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CLASS NUMBERS, THE NOVIKOV CONJECTURE, AND TRANSFORMATION GROUPS

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INTRODUCTION

EWING [7] showed that the G-signature of a smooth G-manifold for $G = Z_p$, p an odd prime, is essentially unrestricted if and only if the class number of the cyclotomic field containing the p-th roots of unity is odd. Katz [9] has elaborated on this, using results from smooth equivariant cobordism theory, to prove precise integrality formulae; in particular, he connects the G-signature mod 4 to the signatures and local representations occurring at components of the fixed set.

In a different direction the author [21] showed that the existence of certain group actions on nonsimply connected manifolds forces their higher structures to vanish. In fact, the Novikov conjecture is equivalent to a statement about group actions.

In yet another direction, S. Cappell and the author constructed [5] certain characteristic classes for semifree PL (locally linear) G-actions; these were applied there to prove a splitting theorem for some classifying spaces. This splitting, crucial for many equivariant existence and classification problems, at the prime 2 depends on understanding peripheral invariants of free group actions on sphere bundles. For locally-nonlinear actions, the results are deduced by means of comparison to the locally linear case.

In this paper, we study the cobordism of homologically trivial actions and use it to unify, extend, and improve our understanding of all the above phenomena. The method is to consider, say, $QHT_n(Z_p, X)$ which is roughly speaking the cobordism group of n-manifolds with rationally homologically trivial Z_p-action, mapping into X. This is *not* a representable functor, and one cannot reduce the problem to stable homotopy theory of some Thom spectrum. (It is not always possible to make a transverse inverse image have a homologically trivial action). The solution (Theorem 1) to this difficulty, similar to that for Poincaré cobordism, measures the deviation through an exact sequence involving $L_n^h(R[\pi_1 X])$ (for an appropriate ring R) as the "third term".

At this point one easily recovers the result that the Novikov conjecture implies the vanishing of higher signatures for homologically trivial actions. This then gives information about which classes in the bordism group $\Omega(B(\pi \times G))$ have such G-actions, while [21] only solves (in some cases) this for $\Omega(B\pi)$. (Of course, the exact sequence also computes how many actions there are corresponding to a given class.)

Now, for X = point, the Atiyah–Singer invariant is essentially a cobordism invariant, and the next order of business is to compute its role in the theory. Using the connection

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between surgery groups, discriminants, and multisignature one recovers a weak form of Katz's formula for PL actions. (It is unreasonable to expect to re-derive his entire result because of ignorance of equivariant PL-cobordism.) Reversing the smooth program further, we study the effect of PL-local linearity and recover Ewing's theorem for PL-locally linear actions. For arbitrary PL actions the multisignature is unrestricted for some primes with even class numbers, and for topologically locally linear actions it is never restricted.† These same ideas give an interpretation of the parity of the second factor of the class number in terms of a refined G-signature for $G = Z_p$.

The divisibility theorems necessary for the PL-classes of [5], at least for Z_p, are now immediate in a much more general form. In the present development the locally linear case is the more difficult, requiring the notion of a simple homologically trivial action, and is derivative of the nonlocally linear case. In addition, away from 2,‡ these classes are given a totally new description using a blocked version of the calculation, which yields a slightly more conceptual definition. (Of course, these results do not yield the desired geometric consequences of [5] without the material on classifying spaces from that paper; we construct obstructions but do not prove that they are the only obstructions for the relevant problems.)

The organization of this paper is as follows: section 1 gives a theoretical calculation of homologically trivial cobordism, and gives the characteristic classes away from 2 inter alia. In section 2 we derive immediate consequences including connections with the Novikov conjecture. An appendix reviews the conjecture and proves a form, in many cases, that is useful for transformation groups. Section 3 deals the cobordism of nilpotent manifolds as a special case, and derives some results on the range of ρ-invariants, relevant to [5] and section 3. Section 4 is brief, introducing simple actions and their cobordism. The last section is devoted to the PL analogues and extensions of the work of Katz and Ewing.

In a future paper we will extend this last application to Z_n for n not a prime. Here there are a number of differences because of the number theory of composite cyclotomic fields which differs from the prime power case. However, much of section 5 is relevant to this case as well.

1. THE EXACT SEQUENCE

We begin with a definition:

DEFINITION. *A free action of* G *on* M *is* R-(*twisted*) *homologically trivial if* (a) $\pi_1(M/G)$ *is isomorphic to* $\pi_1(M) \times G$ *and* (b) *the action of* G *on* $H_*(M; R[\pi_1 M])$ *is trivial.*

It is the aim of this section to compute the cobordism groups of these objects for R a subring of $\mathbf{Q}$ containing $1/|G|$. The Lefshetz fixed point theorem implies that a manifold having a free homologically trivial action of any non-trivial group necessarily has vanishing Euler characteristic. Therefore one makes another:

DEFINITION. $\bar{\Omega}(B\pi)$ *is the cobordism group of (oriented) manifolds with fundamental group* π *and Euler characteristic zero. (Ordinary cobordism is obtained by deleting the bar from the notation.)*

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†Never, at least for $n > 4$. In dimension 4, the topological locally linear result agrees with the smooth result, and it is ANR actions that have unrestricted multisignatures.

‡This is not crucial; a trick from [5] can be combined with the methods here to complete the description, even at 2.

PROPOSITION. *$\bar{\Omega}_n(B\pi) \to \Omega_n(B\pi)$ is an isomorphism for $n \equiv 2, 3 \bmod 4$, an injection for $n \equiv 0 \bmod 4$ and a (split) surjection for $n \equiv 1 \bmod 4$. In the last two cases the cokernel and kernel (respectively) are Z_2.*

The proof is not difficult. The splitting, for $n \equiv 1 \bmod 4$, of $\bar{\Omega}_n(B\pi) \to Z_2$ is given by the rational semicharacteristic (see [21]).

We introduce one last bit of (nonstandard!) notation:

Notation. $\bar{L}_n(R[\pi \times G]) = \operatorname{Ker} L_n^h(R[\pi \times G]) \to L_n^h(R[\pi])$. If $\frac{1}{|G|} \in R$, it can be identified with $L_n^h(\mathscr{I}[\pi])$ where $\mathscr{I}$ is the augmentation ideal of $R[G]$ viewed as a unitary ring with involution $\left(\text{the unit being } \left[1 - \frac{1}{|G|}\sum_{g \in G} g\right], \text{ of course}\right)$. We shall use these notations interchangeably.

DEFINITION. $\mathrm{RHT}_n(G, B\pi)$ *is the n-dimensional cobordism group of manifolds with fundamental group π and with free R-twisted homologically trivial G action.*

Our calculation is:

THEOREM 1. *For $n \geq 5$ there is an exact sequence, assuming $Z\left[\frac{1}{|G|}\right] \subseteq R \subseteq Q$.*

$$\cdots \xrightarrow{\theta} L_{n+1}^h(\mathscr{I}[\pi]) \xrightarrow{a} \mathrm{RHT}_n(G, B\pi) \xrightarrow{f} \bar{\Omega}_n(B(\pi \times G)) \xrightarrow{\theta} L_n^h(\mathscr{I}[\pi])$$

The map from $RHT_n(G, B\pi) \to \bar{\Omega}_n(B(\pi \times G))$ is just given by taking the quotient. The map $\bar{\Omega}_n(B(\pi \times G)) \to L_n^h(\mathscr{I}[\pi])$ is induced by considering the local surgery obstruction of the problem $M/G \to M/G \times BG$. (By [21, II] this is a surgery problem since $\chi(M) = \chi(M/G) = 0$.) Here the obstruction lies in $L_n^h(R[\pi_1(M/G) \times G])$ which we map to $L_n^h(R[\pi_1(M) \times G])$ using the splitting given in the definition of $\bar{\Omega}_n(B(\pi \times G))$. Since $\tilde{H}_*(BG; R) = 0$, this is already an equivalence with $R[\pi_1 M]$ coefficients so the obstruction lies in the correct relative group.

The map from $L_{n+1}^h(\mathscr{I}[\pi])$ to $RHT_n(G, B\pi)$ is given as an action, so it is first logically necessary to show that $RHT_n(G, B\pi)$ is not empty. This follows quite easily from [21, II] or the proof of exactness given below. (Consider $0 \in \bar{\Omega}_n$.) Note however, that for $\frac{1}{|G|} \notin R$ this may already fail. (Think about $R = Z$ and G a large elementary p-group.) (For certain groups $RHT_n(G, \{e\})$ might be empty for $n = 4$.) Now, let $L_{n+1}^h(\mathscr{I}[\pi])$ act on $M/G \xrightarrow{id} M/G$ by the local form of the Wall realization theorem [18]. The "other end" of the cobordism is a manifold $R[\pi \times G]$-homology equivalent to M/G so that its cover corresponding to G has the appropriate twisted homological triviality.

Now, let us consider exactness. $\theta \circ a = 0$ since homological triviality implies that one is computing the obstruction of a homology equivalence. Conversely, if $\theta([M]) = 0$ then applying local surgery one obtains a manifold N, $R[\pi \times G]$ equivalent to $M/G \times BG$. The G action on $\tilde{N}$ is R-homologically trivial. That $f \circ a = 0$ is clear for $n \not\equiv 1 \bmod 4$ since "a" is defined by constructing a cobordism, so the image in $\Omega_n(B(\pi \times G))$ vanishes and $\bar{\Omega}_n(B(\pi \times G)) \to \Omega_n(B(\pi \times G))$ injects. For $n \equiv 1 \bmod 4$ there is no longer an injection, but the additional information is carried by the rational semicharacteristic, but the rational homotopy type of the action is unchanged by the L-group action. On the other hand from an element of RHT which bounds in $\bar{\Omega}$ one can construct the above surgery problem with obstruction in $L^h_{n+1}(\mathscr{I}[\pi])$. We give the reader a hint on checking that the image of this element is the original action (up to RHT cobordism). The difficulty is that one uses different surgery problems in the definitions of the relevant obstructions. Recall that the element dies in $L^h_{n+1}(R[\pi])$ so that if N is a cobordism erected in the definition of a, the map $N/G \times BG \to (\partial/G \times I) \times BG$ has a trivial surgery obstruction (rel ∂). The result of surgery rel ∂ is the desired RHT-cobordism. $a \circ \theta = 0$ for the same sort of reason, and if a acts trivially one can glue in an R-homologically trivial cobordism between the ends to get an element of $\bar{\Omega}_{n+1}(B(\pi \times G))$. Q.E.D.

Remarks. (1) If $\frac{1}{2} \in R$ (e.g., we have already assumed this if G is of even order) then $L^h_n(\mathscr{I}[\pi]) \simeq L^n_h(\mathscr{I}[\pi])$ (see [14] for definitions) and the map θ is easier to understand. It is simply the symmetric signature of M/G pushed into the augmentation ring.

(2) Rather than considering manifolds with fundamental group π one can instead consider manifolds with equivariant maps to X (trivial action on X). Almost identical reasoning yields the exact sequence (obvious notation)

$$\cdots \to L^h_{n+1}(\mathscr{I}[\pi_1 X]) \to RHT_n(G, X) \to \bar{\Omega}_n(X \times BG) \to L^h_n(\mathscr{I}[\pi_1 X])$$

(3) Consider the case where X^k is a manifold of dimension k. It is also of some interest to consider actions that are homologically trivial over X, *because* they are homologically trivial over small pieces of X. There are a number of equivalent ways of formalising this notion. The simplest is to demand the map to X be transverse to some very fine triangulation so that inverse images of simplices of X have a R-homologically trivial action. We call this set $RHT_n(G, X)$. One can also form the obvious Δ-set whose 0-simplices are such objects, 1-simplices such cobordisms, etc. and call it $RHT(G, X)$. The calculation, which is just a formal Δ-version of the above is a fibration.

$$RHT_{n+k}(G, X) \to \bar{\Omega}_n(G)^X \to L^h_n(\mathscr{I})^X$$

where $\bar{\Omega}$ is a spectrum for cobordism with vanishing X and is related to the ordinary spectrum Ω by a fibration $\bar{\Omega} \to \Omega \to \bigvee_k \mathbf{K}(Z_2, 2k)$ and L is a Quinn–Ranicki surgery spectrum. We refer to this fibration as "Fine Theorem 1" or "$\pitchfork$-Theorem 1". If one studies $RHT^{\pitchfork}(G, X)$ with some canonically given ordinary coboundary over X (this can always be made $\pitchfork$ by ordinary transversality) one sees directly from the fine theorem that, away from 2 and $|G|$, this is given by homotopy classes of maps $X \to L^h_{n+1}(\mathscr{I})\frac{1}{2|G|}$. Applying this to the boundary of an equivariant regular neighborhood of the fixed set S of a semifree G action one obtains the characteristic classes (inverting 2 if $|G|$ is odd) of [5]. For more on these, see section 4.

2. FIRST APPLICATIONS

The material of section 1 can be immediately applied to give interesting information without any additional development. We always assume that R is a subring of Q containing $1/|G|$. We begin with what might be the most interesting case, π the trivial group.

COROLLARY 1. $\mathrm{RHT}_*(\mathrm{G}, \{e\})$ *is finitely generated for all* $*$ *if and only if there are only finitely many primes inverted in* R.

Proof. This is the result of well-known calculations of Wall groups. For $G = Z_2$ the relevant calculations are in [11].

COROLLARY 2. *For* $\mathrm{R} \subseteq \mathbf{Q}$, $\mathrm{RHT}_*(\mathrm{G},\mathrm{X}) \to \mathbf{Q}\mathrm{HT}_*(\mathrm{G},\mathrm{X})$ *is an isomorphism away from* 2.

Proof. This follows from Ranicki's localization sequence [15].

The next result shows the relationship between $RHT^{\pitchfork}(G,X)$ and $RHT(G,X)$.

COROLLARY 3. $\mathrm{RHT}^{\pitchfork}(\mathrm{G},\mathrm{X}) \to \mathrm{RHT}(\mathrm{G},\mathrm{X})$ *is an isomorphism if and only if an assembly map* $\mathbf{H}(\mathrm{X}; \mathrm{L}(\mathscr{I})) \to \mathbf{L}_*(\mathscr{I}[\pi_1 \mathrm{X}])$ *is an isomorphism.*

Proof. Apply the 5-Lemma to Theorem 1 and its fine version.

As a result, unless X is aspherical there are almost always transversality obstructions. In the aspherical case, it depends on the Novikov conjecture and on G. From material explained later, (see section five) the reader can easily deduce that for $X = S^1$ and $G = G_p$ and p with odd class number $RHT^{\pitchfork} \to RHT$ is an isomorphism, but that for $p = 29$ it is not.

Another application of this approach to homologically trivial transversality to "higher equivariant signature formulae" will appear in [25].

Before continuing, it is of use to describe the G-signature and ρ-invariants of the appropriate sorts of G-manifolds. If G acts orientation-preservingly on M^{4k} then it acts as a group of isometries on the intersection cup product pairing $H^{2k}(M;R)$. In particular it preserves the positive and negative definite pieces. We let $\sigma(G,M) \in R_+(G)$ denote the difference between the representations of G on these pieces. The $+$ subscript indicates that the character of the representation is real. This is an equivariant cobordism invariant. There is a similar definition possible for manifolds of dimension $4k+2$ except that the character is purely imaginary (see [2]). Key facts are that the G-signature of a free action on a closed manifold is always a multiple of the regular representation and that Novikov additivity holds for computing the G-signature of a manifold obtained by glueing two manifolds with boundary together along some components of their boundary. These allow the definition of the ρ-invariant of an odd dimensional free G-manifold, M. In short, bordism theory shows that for some $r > 0$, $rM = \partial N$ for a free G-manifold N. $\rho(M) = \dfrac{1}{r}\sigma(G,N)$ modulo the regular representation, e.g., $\rho(M) \in \tilde{R}(G) \otimes Q$.

COROLLARY 4. $\mathrm{RHT}(\mathrm{G}, \{\mathrm{e}\}) \otimes \mathrm{Z}[\frac{1}{2}]$ *is detected by the bordism class of the manifold and the* ρ*-invariant of the action (modulo the trivial representation in dimension* $4k-1$).

Proof. That these are cobordism invariants is obvious. That they detect follows from the connection between G-signature and L-groups (see e.g. [13], [19]) and Theorem 1.

In section 4 we shall examine more closely the role of the ρ invariant in homologically trivial cobordism theories to prove integrality formulae for it and for σ.

The methods of this paper also allow a result not $\otimes Z[\frac{1}{2}]$.

The same reasoning, coupled with calculations found in the Appendix one has:

COROLLARY 5. *If* π *is free, free abelian, a surface group or more generally lies in Cappell's class* $\mathscr{C}_1$ *(see [4]) then* $\mathrm{RHT}^{\pitchfork}_n(\mathrm{G}, \mathrm{B}\pi) \to \mathrm{RHT}_n(\mathrm{G}, \mathrm{B}\pi)$ *is an isomorphism after* $\otimes\, \mathbf{Z}[\frac{1}{2}]$. *After tensoring with* $\mathbf{Q}$ *these are detected by the bordism class in* $\Omega_*(\mathrm{B}\pi) \otimes \mathbf{Q}$ *and a "reduced higher* ρ*-invariant (analogous to Novikov's higher signature).*

We leave as an exercise the situation of π finite for comparison.

In terms of realization of these invariants, one knows that everything is realized except for cobordism classes with nontrivial Novikov higher signature, again by examining the exact sequence. The details are similar to (but easier than) the ones in [21, II].

As a last general point one has

COROLLARY 6. *As a function of* G, $\mathrm{RHT}_*(\mathrm{G},\mathrm{X})$ *satisfied induction* (*i.e., after* $\otimes\, \mathrm{Z}_{(p)}$ *one can apply* p-*hyperelementary induction*).

Proof. This, of course, is ultimately a consequence of Dress induction. A complete proof of a similar sort is carried out in Nicas' thesis [12]. Recent (as yet unpublished) work of I. Hambleton, J. Milgram, L. Taylor, and B. Williams provides a context in which such induction results become routine.

APPENDIX TO SECTION 2. A REFINED NOVIKOV CONJECTURE

The Novikov conjecture as it is usually constructed is the homotopy invariance of certain generalized Pontrjagin numbers. More precisely, let M be a manifold with fundamental group identified with π via a map $f: M \to B\pi$. Then one can push the homology L-class of M into the group homology, i.e., if $L(M)$ is the Hirzebruch L-polynomial (considered as a graded cohomology class) in the Pontrjagin classes of M, then one defines

$$\sigma_\pi(M) = f_*(L(M) \cap [M]) \in H_*(B\pi; \mathbf{Q}).$$

The Novikov conjecture states that if $h: M \to N$ is a homotopy equivalence of manifold that preserves orientation and identification of fundamental group then $\sigma_\pi(M) = \sigma_\pi(N)$. Notice that for π the trivial group, $\sigma_\pi(M)$ is just the ordinary signature by Hirzebruch's formula and that homotopy invariance is then trivial.

A link between the Novikov conjecture and surgery theory can be forged as follows. (See the last section of [18] or [4].) There is a natural map (regarding the objects as being Z_4-graded)

$$A: H_*(\pi; \mathbf{Q}) \to L_*(\pi) \otimes \mathbf{Q}.$$

The image of the higher signature in the L-theory is a homotopy invariant. Thus, to prove the Novikov conjecture for a group π it suffices (and, in fact, is also necessary) to show that A is injective. In all the presently known examples, if π is torsion free, A is actually an isomorphism. One can generalize A somewhat. Let S be an arbitrary unitary ring with involution then there is a homomorphism (induced by a map of spectra):

$$A: \mathbf{H}_*(\pi; \mathbf{L}(S)) \to L_*(S\pi).$$

(We have shifted viewpoints a bit now—we view $\bigoplus_{i \equiv n(4)} H_i(\pi; \mathbf{Q})$ as the rationalization of $H_n(B\pi; \mathbf{L}(\mathbf{Z}))$.) It is unreasonable to demand that *this* map be an isomorphism for π torsion free. This is due to the contribution of the Tate cohomology of the K-theory of S. (Think about Shaneson's $\mathbf{Z} \times G$ formula [17], $L_*^h(S[Z]) \simeq L_*^h(S) \times L_{*-1}^p(S)$, so $H(\mathbf{Z}_2; K_0(S))$ messes up the isomorphism.) There are two solutions. The simpler is to $\otimes Z[\frac{1}{2}]$. The more refined is to work with $L^{-\infty}$ rather than any of the usual L-theories. These groups solve the problem of when a surgery problem crossed with some (unspecified) torus can be coborded to a homotopy equivalence. With either of these emendations it might be reasonable to suggest the conjecture that A is an isomorphism.

A case of particular importance is $S = R[G]$. Then A can be viewed as a "partial assembly" via

$$\begin{tikzcd}
\mathbf{H}(G \times \pi;\, \mathbf{L}(R)) \simeq \mathbf{H}(\pi;\mathbf{H}(G;L(R))) \arrow[d, "A_G \text{ (on coefficients)}"] \arrow[dd, bend right, "A_{G\times\pi}"'] \\
\mathbf{H}(\pi;\mathbf{L}(R[G])) \arrow[d, "A_\pi"] \\
L(R[G][\pi]) = L(R[G \times \pi])
\end{tikzcd}$$

Notice that we immediately have

PROPOSITION 7. *Strong Novikov conjectures for* G $\times$ π *follow from the conjectures for* G *and for* π.

For π trivial, the conjecture is trivial, and for $\pi = Z$ this is Shaneson's thesis again (actually the algebraic version due to Ranicki), and hence the conjecture is true for free abelian groups. Using the splitting theorem of Cappell [4] one obtains

THEOREM 8. *Let $\mathscr{C}_1$ be the class of groups obtained from the trivial group from amalgamated free products (along square root closed subgroups),* HNN *extension and direct products. Then for $\pi \in \mathscr{C}_1$ the (more) refined Novikov conjecture holds.* (Recently, S. Ferry and the author have obtained a version of the strong Novikov conjecture for discrete subgroups of semisimple Lie groups.)

Even for the ordinary Novikov conjecture this is a (slightly) stronger result than [4]. Also if the ordinary conjecture holds for G then it does for $\mathscr{C}_1 \times G$. We leave the proof of this and the obvious analogues of other statements from [4] to the reader.

The case relevant for section 2 is $S = \mathscr{I}$ the augmentation ring of RG. We remark that the simple ideas of this appendix can be used to short circuit some calculations in the literature. (For instance, to take a personal example, using $Z^2 \times \pi$ where π has infinite ooze as in [22] one can give a less calculational example of a non-CP homotopy equivalence, using $S = Z[Z^2]$ than one given in [23].)

3. COBORDISM OF NILPOTENT MANIFOLDS

Recall that a space is nilpotent if its fundamental group is, and its fundamental group acts nilpotently on its higher homotopy groups. Thus all simply connected spaces are nilpotent. In particular, elementary surgeries show that every oriented manifold is cobordant, by a simply connected cobordism, to a simply connected manifold, so that ordinary (oriented) cobordism coincides with cobordism of nilpotent manifolds. However, things heat up a bit if one considers just examples with a fixed fundamental group. Here we restrict attention to π an odd p-group, and in fact to $\pi = Z_{p^r}$. The latter restriction is reasonable because we are mainly interested in 2-torsion and Corollary 6 will permit restriction to 2-hyperelementary subgroups which are cyclic. The situation for more general π_1 is unclear.

In order to apply our methods to the problem we invoke:

LEMMA 9. *If $\pi = \pi_1 X$ is a* p*-group, then* X *is nilpotent if and only if π acts trivially on* $H_*\left(\tilde{X};Z\left[\frac{1}{p}\right]\right)$ *where $\tilde{X}$ is the universal cover of* X.

Proof. Note that the action on homotopy groups is nilpotent if and only if π acts trivially on $\pi_* \otimes Z\left[\frac{1}{p}\right]$. If X is nilpotent then localization theory applies to show that

$X\left[\frac{1}{p}\right] \sim \tilde{X}\left[\frac{1}{p}\right] \times B\pi$. $\Big($Here "$\sim$" means "has the $Z\left[\frac{1}{p}\right][Z_p]$ homotopy type".$\Big)$ From this equivalence, the statement on homology follows. Conversely, one can make a p-plus construction ([21, I]) $X \to X^+$ that kills $\pi_1 X$ and is a $Z\left[\frac{1}{p}\right]$ homology isomorphism, that is, $\tilde{X}\left[\frac{1}{p}\right] \to X^+\left[\frac{1}{p}\right]$ is a homotopy equivalence by the homology assumption so that there is an equivalence $X \to \tilde{X} \times B\pi$ so the usual definition of nilpotence can be verified.

Thus, for π a p-group, nilpotent cobordism is just $Z\left[\frac{1}{p}\right]HT(\pi, \{e\})$. Therefore we have an exact sequence

$$\cdots \to \bar{L}^h_{n+1}\left(\mathbf{Z}\left[\frac{1}{p}\right][\mathbf{Z}_{p^r}]\right) \to \mathrm{Nil}_n(B\mathbf{Z}_{p^r}) \to \bar{\Omega}_n(B\mathbf{Z}_{p^r}) \to \bar{L}^h_n\left(\mathbf{Z}\left[\frac{1}{p}\right][\mathbf{Z}_{p^r}]\right) \to \cdots$$

where $\mathrm{Nil}_n(B\mathbf{Z}_{p^r})$ is the n-dimensional cobordism of nilpotent manifolds with $\pi_1 = Z_{p^r}$. Now, for odd n the L-group $\bar{L}^h_n$ vanishes. (This can be proved by the same methods as the analogous result for $\bar{L}^h_{\mathrm{odd}}(\mathbf{Z}\pi)$.) For n even one can compute as follows: $L \to L \otimes \mathbf{Z}_{(2)}$ is an injection so it is safe to $\otimes\, \mathbf{Z}_{(2)}$. Now $\bar{\Omega}(pt) \to \bar{\Omega}(B\mathbf{Z}_{p^r})$ becomes an isomorphism. The map sends $[M]$ to the obstruction of $M \to M \times B\mathbf{Z}_{p^r}\left[\frac{1}{p}\right]$ which vanishes if and only if the ordinary signature of M does (see [21, II]). Thus one has,

PROPOSITION 10. *An element of* $\bar{\Omega}_n(\mathrm{B}\mathbf{Z}_{p^r})$ *has a nilpotent representative if and only if its signature vanishes* (*i.e. automatically if* $n \not\equiv 0 \bmod 4$).

Next we study the 2-torsion in nilpotent cobordism. To identify the part not coming from ordinary cobordism we study only those elements which are trivial in Ω_n. Now we identify several sources of 2-torsion. For n even, of course, there is none.

1. If $n \equiv 1 \bmod 4$, the rational semicharacteristic detects a $\mathbf{Z}_2$.
2. If $n \equiv 1 \bmod 4$, then $\mathbf{Z}\left[\frac{1}{p}\right][\mathbf{Z}_{p^r}]$ breaks up into r components so that L_2 contains $(\mathbf{Z}_2)^{r-1}$ coming from different Arf invariants.
3. For $n \equiv 3 \bmod 4$ then there are $(r-1)$ copies of the Witt group $W(\mathbf{Z}_p)$ coming up. This is $\mathbf{Z}_2 + \mathbf{Z}_2$ or $\mathbf{Z}_4$ depending on p mod 4; and
4. For n odd, there is 2-torsion coming from $\bigoplus_{i \leq r} H(\mathbf{Z}_2; Cl(\mathbf{Z}[\zeta_i]))$ where ζ_i is a primitive p^i-th root of unity, Cl is the ideal class group and $\mathbf{Z}_2$ acts by complex conjugation.

This is just the result of an analysis of the L-theory localization sequence obtained for inverting p. (More details are in section 5.) Thus, the 2-torsion is rather regular except for a part related to the number theory of p, i.e., which vanishes for odd class number. We will discuss this aspect of the calculation more thoroughly in section 5. As examples, $\mathbf{Z}_p$ for $p = 29$ has some "interesting" 2-torsion, but for $p = 163$ does not although it has an even class number. The involution is such that the cohomology vanishes (see [7]).

As a final conclusion we have

PROPOSITION 11. *If* Z_{p^r} *acts nilpotently on* M^{4k+1} *which is a boundary then* $\rho(\mathrm{M}) \in \tilde{R}_{\pm}(\mathbf{Z}_{p^r}) \otimes \mathrm{Q}$ *lies in the image of* $\mathrm{L}^h_2(\mathbf{Z}[\mathbf{Z}_{p^r}]) \otimes \mathbf{Z}\left[\frac{1}{p}\right]$. *For* M^{4k+3} *this is true modulo the trivial representation.*

Proof. The exact sequence and localization for the L-groups (or an argument in [6]) immediately yields this.

By a refinement of this result, or the use of a "$\times D^2$" (with multiplication by $e^{2\pi i/p}$ as action) trick one now has enough of an integrality theorem for ρ-invariants to construct the classes, at the prime 2, for semifree PL $\mathbf{Z}_{p^r}$ actions on manifolds with manifold fixed sets constructed in [5] by other means. Moreover, using the material of the next two sections one can obtain the improved classes that exist for locally linear actions.

4. SIMPLE HOMOLOGICALLY TRIVIAL ACTIONS

This section is little more than a definition, and that definition is taken from Milnor [10].

DEFINITION. *If* G *acts freely and* R*-homologically trivially on* X *then*

$$\tau(\mathrm{G},\mathrm{X}) \in \mathrm{K}_1(\mathcal{S}[\pi])/\pm\{\mathrm{G}\times\pi\} \equiv \mathrm{Wh}(\mathcal{S}[\pi])$$

is the torsion of the based acyclic chain complex $\mathrm{C}_*(\mathrm{X}/\mathrm{G}) \otimes_{\mathrm{R}[\pi\times\mathrm{G}]} \mathcal{S}[\pi]$. *The action is simple if and only if* $\tau = 0$.

Now one can classify simple homologically trivial actions up to cobordism, (calling the equivalence classes $RHT_n^s(G,X)$, of course) and one obtains

THEOREM 12. *For* $n \geqslant 5$ *the following is exact,*

$$\cdots \to L_{n+1}^s(\mathcal{S}[\pi_1 X]) \to RHT_n^s(G,X) \to \tilde{\Omega}_n(X\times BG) \to L_n^s(\mathcal{S}[\pi_1 X]) \to \cdots$$

Proof. The proof is almost the same as that given in section 1. The only subtlety involved is the fact that if S is finitely dominated (in our case, S being $K(G,1)$) and E is finite with $\chi(E)=0$ then $X\times E$ can be given canonically a simple homotopy type. For $E=S^1$ this is due to Ferry and in general to Ranicki [16]. In any case it is easy for our application where one can define the canonical type in terms of the vanishing of the (Reidemeister) torsion. We leave verification to the reader.

Note that there is a Duality theorem for τ so that τ actually defines an element in $H(Z_2;\, Wh(\mathcal{S}[\pi]))$. From the theorem, we derive (see also [6] on simplicity obstructions):

COROLLARY 13. *For* $n \geqslant 4$, *there is an exact sequence*

$$\cdots \to RHT_n^s(G,X) \to RHT_n(G,X) \xrightarrow{\tau} H(Z_2; Wh(\mathcal{S}[\pi])) \to \cdots$$

5. THE TORSION SIGNATURE FORMULA

In this section we derive various formulae for topological invariants such as the ρ-invariant and G-signature in terms of units in number rings and using this to interpret ideal class phenomena of cyclotomic fields geometrically. We begin with some algebraic preliminaries.

To effectively exploit the previous section it is useful to recall the results of [19] computing L^s groups. For a cyclic group Z_n there is the multisignature map $L_{2k}^s(Z_n) \to R(Z_n)$ to the complex representations with image the set of $\lambda \in R(Z_n)$ satisfying

(1) $\lambda = (-1)^k \bar{\lambda}$
(2) $\lambda = 4R(Z_n)$
(3) The coefficient of a real one-dimensional representation is

$$\begin{cases} 0 & \text{if } k \text{ is odd} \\ 0 \bmod 8 & \phantom{\text{if } k \text{ is }} \text{even} \end{cases}.$$

This map is an isomorphism for k even and has kernel Z_2 for k odd corresponding to the Arf invariant. If we work with reduced L-groups the result is easily seen to be the same except that in (3) one makes the coefficient 0 in all cases. Furthermore, in this case, the method of [19] shows that the same holds for $\tilde{L}^s(R[Z_n])$ for any ring $Z \subseteq R \subseteq Q$. (Note the calculation for R the real number ring in that paper.) It now follows from the proof of Proposition 11 and section 4 that

PROPOSITION 14. *The ρ-invariant (modulo the trivial representation) of a free simple* $\mathbf{Q}$*-homologically trivial* $\mathbf{Z}_n$ *action on a manifold* $M^{4k\mp 1}$ *which bounds, lying in* $\tilde{\mathbf{R}}_{\pm}(\mathbf{Z}_n) \otimes \mathbf{Z}\left[\frac{1}{n}\right]$, *is a multiple of* 4.

(This divisibility is exactly what is needed for [5].)

The next step is to analyze the effect of nonsimplicity. However, this is implicit already in section 4's comparison of RHT^s and RHT. As one goes from L^s to L^h some multiples of 4 become slightly less divisible, and one obtains certain multiples of 2. This is controlled by the discriminant map

$$\tilde{L}^h_{2k}(\mathbf{Q}[\mathbf{Z}_n]) \xrightarrow{\delta} H(Z_2; Wh(\mathbf{Q}[\mathbf{Z}_n])).$$

If $\bar{\rho}(M)$ in $\tilde{L}^h_{2k}(\mathbf{Z}_n)$ is the preimage of $(M, \mathbf{Z}_n)$ in the main exact sequence then

$$\delta(\bar{\rho}(M)) = \tau(M, \mathbf{Z}_n) \in H(Z_2; Wh(\mathbf{Q}[\mathbf{Z}_n])).$$

Moreover, this determines the value of $\rho \in 2\tilde{R}_{\pm}(\mathbf{Z}_n)/4\tilde{R}_{\pm}(\mathbf{Z}_n)$. It is interesting to see what this says about the G-signature.

THEOREM 15. (Torsion Signature Theorem). *If* $\mathbf{Z}_n$ *acts semifreely and* PL *locally linearly on* M^{2k} *with fixed set components* F_i *with normal representations* ρ_i *then*

$$\delta(\sigma(\mathbf{Z}_n, M)) = \sum \chi(F_i)\tau(\rho_i)$$

so that the value of the signature mod 4 *and modulo the trivial representation the trivial representation is determined by the Euler characteristic* mod 2 *of fixed set components and their normal representations.*

This theorem, in the smooth case, is subsumed in work of Katz [9] in a different language. Note that $\tau(\rho_i)$ is explicitly computed in [10] and that it is a square (a multiple of 2 in the above additive notation) if ρ_i is, and thus does not contribute to the formula. (This can be seen noncomputationally as well.)

The proof is entirely obvious; one examines ρ for the restriction of the action of Z_n to the boundary of a regular neighborhood of F_i. The torsion is computed in [6] by a direct Mayer–Vietoris argument using local linearity and is, of course, just the right hand expression of the theorem.

For the remainder of the paper we restrict our attention to the case of n an odd prime p. The general case is deferred to a future paper.

We will phrase our result in Ewing's terminology [7]. Here one has a map from smooth (or PL, PL locally linear, topological, etc.) bordism of Z_p-actions, denoted $\mathcal{C}$, to the Witt-group of Z_p-isometries of $\pm$ symmetric bilinear forms over Z (just by examining middle dimensional cup product)

$$ab:\mathcal{C}^{cat}(Z_p) \to W_{(-1)^k}(Z;Z_p)$$

This Witt-group was computed by Alexander, Conner, Hamrick and Vick [1] and is torsion free and detected by multisignature. The image, by straightforward calculation (using the irreducibility of cyclotomic polynomials) is essentially the regular representation, the trivial representation and multiples of 2. Ewing proves for *cat* = smooth

THEOREM (Ewing). ab^{smooth} *is onto if* h(p) *is odd, where* h(p) *is the class number of the cyclotomic field* $\mathbf{Q}(\zeta_p)$, ζ_p *a* p*th root of unity.*

Actually, Ewing states his result slightly differently.† The study of $h(p)$ traditionally breaks up into two pieces based on a factorization $h(p) = h_-(p)h_+(p)$. h_-, sometimes called the first factor, can be described as the order of the -1 eigenspace of complex conjugation on the ideal class group. h_+, the second factor, is much more difficult to analyze and can be viewed as the order of the class group of the real subfield $\mathbf{Q}(\zeta + \zeta^{-1})$. We will return to h_+ in a moment. From the description of h_- given it is clear that h is odd if and only if h_- is. Ewing stated his result in terms of h_-, which was entirely natural given his method of proof. We consider the categories of PL, topological locally linear (t.1.1.) actions, PL locally linear (PL.1.1.) actions, and smooth actions and have the following:

THEOREM 16. (a) $\mathrm{ab}^{t.1.1.}$ *is always onto.*
(b) $\mathrm{ab}^{PL.1.1.}$ *is onto iff* h(p) *is odd (and hence iff* ab^{smooth} *is).*
(c) *In all but the topological category here is a lift*

$$\begin{tikzcd} & \tilde{L}^h_*\left(Z\left[\frac{1}{p},\zeta\right]\right) \arrow[d] \\ \mathcal{C}^{cat}_*(Z_p) \arrow[ur, "ab"] \arrow[r, "ab"] & W_*(Z;Z_p)/\text{trivial rep.} \end{tikzcd}$$

which can be viewed as a refined multisignature. $\overline{ab}^{PL}$ *is always onto, so* ab^{PL} *is onto exactly when the vertical map is.*
(d) $\overline{ab}^{PL.1.1.}$ *and* $\overline{ab}^{smooth}$ *are onto exactly when* $h_+(p)$ *is odd.*

The sequence of proof is (c), (d), then (b) and (a). It is interesting to note that in our proof of (b), the PL.1.1. version of Ewing's result, makes use of (d), so that h_+ phenomena are conceptually tied to what is naturally an h_- phenomenon. (Of course, h_+ even implies h_- even, but it seems that something deeper is going on.)

Proof. (c) The map from L^h is just multisignature, since its image in $R_\pm(Z_p)$ is included in the image of $W_\pm$ by the calculation cited above. $\overline{ab}$ is defined by the ρ-invariant of the

†He also has more precise results on cok ab when it is nontrivial. In another paper we will obtain this from the methods of this section together with Cappell and Shaneson's calculations of torsion in L^h.

boundary of a regular neighborhood.† That $\overline{ab}$ is onto for PL actions is a consequence of the Wall realization theorem, and the surgectivity of the images in the Witt group under the map $L_*^h(Z[Z_p]) \to L_*^h\left(Z\left[\frac{1}{p}, \zeta\right]\right)$. Act on a lens space by an element of $L_{2k}^h(Z[Z_p])$, pass to the universal cover, and cone both boundary components, extending the action radially to get an arbitrary element.

(d) We first recall the interpretation of h_+ in terms of units. According to the Dirichlet unit theorem, the units of $Z[\zeta]$ form a group which is the product of a finite group, clearly Z_{2p} generated by $-\zeta_j$, and a free abelian group of rank $\frac{p-1}{2}$. There are a number of obvious candidates for generators, called the cyclotomic units, namely $\frac{1-\zeta^i}{1-\zeta}$, $1 \leqslant i \leqslant \frac{p+1}{2}$. These are multiplicatively independent and thus generate a subgroup of finite index. The index turns out to be $h_+(p)$ according to a beautiful result of Kummer. For all this, see [20]. In $\mathbf{Z}\left[\frac{1}{p}, \zeta\right]$ the addition unit is $1-\zeta$ (see [6]).

With this preparation we can derive (d) from the torsion signature theorem ideas. From manifolds with even Euler characteristic fixed set components one gets elements of L^s. Thus one has to see which discriminants arise from which normal representations. $\overline{ab}$ will be onto only if all of $H\left(\mathbf{Z}_2; Wh\left(\mathbf{Z}\left[\frac{1}{p}, \zeta\right]\right)\right)$ is realized. Modding out by $1-\xi$ which is realized by rotation by $\frac{2\pi}{p}$ on S^1, one need deal with the units of $Z[\zeta]$. Up to multiplication by a root of unity, they are all real (Kummer's lemma, see [20]). Thus the cohomology is units modulo squares. Now $\tau(\Sigma a_i \theta^i) = \Pi(1-\omega^j)^{a_i}$ where $ij \equiv 1 \mod p$, so one obtains exactly the cyclotomic units. Thus, the map is onto cohomology if and only if cyclotomic units have odd index, i.e., $h_+(p)$ is odd.

Conversely, if $h_+(p)$ is odd then one can do a Wall realization type argument to get the element of L^s. (One looks at, say, $CP^2 \times S^4$ by rotation on S^4, and acts by L^s on a small disc on say $CP^2 \times *$, to obtain a locally linear action.) Now, $p^i x$ representations bound, for some i, and one realizes all discriminants in this way. Smoothly, rather than acting by L^s, one uses the fact that Ewing showed cok ab is always a 2-group, and ab and $\overline{ab}$ differ only at 2 and then use the PL argument.

(b) First, if $h(p)$ is odd then $h_+(p)$ is and $\overline{ab}$ is onto. Now, using calculations of Bak and Scharlau [3], $W_*(\mathbf{Z}; \mathbf{Z}_p)$ is essentially the same as L^p, so one has the comparison $L^h \to L^p \xrightarrow{\delta} H^0\left(\mathbf{Z}_2; K_0\left(\mathbf{Z}\left[\frac{1}{p}\right]\right)\right)$. Now $\tilde{K}_0\left(\mathbf{Z}\left[\frac{1}{p}, \zeta\right]\right) = Cl(Z[\zeta])$ so if h is odd the vertical map is onto as well, so ab is onto.

Conversely, if ab is onto, the vertical map had better be, so $H^0(\mathbf{Z}_2; K_0) = 0$. (The map in the Rothenberg sequence is onto since L_{odd} vanishes.) Now, if this vanishes, $H^1(\mathbf{Z}_2; Cl) = 0$ as well by Herbrand's lemma (see [8]), as Cl is finite, so that we have $\overline{ab}$ is onto, and hence by (d), $h_+(p)$ is odd. In that case $h_-(p)$ must also be odd, for if not the cohomology would be nonzero.

†Since one has an explicit coboundary (i.e., the complement) one need not work mod the trivial representation. In any case, this does not affect cok calculations. For topological locally linear actions there need not be an equivariant regular neighborhood and therefore the construction fails.

(a) One gives the same argument a for $PL.1.1$. (starting from $CP^2 \times$ lens space to get some positive dimensionality) using L^p and one-point compactifying instead of using L^s. Local linearity follows readily from the Quinn regularity theorem as in [24].

We note that these topological locally linear manifolds have arbitrary finiteness obstruction in $\tilde{K}_0(\mathbf{Z}_p)$. This construction works in all even dimensions $\geqslant 6$. In dimension 4 it gives an ANR action. The footnote to the proof of (c) explains why such a point cannot have even an invariant homology sphere near that point.

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