}$ *exists in* $\mathcal{M}_1(\mathbb{M})$ *and is the unique unimodular measure supported on the set of pointed-isomorphism classes of mm-spaces that are isomorphic with* X.

Proof

Let $p_i \in M_i$ be uniformly random. (So $\mu_{M_i} = \mathrm{Law}([M_i, p_i])$.) The two hypotheses on $\{M_i\}_{i=1}^{\infty}$ imply for every $r > 0$, the probability that $B_{X/\Gamma_i}(p_i, r) \subset M_i$ tends to 1 as $i \to \infty$. Moreover, the probability that $B_{X/\Gamma_i}(p_i, r)$ is isomorphic with a ball in X tends to 1 as $i \to \infty$. (The universal cover provides the isometry.) It follows that if μ_∞ is any subsequential limit point of $\{\mu_{M_i}\}_{i=1}^{\infty}$, then μ_∞-almost every $[M, p]$ is such that M is isomorphic with X. Lemma 6.1 implies that μ_∞ is unimodular, and Lemma 6.2 implies that μ_∞ is the *unique* unimodular measure supported on pointed-isomorphism classes of mm-spaces that are isomorphic with X.

It now suffices to show that $\{\mu_{M_i}\}_{i=1}^{\infty}$ is precompact (so that a subsequential limit exists). Let $D \subset X$ be a compact set such that $\Lambda D = X$. Let $p'_i \in X$ be a lift of p_i under the covering map $X \to X/\Gamma_i$. Let $p''_i \in D$ be a point such that $\Lambda p'_i = \Lambda p''_i$. Note that the pointed-isomorphism class of (X, p''_i) depends only on p_i. Therefore $\mathrm{Law}([X, p''_i]) \in \mathcal{M}_1(\mathbb{M})$ is well defined. Because D is compact, $\{\mathrm{Law}([X, p''_i])\}_{i=1}^{\infty}$ is precompact. Fix $r > 0$. As noted before, with probability tending to 1 as $i \to \infty$, $B_{M_i}(p_i, r)$ is isomorphic with $B_X(p''_i, r)$. So $\{\mathrm{Law}([B_{M_i}(p_i, r), p_i])\}_{i=1}^{\infty}$ is precompact in $\mathcal{M}_1(\mathbb{M})$, which implies, since r is arbitrary, that $\{\mu_{M_i}\}_{i=1}^{\infty}$ is precompact. $\square$

7. Proof of Theorem 1.2

We will derive Theorem 1.2 from Theorem 7.1 below, which essentially is a version of Theorem 1.2 for mm-spaces. First we need a few definitions.

Definition 21

Let X be an mm-space, and let $r > 0$. We define the *radius-r Cheeger constant of X* by

$$h_r(X) = \inf_M \frac{\mathrm{vol}_X(N_r(\partial M))}{\mathrm{vol}_X(M)},$$

where the infimum is over all pathwise-connected compact subsets $M \subset X$ with positive volume such that $\mathrm{vol}_X(M) \le \mathrm{vol}_X(X)/2$. (Recall that $\partial M = M \cap \overline{X \setminus M}$ and $N_r(\partial M)$ is the closed radius-r neighborhood of M.)

Definition 22

For any class of groups $\mathcal{F}$, mm-space X, and $r > 0$ let $I_r(X \mid \mathcal{F}) = \inf_\Gamma h_r(X/\Gamma)$ where the infimum is over all geometric $\Gamma < \mathrm{Isom}(X)$ such that Γ is isomorphic to a group in $\mathcal{F}$.

THEOREM 7.1

Let X be a contractible special mm-space (Definition 7). Suppose that

- *there exists a residually finite geometric cocompact lattice $\Lambda < \mathrm{Isom}(X)$ and $b_d^{(2)}(\Lambda) > 0$;*
- *there exists an $\epsilon > 0$ such that every ball of radius less than or equal to 10ϵ in X is strongly convex.*

Then there exists $r > 0$ such that $I_r(X \mid \mathcal{G}_d) > 0$ where $\mathcal{G}_d$ is as in Theorem 1.2.

Example 2

Let $d > 2$, let T_d denote the d-regular tree, and let $X_d = T_d \times T_d$. We could consider T_d to be an mm-space by making each edge isomorphic with the unit interval. Then let $\mathrm{vol}_{X_d} = \mathrm{vol}_{T_d} \times \mathrm{vol}_{T_d}$, and set dist_{X_d} equal to the sum of the distances of its coordinate projections. This makes X_d into a CAT(0) space and therefore every ball is strongly convex. Moreover, $\mathrm{Isom}(X_d)$ equals the automorphism group of $T_d \times T_d$ as a cell-complex.

Because every lattice $\Lambda < \mathrm{Aut}(T_d \times T_d)$ has $b_2^{(2)}(\Lambda) > 0$, it follows from Theorem 7.1 that $I_r(X_d \mid \mathcal{G}_2) > 0$ for some $r > 0$. The second L^2-Betti numbers of free groups vanish. So $h_r(X_d/\Gamma) \geq I_r(X_d \mid \mathcal{G}_2) > 0$ for any free group $\Gamma < \mathrm{Isom}(X_d)$.

The next lemma shows that by passing to a subgroup $\Gamma_i'' < \Gamma_i$ we may substantially simplify the problem. We will need the following definition.

Definition 23 (Asymptotic lower Betti numbers)

Let Γ be a residually finite countable group, and let $d \geq 1$ be an integer. Let

$$\widehat{b_d}(\Gamma) = \liminf_N \frac{b_d(N)}{[\Gamma : N]},$$

where the limit is over the net of finite-index normal subgroups of Γ ordered by reverse inclusion. Equivalently, $\widehat{b_d}(\Gamma)$ is the smallest number x such that for every $\epsilon > 0$ and every finite-index normal subgroup $N \lhd \Gamma$ there exists a finite-index normal subgroup $N' \lhd \Gamma$ with $N' < N$ and

$$\left| x - \frac{b_d(N')}{[\Gamma : N']} \right| < \epsilon.$$

In the special case that Γ has a finite classifying space, $\widehat{b_d}(\Gamma) = b_d^{(2)}(\Gamma)$ by Theorem 5.1.

LEMMA 7.2

Let X be as in Theorem 7.1. Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a sequence of geometric residually finite subgroups $\Gamma_i < \mathrm{Isom}(X)$ such that $\lim_{i\to\infty} h_r(X/\Gamma_i) = 0$ for every $r > 0$.

Then there exist subgroups $\Gamma_i'' < \Gamma_i' < \Gamma_i$ and positive-volume compact subsets $M_i' \subset X/\Gamma_i'$, $M_i'' \subset X/\Gamma_i''$ such that

(a) M_i'' *is a pathwise connected compact subset of X/Γ_i'' for each i;*

(b) $\lim_{i\to\infty}(\mathrm{vol}(N_r(\partial M_i'')))/(\mathrm{vol}(M_i'')) = 0$ *for every r;*

(c) $\lim_{i\to\infty} \mathrm{covrad}(M_i'' \mid X/\Gamma_i'') = \infty$;

(d)

$$\liminf_{i\to\infty} \frac{\widehat{b}_d(\Gamma_i')}{\mathrm{vol}(M_i')} = \liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{\mathrm{vol}(M_i'')}.$$

Proof

By hypothesis, there exist path-connected positive-volume compact sets $M_i \subset X/\Gamma_i$ such that

$$\lim_{i\to\infty} \frac{\mathrm{vol}(N_r(\partial M_i))}{\mathrm{vol}(M_i)} = 0$$

for every $r > 0$ where we have dropped the subscript on $\mathrm{vol}_{X/\Gamma_i}(\cdot)$ for simplicity.

Because X is contractible and Γ_i acts freely and properly discontinuously, we may identify Γ_i with the fundamental group $\pi_1(X/\Gamma_i)$. Let $\Gamma_i' < \Gamma_i$ be the image of $\pi_1(M_i)$ under the natural map from $\pi_1(M_i) \to \pi_1(X/\Gamma_i)$ induced by inclusion $M_i \to X/\Gamma_i$. Let $\phi_i : X/\Gamma_i' \to X/\Gamma_i$ be the covering map, and let M_i' be a path-connected component of $\phi_i^{-1}(M_i)$. The choice of Γ_i' implies that ϕ_i restricted to M_i' is a homeomorphism onto M_i. So M_i' is compact and

$$\lim_{i\to\infty} \frac{\mathrm{vol}(N_r(\partial M_i'))}{\mathrm{vol}(M_i')} = 0 \tag{6}$$

for every $r > 0$.

For each $\gamma \in \Gamma_i'$, let $L_i(\gamma)$ denote the infimum over all numbers r such that there is a $p \in X$ whose image in X/Γ_i' is contained in M_i' and $\mathrm{dist}_X(p, \gamma p) \le r$. This number depends only on the conjugacy class of γ in Γ_i'. So we may think of L_i as a function on the set of conjugacy classes of Γ_i'.

Let $r > 0$. We claim that there are only a finite number of Γ_i'-conjugacy classes $[\gamma]$ with $L_i([\gamma]) \le r$. To obtain a contradiction, suppose that $\gamma_1, \gamma_2, \ldots \in \Gamma_i'$ is an infinite sequence of pairwise nonconjugate elements with $L_i(\gamma_i) \le r$. Let $p_i \in X$ be such that the image of p_i in X/Γ_i' is in M_i' and $\mathrm{dist}_X(p_i, \gamma_i p_i) \le r$. Let $Y_i \subset X$ be a compact set which surjects onto M_i' under the covering map $X \mapsto X/\Gamma_i'$. After conjugating γ_i if necessary, we may assume that $p_i \in Y_i$ for all i. After passing to a subsequence if necessary, we may assume that $\lim_{i\to\infty} p_i = p_\infty$ and $\lim_{i\to\infty} \gamma_i p_i = q_\infty$ exist. It follows that $\lim_{i,j\to\infty} \gamma_j \gamma_i^{-1} q_\infty = q_\infty$. This contradicts the assumption that Γ_i acts properly discontinuously and freely on X. So there are only a finite number of Γ_i'-conjugacy classes $[\gamma]$ with $L_i([\gamma]) \le r$ as claimed.

Because Γ_i is residually finite, Γ_i' is also residually finite. So there is a finite-index normal subgroup $\Gamma_i'' < \Gamma_i'$ such that Γ_i'' does not contain any nontrivial element $\gamma \in \Gamma_i'$ with $L_i(\gamma) \le i$. We may choose Γ_i'' to also satisfy

$$\left|\frac{b_d(\Gamma_i'')}{[\Gamma_i' : \Gamma_i'']} - \widehat{b}_d(\Gamma_i')\right| < \frac{\mathrm{vol}(M_i')}{i}.$$

This implies that

$$\liminf_{i\to\infty} \frac{\widehat{b}_d(\Gamma_i')}{\mathrm{vol}(M_i')} = \liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{[\Gamma_i' : \Gamma_i'']\,\mathrm{vol}(M_i')}. \tag{7}$$

Let $\psi_i : X/\Gamma_i'' \to X/\Gamma_i'$ be the quotient map, and let $M_i'' = \psi_i^{-1}(M_i')$. Because $\pi_1(M_i)$ surjects onto Γ_i' (under the natural map from $\pi_1(M_i) \to \pi_1(X/\Gamma_i)$), it follows that $\pi_1(M_i')$ also surjects onto $\pi_1(X/\Gamma_i') \simeq \Gamma_i'$. This implies that M_i'' is path-connected.

Note that $\mathrm{covrad}(M_i'' \mid X/\Gamma_i'') \ge i/2$. So $\lim_{i\to\infty} \mathrm{covrad}(M_i'' \mid X/\Gamma_i'') = \infty$.

The restriction of ψ_i to $N_r(M_i'')$ is a finite-degree covering map onto $N_r(M_i')$. So

$$\mathrm{vol}\big(N_r(\partial M_i'')\big) = [\Gamma_i' : \Gamma_i'']\,\mathrm{vol}\big(N_r(\partial M_i')\big), \qquad \mathrm{vol}(M_i'') = [\Gamma_i' : \Gamma_i'']\,\mathrm{vol}(M_i').$$

Now (6) and (7) imply that $\lim_{i\to\infty}(\mathrm{vol}(N_r(\partial M_i'')))/(\mathrm{vol}(M_i'')) = 0$ and

$$\liminf_{i\to\infty} \frac{\widehat{b}_d(\Gamma_i')}{\mathrm{vol}(M_i')} = \liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{\mathrm{vol}(M_i'')}. \qquad \square$$

Proof of Theorem 7.1

Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a sequence of geometric residually finite subgroups $\Gamma_i < \mathrm{Isom}(X)$ such that $\lim_{i\to\infty} h_r(X/\Gamma_i) = 0$ for every $r > 0$. Let $\Gamma_i'' < \Gamma_i' < \Gamma_i$, $M_i' \subset X/\Gamma_i'$, and $M_i'' \subset X/\Gamma_i''$ be as in Lemma 7.2. It suffices to show that, for all but finitely many i, $\widehat{b}_d(\Gamma_i') > 0$.

By hypothesis, there exists an $\epsilon > 0$ such that every ball of radius less than or equal to 10ϵ in X is strongly convex. For sufficiently large i, $\mathrm{covrad}(M_i'' \mid X/\Gamma_i'') > 10\epsilon$, which implies that each ball of radius less than or equal to 10ϵ with center in $N_{10\epsilon}(M_i'')$ is strongly convex. Moreover, for any $p \in X$ there is some $r > 0$ such that $B_X(p,r)$ maps isometrically onto the ball of radius r centered at the image of p in X/Γ_i''. This is because Γ_i'' acts properly discontinuously and freely (because Γ_i does and $\Gamma_i'' < \Gamma_i$). So for every $q \in X/\Gamma_i''$ there is some number $\kappa(q)$ such that every ball of radius less than or equal to $\kappa(q)$ centered at q is strongly convex.

Let $S_i \subset X/\Gamma_i''$ be a set, and let $\rho : S_i \to (0,\infty)$ be a function such that

- $S_i \cap M_i''$ is ϵ-separated and 10ϵ-covers M_i'';
- $\rho(s) = 10\epsilon$ for every $s \in S_i \cap M_i''$;
- $\rho(s) \le 10\epsilon$ for all $s \in S_i$;
- $B_{X/\Gamma_i''}(s,r)$ is strongly convex for every $s \in S_i$ and $r \le \rho(s)$;
- $\{B^o_{X/\Gamma_i''}(s,\rho(s)) : s \in S_i\}$ is locally finite and covers X/Γ_i''.

Let

$$U_i = \bigcup \big\{B^o_{X/\Gamma_i''}\big(s,\rho(s)\big) : s \in S_i \cap M_i''\big\}.$$

Observe that $M_i'' \subset U_i$ and U_i is pathwise connected (because M_i'' is pathwise connected). Because $S_i \cap M_i''$ is finite we can choose $0 < \delta_i < \epsilon$ so that if

$$U_i' := \bigcup \big\{B_{X/\Gamma_i''}\big(s,\rho(s)-\delta_i\big) : s \in S_i \cap M_i''\big\},$$

then U_i' is homologically equivalent to U_i (in the sense that they have the same Betti numbers), $\lim_{i\to\infty} \mathrm{vol}(U_i')/(\mathrm{vol}(U_i)) = 1$, and $M_i'' \subset U_i'$. In particular, U_i' is pathwise connected. Note that U_i' is closed while U_i is open. For simplicity we have dropped the subscript on $\mathrm{vol}_{X/\Gamma_i''}(\cdot) = \mathrm{vol}(\cdot)$.

Because $M_i'' \subset U_i' \subset N_{10\epsilon}(M_i'')$ it follows that

- $\lim_{i\to\infty} \mathrm{covrad}(U_i' \mid X/\Gamma_i) = +\infty$ and
- $\limsup_{i\to\infty} (\mathrm{vol}(N_r(\partial U_i')))/(\mathrm{vol}(U_i')) \le \limsup_{i\to\infty} (\mathrm{vol}(N_{r+10\epsilon}(\partial M_i'')))/(\mathrm{vol}(M_i'')) = 0$ for every $r > 0$.

Let p_i be a uniformly random point of U_i', and let $\mu_i = \mathrm{Law}(U_i', p_i) \in \mathcal{M}_1(\mathbb{M})$. By Lemma 6.3, $\lim_{i\to\infty} \mu_i = \mu_\infty$ is the unique unimodular measure supported on pointed-isomorphism classes of mm-spaces that are isomorphic with X.

To apply Theorem 4.1 (to U_i') we need to check a few more hypotheses. We claim that there is a $v_0 > 0$ such that for every $p,q \in X$ if $\mathrm{dist}_X(p,q) \le 10\epsilon - \delta_i$, then $\mathrm{vol}(B_X(q,\epsilon/2) \cap B_X(p,10\epsilon-\delta_i)) > v_0$. If this is false, then there are sequences $\{p_j\}_{j=1}^\infty, \{q_j\}_{j=1}^\infty \subset X$ and $\{i_j\}_{j=1}^\infty \subset \mathbb{N}$ such that $\mathrm{dist}_X(p_j,q_j) \le 10\epsilon - \delta_{i_j}$ and $\lim_{j\to\infty} \mathrm{vol}(B_X(q_j,\epsilon/2) \cap B_X(p_j, 10\epsilon - \delta_{i_j})) = 0$. Let $D \subset X$ be a compact set that surjects onto X/Λ. By replacing p_j, q_j with $g_j p_j, g_j q_j$ for some $g_j \in \Lambda$ if necessary, we may assume that each $p_j \in D$. After passing to a subsequence if necessary, we may assume that $\lim_{j\to\infty} p_j = p_\infty$, $\lim_{j\to\infty} q_j = q_\infty$, and $\lim_{j\to\infty} \delta_{i_j} = \delta_\infty \in [0,\epsilon]$ exist. Let $\eta > 0$. For all sufficiently large j,

$$B_X(q_\infty, \epsilon/2 - \eta) \cap B_X(p_\infty, 10\epsilon - \delta_\infty - \eta) \subset B_X(q_j, \epsilon/2) \cap B_X(p_j, 10\epsilon - \delta_{i_j}).$$

This implies that $\mathrm{vol}_X(B^o_X(q_\infty,\epsilon/2) \cap B^o_X(p_\infty, 10\epsilon - \delta_\infty)) = 0$. However, $B_X(p_\infty, 10\epsilon - \delta_\infty)$ is strongly convex and so there is a geodesic from p_∞ to q_∞ in $B_X(p_\infty, 10\epsilon - \delta_\infty)$. It follows that $B^o_X(q_\infty,\epsilon/2) \cap B^o_X(p_\infty, 10\epsilon - \delta_\infty)$ is a nonempty open set. Since vol_X is fully supported (because X is special), this is a contradiction. This proves the claim. Note that v_0 does not depend on i.

If i is sufficiently large, then $\mathrm{covrad}(U_i' \mid X/\Gamma_i'') > 10\epsilon$, which implies that every $(10\epsilon - \delta_i)$-ball in X/Γ_i'' which lies in U_i' is isometric with a $(10\epsilon - \delta_i)$-ball in X. Therefore, for every $q_i \in U_i'$, $\mathrm{vol}(B_{X/\Gamma_i''}(q_i, \epsilon/2) \cap U_i') = \mathrm{vol}(B_{U_i'}(q_i, \epsilon/2)) > v_0$. Also because X/Λ is compact, there is a $v_1 > 0$ such that $B_X(x, 20\epsilon) < v_1$ for every $x \in X$. This implies that $B_{U_i'}(q_i, 20\epsilon) < v_1$ too. The hypotheses of Theorem 4.1 have now been checked. That result implies that $\lim_{i\to\infty}(b_d(U_i'))/(\mathrm{vol}(U_i'))$ exists.

Because Λ is residually finite, there exists a decreasing sequence $\{\Lambda_i\}_{i=1}^{\infty}$ of finite-index normal subgroups of Λ such that $\bigcap_{i=1}^{\infty} \Lambda_i = \{e\}$. Note that the covering radius of X/Λ_i tends to infinity as $i \to \infty$. So Lemma 6.3 implies that $\lim_{i\to\infty} \mu_{X/\Lambda_i} = \mu_\infty$. Theorem 4.1 now implies that

$$\lim_{i\to\infty} \frac{b_d(U_i')}{\mathrm{vol}(U_i')} = \lim_{i\to\infty} \frac{b_d(X/\Lambda_i)}{\mathrm{vol}(X/\Lambda_i)}.$$

Because X is contractible, X/Λ_i is a classifying space for Λ_i, which implies that $b_d(X/\Lambda_i) = b_d(\Lambda_i)$. Because X/Λ_i is a $[\Lambda : \Lambda_i]$-fold cover of X/Λ, it follows that $\mathrm{vol}(X/\Lambda_i) = [\Lambda : \Lambda_i]\,\mathrm{vol}(X/\Lambda)$. By Theorem 5.1,

$$\lim_{i\to\infty} \frac{b_d(X/\Lambda_i)}{\mathrm{vol}(X/\Lambda_i)} = \lim_{i\to\infty} \frac{b_d(\Lambda_i)}{[\Lambda : \Lambda_i]\,\mathrm{vol}(X/\Lambda)} = \frac{b_d^{(2)}(\Lambda)}{\mathrm{vol}(X/\Lambda)}.$$

So we have established that

$$\lim_{i\to\infty} \frac{b_d(U_i')}{\mathrm{vol}(U_i')} = \frac{b_d^{(2)}(\Lambda)}{\mathrm{vol}(X/\Lambda)}.$$

Because U_i' is homologically equivalent to U_i and $\lim_{i\to\infty} \mathrm{vol}(U_i')/(\mathrm{vol}(U_i)) = 1$, we have that

$$\lim_{i\to\infty} \frac{b_d(U_i)}{\mathrm{vol}(U_i)} = \frac{b_d^{(2)}(\Lambda)}{\mathrm{vol}(X/\Lambda)}. \tag{8}$$

Let

$$\begin{aligned}
W_i &= \bigcup \big\{ B^o_{X/\Gamma_i''}\big(s, \rho(s)\big) : s \in S_i \setminus M_i'' \big\},\\
S_i^V &= (S_i \setminus M_i'') \cup \big\{ s \in S_i \cap M_i'' : B^o_{X/\Gamma_i''}\big(s, \rho(s)\big) \cap W_i \neq \emptyset \big\},\\
V_i &= \bigcup \big\{ B^o_{X/\Gamma_i''}\big(s, \rho(s)\big) : s \in S_i^V \big\}.
\end{aligned}$$

Let K_i be the nerve complex of $\{B^o_{X/\Gamma''_i}(s,\rho(s)) : s \in S_i\}$. Let $K_i^U \subset K_i$ be the nerve complex of $\{B^o_{X/\Gamma''_i}(s,\rho(s)) : s \in S_i \cap M''_i\}$. Similarly, let $K_i^V \subset K_i$ be the nerve complex of $\{B^o_{X/\Gamma''_i}(s,\rho(s)) : s \in S_i^V\}$.

Because each $B^o_{X/\Gamma''_i}(s,\rho(s))$ is strongly convex (for $s \in S_i$), it follows that any nonempty intersection of such balls is also strongly convex and is therefore contractible (see [31]). By [19, Corollary 4G.3], this implies that K_i is homotopic to X/Γ''_i, K_i^U is homotopic to U_i, and K_i^V is homotopic to V_i. Therefore, $b_d(K_i) = b_d(X/\Gamma''_i) = b_d(\Gamma''_i)$ (since X/Γ''_i is a classifying space for Γ''_i since X is contractible), $b_d(K_i^U) = b_d(U_i)$, and $b_d(K_i^V) = b_d(V_i)$.

We claim that $K_i^U \cup K_i^V = K_i$. To see this, suppose that $T \subset S_i$ spans a simplex in K_i. Then either $T \subset K_i^U$ or there exists $s \in T \setminus M''_i$. For any $t \in T$, $B^o_{X/\Gamma''_i}(t,\rho(t)) \cap B^o_{X/\Gamma''_i}(s,\rho(s)) \neq \emptyset$. Since $B^o_{X/\Gamma''_i}(s,\rho(s)) \subset W_i$, this implies that $t \in S_i^V$. Since t is arbitrary, the simplex spanning T is contained in K_i^V. Since T is arbitrary, $K_i^U \cup K_i^V = K_i$.

The Mayer–Vietoris sequence

$$\cdots \to H_d(K_i^U \cap K_i^V) \to H_d(K_i^U) \oplus H_d(K_i^V) \to H_d(K_i) \to \cdots$$

implies that

$$b_d(U_i) = b_d(K_i^U) \le b_d(K_i) + b_d(K_i^U \cap K_i^V) = b_d(\Gamma''_i) + b_d(K_i^U \cap K_i^V). \quad (9)$$

If $s \in M''_i$ and $Z \subset B_{M''_i}(s, 20\epsilon)$ is any ϵ-separated subset, then because

$$v_1 > \operatorname{vol}\bigl(B^o_{M''_i}(q,20\epsilon)\bigr) \ge \operatorname{vol}\bigl(B^o_{M''_i}(q,\epsilon/2)\bigr) > v_0 > 0$$

for every $q \in M''_i$, we must have $v_0|Z| \le v_1$. So $|Z| \le v_1/v_0$. So setting $\Delta := v_1/v_0$, we see that the degree of any vertex of K_i^U is at most Δ. So $b_d(K_i^U \cap K_i^V)$ is at most the number of d-simplices in $K_i^U \cap K_i^V$, which is at most the number of vertices of $K_i^U \cap K_i^V$ multiplied by $\binom{\Delta}{d}$. The vertex set of $K_i^U \cap K_i^V$ is $S'_i = \{s \in S_i \cap M''_i : B^o_{X/\Gamma''_i}(s,\rho(s)) \cap W_i \neq \emptyset\}$. So

$$b_d(K_i^U \cap K_i^V) \le |S'_i| \binom{\Delta}{d}.$$

Note that S'_i is contained in the 20ϵ-neighborhood of $\partial M''_i$. Because S'_i is ϵ-separated and each $(\epsilon/2)$-ball has volume at least v_0 (for some $v_0 > 0$ independent of i), we have $|S'_i| v_0 \le \operatorname{vol}(N_{20\epsilon}(\partial M''_i))$. So

$$b_d(K_i^U \cap K_i^V) \le v_0^{-1} \operatorname{vol}\bigl(N_{20\epsilon}(\partial M''_i)\bigr) \binom{\Delta}{d}.$$

Therefore,

$$\limsup_{i\to\infty} \frac{b_d(K_i^U \cap K_i^V)}{\mathrm{vol}(U_i)} \le \binom{\Delta}{d} v_0^{-1} \limsup_{i\to\infty} \frac{\mathrm{vol}(N_{20\epsilon}(\partial M_i''))}{\mathrm{vol}(U_i)}$$
$$\le \binom{\Delta}{d} v_0^{-1} \limsup_{i\to\infty} \frac{\mathrm{vol}(N_{20\epsilon}(\partial M_i''))}{\mathrm{vol}(M_i'')} = 0.$$

Lemma 7.2, the fact that $M_i'' \subset U_i$, (8), and (9) now imply that

$$\liminf_{i\to\infty} \frac{\widehat{b}_d(\Gamma_i')}{\mathrm{vol}(M_i')} = \liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{\mathrm{vol}(M_i'')} \ge \liminf_{i\to\infty} \frac{b_d(\Gamma_i'')}{\mathrm{vol}(U_i)}$$
$$\ge \liminf_{i\to\infty} \frac{b_d(U_i)}{\mathrm{vol}(U_i)} - \frac{b_d(K_i^U \cap K_i^V)}{\mathrm{vol}(U_i)}$$
$$= \frac{b_d^{(2)}(\Lambda)}{\mathrm{vol}(X/\Lambda)} > 0.$$

So $\widehat{b}_d(\Gamma_i') > 0$ for all but finitely many i. This implies the theorem. □

We now turn to the proof of Theorem 1.2. We will need the following lemma to smooth out the Cheeger submanifolds of X/Γ.

LEMMA 7.3 (Haircutting lemma)
Let M be an infinite-volume complete Riemannian n-manifold. Suppose there is a $\delta > 0$ such that the Ricci curvature of M is at least $-\delta^2(n-1)$ (everywhere). Suppose as well that $h(M) < 1$. Then there exist a pathwise connected compact subset $M'' \subset M$ and a function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for every $R > 0$

$$\frac{\mathrm{vol}(N_R(\partial M''))}{\mathrm{vol}(M'')} \le f(R)h(M). \tag{10}$$

Moreover, f depends only on δ and $\dim(M)$.

Proof
This is contained in [7, Lemma 7.2] except in one detail: M'' is not required to be pathwise connected. However, a small perturbation of the proof yields a pathwise connected subset. To explain this, let us recall the construction of M' from [7]. Let $\epsilon > 0$, and let A be a smooth compact submanifold of M with

$$\frac{\mathrm{area}(\partial A)}{\mathrm{vol}(A)} \le h(M)(1+\epsilon).$$

Let $r > 0$ be a sufficiently small constant (how small depends only on the dimension). Let

$$M' = \{p \in M : \mathrm{vol}(A \cap B_M(p,r)) > (1/2)\,\mathrm{vol}(B_M(p,r))\}.$$

Note that $\partial M' = \{p \in M : \mathrm{vol}(A \cap B_M(p,r)) = (1/2)\,\mathrm{vol}(B_M(p,r))\}$.

(In Buser's notation, $B_M(p,r)$ is denoted by $U(p,r)$, M' is denoted by $\widetilde{A}$, $\partial M'$ is denoted by $\widetilde{X}$, $N_t(\partial M')$ is denoted by $\widetilde{X}^t$, and $\mathrm{area}(\partial A)/(\mathrm{vol}(A))$ is denoted by $\mathcal{H}$.)

Let $K_1, \dots, K_m$ be the components of M'. Observe that

$$\frac{\mathrm{area}(\partial A)}{\mathrm{vol}(A \cap M')} = \sum_{i=1}^{m} \frac{\mathrm{area}(\partial A \cap K_i)}{\mathrm{vol}(K_i \cap A)} \frac{\mathrm{vol}(K_i \cap A)}{\mathrm{vol}(A \cap M')}.$$

In particular, $\mathrm{area}(\partial A)/(\mathrm{vol}(A \cap M'))$ is a convex sum of $\mathrm{area}(\partial A \cap K_i)/(\mathrm{vol}(K_i \cap A))$. So there exists a component K_i such that

$$\frac{\mathrm{area}(\partial A \cap K_i)}{\mathrm{vol}(K_i \cap A)} \leq \frac{\mathrm{area}(\partial A)}{\mathrm{vol}(A \cap M')} \leq (1+\epsilon)h(M)\frac{\mathrm{vol}(A)}{\mathrm{vol}(A \cap M')}.$$

According to [7, equations 4.6 and 4.9], $\mathrm{vol}(A \cap M') \geq c\,\mathrm{vol}(A)$ where $c = 1 - (4\mathcal{H}\beta(4r))/(j(r)\beta(r)) \geq 1/2$ (in Buser's notation). Therefore,

$$\frac{\mathrm{area}(\partial A \cap K_i)}{\mathrm{vol}(K_i \cap A)} \leq \frac{\mathrm{area}(\partial A)}{\mathrm{vol}(A \cap M')} \leq 2(1+\epsilon)h(M).$$

Let M'' be the closure of K_i. It is now possible to replace M' with M'' in the proof of [7, Lemma 7.2] (which is mostly contained in [7, Section 4]) to conclude that M'' satisfies (10). ☐

Proof of Theorem 1.2

Because X/Λ is compact, [8, Theorem 7.9] implies that there exists an $\epsilon > 0$ such that every ball of radius less than or equal to 10ϵ in X is strongly convex. So Theorem 7.1 implies that $I_r(X \mid \mathcal{G}_d) > 0$ for some $r > 0$. Lemma 7.3 implies that if $I(X \mid \mathcal{G}_d) < 1$, then $I_r(X \mid \mathcal{G}_d) \leq f(r)I(X \mid \mathcal{G}_d)$ for some function f which depends only on the dimension of X and a lower bound on its Ricci curvature. Thus $I(X \mid \mathcal{G}_d) > 0$. ☐

8. Applications

In this section we prove Corollary 1.3. The starting point is the following.

LEMMA 8.1

If Λ is a lattice in $\mathrm{Isom}(\mathbb{H}^{2n})$ for some $n \geq 1$, then $b_n^{(2)}(\Lambda) > 0$.

Proof
This is contained in [26, Theorem 5.12]. □

Remark 6
Theorem 5.12 of [26] also shows that if $\Lambda < \mathrm{Isom}(\mathbb{H}^n)$ is a lattice, then $b^{(2)}_d(\Lambda) = 0$ unless $d = n/2$ is an integer.

It now suffices to show the following.

PROPOSITION 8.2
If Γ is a torsion-free lattice in $\mathrm{Isom}(\mathbb{H}^3)$, then $\Gamma \in \mathcal{G}_d$ for all $d > 1$.

Proof
The fact that Γ is residually finite is well known: Γ is linear (since it is a subgroup of $SO(3,1)$) and all finitely generated linear groups are residually finite by [28]. Let $\Gamma' < \Gamma$ be finitely generated. Observe that Γ' is the fundamental group of a hyperbolic 3-manifold (namely, $\mathbb{H}^3/\Gamma'$). By the Scott core theorem [32], Γ' has a finite classifying space. By Lück's approximation theorem (Theorem 5.1), it suffices to show that $b^{(2)}_d(\Gamma') = 0$ for all $d > 1$. This is handled in Lemma 8.9 below. In fact, we will prove something stronger: that Γ is almost treeable, as defined next. □

Definition 24 (Treeability and almost treeability)
Let Γ be a countable discrete group. Let $\binom{\Gamma}{2}$ be the set of all unordered pairs of elements in Γ, and let $\mathcal{G}(\Gamma) = 2^{\binom{\Gamma}{2}}$ be the set of all subsets of $\binom{\Gamma}{2}$ with the product topology. Because Γ is countable, this means that $\mathcal{G}(\Gamma)$ is a compact metrizable space. (In fact, it is homeomorphic to a Cantor set.) Associated to any element $x \in \mathcal{G}(\Gamma)$ is a graph G_x with vertex set Γ and edge set x. Observe that Γ acts on $\mathcal{G}(\Gamma)$ by $gx = \{\{ga, gb\} : \{a,b\} \in x\}$ for $g \in \Gamma$, $x \in \mathcal{G}(\Gamma)$.

Let $\mathcal{F}(\Gamma)$ denote the set of all $x \in \mathcal{G}(\Gamma)$ such that G_x is a forest (i.e., every connected component of G_x is simply connected). Let $\mathcal{T}(\Gamma) \subset \mathcal{F}(\Gamma)$ denote the set of all $x \in \mathcal{F}(\Gamma)$ such that G_x is a tree. The action of Γ preserves both $\mathcal{F}(\Gamma)$ and $\mathcal{T}(\Gamma)$.

We say that Γ is *treeable* if there is a Γ-invariant Borel probability measure on $\mathcal{T}(\Gamma)$. The group Γ is *almost treeable* if for every finite set $F \subset \Gamma$ and every $\epsilon > 0$ there exists a Γ-invariant Borel probability measure μ on $\mathcal{F}(\Gamma)$ such that if $x \in \mathcal{F}(\Gamma)$ is random with law μ, then with probability greater than or equal to $1-\epsilon$ the set F is contained in a connected component of G_x. In particular, if Γ is treeable, then Γ is almost treeable.

Treeability was introduced in [2] and almost treeability first appeared in [16]. The connection between almost treeability and L^2-Betti numbers is furnished by the following.

LEMMA 8.3
If Γ is almost treeable, then $b_k^{(2)}(\Gamma) = 0$ for every $k \geq 2$.

Proof
This is [16, Theorem 0.8]. □

It is technically easier to work in the realm of equivalence relations. So we introduce the following definitions.

Definition 25
Let (X, μ) be a standard Borel probability space, and let $E \subset X \times X$ be a discrete Borel equivalence relation. (*Discrete* means that every equivalence class is at most countable.) We say that E is *treeable* (mod μ) if there exists a Borel subset $H \subset E$ such that H is symmetric (so $(a,b) \in H \Rightarrow (b,a) \in H$) and the graph G_H with vertex set X and edge set $\{\{a,b\} : (a,b) \in H\}$ is such that for μ-almost every $x \in X$ the connected component of G_H containing x is a tree spanning the E-class of x.

We say that E is *almost treeable* (mod μ) if there is a sequence $\{H_i\}_{i=1}^{\infty}$ of symmetric Borel subsets $H_i \subset E$ such that the corresponding graphs G_{H_i} are forests and for almost every $x \in X$ and any y in the E-class of x we have that x and y are contained in the same component of H_i for all but finitely many i.

The connection between equivalence relations and groups is given by the following.

PROPOSITION 8.4
A group Γ is treeable if and only if there is a free probability-measure-preserving (pmp) action $\Gamma \curvearrowright (X, \mu)$ such that if E is the orbit-equivalence relation $E = \{(x, gx) : x \in X, g \in \Gamma\}$, then E is treeable (mod μ). Similarly, Γ is almost treeable if and only if there is a free pmp action $\Gamma \curvearrowright (X, \mu)$ such that the orbit-equivalence relation E is almost treeable (mod μ).

Proof
In the case of treeability, this is [24, Proposition 30.1]. The almost treeable case is similar (and an easy exercise). □

LEMMA 8.5

If Γ is treeable and $\Gamma' < \Gamma$, then Γ' is treeable. Similarly if Γ is almost treeable and $\Gamma' < \Gamma$, then Γ' is almost treeable.

Proof

This is a consequence of Proposition 8.4 and [15, Propositions 5.8 and 5.16]. □

LEMMA 8.6

Treeability and almost treeability are measure-equivalence invariants. Therefore, if Γ_1, Γ_2 are lattices in a locally compact group G and Γ_1 is almost treeable, then Γ_2 is almost treeable.

Proof

In the case of treeability this is [15, Proposition 6.5]. Almost treeability is similar. □

LEMMA 8.7

If Γ is the fundamental group of a surface, then Γ is treeable.

Proof

If Γ is free, then this is obvious as the usual Cayley graph of Γ is a tree. If Γ is amenable, then this is a well-known consequence of the fact that there is a unique hyperfinite II_1-equivalence relation [29] (see also [24, Chapter III, Proposition 30.1] to see the connection). If Γ is the fundamental group of a closed surface of genus greater than or equal to 2, then Γ is measure-equivalent to a free group since Γ can be realized as a lattice in $\mathrm{Isom}(\mathbb{H}^2)$ (and so can any finite-rank nonamenable free group). Lemma 8.6 now implies that Γ is treeable. □

LEMMA 8.8

Lattices in $\mathrm{Isom}(\mathbb{H}^3)$ *are almost treeable.*

Proof

Let $\Lambda < \mathrm{Isom}(\mathbb{H}^3)$ be a lattice such that $\mathbb{H}^3/\Lambda$ is a manifold which fibers over a circle with the fiber being a noncompact surface. It is well known that such lattices exist (see, e.g., [22]). Note that Λ can be expressed as $\Lambda = F_r \rtimes_\theta \mathbb{Z}$ where F_r denotes the free group of some rank $r \geq 2$ and $\theta : F_r \to F_r$ is an automorphism. We can therefore write elements of Λ as pairs (f,n) with $f \in F_r$ and $n \in \mathbb{Z}$ subject to the multiplication rule

$$(f,n)(g,m) = \big(f\theta^n(g), n+m\big).$$

Now let $p > 0$ be an integer, and let i be a uniformly random integer in $\{0,\dots,p-1\}$.

Let $S = \{s_1, \dots, s_r\} \subset F_r$ be a free generating set. Let $E_i \in \mathcal{G}(\Lambda)$ be the set containing

- $\{(f,m),(fs_j,m)\}$ for every $f \in F_r$, $1 \le j \le r$, and $m \in \mathbb{Z}$ with $p \mid (m-i)$;
- $\{(f,m),(f,m+1)\}$ for every $f \in F_r$ and $m \in \mathbb{Z}$ with $p \nmid (m-i-1)$.

Observe that the graph with vertex set Λ and edge set E_i is a forest. Moreover, the law of E_i is an invariant probability measure λ_p on $\mathcal{G}(\Lambda)$. Finally, for any $(f,n),(g,m) \in \Lambda$ with $n \le m$, $(f,n),(g,m)$ are in the same connected component of (Λ, E_i) if and only if there does not exist an integer q with $n \le q < m$ such that $p \nmid (q-i-1)$. This occurs with probability equal to $(p - |m-n|)/p$ if $|m-n| \le p$. In particular, this probability tends to 1 as $p \to \infty$. This implies that Λ is almost treeable.

By Lemma 8.6, it follows that every lattice in $\mathrm{Isom}(\mathbb{H}^3)$ is almost treeable. □

LEMMA 8.9

If Γ' is a subgroup of the fundamental group Γ of a complete finite-volume hyperbolic 3-manifold, then $b_d^{(2)}(\Gamma') = 0$ for every $d \ge 2$.

Proof

This is true because Γ is almost treeable by Lemma 8.8, every subgroup of an almost treeable group is almost treeable by Lemma 8.5, and any almost treeable group Λ has $b_d^{(2)}(\Lambda) = 0$ for every $d \ge 2$ by Lemma 8.3. □

Proof of Corollary 1.3

This follows from Theorem 1.2 and Proposition 8.2. □

Appendices

A. Pointed subsets and measures of a metric space

The purpose of this appendix is to prove Theorem 3.1. We begin by studying pointed measures and pointed subspaces of a given metric space Z and their limits.

Definition 26

A *pointed measure* on a topological space Z is a pair (μ, p) where $p \in Z$ and μ is a Borel measure on Z. A *pointed subset* of Z is a pair (X, p) where $X \subset Z$ and $p \in Z$.

Definition 27

Given a subset F of a metric space Z, let $N_Z^o(F, \epsilon)$ denote the open ϵ-neighborhood of F in Z.

Definition 28
We say that two pointed measures $(\mu_1, p_1), (\mu_2, p_2)$ on a metric space Z are (ϵ, R)-*related* if, for every closed $F_i \subset B_Z(p_i, R)$,

$$\mu_1(F_1) < \mu_2\big(N_Z^o(F_1, \epsilon)\big) + \epsilon, \qquad \mu_2(F_2) < \mu_1\big(N_Z^o(F_2, \epsilon)\big) + \epsilon,$$

and $\mathrm{dist}_Z(p_1, p_2) < \epsilon$. We say two pointed subsets $(X_1, p_1), (X_2, p_2)$ of Z are (ϵ, R)-*related* if $\mathrm{dist}_Z(p_1, p_2) < \epsilon$ and

$$B_Z(p_1, R) \cap X_1 \subset N_Z^o(X_2, \epsilon), \qquad B_Z(p_2, R) \cap X_2 \subset N_Z^o(X_1, \epsilon).$$

A sequence $\{(X_i, p_i)\}_{i=1}^{\infty}$ of pointed closed subsets of Z converges to (X_∞, p_∞) in the *pointed Hausdorff topology* if, for every $\epsilon, R > 0$, there is an I such that $i > I$ implies that (X_i, p_i) and (X_∞, p_∞) are (ϵ, R)-related.

LEMMA A.1
If pointed measures $(\mu_1, p_1), (\mu_2, p_2)$ are (ϵ_1, R_1)-related and $(\mu_2, p_2), (\mu_3, p_3)$ are (ϵ_2, R_2)-related, then $(\mu_1, p_1), (\mu_3, p_3)$ are $(\epsilon_1 + \epsilon_2, R_3)$-related where $R_3 = \min\{R_1 - 2\epsilon_2, R_2 - 2\epsilon_1\}$. Similarly, if $(X_1, p_1), (X_2, p_2)$ are (ϵ_1, R_1)-related pointed subsets and $(X_2, p_2), (X_3, p_3)$ are (ϵ_2, R_2)-related pointed subsets, then (X_1, p_1), (X_3, p_3) are $(\epsilon_1 + \epsilon_2, R_3)$-related.

Proof
Let $F \subset B_Z(p_1, R_3) \subset B_Z(p_1, R_1)$ be closed. Then

$$N_Z^o(F, \epsilon_1) \subset B_Z(p_1, R_3 + \epsilon_1) \subset B_Z(p_2, R_3 + 2\epsilon_1) \subset B_Z(p_2, R_2).$$

Therefore,

$$\begin{aligned}\mu_1(F) &< \mu_2\big(N_Z^o(F, \epsilon_1)\big) + \epsilon_1 < \mu_3\big(N_Z^o\big(N_Z^o(F, \epsilon_1), \epsilon_2\big)\big) + \epsilon_1 + \epsilon_2 \\ &\le \mu_3\big(N_Z^o(F, \epsilon_1 + \epsilon_2)\big) + \epsilon_1 + \epsilon_2.\end{aligned}$$

The other inequality is similar. The result for pointed subsets is similar. □

LEMMA A.2
Let Z be a proper metric space. Let (μ_i, p_i) (for $1 \le i \le \infty$) be pointed Radon measures of Z with $\lim_{i\to\infty} p_i = p_\infty$. Then $\lim_{i\to\infty} \mu_i = \mu_\infty$ in the weak topology if and only if for every $\epsilon, R > 0$ there exists I such that $i > I$ implies that (μ_i, p_i) and (μ_∞, p_∞) are (ϵ, R)-related.*

Proof
Suppose that $\lim_{i\to\infty} \mu_i = \mu_\infty$ in the weak* topology. Let $\epsilon, R > 0$. Let $\mathcal{F} \subset C_c(Z)$

be a finite set such that for every compact subset $F \subset B_Z(p_\infty, R+\epsilon)$ there exists $g \in \mathcal{F}$ such that $g = 1$ on F, $g = 0$ on the complement of $N_Z^o(F,\epsilon)$, and $0 \le g \le 1$ on all of Z. To see that such a set exists, let $\mathcal{O}$ be any finite open cover of $B_Z(p_\infty, R+\epsilon)$ by open balls of radius less than ϵ. Let $\mathcal{F}' = \{g_U : U \in \mathcal{O}\}$ be a partition of unity subordinate to $\mathcal{O}$. Let $\mathcal{F}$ be the set of all sums of the form $\sum\{g_U : U \in \mathcal{O}'\}$ over all subsets $\mathcal{O}' \subset \mathcal{O}$. If $F \subset B_Z(p_\infty, R+\epsilon)$ is compact and $g = \sum\{g_U : U \in \mathcal{O}, U \cap F \neq \emptyset\}$, then $g = 1$ on F, $0 \le g \le 1$, and $g = 0$ on the complement of $N_Z^o(F,\epsilon)$ as required.

Let I be large enough so that $i > I$ implies that $\operatorname{dist}_Z(p_i, p_\infty) < \epsilon$ and $|\mu_i(g) - \mu_\infty(g)| < \epsilon$ for all $g \in \mathcal{F}$. Let $F \subset B_Z(p_i, R)$ be closed. Then $F \subset B_Z(p_\infty, R+\epsilon)$. So there exists $g \in \mathcal{F}$ as above. Observe that

$$\mu_i(F) \le \int g\, d\mu_i < \epsilon + \int g\, d\mu_\infty \le \epsilon + \mu_\infty\big(N_Z^o(F,\epsilon)\big).$$

Similarly, if $F \subset B_Z(p_\infty, R)$, then

$$\mu_\infty(F) \le \int g\, d\mu_\infty < \epsilon + \int g\, d\mu_i \le \epsilon + \mu_i\big(N_Z^o(F,\epsilon)\big).$$

This shows that μ_i, μ_∞ are (ϵ, R)-related.

Now suppose that for every $\epsilon, R > 0$ there exists I such that $i > I$ implies that (μ_i, p_i) and (μ_∞, p_∞) are (ϵ, R)-related. Then there exist sequences $\{\epsilon_i\}_{i=1}^\infty, \{R_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} \epsilon_i = 0$, $\lim_{i\to\infty} R_i = +\infty$, and (μ_i, p_i) and (μ_∞, R_∞) are (ϵ_i, R_i)-related.

CLAIM 1

For any compact $S \subset Z$,

$$\lim_{i\to\infty} \mu_i\big(N_Z^o(S,\epsilon_i)\big) = \mu_\infty(S).$$

Proof

For all sufficiently large i, $S \subset B_Z(p_i, R_i - \epsilon_i) \cap B_Z(p_\infty, R_i - \epsilon_i)$. So

$$\mu_\infty(S) \le \mu_i\big(N_Z^o(S,\epsilon_i)\big) + \epsilon_i \le \mu_\infty\big(N_Z^o(S,2\epsilon_i)\big) + 2\epsilon_i.$$

By taking the limit as $i \to \infty$, the claim follows. This uses the fact that $\mu_\infty(N_Z^o(S, 2\epsilon_i))$ is finite for all sufficiently large i, which is true because μ_∞ is Radon and Z is proper. $\square$

Now let f be a real-valued compactly supported continuous function on Z. It suffices to show that $\lim_{i\to\infty} \mu_i(f) = \mu_\infty(f)$. Let S denote the support of f, and for $\alpha < \beta$ let

$$F(\alpha,\beta) = \{x \in Z : \alpha \le f(x) \le \beta\} \cap S.$$

Let $\{\alpha_t\}_{t=1}^r$ be a sequence of real numbers such that $\alpha_1 < \min\{f(x) : x \in Z\} < \alpha_2 < \cdots < \max\{f(x) : x \in Z\} < \alpha_r$ and $\mu_\infty(F(\alpha_t,\alpha_t)) = 0$ for every $t = 1,\dots,r$.

By Claim 1,

$$\begin{aligned}
\limsup_{i\to\infty} \int f\, d\mu_i &\le \limsup_{i\to\infty} \sum_{t=1}^{r-1} \alpha_{t+1}\mu_i\big(N^o_Z\big(F(\alpha_t,\alpha_{t+1}),\epsilon_i\big)\big)\\
&= \sum_{t=1}^{r-1} \alpha_{t+1}\mu_\infty\big(F(\alpha_t,\alpha_{t+1})\big)\\
&\le \big(\sup_{1\le t<r} \alpha_{t+1}-\alpha_t\big)\mu_\infty(S) + \int f\, d\mu_\infty.
\end{aligned}$$

We now minimize over all such sequences $\{\alpha_t\}_{t=1}^r$ to obtain $\limsup_{i\to\infty}\int f\, d\mu_i \le \int f\, d\mu_\infty$. Similarly,

$$\begin{aligned}
&\liminf_{i\to\infty} \int f\, d\mu_i\\
&\quad\ge \liminf_{i\to\infty} \sum_{t=1}^{r-1} \int_{N^o_Z(F(\alpha_t,\alpha_{t+1}),\epsilon_i)} f\, d\mu_i - 2\|f\|_\infty \sum_{t=1}^{r} \mu_i\big(N^o_Z\big(F(\alpha_t,\alpha_t),\epsilon_i\big)\big)\\
&\quad= \liminf_{i\to\infty} \sum_{t=1}^{r-1} \int_{N^o_Z(F(\alpha_t,\alpha_{t+1}),\epsilon_i)} f\, d\mu_i\\
&\quad\ge \liminf_{i\to\infty} \sum_{t=1}^{r-1} \alpha_t\mu_i\big(N^o_Z\big(F(\alpha_t,\alpha_{t+1}),\epsilon_i\big)\big) = \sum_{t=1}^{r-1} \alpha_t\mu_\infty\big(F(\alpha_t,\alpha_{t+1})\big)\\
&\quad\ge -\big(\sup_{1\le t<r} \alpha_{t+1}-\alpha_t\big)\mu_\infty(S) + \int f\, d\mu_\infty.
\end{aligned}$$

Maximizing over all such sequences $\{\alpha_t\}_{t=1}^r$ and combining with the previous inequality, we obtain $\lim_{i\to\infty}\int f\, d\mu_i = \int f\, d\mu_\infty$. Because f is arbitrary, this implies that $\lim_{i\to\infty}\mu_i = \mu_\infty$ as required. □

B. mm-spaces

We can now define (ϵ, R)-related pointed mm^n-spaces. This will allow us to define open neighborhoods in $\mathbb{M}^n$.

Definition 29 ((ϵ, R)-related mm^n-spaces)
We say that mm^n-spaces $(M_1, p_1), (M_2, p_2)$ are (ϵ, R)-related if there exist a metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that

- $(\varphi_1(M_1), \varphi_1(p_1))$, $(\varphi_2(M_2), \varphi_2(p_2))$ are (ϵ, R)-related as pointed subsets of Z;
- for every $k = 1, \dots, n$, $((\varphi_1)_* \operatorname{vol}^{(k)}_{M_1}, \varphi_1(p_1))$ and $((\varphi_2)_* \operatorname{vol}^{(k)}_{M_2}, \varphi_2(p_2))$ are (ϵ, R)-related as pointed measures of Z.

Let $N_{\epsilon,R}(M, p)$ denote the set of all $[M', p'] \in \mathbb{M}^n$ such that (M', p') is (ϵ', R')-related to (M, p) for some $\epsilon' < \epsilon$ and $R' > R$. We show below that this is an open set.

Definition 30
A *pseudometric* d on a set X is a function $d : X \times X \to [0, \infty)$ satisfying all the properties of a metric with one exception: it may happen that $d(x, y) = 0$ even if $x \neq y$.

LEMMA B.1
Let Z be a set equal to a disjoint union $Z = \bigsqcup_{i=1}^{\infty} M_i$ of its subsets M_i. Suppose that for each i there is a metric $\operatorname{dist}_{M_i}$ on M_i, suppose that there is a collection $\{L_j\}_{j \in J}$ of subsets $L_j \subset Z$, and suppose that for each j there is a pseudometric $\operatorname{dist}_{L_j}$ on L_j. Suppose as well that if $x, y \in L_j \cap M_i$ for some i, j, then $\operatorname{dist}_{M_i}(x, y) = \operatorname{dist}_{L_j}(x, y)$. Lastly, we assume that for any $x, y \in Z$ there is a sequence $x = x_1, x_2, \dots, x_n = y$ such that for each i either $x_i, x_{i+1} \in M_k$ for some k or $x_i, x_{i+1} \in L_j$ for some j. Then there is a pseudometric dist_Z on Z such that

- *$\operatorname{dist}_Z(x, y) = \operatorname{dist}_{M_i}(x, y)$ for any $x, y \in M_i$, for any i;*
- *$\operatorname{dist}_Z(x, y) \leq \operatorname{dist}_{L_j}(x, y)$ for any $x, y \in L_j$ for any j.*

Proof
For each $x, y \in Z$ we define $\operatorname{dist}_Z(x, y) = \inf \sum_{k=1}^{r} \operatorname{dist}_{N_k}(x_k, x_{k+1})$ where the infimum is over all sequences $x = x_1, \dots, x_r = y$ and choices $N_k \in \{M_i\}_{i=1}^{\infty} \cup \{L_j\}_{j \in J}$ such that $x_k, x_{k+1} \in N_k$ for all $1 \leq k < r$. It is easy to check that the conclusions hold. $\square$

LEMMA B.2
Suppose that $\{(M_i, p_i)\}_{i=1}^{\infty}$ is a sequence of mm^n spaces such that (M_i, p_i) and (M_j, p_j) are (ϵ_{ij}, R_{ij})-related for all i, j (where ϵ_{ij}, R_{ij} are positive real numbers). Then there exist a complete separable metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that for all i, j, k

- $(\varphi_i(M_i), \varphi_i(p_i))$, $(\varphi_j(M_j), \varphi_j(p_j))$ *are* (ϵ_{ij}, R_{ij})*-related as pointed subsets of* Z*;*
- $((\varphi_i)_* \operatorname{vol}^{(k)}_{M_i}, \varphi_i(p_i))$ *and* $((\varphi_j)_* \operatorname{vol}^{(k)}_{M_j}, \varphi_j(p_j))$ *are* (ϵ_{ij}, R_{ij})*-related as pointed measures of* Z*.*

Proof
For each i, j, there exist a complete separable metric space Y_{ij} and isometric embeddings $\phi_{ij} : M_i \to Y_{ij}, \psi_{ij} : M_j \to Y_{ij}$ such that
- $(\phi_{ij}(M_i), \phi_{ij}(p_i))$, $(\psi_{ij}(M_j), \psi_{ij}(p_j))$ are (ϵ_{ij}, R_{ij})-related as pointed subsets of Y_{ij};
- $((\phi_{ij})_* \operatorname{vol}^{(k)}_{M_i}, \phi_{ij}(p_i))$ and $((\psi_j)_* \operatorname{vol}^{(k)}_{M_j}, \psi_{ij}(p_j))$ are (ϵ_{ij}, R_{ij})-related as pointed measures of Y_{ij} for every k.

Let Z' be the disjoint union of M_i $(i = 1, 2, \ldots)$. By Lemma B.1 there exists a pseudometric $\operatorname{dist}_{Z'}$ on Z' satisfying the following.
- If $x, x' \in M_i \subset Z'$, then $\operatorname{dist}_{Z'}(x, x') = \operatorname{dist}_{M_i}(x, x')$.
- If $x_i \in M_i, x_j \in M_j$, then $\operatorname{dist}_{Z'}(x_i, x_j) \le \operatorname{dist}_{Y_{ij}}(\phi_{ij}(x_i), \psi_{ij}(x_j))$.

This induces an equivalence relation on Z' by $x \sim y$ if $\operatorname{dist}_{Z'}(x, y) = 0$. Let $Z'' = Z'/\sim$ with the metric $\operatorname{dist}_{Z''}([x], [y]) = \operatorname{dist}_{Z'}(x, y)$. Let $(Z, \operatorname{dist}_Z)$ be the metric completion of $(Z'', \operatorname{dist}_{Z''})$. For each i there is a canonical isometric embedding $\varphi_i : M_i \to Z$ and the union of the images of these embeddings is dense in Z. So Z is separable.

For any i, j, there is a map $\pi_{ij} : \phi_{ij}(M_i) \cup \psi_{ij}(M_j) \to Z$ such that $\pi_{ij}(\phi_{ij}(x_i)) = \varphi_i(x_i)$ if $x_i \in M_i$ and $\pi_{ij}(\psi_{ij}(x_j)) = \varphi_j(x_j)$ if $x_j \in M_j$. This map is distance non-increasing: $\operatorname{dist}_{Y_{ij}}(x, y) \ge \operatorname{dist}_Z(\pi_{ij}(x), \pi_{ij}(y))$. Since $((\phi_{ij})_* \operatorname{vol}^{(k)}_{M_i}, \phi_{ij}(p_i))$ and $((\psi_j)_* \operatorname{vol}^{(k)}_{M_j}, \psi_{ij}(p_j))$ are (ϵ_{ij}, R_{ij})-related this implies that $((\varphi_i)_* \operatorname{vol}^{(k)}_{M_i}, \varphi_i(p_i))$ and $((\varphi_j)_* \operatorname{vol}^{(k)}_{M_j}, \varphi_j(p_j))$ are (ϵ_{ij}, R_{ij})-related. Similarly, $(\varphi_i(M_i), \varphi_i(p_i))$, $(\varphi_j(M_j), \varphi_j(p_j))$ are (ϵ_{ij}, R_{ij})-related as required. $\square$

LEMMA B.3
If $(M_1, p_1), (M_2, p_2)$ *are* (ϵ_1, R_1)*-related and* $(M_2, p_2), (M_3, p_3)$ *are* (ϵ_2, R_2)*-related, then* $(M_1, p_1), (M_3, p_3)$ *are* $(\epsilon_1 + \epsilon_2, R_3)$*-related where* $R_3 = \min(R_1 - 2\epsilon_2, R_2 - 2\epsilon_1)$.

Proof
This follows from Lemmas B.2 and A.1. $\square$

PROPOSITION B.4
A sequence $\{[M_i, p_i]\}_{i=1}^{\infty} \subset \mathbb{M}^n$ *converges to* $[M_\infty, p_\infty] \in \mathbb{M}^n$ *if and only if for every*

$\epsilon, R > 0$ *there exists an* I *such that* $i > I$ *implies that* (M_i, p_i) *is* (ϵ, R)*-related to* (M_∞, p_∞).

Proof

Suppose that $\{[M_i, p_i]\}_{i=1}^\infty \subset \mathbb{M}^n$ converges to $[M_\infty, p_\infty] \in \mathbb{M}^n$. By definition, this means that there exist a complete separable proper metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that $(\varphi_i(M_i), \varphi_i(p_i))$ converges to $(\varphi_\infty(M_\infty), \varphi_\infty(p_\infty))$ in the pointed Hausdorff topology and $(\varphi_i)_* \operatorname{vol}_{M_i}^{(k)}$ converges to $(\varphi_\infty)_* \operatorname{vol}_{M_\infty}^{(k)}$ as $i \to \infty$. The proposition now follows from Lemma A.2.

Let us now assume for every $\epsilon, R > 0$ that there exists I such that $i > I$ implies (M_i, p_i) is (ϵ, R)-related to (M_∞, p_∞). By Lemma B.3 this implies that, for any $i, j > I$, (M_i, p_i) and (M_j, p_j) are $(2\epsilon, R - 2\epsilon)$-related. So there exist positive real numbers ϵ_i, R_i such that

- $\lim_{i\to\infty} \epsilon_i = 0$, $\lim_{i\to\infty} R_i = +\infty$;
- $(M_i, p_i), (M_j, p_j)$ are (ϵ_i, R_i)-related for every $1 \le i < j \le \infty$.

By Lemma B.2, there exist a complete separable metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that for every $1 \le i < j \le \infty$

- $(\varphi_i(M_i), \varphi_i(p_i)), (\varphi_j(M_j), \varphi_j(p_j))$ are (ϵ_i, R_i)-related;
- for every $k = 1, \dots, n$, $((\varphi_i)_* \operatorname{vol}_{M_i}^{(k)}, \varphi_i(p_i)), ((\varphi_j)_* \operatorname{vol}_{M_j}^{(k)}, \varphi_j(p_j))$ are (ϵ_i, R_i)-related.

By replacing Z with the closure of the images of the M_i's, we may assume, without loss of generality, that the union $\bigcup_{i=1}^\infty \varphi_i(M_i)$ is dense in Z. Without loss of generality, we may also assume that each $M_i \subset Z$ and φ_i is the inclusion map. This helps simplify notation.

We claim that Z is proper. It suffices to show that every ball centered at p_∞ is sequentially compact. So let $R > 0$, and let $\{x_i\}_{i=1}^\infty \subset B_Z(p_\infty, R)$. There is a sequence $\{y_i\}_{i=1}^\infty$ such that, for each i, $\operatorname{dist}_Z(x_i, y_i) < 1/i$ and $y_i \in M_{n(i)}$ for some $n(i)$. It suffices to show that a subsequence of $\{y_i\}_{i=1}^\infty$ is convergent. If there is some j such that $\{y_i\}_{i=1}^\infty \cap M_j$ is infinite, then, since M_j is proper, it follows that there is a convergent subsequence. Otherwise, $\lim_{i\to\infty} n(i) = +\infty$.

Observe that

$$\operatorname{dist}_Z(p_i, y_i) \le \operatorname{dist}_Z(p_i, p_\infty) + \operatorname{dist}_Z(p_\infty, x_i) + \operatorname{dist}_Z(x_i, y_i) \le \epsilon_{n(i)} + R + 1/i.$$

In other words, $y_i \in B_Z(p_i, R + 1/i + \epsilon_{n(i)})$. If i is large enough, then $R_{n(i)} > R + 1/i + \epsilon_{n(i)}$. Because $(M_{n(i)}, p_{n(i)}), (M_\infty, p_\infty)$ are $(\epsilon_{n(i)}, R_{n(i)})$-related,

$$B_Z(p_i, R + 1/i + \epsilon_{n(i)}) \cap M_{n(i)} \subset N_Z^o(M_\infty, \epsilon_{n(i)}).$$

So there exists $z_i \in M_\infty$ with $\operatorname{dist}_Z(y_i, z_i) \le \epsilon_{n(i)}$. Note that

$$\mathrm{dist}_Z(z_i, p_\infty) \leq \mathrm{dist}_Z(z_i, y_i) + \mathrm{dist}_Z(y_i, x_i) + \mathrm{dist}_Z(x_i, p_\infty) \leq \epsilon_{n(i)} + 1/i + R.$$

Because M_∞ is proper, this implies that $\{z_i\}_{i=1}^\infty$ has a convergent subsequence. Since $\mathrm{dist}_Z(z_i, x_i) \leq \mathrm{dist}_Z(z_i, y_i) + \mathrm{dist}_Z(y_i, x_i) \leq \epsilon_{n(i)} + 1/i$ tends to zero as $i \to \infty$, this implies that $\{x_i\}_{i=1}^\infty$ has a convergent subsequence as required.

The proposition now follows from Lemma A.2. □

LEMMA B.5
For any $[M, p] \in \mathbb{M}^n$ and $\epsilon, R > 0$, the set $N_{\epsilon,R}(M, p) \subset \mathbb{M}^n$ is open.

Proof
Let $\{[M_i, p_i]\}_{i=1}^\infty$ be a sequence in $\mathbb{M}^n \setminus N_{\epsilon,R}(M, p)$ which converges to $[M_\infty, p_\infty]$. If $[M_\infty, p_\infty] \in N_{\epsilon,R}(M, p)$, then there exist an $\epsilon' < \epsilon$ and $R' > R$ such that (M_∞, p_∞) and (M, p) are (ϵ', R')-related. Choose $\epsilon'', R'' > 0$ so that $\epsilon'' + \epsilon' < \epsilon$ and $R < \min(R' - 2\epsilon'', R'' - 2\epsilon')$. By Proposition B.4, there is an i such that (M_i, p_i) and (M_∞, p_∞) are (ϵ'', R'')-related. Lemma B.3 now implies that $[M_i, p_i] \in N_{\epsilon,R}(M, p)$. This contradiction proves that the complement of $N_{\epsilon,R}(M, p)$ is closed. □

We can now prove Theorem 3.1, which states that $\mathbb{M}^n$ is separable and metrizable.

Proof of Theorem 3.1
First we show that $\mathbb{M}^n$ is metrizable. For $[M, p], [M', p'] \in \mathbb{M}^n$, let

$$\rho\big([M, p], [M', p']\big) = \inf \epsilon + \frac{1}{R + 2\epsilon},$$

where the infimum is over all $\epsilon, R > 0$ such that $[M, p]$ and $[M', p']$ are (ϵ, R)-related. In order to check the triangle inequality, let $[M_i, p_i] \in \mathbb{M}^n$ (for $i = 1, 2, 3$), and suppose that $(M_1, p_1), (M_2, p_2)$ are (ϵ_1, R_1)-related and $(M_2, p_2), (M_3, p_3)$ are (ϵ_2, R_2)-related for some $\epsilon_1, \epsilon_2, R_1, R_2 > 0$. By Lemma B.3,

$$\begin{aligned}
\rho\big([M_1, p_1], [M_3, p_3]\big) &\leq \epsilon_1 + \epsilon_2 + \frac{1}{\min\{R_1 - 2\epsilon_2, R_2 - 2\epsilon_1\} + 2\epsilon_1 + 2\epsilon_2} \\
&= \epsilon_1 + \epsilon_2 + \frac{1}{\min\{R_1 + 2\epsilon_1, R_2 + 2\epsilon_2\}} \\
&\leq \Big(\epsilon_1 + \frac{1}{R_1 + 2\epsilon_1}\Big) + \Big(\epsilon_2 + \frac{1}{R_2 + 2\epsilon_2}\Big).
\end{aligned}$$

By minimizing the right-hand side over all $\epsilon_1, \epsilon_2, R_1, R_2$ such that $(M_1, p_1), (M_2, p_2)$ are (ϵ_1, R_1)-related and $(M_2, p_2), (M_3, p_3)$ are (ϵ_2, R_2)-related, we see that ρ satisfies the triangle inequality. It is therefore a metric on $\mathbb{M}^n$. It is continuous by Lemma B.5. So $\mathbb{M}^n$ is metrizable.

To show that $\mathbb{M}^n$ is separable, let $\mathbb{F}^n_{\mathbb{Q}}$ be the set of all $[M, p] \in \mathbb{M}^n$ such that M is a finite set and $\mathrm{dist}_M, \mathrm{vol}^{(1)}_M, \dots, \mathrm{vol}^{(n)}_M$ are rational-valued. Note that $\mathbb{F}^n_{\mathbb{Q}}$ is countable. We claim that $\mathbb{F}^n_{\mathbb{Q}}$ is dense in $\mathbb{M}^n$. Let $\mathbb{F}^n$ be the set of all $[M, p] \in \mathbb{M}^n$ such that M is finite. An exercise shows that the closure of $\mathbb{F}^n_{\mathbb{Q}}$ contains $\mathbb{F}^n$. So it suffices to show that $\mathbb{F}^n$ is dense in $\mathbb{M}^n$. For this purpose, let $[M, p] \in \mathbb{M}^n$. Let $\mathcal{M}^{\mathbb{F}}(M)$ denote the set of all measures $\mu \in \mathcal{M}(M)$ with finite support. It is well known that $\mathcal{M}^{\mathbb{F}}(M)$ is dense in the space of Radon measures on M in the weak* topology. So there exist measures $\mu^{(k)}_i \in \mathcal{M}^{\mathbb{F}}(M)$ such that $\lim_{i\to\infty} \mu^{(k)}_i = \mathrm{vol}^{(k)}_M$ for every k. Let X_i be a finite subset of M containing $\{p\} \cup \bigcup_{k=1}^n \mathrm{supp}(\mu^{(k)}_i)$ such that $\bigcup_{i=1}^\infty X_i$ is dense in M. We may regard (X_i, p) as an mm^n-space with distance dist_{X_i} equal to the restriction of dist_M to X_i and measures $\mathrm{vol}^{(k)}_{X_i}$ equal to $\mu^{(k)}_i$. By definition $[X_i, p]$ converges to $[M, p]$ in $\mathbb{M}^n$ as $i \to \infty$. So $\mathbb{F}^n$ and therefore $\mathbb{F}^n_{\mathbb{Q}}$ are dense in $\mathbb{M}^n$ as claimed. □

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