On the stable Cannon Conjecture

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ON THE STABLE CANNON CONJECTURE

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Abstract. The Cannon Conjecture for a torsionfree hyperbolic group G with boundary homeomorphic to S^2 says that G is the fundamental group of an aspherical closed 3-manifold M. It is known that then M is a hyperbolic 3-manifold. We prove the stable version that for any closed manifold N of dimension greater or equal to 2 there exists a closed manifold M together with a simple homotopy equivalence $M \to N \times BG$. If N is aspherical and $\pi_1(N)$ satisfies the Farrell-Jones Conjecture, then M is unique up to homeomorphism.

0. Introduction

0.1. **The motivating conjectures by Wall and Cannon.** This paper is motivated by the following two conjectures which will be reviewed in Sections 1 and Sections 2.

Conjecture 0.1 (Wall's Conjecture on Poincaré duality groups and aspherical closed 3-manifolds)**.** *Every Poincaré duality group of dimension* 3 *is the fundamental group of an aspherical closed* 3*-manifold.*

Conjecture 0.2 (Cannon Conjecture in the torsionfree case)**.** *Let G be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to S^2.*

Then G is the fundamental group of a hyperbolic closed 3*-manifold.*

We want to investigate, whether these conjecture are true stably in the sense, that we ask whether for any closed smooth manifold N of dimension ≥ 2 the product $BG \times N$ is simply homotopy equivalent to a closed smooth manifold, and analogously in the PL and topological category.

0.2. **The main results.** In the sequel $\underline{\mathbb{R}^a}$ denotes the trivial a-dimensional vector bundle.

Theorem 0.3 (Vanishing of the surgery obstruction)**.** *Let G be a hyperbolic* 3*-dimensional Poincaré duality group.*

Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "f"] & BG \end{tikzcd}$$

satisfying

1. *The space BG is a finite* 3*-dimensional CW-complex;*
2. *The map $H_n(f,\mathbb{Z})\colon H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(BG;\mathbb{Z})$ is bijective for all $n \geq 0$;*
3. *The simple algebraic surgery obstruction $\sigma(f,\overline{f}) \in L_3^s(\mathbb{Z}G)$ vanishes.*

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Notice that the vanishing of the surgery obstruction does not imply that we can arrange by surgery that f is a simple homotopy equivalence since this works only in dimensions ≥ 5. In dimension 3 we can achieve at least a homology equivalence.

However, if we cross the normal map with a closed manifold N of dimension ≥ 2, the resulting normal map has also vanishing surgery obstruction by the product formula and hence can be transformed by surgery into a simple homotopy equivalence. Thus Theorem 0.3 implies assertion (1) of Theorem 0.4 below, the proof of assertion (2) of Theorem 0.4 below will require more work.

Theorem 0.4 (Stable Cannon Conjecture)**.** *Let G be a hyperbolic 3-dimensional Poincaré duality group. Let N be any smooth, PL or topological manifold respectively which is closed and whose dimension is ≥ 2.*

Then there is a closed smooth, PL or topological manifold M and a normal map of degree one

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\underline{f}"] \arrow[d] & \xi \times TN \arrow[d] \\ M \arrow[r, "f"] & BG \times N \end{tikzcd}$$

satisfying

(1) *The map f is a simple homotopy equivalence;*
(2) *Let $\widehat{M} \to M$ be the G-covering associated to the composite of the isomorphism $\pi_1(f)\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ with the projection $G \times \pi_1(N) \to G$. Suppose additionally that N is aspherical and* $\dim(N) \geq 3$.

    Then $\widehat{M}$ is homeomorphic to $\mathbb{R}^3 \times N$. Moreover, there is a compact topological manifold $\overline{\widehat{M}}$ whose interior is homeomorphic to $\widehat{M}$ and for which there exists a homeomorphism of pairs $(\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$.

We call a group G a *Farrell-Jones-groups* if it satisfies the Full Farrell-Jones Conjecture. We will review what is known about the class of Farrell-Jones groups in Theorem 4.1. At least we mention already here that every hyperbolic group, every CAT(0)-group, and the fundamental group of any (not necessarily compact) 3-manifold (possibly with boundary) is a Farrell-Jones groups.

We have the following uniqueness statement.

Theorem 0.5 (Borel Conjecture)**.** *Let M_0 and M_1 be two aspherical closed manifolds of dimension n satisfying $\pi_1(M) \cong \pi_1(N)$. Suppose one of the following conditions hold:*

- *We have $n \leq 3$;*
- *We have $n = 4$ and $\pi_1(M)$ is a Farrell-Jones group, which is good in the sense of Freedman* [26]*;*
- *We have $n \geq 5$ and $\pi_1(M)$ is a Farrell-Jones group.*

Then any map $f\colon M_0 \to M_1$ inducing an isomorphism on the fundamental groups is homotopic to a homeomorphism.

Proof. Obviously the Borel Conjecture is true im dimension $n \leq 1$. The Borel Conjecture is true in dimension ≤ 2 by the classification of closed manifolds of dimension 2. It is true in dimension 3 since Thurston's Geometrization Conjecture holds. This follows from results of Waldhausen (see Hempel [30, Lemma 10.1 and Corollary 13.7]) and Turaev, see [54], as explained for instance in [36, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [35, 42] following ideas of Perelman. The Borel Conjecture follows from surgery theory in dimension ≥ 4, see for instance [4, Proposition 0.3]. $\square$

One cannot replace homeomorphism by diffeomorphism in Theorem 0.5. The torus T^n for $n \geq 5$ is a counterexample, see [57, 15A]. Other counterexamples involving negatively curved manifolds are constructed by Farrell-Jones [24, Theorem 0.1].

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The paper is organized as follows:

Contents

1. Short review of Poincaré duality groups

Definition 1.1 (Poincaré duality group)**.** A *Poincaré duality group G of dimension n* is a group satisfying:

- G is of type FP;

- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

1.1. Basic facts about Poincaré duality groups.

- A Poincaré duality group is finitely generated and torsionfree;
- For $n \geq 4$ there exists n-dimensional Poincaré duality groups which are not finitely presented, see [18, Theorem C];
- A Poincar'e duality group of dimension $n \geq 3$ is a finitely dominated n-dimensional Poincaré complex in the sense of Wall [56] if and only if it is finitely presented, see [31, Theorem 1];
- If $n \geq 3$ and G is a finitely presented Poincaré duality group of dimension n such that $\widetilde{K}_0(\mathbb{Z}G)$ vanishes, then BG is homotopy equivalent to finite n-dimensional CW-complex, see [55, Theorem F];
- If G is the fundamental group of an aspherical closed manifold of dimension n, then BG is homotopy equivalent to a finite n-dimensional CW-complex and in particular G is finitely presented;
- To our knowledge there exists in the literature no example of a 3-dimensional Poincaré duality group, which is not homotopy equivalent to a finite 3-dimensional CW-complex;
- Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface. This result is due to Bieri, Eckmann and Linnell, see for instance [22].

1.2. Some prominent conjectures and results about Poincaré duality groups.

Conjecture 1.2 (Poincaré duality groups and aspherical closed manifolds)**.** *Every finitely presented Poincaré duality group is the fundamental group of an aspherical closed topological manifold.*

A weaker version is

Conjecture 1.3 (Poincaré duality groups and aspherical closed homology ANR-manifolds)**.** *Every finitely presented Poincaré duality group is the fundamental group of an aspherical closed homology* ANR*-manifold.*

Michel Boileau informed us about the following two facts:

Theorem 1.4. *A Poincaré duality group G of dimension* 3 *is the fundamental group of an aspherical closed* 3*-manifold if and only if G contains a subgroup H, which is the fundamental group of an aspherical closed* 3*-manifold.*

Proof. Let H be a subgroup of G which is the fundamental group of an aspherical closed 3-manifold. Next we show that the index of H in G is finite. Suppose that it is infinite. Then the cohomological dimension of H is smaller than the cohomological dimension of G by [53]. Since the cohomological dimension of both H and G is three, we get a contradiction. Hence the index of H in G is finite, The solution of Thurston's Geometrization Conjecture by Perelman, see [42], implies that G is the fundamental group of an aspherical closed 3-manifold, see for instance [27, Theorem 5.1]. $\square$

Moreover, Theorem 1.4 and the works of Cannon-Cooper [14], Eskin-Fisher-Whyte [23], Kapovich-Leeb [34], and Rieffel [49] imply

Theorem 1.5. *A Poincaré duality group G of dimension* 3 *is the fundamental group of an aspherical closed* 3*-manifold if and only if it is quasiisometric to the fundamental group of an aspherical closed* 3*-manifold.*

The next result is due to Bowditch [11, Corollary 0.5].

Theorem 1.6. *If a Poincaré duality group of dimension* 3 *contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert* 3*-manifold.*

The following result follows from the algebraic torus theorem of Dunwoody-Swenson [21].

Theorem 1.7. *Let G be a* 3*-dimensional Poincaré duality group. Then precisely one of the following statements are true:*

(1) It is the fundamental group of a closed Seifert 3*-manifold;*
(2) It splits over a subgroup $\mathbb{Z} \oplus \mathbb{Z}$;
(3) It is atoroidal, i.e., it contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Conjecture 1.8 (Weak hyperbolization Conjecture)**.** *An atoroidal* 3*-dimensional Poincaré duality group is hyperbolic.*

The next result is due to Kapovich-Kleiner [33, Theorem 2].

Theorem 1.9. *A* 3*-dimensional Poincaré duality group, which is a CAT(0)-group and atoroidal, is hyperbolic.*

We conclude from [9, Theorem 2.8 and Remark 2.9].

Theorem 1.10. *Let G be a hyperbolic* 3*-dimensional Poincaré duality group. Then its boundary is homeomorphic to S^2.*

1.3. **High-dimensions.**

Theorem 1.11 (Poincaré duality groups and homology ANR-manifolds)**.** *Let G be a finitely presented torsionfree group which is a Farrell-Jones group.*

(1) Then for $n \geq 6$ the following are equivalent:
   (a) G is a Poincaré duality group of dimension n;
   (b) There exists a closed homology ANR*-manifold M homotopy equivalent to BG. In particular, M is aspherical and $\pi_1(M) \cong G$;*
(2) If the statements in assertion (1) *hold, then the closed homology* ANR*-manifold M appearing there can be arranged to have the DDP, see Definition 6.2;*
(3) If the statements in assertion (1) *hold, then the closed homology* ANR*-manifold M appearing there is unique up to s-cobordism of homology* ANR*-manifolds;*

Proof. See Bartels-Lück-Weinberger [8, Theorem 1.2]. It relies strongly on the surgery theory for homology ANR-manifolds, see for instance [13] □

The question whether a closed homology ANR-manifold, which has dimension ≥ 5 and has the DDP, is a topological manifold is decided by Quinn's obstruction, see Section 8.

More information about Poincaré duality groups can be found for instance [19] and [58].

2. Short review of the Cannon Conjecture

The following conjecture is taken from [15, Conjecture 5.1].

Conjecture 2.1 (Cannon Conjecture)**.** *Let G be a hyperbolic group. Suppose that its boundary is homeomorphic to S^2.*

Then G acts properly cocompactly and isometrically on the 3*-dimensional hyperbolic space.*

If G is torsionfree, then the Cannon Conjecture 2.1 reduces to the Cannon Conjecture for torsionfree groups 0.2.

Remark 2.2. We mention that Conjecture 0.2 is open and does not follow from Thurston's Geometrization Conjecture which is known to be true by the work of Perelman, see Morgan-Tian [42].

The next result is due to Bestvina-Mess [10, Theorem 4.1] and says that for the Cannon Conjecture one just have to find some aspherical closed 3-manifold with G as fundamental group.

Theorem 2.3. *Let G be a hyperbolic group which is the fundamental group of an aspherical closed 3-manifold M.*

Then the universal covering $\widetilde{M}$ of M is homeomorphic to $\mathbb{R}^3$ and its compactification by ∂G is homeomorphic to D^3, and the Geometrization Conjecture of Thurston, implies that M is hyperbolic and G satisfies Cannon's Conjecture 0.2.

Ursula Hamenstädt informed us that she has a proof for the following result.

Theorem 2.4 (Hamenstädt)**.** *Let G be a hyperbolic group G whose boundary is homeomorphic to S^{n-1}.*

Then G acts properly and cocompactly on $S^{n-1} \times \mathbb{R}^n$.

Hamenstädt's result is proved by completely different methods and does not need the assumption that G is torsionfree. It aims for $n = 3$ at construction of the sphere tangent bundle of the universal covering of the conjectured hyperbolic 3-manifold M appearing in the Cannon Conjecture 2.1, where we aim at constructing $M \times N$ for any closed manifold N with $\dim(N) \geq 2$.

2.1. **The high-dimensional analogue of the Cannon Conjecture.** The following result is taken from [8, Theorem A].

Theorem 2.5 (High-dimensional Cannon Conjecture)**.** *Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. The following statements are equivalent:*

1. *The boundary ∂G is homeomorphic to S^{n-1};*
2. *There is a aspherical closed topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n;*

Moreover, the aspherical manifold M appearing in assertion (2) *is unique up to homeomorphism.*

In high dimensions there are exotic examples of hyperbolic n-dimensional Poincaré duality groups G, see [8, Section 5]. For instance, for any integer $k \geq 2$ there are examples satisfying $\partial G = S^{4k+1}$ such that G is the fundamental group of an aspherical closed topological manifold, but not of an aspherical closed smooth manifold. For $n \geq 6$ there exists an aspherical closed topological manifold whose fundamental group is hyperbolic and which cannot be triangulated, see [20, page 200].

We mention without giving the details that using the method of this paper one can prove Theorem 2.5 also in the case $n = 5$.

2.2. **The Cannon Conjecture 0.2 in the torsionfree case implies Theorem 0.3 and Theorem 0.4.** Let G be hyperbolic 3-dimensional Poincaré duality group. We want to show that then all claims in Theorem 0.3 and Theorem 0.4 are obviously true, provided that the Cannon Conjecture 0.2 in the torsionfree case holds for G.

We know already that there is a 3-dimensional finite model for BG and ∂G is S^2. By the Cannon Conjecture 0.2 we can find a hyperbolic closed 3-manifold together with a homotopy equivalence $f\colon M \to BG$. Since G is a Farrell-Jones group, f is a simple homotopy equivalence. We obviously can cover f by a bundle map $\overline{f}\colon TM \to \xi$ if we take ξ to be $(f^{-1})^*TM$ for some homotopy inverse $f^{-1}\colon BG \to M$ of f. Hence we get Theorem 0.3 and assertion (1) of Theorem 0.4. It remains to prove assertion (2) of Theorem 0.4.

The universal covering $\widetilde{M}$ is the hyperbolic 3-space. Hence it is homeomorphic to $\mathbb{R}^3$ and the compactification $\overline{\widetilde{M}} = \widetilde{M} \cup \partial G$ is homeomorphic to D^3. In particular $\overline{\widetilde{M}}$ is a compact manifold whose interior is $\widetilde{M}$ and whose boundary is S^2. Hence $\overline{\widetilde{M}} \times N$ is a compact manifold and there is a homeomorphism $(\overline{\widetilde{M}} \times N, \partial(\overline{\widetilde{M}} \times N)) \xrightarrow{\cong} (D^3 \times N, S^2 \times N)$.

2.3. **When does the Cannon Conjecture 0.2 in the torsionfree case follows from Theorem 0.4.** Next we discuss what would be needed to conclude the Cannon Conjecture 0.2 in the torsionfree case from Theorem 0.4.

Let G by a hyperbolic group such that ∂G is S^2. Then G is a 3-dimensional Poincaré duality group by Bestvina-Mess [10, Corollary 1.3]. Fix any aspherical closed manifold N of dimension ≥ 2 such that $\pi_1(N)$ is a Farrell-Jones group.

We get from Theorem 0.4 an aspherical closed $(3+\dim(N))$-dimensional manifold M together with a homotopy equivalence $f\colon M \to BG \times N$. Let $\alpha\colon \pi_1(M) \xrightarrow{\cong} G \times \pi_1(N)$ be the isomorphism $\pi_1(f)$. If M' is any other aspherical closed manifold together with an isomorphism $\alpha'\colon \pi_1(M') \xrightarrow{\cong} G \times \pi_1(N)$, then we conclude from Theorem 4.1 (1a) and (2b) that $\pi_1(M) \cong G \times \pi_1(N)$ is a Farrell-Jones group and from Theorem 0.5 that there exists a homeomorphism $u\colon M \to M'$ such that $\alpha' \circ \pi_1(u)$ and α agree (up to inner automorphisms). Hence the pair (M, α) is unique and thus an invariant depending on G and N only.

What does the Cannon Conjecture 0.2 tell us about (M, α) and what do we need to know about (M, α) in order to prove the Cannon Conjecture 0.2? This is answered by the next result.

Lemma 2.6. *The following statements are equivalent*

(1) The Cannon Conjecture 0.2 holds for G;

(2) There is a closed 3-manifold M' and a homeomorphism $h\colon M \xrightarrow{\cong} M' \times N$ such that for the projection $p\colon M' \times N \to N$ the map $\pi_1(p \circ h)$ agrees with the composite $\pi_1(M) \xrightarrow{\alpha} G \times \pi_1(N) \xrightarrow{\mathrm{pr}} \pi_1(N)$ for pr the projection;

(3) There is a closed 3-manifold M' and a map $p\colon M \to N$ with homotopy fiber M' such that $\pi_1(p)$ agrees with the composite $\pi_1(M) \xrightarrow{\alpha} G \times \pi_1(N) \xrightarrow{\mathrm{pr}} \pi_1(N)$ for pr the projection.

Proof. (1) $\implies$ (2). By the Cannon Conjecture 0.2 there exists a hyperbolic closed 3-manifold M' with $\pi(M') = G$. We can find a homotopy equivalence $h\colon M \to M' \times N$ with $\pi_1(h) = \alpha$. By Theorem 0.5 we can assume that h is a homeomorphism.

(2) $\implies$ (3) This is obvious.

(3) $\implies$ (1) The long exact homotopy sequence associated to p implies that $\pi_1(M') \cong G$ and M' is aspherical. We conclude from Theorem 2.3 that M' is a hyperbolic closed 3-manifold. Hence G satisfies the Cannon Conjecture 0.2. □

2.4. **The special case $N = T^k$.** Now suppose that in the situation of Subsection 2.3 we take $N = T^k$ for some $k \geq 2$. Then we get a criterion, where α does not appear anymore.

Lemma 2.7. *Fix an integer $k \geq 2$. Let M be an aspherical closed $(3+k)$-dimensional manifold with fundamental group $G \times \mathbb{Z}^k$. Then the following statements are equivalent*

1. *The Cannon Conjecture 0.2 holds for G;*
2. *There is closed 3-manifold M' together with a homeomorphism $h\colon M \xrightarrow{\cong} M' \times T^k$;*
3. *There is a closed 3-manifold M' and a map $p\colon M \to T^k$ with homotopy fiber M'.*

Proof. (1) $\implies$ (2) This follows from Theorem 2.6.

(2) $\implies$ (3) This is obvious.

(3) $\implies$ (1) First we explain that we can assume that $\pi_1(p)\colon \pi_1(M) \to \pi_1(T^k)$ is surjective. Since M' is compact and has only finitely many path components, we conclude from the long homotopy sequence that the image of $\pi_1(p)\colon \pi_1(M) \to \pi_1(T^k)$ has finite index. Let $q\colon T^k \to T^k$ be a finite covering such that the image of $\pi_1(p)$ and $\pi_1(q)$ agree. Then we can lift $p\colon M \to T^k$ to a map $p'\colon M \to T^k$ such that $q \circ p' = p$. One easily checks that that $\pi_1(p')$ is surjective and the homotopy fiber of p' fiber is a finite covering of M' and in particular a closed 3-manifold. Hence we assume without loss of generality that $\pi_1(p)$ is surjective, otherwise replace p by p'.

Let K be the kernel of the map $\pi_1(p)\colon \pi_1(M) \cong G \times \mathbb{Z}^k \to \pi_1(T^k) \cong \mathbb{Z}^k$. Since M and T^k are aspherical, the homotopy fiber of p is homotopy equivalent to BK. Hence K is the fundamental group of an aspherical closed 3-manifold M'. Define $K' := K \cap \{1\} \times \mathbb{Z}^k$. This is a normal subgroup of both K and $\mathbb{Z}^k$ if we identify $\{1\} \times \mathbb{Z}^k = \mathbb{Z}^k$.

We begin with the case, where K' is trivial. Then the projection $\mathrm{pr}\colon G \times \mathbb{Z}^k \to G$ induces an isomorphism $K \xrightarrow{\cong} L$ for $L = \mathrm{pr}(K) \subseteq G$. We conclude from Theorem 1.4 that G is the fundamental group of a closed 3-manifold. Theorem 2.3 implies that G is the fundamental group of a hyperbolic closed 3-manifold.

Next we consider the case where K' is non-trivial. Consider the following commutative diagram

$$\begin{tikzcd}
 & \{1\} \arrow[d] & \{1\} \arrow[d] & \{1\} \arrow[d] & \\
\{1\} \arrow[r] & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K/K' \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathbb{Z}^k \arrow[r] \arrow[d] & G \times \mathbb{Z}^k \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \{1\} \\
\{1\} \arrow[r] & \mathbb{Z}^k/K' \arrow[r] \arrow[d] & \mathbb{Z}^k \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \{1\} \\
 & \{1\} & \{1\} & \{1\} &
\end{tikzcd}$$

where the upper and the middle row and the left and the middle column are the obvious exact sequences, the map $\mathbb{Z}^k/K' \to \mathbb{Z}^k$ is the map making the diagram commutative, Q is defined to be the cokernel of the map $\mathbb{Z}^k/K' \to \mathbb{Z}^k$, and all other arrows are uniquely determined by the property that the diagram commutes. An easy diagram chase shows that all rows and columns are exact.

The center of K contains a copy of $\mathbb{Z}$, since $K' \subseteq \operatorname{cent}(K)$ and K is torsionfree. We conclude from Theorem 1.6 that there is an aspherical closed Seifert 3-manifold S such that $K = \pi_1(N)$. There exists a finite covering $\overline{S} \to S$ such that $\overline{S}$ is orientable, there is a principal S^1-fiber bundle $S^1 \to \overline{S} \to F_g$ for a closed orientable surface of genus $g \geq 1$, see [52, page 436 and Theorem 2.3], and we obtain a short exact sequence $\{1\} \to \pi_1(S^1) \to \pi_1(\overline{S}) \to \pi_1(F_g) \to \{1\}$. The center of $\pi_1(\overline{S})$ contains the image of $\pi_1(S^1) \to \pi_1(\overline{S^1})$. The center cannot be larger if $g \geq 2$ since $\operatorname{cent}(\pi_1(F_g))$ is trivial for $g \geq 2$. If the center is larger and $g = 1$, the extension has to be trivial, after possibly passing to a finite covering of $\overline{S}$. Hence we can arrange that there is a subgroup $\overline{K} \subseteq K$ of finite index such that $\operatorname{cent}(\overline{K}) \cong \mathbb{Z}$ and $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ holds for some $g \geq 1$, or we have $\overline{K} \cong \mathbb{Z}^3$, just take $\overline{K} = \pi_1(\overline{S})$.

Next we show that $\operatorname{cent}(\overline{K})$ must be infinite cyclic. If $\operatorname{cent}(\overline{K})$ is not infinite cyclic, then $\overline{K}$ has to be $\mathbb{Z}^3$. We conclude that K and hence also K/K' are virtually finitely generated abelian. Since Q is abelian, we have the exact sequence $1 \to K/K' \to G \to Q \to 1$ and G has cohomological dimension 3, the group G cannot be hyperbolic, a contradiction. Hence $\operatorname{cent}(\overline{K})$ must be infinite cyclic and and $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ for some $g \geq 1$.

We have $\{0\} \neq K' \subseteq \operatorname{cent}(K)$ and $\operatorname{cent}(K) \cap \overline{K} \subseteq \operatorname{cent}(\overline{K}) \cong \mathbb{Z}$. Since K' is torsionfree and $[K : \overline{K}]$ is finite, $\operatorname{cent}(K)$ is a non-trivial torsionfree virtually cyclic group and hence $\operatorname{cent}(K)$ is infinite cyclic. Since $\operatorname{cent}(K)/K'$ is a finite subgroup of K/K' and K/K' is isomorphic to a subgroup of the torsionfree group G, we have $K' = \operatorname{cent}(K)$. The group $\overline{K}/(\overline{K} \cap \operatorname{cent}(K))$ is a subgroup of $K/K' = K/\operatorname{cent}(K)$ of finite index and admits an epimorphism onto $\overline{K}/\operatorname{cent}(\overline{K}) \cong \pi_1(F_g)$ whose kernel $\operatorname{cent}(\overline{K})/(\overline{K} \cap \operatorname{cent}(K))$ is finite. Since $\overline{K}/(\overline{K} \cap \operatorname{cent}(K))$ is isomorphic to a subgroup of the torsionfree group G, this kernel is trivial and hence $\overline{K}/(\overline{K} \cap \operatorname{cent}(K)) \cong \pi_1(F_g)$.

Since K' is infinite cyclic, Q contains a copy of $\mathbb{Z}$ of finite index. Hence we can find a subgroup G' of G of finite index together with a short exact sequence $\{1\} \to K/K' \to G' \to \mathbb{Z} \to \{1\}$. So there exists an automorphism $\phi\colon K/K' \to K/K'$ such that G' is isomorphic to the semi-direct product $K/K' \rtimes_\phi \mathbb{Z}$. If we put $L := \overline{K}/(\overline{K} \cap \operatorname{cent}(K))$, then $L \cong \pi_1(F_g)$ and L is a subgroup of the finitely generated group K'/K of finite index. Then $L' = \bigcap_{n \in \mathbb{Z}} \phi^n(L)$ is a subgroup of K/K' of finite index again which satisfies $L' \subseteq L$ and $\phi(L') = L'$ and for which there is an isomorphism $u\colon L' \xrightarrow{\cong} \pi_1(F_{g'})$ for some $g' \geq 1$. Let $\phi'\colon L' \to L'$ be the automorphism induced by ϕ. Then $G'' := L' \rtimes_{\phi'} \mathbb{Z}$ is isomorphic to a subgroup of G of finite index in G. Choose a homeomorphism $h'\colon F_{g'} \to F_{g'}$ satisfying $\pi_1(h') = u^{-1} \circ \phi' \circ u$. The mapping torus $T_{h'}$ is an aspherical closed 3-manifold with $\pi_1(T_{h'}) \cong G''$. Theorem 1.4 shows that G is the fundamental group of a closed 3-manifold. Theorem 2.3 implies that G is the fundamental group of a hyperbolic closed 3-manifold. □

Remark 2.8 (MAF)**.** Some evidence for Theorem 2.7 comes from the conclusion of [25, Theorem 1.8] that one can find for any epimorphism $\alpha\colon \pi_1(M) \to \pi_1(T^k)$ at least a MAF (manifold approximate fibration) $p\colon M \to T^k$ such that $\pi_1(p) = \alpha$.

3. The existence of a normal map of degree one

We call a connected finite CW-complex *oriented* if we have chosen a generator $[X]$ of the infinite cyclic group $H_n^{\pi_1(X)}(\widetilde{X}; \mathbb{Z}^{w_1(X)})$. In this section we show

Theorem 3.1 (Existence of a normal map)**.** *Let X be a connected finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle ξ over BG*

and a normal map of degree one

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & \underline{\xi} \arrow[d] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

Proof. Any element $c \in H^k(BO;\mathbb{Z}/2)$ determines an up to homotopy unique map $\widehat{c}\colon BSG \to K(\mathbb{Z}/2,k)$. It is characterized by the property that $c = H^k(\widehat{c};\mathbb{Z}/2)(\iota_k)$ for the canonical element $\iota_k \in H^k(K(\mathbb{Z}/2,k);\mathbb{Z}/2)$ which corresponds to $\mathrm{id}_{\mathbb{Z}/2}$ under the isomorphism

$$\begin{aligned} H^k(K(\mathbb{Z}/2,k);\mathbb{Z}/2) &\cong \hom_{\mathbb{Z}}(H_k(K(\mathbb{Z}/2,k);\mathbb{Z}),\mathbb{Z}/2) \\ &\cong \hom_{\mathbb{Z}}(\pi_k(K(\mathbb{Z}/2,k)),\mathbb{Z}/2) \cong \hom_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2). \end{aligned}$$

Next we claim that the product of the maps given by the first and second Stiefel-Whitney classes $w_1 \in H^1(BO;\mathbb{Z}/2)$ and $w_2 \in H^2(BO;\mathbb{Z}/2)$

$$\widehat{w_1} \times \widehat{w_2}\colon BO \to K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \tag{3.2}$$

is 4-connected. Since BO is connected, $\pi_1(BO) \cong \pi_2(BSO) = \mathbb{Z}/2$ and $\pi_3(BSO) = 0$, it suffices to show that $\pi_k(\widehat{w_k})\colon \pi_k(BO) \to \pi_k(K(\mathbb{Z}/2,k)$ is non-trivial for $k = 1,2$. This is easily proved using the fact the Hopf fibration $S^1 \to S^3 \to S^2$ has non-trivial second Stiefel-Whitney class. Hence for any 3-dimensional complex X stable vector bundles over X are stably classified by w_1 and w_2. Namely, the map induced by composition with the map (3.2)

$$[X,BSO] \to [X,K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)] = H^1(X;\mathbb{Z}/2) \times H^2(X;\mathbb{Z}/2),$$

is bijective and sends for a vector bundle ξ with classifying map f_ξ the class $[f_\xi]$ to $(w_1(\xi),w_2(\xi))$.

We conclude from [29, page 44] that there is a closed manifold M together with a map $f\colon M \to X$ such that $w_1(M) = f^*w_1(X)$ and the induced map $H_3^{\pi_1(M)}(\widetilde{M};\mathbb{Z}^{w_1(M)}) \xrightarrow{\cong} H_3^{\pi_1(X)}(\widetilde{X};\mathbb{Z}^{w_1(X)})$ is an isomorphism of infinite cyclic groups. The proof in the general case is a variation of the one for trivial $w_1(X)$ which we sketch next. Namely, by the Atiyah-Hirzebruch spectral sequences applied to the homology theory Ω_* given by oriented bordism yields an epimorphism

$$\Omega_3(BG) \xrightarrow{\cong} H_3(BG;\mathbb{Z}), \quad [f\colon M \to BG] \mapsto f_*([M])$$

since the projection $BG \to \{\bullet\}$ induces an epimorphism $\Omega_3(BG) \to \Omega(\{\bullet\})$ and there is a map of degree one from BG to S^3 by [57, Proposition 1.3]). We can choose the fundamental class $[M] \in H_3(M;\mathbb{Z}^{w_1(M)})$ such that it is mapped to $[X] \in H_3(M;\mathbb{Z}^{w_1(X)})$ under the isomorphism $H_3^{\pi_1(M)}(\widetilde{M};\mathbb{Z}^{w_1(M)}) \xrightarrow{\cong} H_3^{\pi_1(X)}(\widetilde{X};\mathbb{Z}^{w_1(X)})$.

Choose a vector bundle ξ over X with $w_1(\xi) = w_1(X)$ and $w_2(\xi) = w_1(X) \cup w_1(X)$. Its pull back $f^*\xi$ satisfies $w_2(f^*\xi) = w_1(M) \cup w_1(M)$ and $w_1(f^*\xi) = w_1(M)$. The Wu formula, see for instance [41, Theorem 11.14 on page 132], implies $w_2(TM) = w_1(TM) \cup w_1(TM)$ and hence $w_2(f^*\xi) = w_2(TM)$ and $w_1(f^*\xi) = w_2(TM)$. Therefore TM and $f^*\xi$ are stably isomorphic. Hence we can cover $f\colon M \to X$ by a bundle map $\overline{f}\colon TM \oplus \underline{\mathbb{R}^a} \to \xi$ after possibly replacing ξ by $\xi \oplus \underline{\mathbb{R}^b}$. $\square$

Notice that the sphere bundle of ξ is necessarily the Spivak normal bundle of X. Hence we see that the Spivak normal fibration of BG has a vector bundle reduction.

Next we want to figure out the simple surgery obstruction

$$\sigma^s(f;\overline{f}) \in L_3^s(\mathbb{Z}[\pi_1(X)],w_1(X))$$

of the normal one map of degree one appearing in Corollary 3.1. The goal is to find one $(f, \overline{f})$ such that $\sigma^s(f, \overline{f})$ vanishes. Notice that the definition of the surgery obstruction makes sense in all dimensions, in particular also in dimension 3. For this purpose we will need the Full Farrell-Jones Conjecture.

4. Short review of Farrell-Jones groups

Recall that a group G is called a Farrell-Jones group if it satisfies the Full Farrell-Jones Conjecture which means that it satisfies both the K-theoretic and the L-theoretic Farrell-Jones Conjecture with coefficients in additive categories and with finite wreath products. A detailed exposition on the Farrell-Jones Conjecture will be given in [39].

The reader does not need to know any details about the Full Farrell-Jones Conjecture since this paper is written such that it can be used as a black box and we mention the consequences, which we need in this paper, when they appear. At least we record the following important consequences for a torsionfree Farrell-Jones group G.

- The projective class group $\widetilde{K}_0(\mathbb{Z}G)$ vanishes. This implies that any finitely presented n-dimensional Poincaré duality group has a finite n-dimensional model for BG;
- The Whitehead group $\mathrm{Wh}(G)$ vanishes. Hence any homotopy equivalence of finite CW-complexes with G as fundamental group is a simple homotopy equivalence and every h-cobordism of dimension ≥ 6 with G as fundamental group is trivial;
- The negative K-groups $K_n(\mathbb{Z}G)$ for $n \leq -1$ all vanish. Hence the decorations $L_n^\epsilon(\mathbb{Z}G)$ in the L-groups do not matter;
- The L-theoretic assembly map, see (5.2),
$$\mathrm{asmb}_n^\epsilon(G,w)\colon H_n^G(EG;\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to H_n^G(\{\bullet\};\mathbf{L}^\epsilon_{\mathbb{Z},w}) = L_n^\epsilon(\mathbb{Z}G,w)$$
is an isomorphism for $n \in \mathbb{Z}$ and all decorations ϵ;
- The Borel Conjecture holds for aspherical closed manifolds of dimension ≥ 5 whose fundamental group is G.

The reader may appreciate the following status report.

Theorem 4.1 (The class $\mathcal{FJ}$)**.** *Let class $\mathcal{FJ}$ of Farrell-Jones groups has the following properties.*

(1) The following classes of groups belong to $\mathcal{FJ}$:
  - *(a) Hyperbolic groups;*
  - *(b) Finite dimensional* CAT(0)*-groups;*
  - *(c) Virtually solvable groups;*
  - *(d) (Not necessarily cocompact) lattices in second countable locally compact Hausdorff groups with finitely many path components;*
  - *(e) Fundamental groups of (not necessarily compact) connected manifolds (possibly with boundary) of dimension ≤ 3;*
  - *(f) The groups $GL_n(\mathbb{Q})$ and $GL_n(F(t))$ for $F(t)$ the function field over a finite field F;*
  - *(g) S-arithmetic groups;*
  - *(h) mapping class groups;*

(2) The class $\mathcal{FJ}$ has the following inheritance properties:
  - *(a)* Passing to subgroups
    Let $H \subseteq G$ be an inclusion of groups. If G belongs to $\mathcal{FJ}$, then H belongs to $\mathcal{FJ}$;

(b) Passing to finite direct products
If the groups G_0 and G_1 belong to $\mathcal{FJ}$, then also $G_0 \times G_1$ belongs to $\mathcal{FJ}$;

(c) Group extensions
Let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{FJ}$ and that the group Q belongs to $\mathcal{FJ}$.
Then G belongs to $\mathcal{FJ}$;

(d) Directed colimits
Let $\{G_i \mid i \in I\}$ be a direct system of groups indexed by the directed set I (with arbitrary structure maps). Suppose that for each $i \in I$ the group G_i belongs to $\mathcal{FJ}$.
Then the colimit $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;

(e) Passing to finite free products
If the groups G_0 and G_1 belong to $\mathcal{FJ}$, then $G_0 * G_1$ belongs to $\mathcal{FJ}$;

(f) Passing to overgroups of finite index
Let G be an overgroup of H with finite index $[G : H]$. If H belongs to $\mathcal{FJ}$, then G belongs to $\mathcal{FJ}$;

Proof. See [1, 2, 3, 4, 6, 7, 32, 51, 59, 60]. □

5. The total surgery obstruction

The results of this section are inspired and motivated by Ranicki's total surgery obstruction, see for instance [37, 45, 48]. Since we consider only aspherical Poincaré complexes whose fundamental groups are Farrell-Jones groups, the exposition simplifies drastically and we get some extra valuable information. Moreover, we get a version of Quinn's resolution obstruction which does not require the structure of an homology ANR-manifold on the relevant Poincaré complexes, and the total surgery obstruction and hence Quinn's resolution obstruction are already determined by the symmetric signature of the finite Poincaré complex.

The main result of this section will be

Theorem 5.1. *Let G be a finitely presented 3-dimensional Poincaré duality group which is a Farrell-Jones group. Then there is a finite 3-dimensional model X for BG and the following statements are equivalent:*

1. *There exists an aspherical closed topological manifold N_0 such that $BG \times N_0$ is homotopy equivalent to a closed topological manifold;*
2. *Let N be any closed smooth manifold, closed Pl-manifold, or closed topological manifold respectively of dimension ≥ 2. Then there is exists a normal map of degree one for some vector bundle ξ over X*

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & (\xi \times TN) \oplus \underline{\mathbb{R}^b} \arrow[d] \\ M \arrow[r, "f"] & X \times N \end{tikzcd}$$

such that M is a smooth manifold, Pl-manifold, or topological manifold respectively and f is a simple homotopy equivalence;

5.1. **The quadratic total surgery obstruction.** Let G be a group together with an orientation homomorphism $w\colon G \to \{\pm 1\}$. Then there is a covariant functor

$$\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

from the orbit category to the category of spectra, where the so called decoration ϵ is $\langle i \rangle$ for some $i \in \{2, 1, 0, -1, \ldots\} \amalg \{-\infty\}$, see [48, Definition 4.1 on page 145].

Notice that the decoration $\langle i \rangle$ for $i = 2,1,0$ is also denoted by s,h,p in the literature. From $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}$ we obtain a G-homology theory on the category of G-CW-complexes $H^G_*(-;\mathbf{L}^{\epsilon}_{\mathbb{Z},w})$ such that for every subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ we have identifications

$$H^G_n(G/H;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \cong \pi_n(\mathbf{L}^{\epsilon}_{\mathbb{Z},w}(G/H)) \cong L^{\epsilon}_n(\mathbb{Z}H, w|_H),$$

where $L^{\epsilon}_n(\mathbb{Z}H, w|_H)$ denotes the n-th quadratic L-group with decoration ϵ of $\mathbb{Z}G$ with the w-twisted involution, see [17, Section 4 and 7]. The projection $EG \to \{\bullet\}$ induces the so called assembly map

$$\text{(5.2)} \qquad \operatorname{asmb}^{\epsilon}_n(G,w)\colon H^G_n(EG;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \to H^G_n(\{\bullet\};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) = L^{\epsilon}_n(\mathbb{Z}G,w),$$

which is induced by the projection $EG \to \{\bullet\}$.

In the sequel we denote for a spectrum $\mathbf{E}$ by $\mathbf{i}(\mathbf{E})\colon \mathbf{E}\langle 1\rangle \to \mathbf{E}$ its 1-connective cover which is a map of spectra such that $\pi_n(\mathbf{i}(\mathbf{E}))$ is an isomorphism for $n \geq 1$ and $\pi_n(\mathbf{E}(\langle 1\rangle)) = 0$ for $n \leq 0$. We claim that there is a functorial construction of the 1-connective cover so that we get from the covariant functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ another covariant functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ together with a natural transformation $\mathbf{i}\colon \mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle \to \mathbf{L}^{\epsilon}_{\mathbb{Z},w}$ such that $\mathbf{i}(G/H)$ is a cofibration of spectra. Then we can also define a functor $\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$ together with a natural transformation $\mathbf{pr}\colon \mathbf{L}^{\epsilon}_{\mathbb{Z},w} \to \mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle$ such that for every object G/H in $\mathsf{Or}(G)$ we obtain a cofibration sequence of spectra

$$\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H) \xrightarrow{\mathbf{i}(G/H)} \mathbf{L}^{\epsilon}_{\mathbb{Z},w}(G/H) \xrightarrow{\mathbf{pr}(G/H)} \mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H).$$

It induces for every G-CW-complex Y a long exact sequence

$$\text{(5.3)} \qquad \cdots \to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \\ \to H^G_n(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \to H^G_{n-1}(Y;\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \to \cdots$$

and we have

$$\pi_n(\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle(G/H)) \cong \begin{cases} L^{\epsilon}_n(H;w|_H) & n \leq 0; \\ 0 & n \geq 1. \end{cases}$$

Now consider an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphisms $w = w_1(X)\colon G \to \{\pm 1\}$ in the sense of [31]. We mention that w can be read of from the underlying CW-complex X.

The equivariant version of the Atiyah-Hirzebruch spectral sequence shows that $H^G_{n+1}(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) = 0$ and yields an isomorphism

$$\text{(5.4)} \qquad H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{\cong} H^G_n(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})^w),$$

where for any abelian group A we denote by A^w the $\mathbb{Z}G$-module whose underlying abelian group is A and on which $g \in G$ acts by multiplication with $w(g)$. Poincaré duality yields an isomorphism

$$\text{(5.5)} \qquad H^G_n(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})^w) \xrightarrow{\cong} H^0_G(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})),$$

where G acts trivially on $L^{\epsilon}_0(\mathbb{Z})$ in $H^0_G(\widetilde{X};L_0(\mathbb{Z})^{\epsilon})$. There is an obvious isomorphism

$$\text{(5.6)} \qquad H^0_G(\widetilde{X};L^{\epsilon}_0(\mathbb{Z})) \xrightarrow{\cong} H^0(X;L^{\epsilon}_0(\mathbb{Z})) \cong L^{\epsilon}_0(\mathbb{Z}).$$

Notice that $L^{\epsilon}_0(\mathbb{Z})$ is independent of the decoration ϵ and hence we abbreviate $L_0(\mathbb{Z}) = L^{\epsilon}_0(\mathbb{Z})$. We obtain from (5.4), (5.5), and (5.6) an isomorphism

$$\text{(5.7)} \qquad H^G_n(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}/\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{\cong} L_0(\mathbb{Z}).$$

Its composition with $H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}/\mathbf{L}^\epsilon_{\mathbb{Z},w}\langle 1\rangle)$ is denoted by

$$\lambda_n^\epsilon(X)\colon H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \to L_0(\mathbb{Z}). \tag{5.8}$$

From the exact sequence (5.3) we obtain a short exact sequence

$$0 \to H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{H_n^G(\mathrm{id};\mathbf{i})} H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \xrightarrow{\lambda_n^\epsilon(X)} L_0(\mathbb{Z}). \tag{5.9}$$

For every ϵ there is a natural transformation

$$\mathbf{e}^\epsilon\colon \mathbf{L}^\epsilon \to \mathbf{L}^{\langle -\infty\rangle}$$

such that $e_n^\epsilon := \pi_n(\mathbf{e}^\epsilon)\colon L_n^\epsilon(\mathbb{Z}G,w) \to L_n^{\langle -\infty\rangle}(\mathbb{Z}G,w)$ is the classical change of decoration homomorphism and the following diagram

$$\begin{tikzcd}
H_n^G(\widetilde{X};\mathbf{L}^\epsilon_{\mathbb{Z},w}) \arrow[r, "\mathrm{asmb}_n^\epsilon(X)"] \arrow[d, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{e}^\epsilon)"'] & H_n^G(\{\bullet\};\mathbf{L}^\epsilon_{\mathbb{Z},w}) = L_n^\epsilon(\mathbb{Z}G,w) \arrow[d, "e_n^\epsilon"] \\
H_n^G(\widetilde{X};\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z},w}) \arrow[r, "\mathrm{asmb}_n^{\langle -\infty\rangle}(X)"'] & H_n^G(\{\bullet\};\mathbf{L}^{\langle -\infty\rangle}_{\mathbb{Z},w}) = L_n^{\langle -\infty\rangle}(\mathbb{Z}G,w)
\end{tikzcd}$$

commutes. Recall that G has to be torsionfree. If G is a Farrell-Jones group, then $\mathrm{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_m(\mathbb{Z}G)$ for $m \leq -1$ vanish and hence all maps in the commutative diagram are isomorphisms, in particular, the choice of the decoration ϵ does not matter.

Let $\mathcal{N}(X)$ be the set of normal bordism classes of normal maps of degree one with target X. Suppose that $\mathcal{N}(X)$ is not empty. Consider a normal map $(f,\overline{f})$ of degree one with target X

$$\begin{tikzcd}
TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & \xi \arrow[d] \\
M \arrow[r, "f"] & X
\end{tikzcd}$$

One can assign to it its simple surgery obstruction $\sigma^s(f,\overline{f}) \in L_n^s(\mathbb{Z}G,w)$. (This makes sense for all dimension n.) Fix a normal map $(f_0,\overline{f_0})$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathcal{N}(X) \arrow[r, "\sigma^s(-{,}-)-\sigma^s(f_0{,}\overline{f_0})"] \arrow[d, "s_0"', "\cong"] & L_n^s(\mathbb{Z}G,w) \\
H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})"'] & H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}) \arrow[u, "\cong"', "\mathrm{asmb}_n^s(X)"]
\end{tikzcd} \tag{5.10}$$

whose vertical arrows are bijections and the upper arrow sends the class of $(f,\overline{f})$ to the difference $\sigma^s(f,\overline{f}) - \sigma^s(f,\overline{f_0})$. This follows from the work of Ranicki [48, Proof of Theorem 17.4 on pages 191ff]. A detailed and careful exposition of the proof of the existence of the diagram above can be found in [37, Proposition 14.18].

Now consider the composite

$$\mu_n^s(X)\colon \mathcal{N}(X) \xrightarrow{\sigma^s} L_n^s(\mathbb{Z}G,w) \xrightarrow{\mathrm{asmb}_n^s(X)^{-1}} H_n^G(\widetilde{X};\mathbf{L}^s_{\mathbb{Z},w}) \xrightarrow{\lambda_n^\epsilon(X)} L_0(\mathbb{Z}), \tag{5.11}$$

where the map $\lambda_n^\epsilon(X)$ has been defined in (5.8). From the exact sequence (5.9) and the diagram 5.10 we conclude that there is precisely one element, called the *quadratic total surgery obstruction*,

$$s(X) \in L_0(\mathbb{Z}) \tag{5.12}$$

such that for any element $[(f, \overline{f})]$ in $\mathcal{N}(X)$ its image under $\mu_n^s(X)$ is $s(X)$. Moreover, we get

Theorem 5.13 (The quadratic total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphisms $w = w_1(X)\colon G \to \{\pm 1\}$. Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty. Then:*

(1) There exists a normal map of degree one $(f, \overline{f})$ with target X whose simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G, w)$ vanishes, if and only if $s(X) \in L_0(\mathbb{Z})$ vanishes;

(2) If X is homotopy equivalent to a closed topological manifold, then $s(X) \in L_0(\mathbb{Z})$ vanishes.

Proof. 1 The "only if"-statement is obvious. For the "if"- observe that the vanishing of $s(X) \in L_0(\mathbb{Z})$ implies that the element $-\sigma^s(f_0, \overline{f_0})$ lies in the image of the upper horizontal arrow of the diagram 5.10 because of the exact sequence (5.9).

2 If X is homotopy equivalent to a closed topological manifold, then there exists an element in $[(f, \overline{f})]$ in $\mathcal{N}(X)$ with $\sigma^s(f, \overline{f}) = 0$. Now apply assertion 1. □

Notice that Theorem 5.13 (1) holds also in dimensions $n \leq 4$. We are *not* claiming in Theorem 5.13 (1) that that we can arrange f to be a simple homotopy equivalence. This conclusion from the vanishing of the simple surgery obstruction does require $n \geq 5$.

5.2. **The symmetric total surgery obstruction.** There is also a symmetric version of the material of Subsection 5.1. There is a covariant functor

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

from the orbit category to the category of spectra such that for every subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ we have identifications

$$H_n(G/H; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}) \cong \pi_n(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H)) \cong L_\epsilon^n(\mathbb{Z}H, w|_H),$$

where $L_\epsilon^n(\mathbb{Z}H, w|_H)$ denotes the 4-periodic n-th symmetric L-group with decoration ϵ of $\mathbb{Z}G$ with the w-twisted involution. The projection $\widetilde{X} \to \{\bullet\}$ induces the symmetric assembly map

$$\mathrm{asmb}_n^{\epsilon,\mathrm{sym}}(X)\colon H_n^G(EG; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}) \to H_n^G(\{\bullet\}; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}) = L_\epsilon^n(\mathbb{Z}G, w), \tag{5.14}$$

which is induced by the projection $\widetilde{X} \to \{\bullet\}$.

There is a natural transformation called symmetrization of covariant functors $\mathsf{Or}(G) \to \mathsf{SPECTRA}$

$$\mathbf{sym}^\epsilon\colon \mathbf{L}_{\mathbb{Z},w}^\epsilon \to \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}. \tag{5.15}$$

It induces the classical symmetrization homomorphisms on homotopy groups

$$\mathrm{sym}_n^\epsilon(G/H)\colon L_n^\epsilon(\mathbb{Z}H, w|_H) \to L_\epsilon^n(\mathbb{Z}H, w|_H), \tag{5.16}$$

which are isomorphism after inverting 2. We obtain a natural transformation of G-homology theories

$$\mathrm{sym}_*^{G,\epsilon}\colon H_*^G(-; \mathbf{L}_{\mathbb{Z},w}^\epsilon) \to H_*^G(-; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}) \tag{5.17}$$

satisfying

Theorem 5.18. *For every $n \in \mathbb{Z}$ and every G-CW-complex X the maps*

$$\mathrm{sym}_n^{G,\epsilon}\colon H_n^G(X; \mathbf{L}_{\mathbb{Z},w}^\epsilon) \to H_n^G(X; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}})$$

are isomorphisms after inverting 2.

The following diagram commutes

$$\begin{tikzcd}
H_n^G(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}^{\epsilon}_n(X)"] \arrow[d, "H_n^G(EG{,}\mathbf{sym}^{\epsilon})"'] & L^{\epsilon}_n(\mathbb{Z}G,w) \arrow[d, "\operatorname{sym}^{\epsilon}_n(G/G)"] \\
H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[r, "\operatorname{asmb}^{\epsilon,\mathrm{sym}}_n(X)"'] & L^n_{\epsilon}(\mathbb{Z}G,w),
\end{tikzcd} \tag{5.19}$$

There is an obvious symmetric analog of the map (5.8)

$$\lambda^{\epsilon,\mathrm{sym}}_n(X)\colon H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \to L^0(\mathbb{Z}), \tag{5.20}$$

and of the short exact sequence (5.9)

$$0 \to H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}\langle 1\rangle) \xrightarrow{H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})} H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \xrightarrow{\lambda^{\epsilon,\mathrm{sym}}_n(X)} L^0(\mathbb{Z}). \tag{5.21}$$

The following diagram
(5.22)

$$\begin{tikzcd}
0 \arrow[r] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})"] \arrow[d, "H_n^G(\mathrm{id};\mathbf{sym}^{\epsilon}\langle 1\rangle)"] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon}_{\mathbb{Z},w}) \arrow[r, "\lambda^{\epsilon}_n(X)"] \arrow[d, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{sym}^{\epsilon})"] & L_0(\mathbb{Z}) \arrow[d, "\operatorname{sym}_0"] \\
0 \arrow[r] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}\langle 1\rangle) \arrow[r, "H_n^G(\mathrm{id}_{\widetilde{X}};\mathbf{i})"] & H_n^G(\widetilde{X};\mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[r, "\lambda^{\epsilon,\mathrm{sym}}_n(X)"] & L^0(\mathbb{Z})
\end{tikzcd}$$

commutes, has exact rows, and all its vertical arrows are bijections after inverting 2. Under the standard identifications

$$h_0\colon L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; \tag{5.23}$$

$$h^0\colon L^0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, \tag{5.24}$$

the map $\operatorname{sym}_0\colon L_0(\mathbb{Z}) \to L^0(\mathbb{Z})$ becomes $8\cdot\mathrm{id}\colon \mathbb{Z}\to\mathbb{Z}$ and hence is injective. Define the *symmetric total surgery obstruction*

$$s^{\mathrm{sym}}(X) \in L^0(\mathbb{Z}) \tag{5.25}$$

to be the image of $s(X)$ defined in (5.12) under the injection $\operatorname{sym}_0\colon L_0(\mathbb{Z})\to L^0(\mathbb{Z})$. Theorem 5.13 implies

Theorem 5.26 (The symmetric total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X}\to X$, fundamental group $G=\pi_1(X)$ and orientation homomorphisms $w=w_1(X)\colon G\to\{\pm 1\}$.Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty. Then*

1. *There exists a normal map of degree one $(f,\overline{f})$ with target X whose simple surgery obstruction $\sigma^s(f,\overline{f})\in L_n^s(\mathbb{Z}G,w)$ vanishes, if and only if $s^{\mathrm{sym}}(X)\in L^0(\mathbb{Z})$ vanishes;*
2. *If X is homotopy equivalent to a closed topological manifold, then $s^{\mathrm{sym}}(X)\in L^0(\mathbb{Z})$ vanishes.*

Now we study the main properties of the symmetric total surgery obstruction.

If A is an abelian group, denote by $A/2\text{-}\mathrm{tors}$ its quotient by the abelian subgroup of elements in A, whose order is finite and a power of two. For an element $a\in A$ denote by $\overline{a}$ its image under the projection $A\to A/2\text{-}\mathrm{tors}$.

Next we show that $s^{\mathrm{sym}}(X)$ and $s(X)$ are determined by the image $\overline{\sigma^{s,\mathrm{sym}}_G(\widetilde{X})}$ of the symmetric signature $\sigma^{s,\mathrm{sym}}_G(\widetilde{X})$ under $L^n_s(\mathbb{Z}G,w)\to L^n_s(\mathbb{Z}G,w)/2\text{-}\mathrm{tors}$.

Theorem 5.27. *Let X be an aspherical oriented finite n-dimensional Poincaré complex X with universal covering $\widetilde{X} \to X$, fundamental group $G = \pi_1(X)$ and orientation homomorphisms $w = w_1(X)\colon G \to \{\pm 1\}$. Suppose that G is a Farrell-Jones group and that $\mathcal{N}(X)$ is non-empty.*

Then there is precisely one element $u \in H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors}$ such that the injective map

$$\mathrm{asmb}_n^{s,\mathrm{sym}}(X)/2\text{-}\mathrm{tors}\colon H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors} \to L^n_s(\mathbb{Z}G, w)/2\text{-}\mathrm{tors}$$

sends u to the element $\overline{\sigma_G^{s,\mathrm{sym}}(\widetilde{X})}$ associated to the symmetric signature $\sigma_G^{s,\mathrm{sym}}(\widetilde{X})$, and the composite

$$H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors} \xrightarrow{\lambda^{s,\mathrm{sym}}(\widetilde{X})/2\text{-}\mathrm{tors}} L^0(\mathbb{Z})/2\text{-}\mathrm{tors} \xrightarrow{h_0/2\text{-}\mathrm{tors}} \mathbb{Z}/2\text{-}\mathrm{tors} = \mathbb{Z}$$

sends u to $-h^0(s^{\mathrm{sym}}(X)) + 1 = -8 \cdot h_0(s(X)) + 1$.

Proof. Since G is a Farrell-Jones group, the assembly map asmb of (5.2) is bijective for all $n \in \mathbb{Z}$. We conclude from the commutative diagram (5.22) that the assembly map

$$\mathrm{asmb}_n^{\epsilon,\mathrm{sym}}(\widetilde{X})\colon H_n^G(\widetilde{X}; \mathbf{L}^{\epsilon,\mathrm{sym}}_{\mathbb{Z},w}) \to H_n^G(\{\bullet\}; \mathbf{L}^{\epsilon,\mathrm{sym}}) = L^n_\epsilon(\mathbb{Z}G, w)$$

of (5.14) is an isomorphism after inverting 2. Hence the map

$$\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-}\mathrm{tors}\colon H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors} \to L^n_s(\mathbb{Z}G, w)/2\text{-}\mathrm{tors}$$

is injective.

Consider a normal map$(f, \overline{f})$ of degree one from M to X. Then the homomorphism $\mathrm{sym}_n^s\colon L_n^s(\mathbb{Z}G, w) \to L^n_s(\mathbb{Z}G, w)$ sends $\sigma^s(f, \overline{f})$ to $\sigma_G^{s,\mathrm{sym}}(\overline{M}) - \sigma_G^{s,\mathrm{sym}}(\widetilde{X})$, where $\overline{M} \to M$ is the pull back of the G-covering $\widetilde{X} \to BG$ by f, see [47, Section 6], We conclude from diagram (5.22) that there is precisely one element $u' \in H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors}$ such that its image under $\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-}\mathrm{tors}$ is $\sigma_G^{s,\mathrm{sym}}(\overline{M}) - \sigma_G^{s,\mathrm{sym}}(\widetilde{X})$ and the image of u' under the composite

$$H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors} \xrightarrow{\lambda^{s,\mathrm{sym}}(\widetilde{X})/2\text{-}\mathrm{tors}} L^0(\mathbb{Z})/2\text{-}\mathrm{tors} \xrightarrow{h_0/2\text{-}\mathrm{tors}} \mathbb{Z}/2\text{-}\mathrm{tors} = \mathbb{Z}$$

is $h^0(s^{\mathrm{sym}}(\widetilde{X}))$. We have $8 \cdot h_0(s(\widetilde{X})) = h^0(s^{\mathrm{sym}}(\widetilde{X}))$. Hence it suffices to show that there is an element $u'' \in H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors}$ such that its image under the map $\mathrm{asmb}_n^{s,\mathrm{sym}}(\widetilde{X})/2\text{-}\mathrm{tors}$ is $\overline{\sigma_G^{s,\mathrm{sym}}(\overline{M})}$ and the image of u'' under the composite

$$H_n^G(\widetilde{X}; \mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors} \xrightarrow{\lambda_n^{s,\mathrm{sym}}(X)/2\text{-}\mathrm{tors}} L^0(\mathbb{Z})/2\text{-}\mathrm{tors} \xrightarrow{h^0/2\text{-}\mathrm{tors}} \mathbb{Z}/2\text{-}\mathrm{tors} = \mathbb{Z}$$

is 1. For simplicity we give the proof of the existence of the element u'' only in the special case, where w is trivial. The symmetric signature defines for every $n \geq 0$ and every connected CW-complex X a map, see [47, Proposition 6.3],

$$\sigma_n^{s,\mathrm{sym}}(X)\colon \Omega_n(X) \to L^n_s(\mathbb{Z}[\pi_1(X)]), \quad [f\colon M \to X] \mapsto \sigma_G^{s,\mathrm{sym}}(\overline{M}).$$

Without giving the details of the proof, we claim that this natural transformation of functors from the category of CW-complexes to the category of $\mathbb{Z}$-graded abelian groups can be implemented as a functor from the category of CW-complexes to the category of spectra. We conclude from the generell theory about assembly maps, see [17, Section 6] or [61], that we can lift $\sigma_n^{s,\mathrm{sym}}(X)$ over $\mathrm{asmb}_n^{s,\mathrm{sym}}(X)$ to a map

$\tau_n^{s,\mathrm{sym}}(X)$

$$\begin{tikzcd}
 & H_n^G(\widetilde{X};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[d, "\mathrm{asmb}_n^{s\,\mathrm{sym}}(X)"] \\
\Omega_n(X) \arrow[ur, "\tau_n^{s,\mathrm{sym}}(X)"] \arrow[r, "\sigma_n^{s,\mathrm{sym}}(X)"'] & L^n_h(\mathbb{Z}G)
\end{tikzcd}$$

such that $\tau_*^{s,\mathrm{sym}}(-)$ is a transformation of homology theories. Consider the map

$$\nu_n(X)\colon \Omega_n(X) \xrightarrow{d_n} H_n(X,\mathbb{Z}) \xrightarrow{-\cap[X]} H^0(X;\mathbb{Z}) \to \mathbb{Z},$$

where the first map d_n sends $[f\colon M \to X]$ to $f_*([M])$. The naturality of the Atiyah-Hirzebruch spectral sequence implies that the following diagram

$$\begin{tikzcd}
\Omega_n(X) \arrow[r, "\nu_n(X)"] \arrow[d, "\tau_n^{s,\mathrm{sym}}(X)"'] & \mathbb{Z} \\
H_n^G(\widetilde{X};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w}) \arrow[r, "\lambda^{s,\mathrm{sym}}(X)"'] & L^0(\{\bullet\}) \arrow[u, "h^0"']
\end{tikzcd}$$

commutes. Define u'' as the image of $f\colon M \to X$ under the composite

$$\Omega_n(X) \xrightarrow{\tau_n^{s,\mathrm{sym}}(X)} H_n^G(\widetilde{X};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w}) \to H_n^G(\widetilde{X};\mathbf{L}^{s,\mathrm{sym}}_{\mathbb{Z},w})/2\text{-}\mathrm{tors}.$$

Since the degree of $f\colon M \to BG$ is one, the image of $[f\colon M \to X]$ under $\nu_n(X)$ is 1. Now an easy diagram chase shows that u'' has the desired properties. This finishes the proof of Theorem 5.27. $\square$

Theorem 5.28 (Homotopy invariance of the total surgery obstruction)**.** *Let X be an aspherical oriented finite n-dimensional Poincaré complex such that $\pi_1(X)$ is a Farrell-Jones group and $\mathcal{N}(X)$ is non-empty. Let Y be a finite n-dimensional CW-complex which is homotopy equivalence to X.*

Then Y is an aspherical oriented finite n-dimensional Poincaré complex such that $\pi_1(Y)$ is a Farrell-Jones group and that $\mathcal{N}(Y)$ is non-empty and we get

$$\begin{aligned} s(X) &= s(Y); \\ s^{\mathrm{sym}}(X) &= s^{\mathrm{sym}}(Y). \end{aligned}$$

Proof. Choose a homotopy equivalence $f\colon X \to Y$. Then $w_1(Y)\circ\pi_1(f) = w_1(X)$ and the induced isomorphism

$$L^n_s(\mathbb{Z}[\pi_1(X)], w_1(X)) \xrightarrow{\cong} L^n_s(\mathbb{Z}[\pi_1(Y)], w_1(Y))$$

sends $\sigma^{s,\mathrm{sym}}_{\pi_1(X)}(\widetilde{X})$ to $\sigma^{s,\mathrm{sym}}_{\pi_1(Y)}(\widetilde{Y})$. Now apply Theorem 5.27. $\square$

Next we show a product formula.

Theorem 5.29 (Product formula)**.** *For $i = 0,1$, let X_i be an aspherical oriented finite n_i-dimensional Poincaré complex with fundamental group $G_i = \pi_1(X_i)$ and orientation homomorphisms $v_i := w_1(X_i)\colon G_i \to \{\pm 1\}$ such that G_i is a Farrell-Jones group and that $\mathcal{N}(X_i)$ is non-empty.*

Then $X_0 \times X_1$ is an aspherical oriented finite (n_0+n_1)-dimensional Poincaré complex with fundamental group $G_0 \times G_1$ and orientation homomorphisms $v := w_1(X \times N)\colon G_0 \times G_1 \to \{\pm 1\}$ sending (g_0,g_1) to $v_0(g_0)\cdot v_1(g_1)$ such that $G_0 \times G_1$ is a Farrell-Jones group and that $\mathcal{N}(X_0 \times X_1)$ is non-empty, and we get in $\mathbb{Z}$

$$(-h^0(s(X_0 \times X_1)) + 1) = (-h^0(s(X_0)) + 1)\cdot(-h^0(s(X_1)) + 1).$$

Proof. The product $G_0 \times G_1$ is a Farrell-Jones group by Theorem 4.1 (2b).

The tensor product gives a pairing, see [46, Section 8],

$$\otimes\colon L_s^{n_0}(\mathbb{Z}G_0, v_0) \otimes L_s^{n_1}(\mathbb{Z}G_1, v_1) \to L^{n_0+n_1}(\mathbb{Z}[G_0, \times G_1], v). \tag{5.30}$$

Now we claim that there is a pairing

$$\cup\colon H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{s,\mathrm{sym}}) \otimes H_{n_1}^{G_1}(\widetilde{X_1}; \mathbf{L}_{\mathbb{Z},v_1}^{s,\mathrm{sym}}) \to H_{n_0+n_1}^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},v}^{s,\mathrm{sym}})$$

such that the following diagram commutes

(5.31)

$$\begin{tikzcd}
L_s^{n_0}(\mathbb{Z}G_0, v_0) \otimes L_s^{n_1}(\mathbb{Z}G_1, v_1) \arrow[r, "\otimes"] & L^{n_0+n_1}(\mathbb{Z}[G_0, \times G_1], v) \\
H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{s,\mathrm{sym}}) \otimes H_{n_1}^{G_1}(\widetilde{X_1}; \mathbf{L}_{\mathbb{Z},v_1}^{s,\mathrm{sym}}) \arrow[r, "\cup"] \arrow[u, "\mathrm{asmb}_{n_0}^{\epsilon,\mathrm{sym}}(\widetilde{X_0}) \otimes \mathrm{asmb}_{n_1}^{\epsilon,\mathrm{sym}}(\widetilde{X_1})"] \arrow[d, "\lambda_{n_0}^{s,\mathrm{sym}}(X_0) \otimes \lambda_{n_1}^{s,\mathrm{sym}}(X_1)"'] & H_{n_0+n_1}^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},v}^{s,\mathrm{sym}}) \arrow[u, "\mathrm{asmb}_{n_0+n_1}^{\epsilon,\mathrm{sym}}(\widetilde{X_0 \times X_1})"'] \arrow[d, "\lambda_{n_0+n_1}^{s,\mathrm{sym}}(X_0 \times X_1)"] \\
L^0(\mathbb{Z}) \otimes L^0(\mathbb{Z}) \arrow[r, "\otimes"] \arrow[d, "h^0 \otimes h^0"', "\cong"] & L^0(\mathbb{Z}) \arrow[d, "\cong"', "h^0"] \\
\mathbb{Z} \otimes \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

where the lowermost horizontal arrow is the multiplication on $\mathbb{Z}$. In order to get this diagram, one has firstly to promote the functor

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}$$

to a functor

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}\colon \mathsf{Or}(G) \to \mathsf{SPECTRA}^{\mathrm{sym}}$$

to the category $\mathsf{SPECTRA}^{\mathrm{sym}}$ of symmetric spectra. Notice that the advantage of $\mathsf{SPECTRA}^{\mathrm{sym}}$ in comparison with $\mathsf{SPECTRA}$ is that $\mathsf{SPECTRA}^{\mathrm{sym}}$ has a functorial smash product $\wedge$. In the second step one has to construct a map of spectra

$$\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_0) \wedge \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_1) \to \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G \times G/H_0 \times H_1),$$

which on homotopy groups induces the map

$$\otimes\colon L_s^{n_0}(\mathbb{Z}H_0, v_0|_{H_0}) \otimes L_s^{n_1}(\mathbb{Z}H_1, v_1|_{H_1}) \to L_s^{n_0+n_1}(\mathbb{Z}[H_0 \times H_1], v|_{H_0 \times H_1})$$

under the identifications

$$\begin{aligned}
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_0)) &\cong L_s^k(\mathbb{Z}H_0, v_0|_{H_0}); \\
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G/H_1)) &\cong L_s^k(\mathbb{Z}H_1, v_1|_{H_1}); \\
\pi_k(\mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}}(G \times G/H_0 \times H_1)) &\cong L_s^k(\mathbb{Z}[H_0 \times H_1], v|_{H_0 \times H_1}),
\end{aligned}$$

and are natural in G/H_0 and G/H_1. We omit the details of this construction, see also Remark 5.32. Now the claim follows from Theorem 5.27 and the product formula for the symmetric signature, see [47, Section 8], which says that the pairing (5.30) sends $\sigma_{G_0}^{s,\mathrm{sym}}(\widetilde{X_0}) \otimes \sigma_{G_1}^{s,\mathrm{sym}}(\widetilde{X_1})$ to $\sigma_{G_0 \times G_1}^{s,\mathrm{sym}}(\widetilde{X_0 \times X_1})$. □

Remark 5.32 (Special case of Theorem 5.29)**.** In the proof of Theorem 5.29 we have not given the details of the proof of the existence of the commutative diagram (5.31). We will need Theorem 5.29 only in the special case, where $n_0 = 3$ and X_1 is a closed n-dimensional manifold and then the desired assertion is

$$s^{\mathrm{sym}}(X_0) = s^{\mathrm{sym}}(X_0 \times X_1).$$

For the reader's convenience we give a direct complete proof in this special case. We have $L^0(\mathbb{Z}) \cong \mathbb{Z}$, $L^1(\mathbb{Z}) \cong \mathbb{Z}/2$ and $L^i(\mathbb{Z}) = 0$ for $i = 1, 2$. The Atiyah-Hirzebruch spectral sequence shows that the map $\lambda_n^{\epsilon,\mathrm{sym}}(X_0)$ of (5.20) induces an isomorphism

$$\lambda_{n_0}^{\epsilon,\mathrm{sym}}(X_0)/2\text{-}\mathrm{tors}\colon H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}})/2\text{-}\mathrm{tors} \to L^0(\mathbb{Z})/2\text{-}\mathrm{tors} = L^0(\mathbb{Z}).$$

since we assume $n_0 = 3$. We have already shown in Theorem 5.27 that

$$\mathrm{asmb}_{n_0}^{s,\mathrm{sym}}(X_0)/2\text{-}\mathrm{tors}\colon H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-}\mathrm{tors} \to L_s^{n_0}(\mathbb{Z}G_0, v_0)/2\text{-}\mathrm{tors}$$

is injective and that there is a unique element $u_0 \in H_n^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{s,\mathrm{sym}})$ which is mapped to $-h^0(s^{\mathrm{sym}}(X_0)) + 1$ and to $\overline{\sigma_{G_0}^{s,\mathrm{sym}}(\widetilde{X_0})}$ under these maps. Let $(f_0, \overline{f_0})$ be a normal map from a closed 3-manifold M_0 to X_0. We have explained in the proof of Theorem 5.27 that there is an element $u_0' \in H_{n_0}^{G}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})$ whose image under $\lambda_{n_0}^{\epsilon,\mathrm{sym}}(X_0)/2\text{-}\mathrm{tors}\colon H_{n_0}^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{\epsilon,\mathrm{sym}})/2\text{-}\mathrm{tors} \to L^0(\mathbb{Z})$ is $\sigma_{G_0}^{s,\mathrm{sym}}(\overline{M_0})$ for the G_0-covering $\overline{M_0} \to M_0$ given by the pullback of $\widetilde{X_0} \to X_0$ with f_0 and whose image under the isomorphism

$$h^0 \circ \lambda_{n_0}^{\epsilon,\mathrm{sym}}(X_0)/2\text{-}\mathrm{tors}\colon H_n^{G_0}(\widetilde{X_0}; \mathbf{L}_{\mathbb{Z},v_0}^{\epsilon,\mathrm{sym}})/2\text{-}\mathrm{tors} \xrightarrow{\cong} \mathbb{Z}$$

is 1. Hence we get

$$\sigma_{G_0}^{s,\mathrm{sym}}(\widetilde{X}_0) = (-h^0(s^{\mathrm{sym}}(X_0)) + 1) \cdot \sigma_{G_0}^{s,\mathrm{sym}}(\overline{M_0}).$$

We conclude from the product formula for the symmetric signature, see [47, Section 8],

$$\begin{aligned} (5.33)\quad \sigma_{G_0 \times G_1}^{s,\mathrm{sym}}(\widetilde{X_0 \times X_1}) &= \sigma_{G_0}^{s,\mathrm{sym}}(\widetilde{X_0}) \otimes \sigma_{G_1}^{s,\mathrm{sym}}(\widetilde{X_1}) \\ &= (-h^0(s^{\mathrm{sym}}(X_0) + 1) \cdot \sigma_{G_0}^{s,\mathrm{sym}}(\overline{M_0})) \otimes \sigma_{G_1}^{s,\mathrm{sym}}(X_1) \\ &= (-h^0(s^{\mathrm{sym}}(X_0)) + 1) \cdot \sigma_{G_0 \times G_1}^{s,\mathrm{sym}}(\overline{M_0} \times \widetilde{X_1}). \end{aligned}$$

As we have explained in the proof of Theorem 5.27, there exists a unique element $u'' \in H_n^{G_0 \times G_1}(\widetilde{X_0} \times \widetilde{X_1}; \mathbf{L}_{\mathbb{Z},w}^{s,\mathrm{sym}})/2\text{-}\mathrm{tors}$, whose image under

$$\lambda_n^{\epsilon,\mathrm{sym}}(X_0 \times X_1)/2\text{-}\mathrm{tors}\colon H_n^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}})/2\text{-}\mathrm{tors} \to L^0(\mathbb{Z})$$

is $\sigma_{G_0 \times G_1}^{s,\mathrm{sym}}(\overline{M_0} \times \widetilde{X_1})$ and whose image under

$$h^0 \circ \lambda_n^{\epsilon,\mathrm{sym}}(X_0 \times X_1)/2\text{-}\mathrm{tors}\colon H_n^{G_0 \times G_1}(\widetilde{X_0 \times X_1}; \mathbf{L}_{\mathbb{Z},w}^{\epsilon,\mathrm{sym}})/2\text{-}\mathrm{tors} \to \mathbb{Z}$$

is 1. Here we use that X_1 and hence $M_0 \times X_1$ is a closed manifold. Theorem 5.27 together with (5.33) implies

$$(-h^0(s^{\mathrm{sym}}(X_0 \times X_1))) + 1 = (-h^0(s^{\mathrm{sym}}(X_0))) + 1.$$

Hence we get $s^{\mathrm{sym}}(X_0) = s^{\mathrm{sym}}(X_0 \times X_1)$.

5.3. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. Recall from Subsection 1.1 that there is a finite 3-dimensional Poincaré complex model X for BG and from Theorem 3.1 that $\mathcal{N}(BG)$ is non-empty. The implication $2 \implies 1$ is obviously true, the implication $1 \implies 2$ is proved as follows.

By assumption there are aspherical closed topological manifolds M_0 and N_0 and a homotopy equivalence $f\colon M_0 \to X \times N_0$. We conclude from Theorem 5.28 that $s^{\mathrm{sym}}(M) = s^{\mathrm{sym}}(X \times N_0)$. Since M_0 and N_0 are aspherical closed topological manifolds, $s^{\mathrm{sym}}(M_0)$ and $s^{\mathrm{sym}}(N_0)$ vanish by Theorem 5.26. We conclude from Theorem 5.29, or just from Remark 5.32, that $s^{\mathrm{sym}}(X) = 0$. From Theorem 5.26 we obtain a normal map of degree one $(f, \overline{f})$ with target X and vanishing simple surgery obstruction $\sigma^s(f, \overline{f}) \in L_n^s(\mathbb{Z}G, w_1(G))$. Let N be a closed smooth manifold, closed Pl-manifold, or closed topological manifold respectively of dimension ≥ 2. By the product formula for the surgery obstruction, see [47, Section 8], the surgery obstruction of the normal map of degree one $(f \times \mathrm{id}_N, \overline{f} \times \mathrm{id}_{TN})$ obtained by crossing $(f, \overline{f})$ with N is trivial, Since the dimension of $X \times N$ is greater or equal to 5, we can do surgery in the smooth, PL, or topological category respectively to arrange that $f \times \mathrm{id}_N$ is a simple homotopy equivalence with a closed smooth manifold, closed Pl-manifold, or closed topological manifold respectively as source. □

6. Short review of homology ANR-manifolds

A topological space X is called an *absolute neighborhood retract* or briefly an ANR if it is normal and for every normal space Z, every closed subset $Y \subseteq Z$ and every (continuous) map $f\colon Y \to X$ there exists an open neighborhood U of Y in Z together with an extension $F\colon U \to X$ of f to U. Notice that a normal space with a countable basis for its topology is metrizable by the Urysohn Metrization Theorem, see [43, Theorem 4.1 in Chapter 4-4 on page 217], and is separable, i.e., contains a countable dense subset [43, Theorem 4.1].

Definition 6.1 (Homology ANR-manifold)**.** A *n-dimensional homology* ANR*-manifold X (without boundary)* is an ANR satisfying:

- X has a countable base for its topology;
- the topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ the i-th singular homology group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$.

We call X *closed* if it is compact.

A homology ANR-manifold in the sense of Definition 6.1 is the same as a generalized manifold in the sense of Daverman [16, page 191], as pointed out in [8, page 3]. Every closed n-dimensional topological manifold is a closed n-dimensional homology ANR-manifold (see [16, Corollary 1A in V.26 page 191]).

Definition 6.2 (DDP)**.** A homology ANR-manifold M is said to have the *disjoint disk property (DDP)*, if for one (and hence any) choice of metric on M, any $\epsilon > 0$ and any maps $f, g\colon D^2 \to M$, there are maps $f', g'\colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$,

Definition 6.3 (Homology ANR-manifold with boundary)**.** An *n-dimensional homology* ANR*-manifold X with boundary ∂X* is an ANR which is a disjoint union $X = \operatorname{int} X \cup \partial X$, where

- $\operatorname{int} X$ is an n-dimensional homology ANR-manifold;
- ∂X is an $(n-1)$-dimensional homology ANR-manifold;
- for every $z \in \partial X$ the singular homology group $H_i(X, X \setminus \{z\})$ vanishes for all i.

7. A stable ANR-version of the Cannon Conjecture

Theorem 7.1 (Stable ANR-version of the Cannon Conjecture)**.** *Let G be a torsionfree hyperbolic group. Suppose that its boundary is homeomorphic to S^{n-1}. Let Γ be any d-dimensional Poincaré group for some natural number d satisfying $n + d \geq 6$ which is a Farrell-Jones group.*

Then there is an aspherical closed homology ANR*-manifold X of dimension $(n+d)$ which has the DDP and satisfies $\pi_1(X) \cong G \times \Gamma$.*

Proof. We conclude that $G \times \Gamma$ is a Farrell-Jones group from Theorem 4.1 (1a) and (2b). Since G is a Poincaré duality groups of dimension 3 by [10, Corollary 1.3], the product $G \times \Gamma$ is a Poincaré duality groups of dimension $n + d$. Since by assumption $n + d \geq 6$, we can apply Theorem 1.11. $\square$

8. Short review of Quinn's obstruction

In order to replace homology ANR-manifolds by topological manifolds, we will later use the following result that combines work of Edwards and Quinn, see [16, Theorems 3 and 4 on page 288], [44].

Theorem 8.1 (Quinn's obstruction)**.** *There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$, known as the* Quinn obstruction*, for homology* ANR*-manifolds with the following properties:*

(1) If $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;
(2) Let M be a homology ANR*-manifold of dimension ≥ 5. Then the following are equivalent*
   - *M has the DDP and $\iota(M) = 1$;*
   - *M is a topological manifold.*

The elementary proof of the following result can be found in [8, Corollary 1.6].

Lemma 8.2. *Let M be an homology* ANR*-manifold with boundary ∂M. If ∂M is a manifold, then $\iota(\operatorname{int} M) = 1$.*

Although we do not need the next result in this paper, we mention that it follows from [48, Proposition 25.8 on page 293] using Theorem 5.27, since we assume aspherical.

Theorem 8.3 (Relating the total surgery obstruction and Quinn's obstruction)**.** *Let X be an aspherical finite n-dimensional Poincaré complex which is homotopy equivalent to an n-dimensional closed homology* ANR*-manifold. Suppose that $\pi_1(X)$ is a Farrell-Jones group.*

Then we get

$$i(X) = 8 \cdot h_0(s(X)) + 1 = h^0(s^{\mathrm{sym}}(X)) + 1.$$

Notice that in the situation of Theorem 8.3 the total surgery obstruction $s(X)$ is defined without the assumption that X is homotopy equivalent to an n-dimensional closed homology ANR-manifold and therefore does make sense for any aspherical 3-dimensional Poincaré complex, and moreover, that $s(X)$ is a homotopy invariant, see Theorem 5.28.

Remark 8.4. There is no example in the literature of an aspherical closed homology ANR-manifold which is not homotopy equivalent to a closed topological manifold.

9. Z-sets

Definition 9.1 (Z-set)**.** A closed subset Z of a compact ANR X is called a *Z-set or a set of infinite deficiency* if for every open subset U of X the inclusion $U \setminus (U \cap Z) \to U$ is a homotopy equivalence.

Any closed subset of the boundary ∂M of a compact topological manifold M is a Z-set of M. According to [10, page 470] each of the following properties characterizes Z-sets:

(1) For every $\epsilon > 0$ there is a map $X \to X \setminus Z$ which is ϵ-close to the identity where we have equipped X with some metric.
(2) For every closed subset $A \subseteq Z$, there exists a homotopy $H \colon X \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$, $H_t|_A$ is the inclusion $A \to X$ and $H_t(X \setminus A) \subseteq X \setminus Z$ for $t > 0$.

The next result is taken from [8, Proposition 2.5].

Lemma 9.2. *Let M be a finite dimensional locally compact* ANR *which is the disjoint union of an n-dimensional homology* ANR*-manifold* $\operatorname{int} M$ *and an $(n-1)$-dimensional homology* ANR*-manifold ∂M such that ∂M is a Z-set in M. Then M is an homology* ANR*-manifold with boundary ∂M.*

Definition 9.3 (Compact sets become small at infinity)**.** Consider a pair $(\overline{Y}, Y)$ of G-spaces. We say that *compact subsets of Y become small at infinity*, if, for every $y \in \partial Y := \overline{Y} \setminus Y$, open neighborhood $U \subseteq \overline{Y}$ of y, and compact subset $K \subseteq Y$, there exists an open neighborhood $V \subseteq \overline{Y}$ of y with the properties:

- For every $g \in G$ we have the implication $g \cdot K \cap V \neq \emptyset \implies g \cdot K \subseteq U$;
- $V \subseteq U$.

In the sequel we will choose l large enough such that the following claims are true for the torsionfree hyperbolic group G with boundary S^2 and its Rips complex $P_l(G)$.

(1) The projection $P_l(G) \to P_l(G)/G$ is a model for the universal principal G-bundle $EG \to BG$ and $P_l(G)/G$ is a finite CW-complex:
(2) One can construct a compact topological space ∂G and a compactification $\overline{P_l(G)}$ of $P_l(G)$ such that $\partial G = \overline{P_l(G)} \setminus P_l(G)$ holds, and $P_l(G)$ is open and dense in $\overline{P_l(G)}$;
(3) $\overline{P_l(G)}$ is a compact metrizable ANR such that $\partial G \subset \overline{P_l(G)}$ is Z-set and $\overline{P_l(G)}$ has finite topological dimension;
(4) Compact subsets of Y become small at infinity for the pair $(\overline{P_l(G)}, P_l(G))$.

The first claim is proved for instance in [40]. The second claim follows from [12, III.H.3.6 on page 429, III.H.3.7(3) and (4) on page 430, III.H.3.7(4) on page 430 and III.H.3.18(4) on page 433] and [5, 9.3.(ii)]. The third claim is due to Bestvina-Mess [10, Theorem 1.2], see also [50, Theorem 3.7]. The fourth assertion is for instance proved in [50, page 531].

10. Pulling back boundaries

We will need for later proofs the following construction which may be interesting in its own right.

Let $(\overline{Y}, Y)$ be a topological pair. Put $\partial Y := \overline{Y} \setminus Y$. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Define a topological pair $(\overline{X}, X)$ and a continuous map $\overline{f}\colon \overline{X} \to X$ as follows. The underlying set of $\overline{X}$ is the disjoint union $X \amalg \partial Y$. We define the map of sets $\overline{f}\colon \overline{X} \to \overline{Y}$ to be $f \cup \mathrm{id}_{\partial Y}$. A subset W of $\overline{X}$ is declared to be open if there exists open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$ such that $W = \overline{f}^{-1}(U) \cup V$. This defines indeed a topology. Obviously $\overline{X}$ and $\emptyset$ are open. Given a collection of open subsets $\{W_i \mid i \in I\}$, their union is again open by the following equality, if we write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$ and define open subsets $U := \bigcup_{i \in I} U_i \subseteq \overline{Y}$ and $V := \bigcup_{i \in I} V_i \subseteq X$:

$$\begin{aligned}\bigcup_{i \in I} W_i = \bigcup_{i \in I} \left(\overline{f}^{-1}(U_i) \cup V_i\right) &= \bigcup_{i \in I} \overline{f}^{-1}(U_i) \cup \bigcup_{i \in I} V_i \\ &= \overline{f}^{-1}\left(\bigcup_{i \in I} U_i\right) \cup \bigcup_{i \in I} V_i = \overline{f}^{-1}(U) \cup V.\end{aligned}$$

Given two open subsets W_1 and W_2, their intersection is again open by the following equality, if we write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$ for

$i = 1, 2$ and define open subsets $U := U_1 \cap U_2 \subseteq \overline{Y}$ and $V := \big(f^{-1}(U_1 \cap Y) \cap V_2\big) \cup \big(V_1 \cap f^{-1}(U_2 \cap Y)\big) \cup \big(V_1 \cap V_2\big) \subseteq X$:

$$\begin{aligned} & W_1 \cap W_2 \\ = \;& \big(\overline{f}^{-1}(U_1) \cup V_1\big) \cap \big(\overline{f}^{-1}(U_2) \cup V_2\big) \\ = \;& \big(\overline{f}^{-1}(U_1) \cap \overline{f}^{-1}(U_2)\big) \cup \big(\overline{f}^{-1}(U_1) \cap V_2\big) \cup \big(V_1 \cap \overline{f}^{-1}(U_2)\big) \cup \big(V_1 \cap V_2\big) \\ = \;& \overline{f}^{-1}(U_1 \cap U_2) \cup \Big(\big(f^{-1}(U_1 \cap Y) \cap V_2\big) \cup \big(V_1 \cap f^{-1}(U_2 \cap Y)\big) \cup \big(V_1 \cap V_2\big)\Big) \\ = \;& \overline{f}^{-1}(U) \cup V. \end{aligned}$$

Definition 10.1 (Pulling back the boundary)**.** We say that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by *pulling back the boundary with f*.

Notice that this is the smallest topology on the set $\overline{X} = X \amalg \partial Y$ for which $\overline{f}$ is continuous and $X \subseteq \overline{X}$ is an open subset. This leads to the following universal property of the construction "pulling back the boundary".

Lemma 10.2. *Let $(\overline{Y}, Y)$ be a topological pair. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f. Consider any pair of spaces $(\overline{\overline{X}}, X)$ and map of pairs $(\overline{\overline{f}}, f)\colon (\overline{\overline{X}}, X) \to (\overline{Y}, Y)$ such that X is an open subset of $\overline{\overline{X}}$ and $\overline{\overline{f}}$ induces a map $\overline{\overline{X}} \setminus X \to \partial Y := \overline{Y} \setminus Y$.*

Then there is precisely one map $u\colon \overline{\overline{X}} \to \overline{X}$ which induces the identity on X and satisfies $\overline{f} \circ u = \overline{\overline{f}}$.

Proof. As a map of sets u exists and is uniquely determined by the properties that u induces the identity on X and $\overline{f} \circ u = \overline{\overline{f}}$. Namely, for $x \in X$ define $u(x) = x$ and for $x \in \overline{\overline{X}} \setminus X$ define $u(x)$ by $\overline{\overline{f}}(x) \in \partial\overline{Y} = \partial\overline{X} \subseteq \overline{X}$. We have to show that u is continuous, i.e., $u^{-1}(W) \subseteq \overline{\overline{X}}$ is open for every open subset $W \subseteq \overline{X}$. By definition there are open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$ such that $W = \overline{f}^{-1}(U) \cup V$. Then $u^{-1}(W) = \overline{\overline{f}}^{-1}(U) \cup V$. Since $\overline{\overline{f}}$ is continuous, $\overline{\overline{f}}^{-1}(U) \subseteq \overline{\overline{X}}$ is open. Since X is open in $\overline{\overline{X}}$ and the topology on X is the subspace topology of $X \subseteq \overline{\overline{X}}$, we conclude that for any open subset $V \subseteq X$ the subset $V \subseteq \overline{\overline{X}}$ is open. Hence $u^{-1}(W) \subseteq \overline{\overline{X}}$ is open. $\square$

Lemma 10.3. *Let $(\overline{Y}, Y)$ be a topological pair. Let X be a topological space and $f\colon X \to Y$ be a continuous map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f.*

1. *If $Y \subseteq \overline{Y}$ is dense and the closure of the image of f in $\overline{Y}$ contains ∂Y, then $X \subseteq \overline{X}$ is dense;*
2. *Suppose that $\overline{Y}$ is compact, $Y \subseteq \overline{Y}$ is open and $f\colon X \to Y$ is proper. Then $\overline{X}$ is compact;*
3. *We have for the topological dimension of $\overline{X}$*
$$\dim(\overline{X}) \le \dim(X) + \dim(\overline{Y}) + 1;$$
4. *The map $\overline{f}\colon \overline{X} \to \overline{Y}$ given by $f \cup \mathrm{id}_{\partial Y}$ is continuous;*
5. *The induced map $\overline{f}$ induces a homeomorphism $\partial f\colon \partial X \to \partial Y$;*
6. *Let $g\colon Z \to X$ be a map. Suppose that $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ and $(\overline{f \circ g}, f \circ g)\colon (\overline{Z}, Z) \to (\overline{Y}, Y)$ respectively are obtained by pulling back the boundary of $(\overline{Y}, Y)$ with f and $f \circ g$ respectively. Let $\overline{g}\colon (\overline{\overline{Z}}, Z) \to (\overline{X}, X)$ be obtained by pulling back the boundary of $(\overline{X}, X)$ with g.*

Then we get an equality of topological spaces $\overline{\overline{Z}} = \overline{Z}$ *and of maps* $\overline{f} \circ \overline{g} = \overline{f \circ g}$*;*

Proof. (1) Consider $x \in \partial X$ and a neighborhood W of x in $\overline{X}$. We have to show $X \cap W \neq \emptyset$. We can write $W = \overline{f}^{-1}(U) \cup V$ for open subsets $U \subset \overline{Y}$ and $V \subseteq X$. Without loss of generality we can assume $V = \emptyset$, or, equivalently $W = \overline{f}^{-1}(U)$ for open subset $U \subset \overline{Y}$. Obviously U is an open neighborhood of $\overline{f}(x) \in \overline{Y}$. Since by assumption the closure of the image of f in $\overline{Y}$ contains ∂Y, we have $\operatorname{im}(f) \cap U \neq \emptyset$ and hence $X \cap W \neq \emptyset$.

(2) Let $\{W_i \mid i \in I\}$ be an open covering of $\overline{X}$. We can write $W_i = \overline{f}^{-1}(U_i) \cup V_i$ for open subsets $U_i \subset \overline{Y}$ and $V_i \subseteq X$. Then $\{U_i \cap \partial Y \mid i \in I\}$ is an open covering of ∂Y. Since $\partial Y \subseteq Y$ is closed and $\overline{Y}$ is compact by assumption, ∂Y is compact. Hence there is a finite subset $J \subseteq I$ with $\partial Y \subseteq \bigcup_{i \in J} U_i$. The set $\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)$ is closed in $\overline{Y}$ and hence compact. Since $\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)$ is contained in Y and $f \colon X \to Y$ is by assumption proper, the preimage $f^{-1}\left(\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)\right)$ is also compact. Hence there is a finite subset $J' \subseteq I$ such that $\{W_i \mid j \in J'\}$ covers $f^{-1}\left(\overline{Y} \setminus \left(\bigcup_{i \in J} U_i\right)\right)$. Hence $\{W_i \mid i \in J \cup J'\}$ covers $\overline{X}$. This shows that $\overline{X}$ is compact.

(3) Consider any open covering $\mathcal{W} = \{W_i \mid i \in I\}$ of $\overline{X}$. By definition there are $U_i \subseteq \overline{Y}$ and $V_i \subseteq X$ such that $W_i = \overline{f}^{-1}(U_i) \cup V_i$. Now put

$$\begin{aligned} \mathcal{W}_{\partial X} &:= \{\overline{f}^{-1}(U_i) \mid i \in I\}; \\ \mathcal{W}_X &:= \{W_i \cap X \mid i \in I\}. \end{aligned}$$

Then $\mathcal{W}_{\partial X} \cup \mathcal{W}_X$ is an open covering of $\overline{X}$, which is a refinement of $\mathcal{W}$. Moreover, $\mathcal{W}_X$ is an open covering of X and the union of the elements in $\mathcal{W}_{\partial X}$ contains ∂X. We can find an open covering $\mathcal{V}_X$ whose covering dimension is less or equal to $\dim(X)$ and which refines $\mathcal{W}_X$. We obtain an open covering $\{U_i \mid i \in I\} \cup \{Y\}$ of $\overline{Y}$, since ∂Y is contained in $\bigcup_{i \in I} U_i$. We can find an open covering $\mathcal{V}_{\overline{Y}}$ of $\overline{Y}$ which is a refinement of $\{U_i \mid i \in I\} \cup \{Y\}$ and has dimension $\leq \dim(\overline{Y})$. Put

$$\mathcal{V}_{\partial Y} := \{V \in \mathcal{V}_{\overline{Y}} \mid V \cap \partial Y \neq \emptyset\}.$$

Then $\mathcal{V}_{\partial Y}$ is a refinement of $\{U_i \mid i \in I\}$, has covering dimension $\leq \dim(\overline{Y})$ and the union of the elements in $\mathcal{V}_{\partial Y}$ contains ∂Y. Define $\overline{f}^* \mathcal{V}_{\partial X}$ to be the collection of open subsets of $\overline{X}$ given by $\{\overline{f}^{-1}(V) \mid V \in \mathcal{V}_{\partial Y}\}$. Then $\overline{f}^* \mathcal{V}_{\partial X}$ is a refinement of $\mathcal{W}_{\partial X}$, has covering dimension $\leq \dim(\overline{Y})$ and the union of its elements contains $\partial X = \partial Y$. Put

$$\mathcal{V} = \mathcal{W}_X \cup \overline{f}^* \mathcal{V}_{\partial X}.$$

Then $\mathcal{V}$ is an open covering of $\overline{X}$ which refines $\mathcal{W}$. It covering dimension satisfies

$$\dim(\mathcal{V}) \leq \dim(\mathcal{V}_X) + \dim(\overline{f}^* \mathcal{V}_{\partial X}) + 1 \leq \dim(X) + \dim(\overline{Y}) + 1.$$

(4) If $U \subseteq \overline{Y}$ is open, then by definition $\overline{f}^{-1}(U) \subseteq \overline{X}$ is open.

(5) Obviously $\overline{f} \colon \overline{X} \to \overline{Y}$ induces a bijective continuous map $\partial f \colon \partial X \to \partial Y$. We have to show that it is open. An open subset of ∂X is of the form $\left(\overline{f}^{-1}(U) \cup V\right) \cap \partial X$ for some open subsets $U \subseteq \overline{Y}$ and $V \subseteq X$. Its image under ∂f is $U \cap \partial Y$ and hence an open subset of ∂Y.

(6). Notice that as sets $\overline{\overline{Z}}$ and $\overline{Z}$ agree, both look like $Z \amalg \partial Y$. Next we show that the two topologies agree. A subset W of $\overline{\overline{Z}}$ is open if there are open subsets $U \subseteq \overline{Y}$ and $V_2 \subseteq Z$ with $W = \overline{f \circ g}^{-1}(U) \cup V_2$. A subset $W_1 \subseteq \overline{X}$ is open if there exist open subsets $U \subseteq \overline{Y}$ and $V_1 \subseteq X$ with $W_1 = \overline{f}^{-1}(U) \cup V_1$. A subset W_2 of $\overline{Z}$ is open, if there exist open subsets $W_1 \subseteq \overline{X}$ and $V_2 \subseteq Z$ such that W_2 looks like

$\overline{g}^{-1}(W_1) \cup V_2$. This is equivalent to the existence of open subsets $U \subseteq \overline{Y}$, $V_1 \subseteq X$ and $V_2 \subseteq Z$ such that

$$W_2 = \overline{g}^{-1}\big(\overline{f}^{-1}(U) \cup V_1\big) \cup V_2.$$

Since

$$\overline{g}^{-1}\big(\overline{f}^{-1}(U) \cup V_1\big) \cup V_2 = \overline{f \circ g}^{-1}(U) \cup \big(g^{-1}(V_1) \cup V_2\big)$$

and $g^{-1}(V_1) \cup V_2$ is an open subset of Z, the topology on $\overline{\overline{Z}}$ is finer than the topology on $\overline{Z}$. So it remains to show that the topology on $\overline{Z}$ is finer than the topology on $\overline{\overline{Z}}$. This follows from the observation that for open subsets $U \subseteq \overline{Y}$ and $V_2 \subseteq Z$ we get

$$\overline{f \circ g}^{-1}(U) \cup V_2 = \overline{g}^{-1}(\overline{f}^{-1}(U) \cup \emptyset) \cup V_2.$$

□

Example 10.4 (One-point-compactification)**.** Let X and Y be locally compact spaces. Denote by X^c and Y^c their one-point-compactification. Let $f\colon X \to Y$ be a map. Denote by $(\overline{X}, X)$ the space obtained from (Y^c, Y) by pulling back the boundary with f.

Consider first the case where f is proper. Recall that a subset $W \subseteq Y^c = Y \cup \{\infty\}$ is open if it belongs to Y and is open in Y or there is a compact subset $C \subseteq Y$ such that $W = Y^c \setminus C$. This is indeed a topology, see [43, page 184]. By construction the underlying sets for $\overline{X}$ and X^c agree, namely, they are both given by $X \amalg \{\infty\}$. Next we compare the topologies.

Consider an open subset W of $\overline{X}$. We want to show that $W \subseteq X^c$ is open. We can write $W = \overline{f}^{-1}(U) \cup V$ for open subsets $U \subseteq Y^c$ and $V \subseteq X$. If ∞ does not belong to U, then U is an already open subset of Y and $\overline{f}^{-1}(U) = f^{-1}(U)$ is an open subset of X which implies that $W \subseteq X$ and hence $W \subseteq X^c$ are open. It remains to treat the case $\infty \in U$. From the definitions we conclude that we can write $W = \overline{f}^{-1}(Y^c \setminus C) \cup V$ for some compact subset $C \subseteq Y$ and an open subset V of Y. Since

$$\overline{f}^{-1}(Y^c \setminus C) = \overline{X} \setminus f^{-1}(C)$$

and by the properness of f the set $f^{-1}(C) \subseteq X$ is compact, W is open regarded as a subset of X^c.

This shows that the identity induces a continuous bijective map $X^c \to \overline{X}$. (One can also deduce this directly from Lemma 10.2.)

Since X^c is compact and $\overline{X}$ is Hausdorff, this is a homeomorphism, see [43, Theorem 5.6 in Chapter III on page 167]. Hence we get an equality of topological spaces $\overline{X} = X^c$ and of maps $\overline{f} = f^c$.

Now consider the case where f is the constant map onto some point $y_0 \in Y$. Suppose that X is not compact, or, equivalently, that the constant map f is not proper. The set $Y^c \setminus \{y_0\}$ is open in Y^c. Hence $\partial X = \{\infty\} = \overline{f}^{-1}(Y^c \setminus \{y_0\})$ is an open subset of $\overline{X}$. Since also $X \subseteq \overline{X}$ is open, $\overline{X}$ is, as a topological space, the disjoint union $X \amalg \{\infty\}$. Since X is not compact, its one-point compactification is not homeomorphic to $\overline{X}$.

Remark 10.5 (Dependency on f)**.** Example 10.4 shows that $\overline{X}$ does depend on the choice of f. So the reader should be careful when we just write $\overline{X}$ and not include f in the notation.

Lemma 10.6. *Consider a pair $(\overline{Y}, Y)$ of G-spaces such that compact subsets of Y become small at infinity in the sense of Definition 9.3. Let $f\colon X \to Y$ be a G-map. Suppose that $(\overline{X}, X)$ is obtained from $(\overline{Y}, Y)$ by pulling back the boundary with f.*

Then compact subsets of Y become small at infinity.

Proof. Consider an element $x \in \partial X$, an open neighborhood $U \subseteq \overline{X}$ of x, and a compact subset $K \subseteq X$. We can find an open neighborhood $U' \subseteq Y$ of $x \in \partial X = \partial Y$ and an open subset $W \subseteq X$ such that $U = \overline{f}^{-1}(U') \cup W$. Put $L = f(K)$. Then $L \subseteq Y$ is compact. By assumption we can find an open neighborhood $V' \subseteq Y$ of $x \in \partial Y$ with $V' \subseteq U'$ such that the implication $g \cdot L \cap V' \neq \emptyset \implies g \cdot L \subseteq U'$ holds for every $g \in G$. Put $V = \overline{f}^{-1}(V')$. This is an open neighborhood of $x \in \partial X$ with $V \subseteq U$. Moreover we get for every $g \in G$

$$\begin{aligned} g \cdot K \cap V \neq \emptyset \implies & g \cdot L \cap V' \neq \emptyset \\ & \implies g \cdot L \subseteq U' \implies g \cdot f^{-1}(L) \subseteq f^{-1}(U') \implies g \cdot K \subseteq U. \end{aligned}$$

□

Definition 10.7 (Continuously controlled)**.** Consider a pair $(\overline{Y}, Y)$ of spaces and a homotopy equivalence $f \colon X \to Y$. We call f *continuously controlled* if there exists a map $u \colon Y \to X$ and homotopies $h \colon f \circ u \simeq \mathrm{id}_Y$ and $k \colon u \circ f \simeq \mathrm{id}_X$ with the following property: For every $z \in \partial Y = \overline{Y} \setminus Y$ and neighborhood U of z in $\overline{Y}$ there is an open neighborhood V of z in $\overline{Y}$ with $V \subseteq U$ such that the implications $y \in V \implies h(\{y\} \times [0,1]) \subseteq U$ and $x \in X, f(x) \in V \implies f \circ k(\{x\} \times [0,1]) \subseteq U$ are true.

Lemma 10.8. *Let $f \colon X \to Y$ be a G-map of proper free G-spaces. Suppose that X is cocompact.*

Then f is proper.

Proof. We have the following pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X/G \arrow[r, "f/G"] & Y/G \end{tikzcd}$$

where the vertical maps are principal G-bundles. Since X/G is compact. f/G is proper. Hence f is proper by [38, Lemma 1.16 on page 14]. □

Lemma 10.9. *Consider a pair $(\overline{Y}, Y)$ of spaces such that $\overline{Y}$ is a* ANR *and ∂Y is a Z-set in $\overline{Y}$. Consider a homotopy equivalence $f \colon X \to Y$ which is continuously controlled. Let $(\overline{f}, f) \colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary along f.*

Then $\overline{X}$ is an ANR *and $\partial X \subseteq \overline{X}$ is a Z-set.*

Proof. We will use characterization (b) of Z-set on the first page of [9]. This characterization says that if $\overline{X} = X \cup \partial X$ with X an ANR and if there is a homotopy $h_t : \overline{X} \to \overline{X}$ with $h_0 = \mathrm{id}$ and $h_t(\overline{X}) \subset X$ for all $t > 0$, then $\overline{X}$ is an ANR and ∂X is a Z-set in $\overline{X}$.

The statement in [9] assumes that $\overline{X}$ is an ANR, but this is unnecessary, since Hanner's criterion, Thm 7.2 of[28], says that a compact metric space is an ANR if it is ϵ-dominated by ANRs for every $\epsilon > 0$. The homotopy h_t above shows that the ANR X ϵ-dominates $\overline{X}$ for every $\epsilon > 0$.

Let $c_t : \overline{Y} \to \overline{Y}$ be a homotopy so that $c_0 = \mathrm{id}_{\overline{Y}}$ and $c_t(\overline{Y}) \subset Y$ for all $t > 0$. The homotopy equivalence f has a homotopy inverse $g : Y \to X$. The continuous control condition means that f extends continuously by the identity on $\partial X = \partial Y$ to $\bar{f} : \overline{X} \to \overline{Y}$, g extends continuously by the identity to $\bar{g} : \overline{Y} \to \overline{X}$ and there are homotopies h_t from id_Y to $f \circ g$ and k_t from id_X to $g \circ f$ which extend continuously by the identity to $\bar{h}_t$ and $\bar{k}_t$.

For $x \in \bar{X}$, let $\alpha(x) = \min(\operatorname{diam}(\{\bar{k}_t(x), 0 \le t \le 1\}), \frac{1}{2})$. Set

$$\bar{e}_t = \begin{cases} \bar{k}_{t/\alpha(x)}(x) & 0 \le t \le \alpha(x),\ \alpha(x) \neq 0; \\ \bar{g} \circ c_{t-\alpha(x)} \circ \bar{f}(x) & \alpha(x) \le t \le 1 \text{ or } \alpha(x) = 0. \end{cases}$$

For $t = 0$ and $\alpha(x) \neq 0$, we have $\bar{e}_0(x) = \bar{k}_0(x) = x$. If $t = 0$ and $\alpha(x) = 0$, we have $\bar{e}_0(x) = \bar{g} \circ c_0 \circ \bar{f}(x) = x$, since $\alpha(x) = 0$ implies that $\bar{k}_t(x) = x$ for all $0 \le t \le 1$. If $t = \alpha(x) \neq 0$, $\bar{e}_t(x) = \bar{g} \circ \bar{f}(x)$ with either definition. If $t = \alpha(x)$, then both definitions give $\bar{g} \circ \bar{f}(x)$. □

Lemma 10.10. *Consider a G-homotopy equivalence of $f\colon X \to Y$ of cocompact proper free G-spaces. Suppose that Y is a subspace of the compact G-space $\overline{Y}$ such that compact subsets become small at infinity for $(\overline{Y}, Y)$.*

Then f is continuously controlled at infinity.

Proof. Choose a G-map $u\colon Y \to X$ and G-homotopies $h\colon f \circ u \simeq \mathrm{id}_Y$ and $k\colon u \circ f \simeq \mathrm{id}_X$. Choose a compact subset $C \subseteq X$ such that $G \cdot C = X$ holds.

Fix a point $z \in \partial Y = \overline{Y} \setminus Y$ and an open neighborhood U of z in $\overline{Y}$.

Since compact subsets become small at infinity for $(\overline{Y}, Y)$, we can find an open neighborhood V of z in $\overline{Y}$ with $V \subseteq U$ such that for every $g \in G$ we have the implication $g \cdot h(C \times [0,1]) \cap V \neq \emptyset \implies g \cdot h(C \times [0,1]) \subseteq U$.

Consider $y \in V$. We can find $g \in G$ with $y \in g \cdot C$. Since $y = h(y,1) \in g \cdot h(C \times [0,1])$, we get $g \cdot h(C \times [0,1]) \cap V \neq \emptyset$. This implies $g \cdot h(C \times [0,1]) \subseteq U$ and in particular $h(\{y\} \times [0,1]) \subseteq U$.

The map f is proper by Lemma 10.8. Hence $f^{-1}(C) \subseteq \widehat{X}$ is compact. Let $(\overline{f}, f)\colon (\overline{X}, X) \to (\overline{Y}, Y)$ be obtained by pulling back the boundary with f. Since compact subsets become small at infinity for $(\overline{X}, X)$ by Lemma 10.6, we can find an open neighborhood V' of z in $\overline{X}$ with $V' \subseteq \overline{f}^{-1}(U)$ such that for every $g \in G$ we have the implication $g \cdot k(f^{-1}(C) \times [0,1]) \cap V' \neq \emptyset \implies g \cdot k(f^{-1}(C) \times [0,1]) \subseteq f^{-1}(U)$. Choose an open subset $V \subseteq \overline{X}$ and an open subset $W \subseteq X$ with $V' = \overline{f}^{-1}(V) \cup W$. Since the implication above remains true if we shrink V', we can assume without loss of generality that $V' = \overline{f}^{-1}(V)$. In particular V is an open neighborhood of $z \in \overline{Y}$.

Consider $x \in X$ with $f(x) \in V$. Then $x \in \overline{f}^{-1}(V)$. We can find $g \in G$ with $x \in g \cdot f^{-1}(C)$. Since $x = k(x,1) \in g \cdot k(f^{-1}(C) \times [0,1])$, we get $g \cdot k(f^{-1}(C) \times [0,1]) \cap \overline{f}^{-1}(V) \neq \emptyset$. This implies $g \cdot k(f^{-1}(C) \times [0,1]) \subseteq \overline{f}^{-1}(U)$ and hence $f \circ k(\{x\} \times [0,1]) \subseteq U$.

□

11. Recognizing the structure of a manifold with boundary

Recall that we have discussed the basic properties of the Rips complex $P_l(G)$ before in Section 9.

Theorem 11.1. *Let G be torsionfree hyperbolic group G with boundary S^2. Consider a homotopy equivalence $f\colon M \to P_l(G)/G \times N$, where M is a closed homology* ANR*-manifold, and N is a closed topological manifold of dimension ≥ 2. Denote by $p_G\colon P_l(G) \to P_l(G)/G$ the canonical projection. Let the G-covering $\widehat{M} \to M$ be the pullback with f of the G-covering $p_G \times \mathrm{id}_N\colon P_l(G) \times N \to P_l(G)/G \times N$ and $\widehat{f}\colon \widehat{M} \to P_l(G) \times N$ be the induced G-homotopy equivalence. Let $(\overline{\widehat{f}}, \widehat{f})\colon (\overline{\widehat{M}}, \widehat{M}) \to (\overline{P_l(G)}, P_l(G))$ be obtained by pulling back the boundary along $\widehat{f}$.*

Then $\overline{\widehat{M}}$ is a compact homology ANR*-manifold whose boundary $\partial\overline{\widehat{M}}$ is $S^2 \times N$ and a Z-set.*

Proof. Recall from Section 9 that $P_l(G) \to P_l(G)/G$ is a model for the universal principal G-bundle $EG \to BG$ and $P_l(G)/G$ is a finite CW-complex. Hence $P_l(G)$ is a cocompact free proper G-space. Compact subsets of $P_l(G)$ become small at infinity for the pair $(\overline{P_l(G)}, P_l(G))$. The space $\overline{P_l(G)}$ is a compact metrizable ANR and $\partial P_l(G) \subseteq \overline{P_l(G)}$ is a Z-set. We conclude from Lemma 10.9 and Lemma 10.10 that $\partial\overline{\widehat{M}} \subseteq \overline{\widehat{M}}$ is a Z-set and $\overline{\widehat{M}}$ is an ANR. We conclude that $\overline{\widehat{M}}$ is compact and has finite dimension from Lemma 10.3 (2) and (3), and Lemma 10.8. Lemma 9.2 implies that $\overline{\widehat{M}}$ is a homology ANR-manifold with boundary in the sense of Definition 6.3. □

12. Proof of Theorem 0.3 and Theorem 0.4

This section is entirely devoted to the proof of Theorem 0.3 and Theorem 0.4. We begin with the following considerations.

Consider a hyperbolic 3-dimensional Poincaré duality group G. Then G is torsionfree and ∂G is S^2 by Theorem 1.10. Let N be an aspherical closed topological manifold of dimension ≥ 3 with fundamental group π. Suppose that π is a Farrell-Jones group. Then $G \times \pi$ is a finitely presented $(3+n)$-dimensional Poincaré duality group. We conclude that $G \times \pi$ is a Farrell-Jones group from Theorem 4.1 (1a) and (2b). Since $3 + n \geq 6$, we conclude from Theorem 1.11 that there is a closed homology ANR-manifold M having the DDP and a homotopy equivalence $M \to BG \times N$.

Denote by $p_G\colon P_l(G) \to P_l(G)/G$ the canonical projection. Let the G-covering $\widehat{M} \to M$ be the pullback with f of the G-covering $p_G \times \mathrm{id}_N\colon P_l(G) \times N \to P_l(G)/G \times N$ and $\widehat{f}\colon \widehat{M} \to P_l(G) \times N$ be the induced G-homotopy equivalence. Let $(\overline{\widehat{f}}, \widehat{f})\colon (\overline{\widehat{M}}, \widehat{M}) \to (\overline{P_l(G)} \times N, P_l(G) \times N)$ be obtained by pulling back the boundary along $\widehat{f}$. Theorem 11.1 implies that $\overline{\widehat{M}}$ is a compact homology ANR-manifold whose boundary $\partial\overline{\widehat{M}}$ is $S^2 \times N$ and a Z-set in $\overline{\widehat{M}}$. We conclude from Lemma 8.2 that $i(\overline{\widehat{M}}) = 1$. Theorem 8.1 (1) implies $i(M) = 1$. We conclude from Theorem 8.1 (2) and a result due to Ferry and Seebeck, which can be found in [16, Theorem 1 in Section 40 on page 285], that $\overline{\widehat{M}}$ is a compact topological manifold with boundary $\partial\overline{\widehat{M}} = S^2 \times N$ and M is a closed topological manifold.

Since $\partial P_l(G)$ is a Z-set in $\overline{P_l(G)}$, $\partial P_l(G) \times N$ is a Z-set in $\overline{P_l(G)} \times N$. We know already that $\partial\overline{\widehat{M}}$ is a Z-set in $\overline{\widehat{M}}$. Since $P_l(G)$ is contractible, $\overline{P_l(G)}$ is contractible. Since $\widehat{f}$ is a homotopy equivalence, $\overline{\widehat{f}}$ is a homotopy equivalence. Hence there is a homotopy equivalence $(U, u)\colon (\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N)$ such that u is a homeomorphism. Since $\pi_1(D^3 \times N) \cong \pi$ is a Farrell-Jones group and $3+n \geq 6$, the relative Borel Conjecture holds, i.e., we can change (U, u) up to homotopy relative $\partial\overline{\widehat{M}}$ such that we obtain a homeomorphism of pairs

$$(U, u)\colon (\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N).$$

Proof of Theorem 0.3. The considerations above applied in the special case $N = T^3$ show that $BG \times T^3$ is homotopy equivalent to a closed topological manifold, since $\mathbb{Z}^3$ is a Farrell-Jones group by Theorem 4.1 (1b). Hence $s^{\mathrm{sym}}(BG \times T^3)$ vanishes by Theorem 5.26 (2). We conclude from Theorem 5.29 or directly from Remark 5.32 that $s^{\mathrm{sym}}(BG)$ vanishes. Now Theorem 0.3 follows from Theorem 5.26 1. □

Proof of Theorem 0.4. The considerations above applied in the special case $N = T^3$ show that $BG \times T^3$ is homotopy equivalent to a closed topological manifold.

Now let N be any closed smooth manifold, closed Pl-manifold, or closed topological manifold respectively of dimension ≥ 2. We conclude from Theorem 5.1 that there exists a normal map of degree one for some vector bundle ξ over BG

$$\begin{tikzcd} TM \oplus \underline{\mathbb{R}^a} \arrow[r, "\overline{f}"] \arrow[d] & (\xi \times TN) \oplus \underline{\mathbb{R}^b} \arrow[d] \\ M \arrow[r, "f"] & BG \times N \end{tikzcd}$$

such that M is a smooth manifold, Pl-manifold, or topological manifold respectively and f is a simple homotopy equivalence. The considerations above applied in the case, where N is aspherical and $n \geq 3$, imply the existence of a homeomorphism

$$(U, u)\colon (\overline{\widehat{M}}, \partial\overline{\widehat{M}}) \to (D^3 \times N, S^2 \times N).$$

This finishes the proof of Theorem 0.4 $\square$

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