The work of Sylvain Cappell.

Splitting Theorems
Codimension two
Intermission (some shorter works)
Nonlinear Similarity
Characteristic Classes of singular spaces and
singular maps
Counting Lattice Points
Clean-up (some things there's no time for)

SPLITTING THEOREMS

When is being a connected sum a homotopy invariant property?

1. For simply connected manifolds (Browder)
2. If the fundamental group has no 2-torsion (Cappell)

Theorem 1. *Let Y be a closed manifold or Poincaré complex of dimension $n+1$, $n \geqq 5$ with $\pi_1(Y)=G$ and X a closed submanifold or sub-Poincaré complex of dimension n of Y with trivial normal bundle and $\pi_1(X)=H \subset G$ a square-root closed subgroup*[1]. *Assume $Y-X$ has two components* (respectively; *one component*) *with fundamental groups G_i and $\xi_i\colon H \to G_i$* (resp; *group J and $\xi_i\colon H \to J$*), *$i=1,2$ the induced maps.*

(i) *If* $\xi_{1_*}-\xi_{2_*}\colon \tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)$ (resp; $\xi_{1_*}-\xi_{2_*}\colon \tilde{K}_0(H) \to \tilde{K}_0(J)$) *is injective or even just*

$$H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)))=0$$

$$(\text{resp; } H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1_*}-\xi_{2_*}} \tilde{K}_0(J)))=0)$$

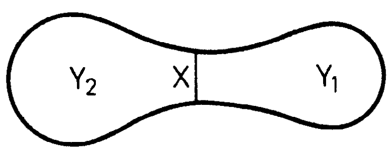
then for any homotopy equivalence $f\colon W \to Y$, W a closed manifold, W is h-cobordant to a manifold W' with the induced homotopy equivalence $f'\colon W' \to Y$ splittable.

(ii) *If* $\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G)$ (resp; $\mathrm{Wh}(J) \to \mathrm{Wh}(G)$) *is surjective, then every homotopy equivalence $f\colon W \to Y$, W a closed manifold, is splittable.*

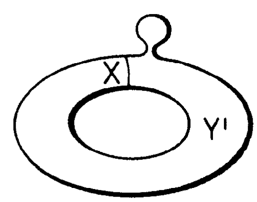
3. Not in general:

THEOREM 1. *There is a closed differentiable $4k+1$ dimensional manifold W, simple homotopy equivalent to $RP^{4k+1} \# RP^{4k+1}$, $k \geq 1$, which is not as a differentiable, piecewise-linear or even as a topological manifold a non-trivial connected sum.*

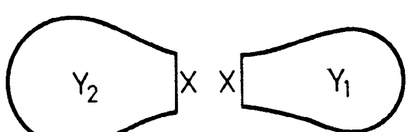
4. There is a general obstruction theory to solving this and other codimension one splitting problems that involves new groups called UNil.



Y in case A



Y in case B



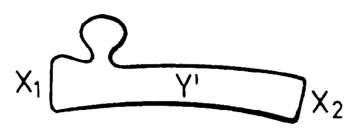
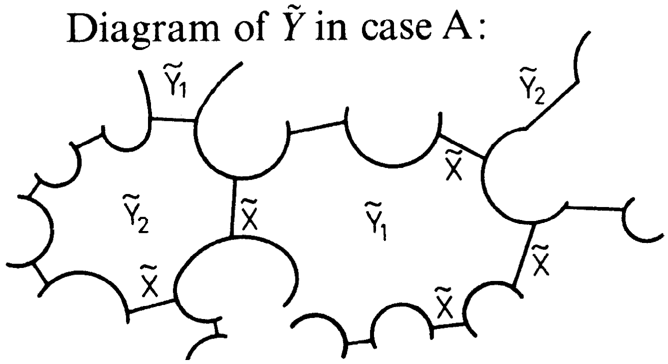


Diagram of $\tilde{Y}$ in case A:



And, his work includes interesting vanishing theorems for this group.

1. The subgroup is square root closed
2. $\frac{1}{2} \in R$ (surgery up to homology equivalence.)
3. Just inverting 2 in the L-group.

If the Farrell-Jones conjecture is correct, then computing UNil groups will be the (may I say) *ineffable* part of the classification of manifolds.

Some implications.

1. Novikov conjecture for a large class of groups. (e.g. fundamental groups of sufficiently large three-manifolds, or almost flat manifolds).

2. Counterexamples to "equivariant Borel conjecture": There are involutions on a torus that are homotopy affine, but not topologically affine.

3. Remark: It is only in the past few years that Banagl-Ranicki and Connolly-Davis have computed the UNil for the infinite dihedral group (started by Cappell).

Moreover, Connolly-Davis established that connected sum is homotopy invariant (for oriented manifolds) in half the dimensions.

CODIMENSION TWO

(joint work with Julius Shaneson)

In Chapter II and § 12 we study the question of when two sufficiently close embeddings f_0 and f_1 of M^n in W^{n+2} are concordant, at least up to taking connected sum with a knot. If f_1 is sufficiently close to f_0, it will lie in a bundle neighborhood. We therefore consider cobordism classes of embeddings of M, in the total space $E(\xi)$ of the disk bundle of a 2-plane bundle ξ over M, homotopic to the zero-section. Such an embedding is called a *local knot* of M in ξ, and the set of cobordism classes is denoted $C(M, \xi)$ or just $C(M)$ if ξ is trivial. (In the sequel, we write $C_0(M; \xi)$, $C_{\mathrm{PL}}(M; \xi)$, $C_{\mathrm{TOP}}(M; \xi)$ to distinguish the various categories; in the last two, local flatness is understood.) $C(M, \xi)$ is a monoid; the operation is called *composition* or *tunnel sum* and is defined as the composition $\hat{\iota}_1\iota_2$, where ι_1 and ι_2 are local knots of M in ξ and $\hat{\iota}_1$ is a thickening of ι_1 to an embedding of $E(\xi)$ in itself. We discuss only the case when M is closed; various relativizations exist.

THEOREM. (See 4.5, 5.3, 6.2, 6.3, and 6.5.) *For $n = \dim M \geqq 3$, $C(M, \xi)$ is a group under composition, and for $n \geqq 4$ it is abelian. For $n \neq 1$, $C(S^n)$ is isomorphic to the n-dimensional knot cobordism group. Connected sum with the zero-section defines a homomorphism $\alpha\colon C(S^n) \to C(M, \xi)$. In the* P.L. *or topological case, α is a monomorphism onto a direct summand, provided ξ is trivial, and $n \geqq 4$.*

We will compute $C(M, \xi)$ in terms of an exact sequence involving the Γ-groups to be described below. In particular, for $\dim M \equiv 1 \pmod 2$, it is caught in an exact sequence of Wall surgery groups and hence tends to be fairly small. For example, one has

THEOREM (7.2). *For $n = \dim M \geqq 4$ even, there is an injection*

$$\bar{\rho}\colon C(M) \longrightarrow L^h_{n+1}(\pi_1 M) \ .$$

This is all based on their "homology surgery theory". Their paper went a long way towards creating the concept of what it means to have a "surgery theory".

On the other hand, for M odd-dimensional, $C(M, \xi)$ is not, in general, finitely generated. For simply-connected M, the main result is the following:

THEOREM (6.5 and 6.6). *Let M be a simply-connected closed n-manifold, $n \geqq 4$, and let ξ be a 2-plane bundle over M. Then $\alpha: C(S^n) \to C(M, \xi)$ is onto, and is an isomorphism for ξ trivial, in the* P.L. *and topological cases. For n even, $C(M, \xi) = 0$.*

We draw some consequences for the study of close embeddings.

THEOREM. (See 12.1 and following discussion.) *Let $f_0: M^n \to W^{n+2}$ be an embedding (locally flat, of course) of the closed, simply-connected manifold M in the (not necessarily compact) manifold W. Assume $n \geqq 5$. Let f be another embedding, sufficiently close to f_0 in the C_0 topology. Then if the normal bundle ξ of f is trivial, or if n is even and the Euler class of ξ is not divisible by two, or if $n \equiv 2 \pmod 4$ and the Euler class of ξ is divisible only by two; then, after composition with a homeomorphism (or diffeomorphism or* P.L. *homeomorphism) of M homotopic to the identity, f is concordant to f_0, for n even and to the connected sum of f_0 with a knot, for n odd.*

The importance of simple connectivity of M is demonstrated by the next result.

THEOREM (7.3 and 14.5). *Let $T^n = S^1 \times \cdots \times S^1$, $n \geqq 4$ and even. Then, in the* P.L. *category,*

$$C(T^n) = [\Sigma(T^n - pt); G/\mathrm{PL}] ,$$

and every element of $C(T^n)$ can be represented by an embedding arbitrarily close to the zero-section $T^n \subset T^n \times D^2$. $C(T^n)$ is generated by products with various $T^{n-i} \subset T^n$ of the connected sum of $T^i \subset T^i \times D^2$ with knots of dimension i.

Subsequent work of J.Smith and of P.Vogel has clarified some aspects of the computation of the relevant surgery groups, but they remain mysterious and relevant to other problems[1].

Applications, also to link cobordism.

(CS) multi-component links that are not cobordant to split links.[2]

Subsequent work by Cochran and Orr applied this to even give links not cobordant to boundary links.[3] Le Dimet used the CS machinery to give a theoretical calculation of “disk link concordance”.

Worth pointing out that homology surgery doesn’t do everything that you would want: categories of modules with particular properties...

Another breathtaking paper:

---

[1] Such as cut paste invariance of higher signatures.

[2] This was also done by Kawauchi

[3] Here the key open problem is whether every even dimensional link is cobordant to a boundary link.

PIECEWISE LINEAR EMBEDDINGS AND THEIR SINGULARITIES.

THEOREM (see 6.1). *Let $f: M^n \to W^{n+2}$ be a homotopy equivalence, M and W compact connected orientable* PL *manifolds, M closed, $n \geqq 3$. If n is even, assume $\pi_1 W = 0$. Then f is homotopic to a* PL *embedding.*

When the conclusion of this theorem is satisfied, M (more precisely, its image under a PL embedding homotopic to f) is called a spine of W. This result is false if $\pi_1 W$ is non-trivial abelian, for n even. In [CS7] we constructed "totally spineless" manifolds of the homotopy type of M^n when n is even, $n \geqq 4$, and $\pi_1 M$ is any non-trivial abelian group. (See [Ma] for the case $n = 2$.)

Matsumoto gave an example of a topological spine in a PL spineless manifold.

Remark: These results had a strong motivating impact on the "replacement theorems for group actions" discussed at the recent AMS meeting.

Conjecture: Topological wild embeddings are even nicer. (The value of inclusiveness.)

PROPOSITION (see 6.8). *If* $g\colon M \to W$ *is a* PL *embedding homotopic to* f, *let* $S_k(g)$ *be the set of non-locally flat points of intrinsic codimension less than* k, *with closure* $\overline{S_k(g)} \subset M$. *Let*

$$L(f) = L(M)L(\xi_f) - g^*L(W) ,$$

and let $L_i(f)$ *be the homogeneous portion of* $L(f)$ *of degree* $4i$, i.e., $L_i(f) \in H^{4i}(M; Q)$. *Let* $D\colon H^*(M; Q) \to H_{n-*}(M; Q)$ *be Poincaré duality. Then, for* $i < k$, $DL_i(f)$ *is in the image of the map*

$$H_{n-i}(\overline{S_k(f); Q}) \longrightarrow H_{n-i}(M; Q)$$

induced by inclusion.

They establish a classifying space for PL wild embeddings and then analyze it.

COROLLARY 4.9. $(\chi, \eta)\colon BSRN_2 \to BSO_2 \times G/\mathrm{PL}$ *has a cross-section.*

The homotopy groups of $BSRN_2$ are closely related to knot cobordism. The formula above gives quantitative weight to these abstract theorems.

PROBLEM: Is there a PL embedding with only even codimensional singularities (analogous to a complex algebraic variety)?

SOME SIMPLE PIECES.

There exist inequivalent knots with the same complement

By SYLVAIN E. CAPPELL* AND JULIUS L. SHANESON*

The study and classification of knots has been based upon invariants of the knot complement. In this paper we give examples of inequivalent smooth spherical knots with diffeomorphic complements. These examples

(Still unknown whether these exist in every dimension >3. Best known is Suciu giving in all dimensions =3,4 mod 8.)

Fake RP^4. Also fake 4-dim'l s-cobordisms, and stable surgery for smooth four-manifolds. (all CS)

THEOREM (Existence of exotic involutions). *There is a smooth, free involution on a homotopy 4-sphere Σ^4, which has no equivariant diffeomorphism or piecewise linear homeomorphism with a linear action on S^4.*

The quotient space $Q = \Sigma^4/Z_2$ of the involution is a smooth manifold of the (simple) homotopy type of real projective 4-space P^4, but not diffeomorphic or even piecewise linear (PL) homeomorphic to P^4. As a corollary

We don't have enough of a picture of smooth four-dimensional topology, even now, to understand why these manifolds exist and a fake $S^1 \times S^3$ does not (predicted by the same heuristic).

We also don't understand much (aside from one theorem of Bauer-Furuta) about how much stabilization is necessary in 4 dim'l topology.

And we don't know whether the Cappell-Shaneson fake s-cobordisms are smoothable.

The Oozing problem, Pseudofree actions, Browder-Livesay groups[4], and the calculation of L^h groups.

I will mention just one theorem from this body of work, that changed our viewpoint on the interaction between the geometry of characteristic classes and the arithmetic of the fundamental group in the finite case.

Theorem (CS): The surgery obstruction of the product

$$S^3/Q_8 \times (T^2 \to S^2)$$

is nonzero.

This direction was very substantially completed by Hambleton-Milgram-Taylor-Williams, but there are twisted variants for which we have a burning "need to know" where nothing has been worked out. Like, for instance,

$$H_*(B\mathbf{Z_2}\,;\,\mathbf{L}(\mathbf{Z_2})) \to \mathbf{L}_*(\mathbf{Z_4}).$$

[4] These groups capture difficulties in codimension one splitting when the submanifold has nontrivial normal bundle.

Geometric interpretation of Siebenmann periodicity.

Theorem (Cappell w/o Shaneson) The composite map

$$\begin{aligned} S(M) &\to S(M\times D^3, M) && \text{(BCHSW)} \\ &\to S^{S^1}(M\times D^4) && \text{(Branched fibration)} \\ &\to S(M\times D^4) && \text{(forgetful map)} \end{aligned}$$

is an isomorphism.

Corollary (of proof): If n>4, infinitely many homotopy CP^ns have locally smooth S^1 actions.

That this cannot happen smoothly is called the Petrie conjecture, and is still open.

NONLINEAR SIMILARITY. (CS)

When are two linear transformations topologically conjugate as dynamical systems? Kuiper and Robin answered it modulo a positive analysis for periodic matrices.

Example (in which $h = h_q$). Let k and j be relatively prime to $4q$, and let B be any matrix with -1 as an eigenvalue. Let $A(k)$ be the 2×2 matrix

$$\begin{pmatrix} \cos\frac{k\pi}{2q} & \sin\frac{k\pi}{2q} \\ -\sin\frac{k\pi}{2q} & \cos\frac{k\pi}{2q} \end{pmatrix},$$

and define $A(j)$ similarly. Let

$$A_\varepsilon = \begin{pmatrix} \varepsilon A(k) & & & & 0 \\ & \varepsilon A(k) & & & \\ & & \varepsilon A(j) & & \\ & & & \varepsilon A(j) & \\ 0 & & & & B \end{pmatrix}$$

be the indicated block sum, $\varepsilon = \pm 1$. Then A_1 and A_{-1} will be topologically similar. However, $-A_1$ and $-A_{-1}$ will usually *not* be topologically similar (e.g., if $B = (-1)$) as the next result easily implies.

NONLINEAR SIMILARITY BEGINS IN DIMENSION SIX

By SYLVAIN E. CAPPELL,* JULIUS L. SHANESON,* MARK STEINBERGER* and JAMES E. WEST*

Also, "topological rationality principle for finite groups" (relies on the "odd order group theorem of Hsiang-Pardon and Madsen-Rothenberg). Also, compact Lie groups.

Also led to the disproof of Smith's conjecture about representations at fixed points for nice smooth group actions on the sphere. *Also beautiful connections to number theory*.

This problem was finally solved for cyclic groups by Hambleton-Pedersen. But more importantly, it led to the development of the modern theory of stratified spaces in general, and much work in transformation groups in particular, partly joint work of Cappell and

Another shorter development.

Theorem (C w/o S).
Suppose that X is a manifold with singularities $\cong \Sigma$, and that Σ has simply connected local links. Then

$$S(X \text{ rel } \Sigma) \cong S^{alg}(X),$$

where the right hand side is what is formally predicted by surgery theory.

Moreover, if all the strata of X are even codimensional, then $S(X) \otimes \mathbf{Z}[1/2]$ can be analyzed completely in terms of Intersection Homology.

This then becomes part of a general thrust of (CS) and other collaborators to understand invariants of stratified spaces and their connections to singular maps, and algebraic geometry and to group actions.

Only some of this can be discussed below.

CHARACTERISTIC CLASSES FOR SINGULAR SPACES, EMBEDDINGS, AND MAPS.

For singular algebraic hypersurfaces, or more general "even codimensional embeddings" there is a formula:

THEOREM 1. *Assume* $\check{X} \subset M$ *stratifies* X *with only even-codimension strata. Let* $\mathscr{C}$ *be the components* V *with* $\dim V < \dim X$. *Assume* $\pi_1(V - V \cap Y)$ *is trivial*, $V \in \mathscr{C}$. *Then*

$$L(M, X) = L(X) + \sum_{\sigma} \sigma(\mathrm{lk}(V))L(\overline{V}).$$

For isolated singularities, σ will just be the classical signature of a Seifert surface with boundary G_V in the knot pair $lk(V) = (F_V, G_V)$. To describe σ in general, the inclusion $G_V \subset F_V$ stratifies G_V, and hence also F_V, with top stratum $F_V - G_V$. By Alexander duality (recall F_V is p.l. homeomorphic to S^{n-1}), $\Lambda = \mathbb{Q}[t, t^{-1}, (1-t)^{-1}]$ becomes a system of local coefficients over $F_V - G_V$ (see *Section 3*). Hence by using ref. 5, the intersection homology groups $IH_j^{\overline{m}}(F_V; \Lambda)$ are defined, where $\overline{m} = (0, 0, 1, 1, \ldots)$ is the (lower) middle perversity. These turn out to be finite-dimensional rational vector spaces. Further in *Section 3* it will be shown that for $j = (n - \dim V - 2)/2$, this group has a nonsingular $(-1)^j$–symmetric bilinear form over $\mathbb{Q}$, ϕ_V say. We then define $\sigma(lk(V))$ to be zero in the skew-symmetric case and the usual signature of ϕ in the symmetric case—i.e., the dimension of the positive define space less that of the negative definite one.

The key hard part is defining a version of "knot invariants" for singular knots.

Easier, and of very general application is the following.

(CS) Suppose that f: X → V is a "stratified map" so that every stratum has even codimension, then

Theorem (see (5.8)). *Assume that each $V \in \mathscr{V}$ is simply connected. Let y_V be a chosen point of V. Then*

$$f_* L_i(X) = \sum_{\mathscr{V}} \sigma(E_{y_V}) j_* L_i(\overline{V}).$$

The simple connectivity is just there to ensure that monodromies are trivial. Banagl-Cappell-Shaneson removed that condition.

These formulae all have algebraic geometric extensions. Some are part of the joint work of C-Maxim-S and were discussed by Maxim at this meeting.

Let me digress for a moment to discuss how this relates to a theorem about group actions:

Theorem (Cappell, W, Yan) : Suppose G is a compact Lie group that acts "locally smoothly", so that the normal representation is 2×complex representation, then any manifold ~ F is the fixed set is the fixed set of G acting on a homotopy equivalent manifold.

For $G = S^1$ this condition is strongly motivated by the previous formula: for odd multiples, the complement determines the characteristic classes of the fixed set.

$\sqcup$

3. Intersection homology Euler characteristics

Let X be a n-dimensional complex algebraic variety which is not necessarily compact (i.e., a reduced separated scheme of finite type over the complex numbers). Let IC_X^{top} be the sheaf complex defined by $(IC_X^{top})^k(U) = IC_{-k}^{BM}(U)$ for $U \subset X$ open (cf. [10]), where $IC_\bullet^{BM}$ is the complex of locally-finite chains with respect to the middle-perversity [9]. Let $IC_X := IC_X^{top}[-n]$, so the middle-perversity intersection cohomology group $IH^k(X;\mathbb{Q})$ is the hypercohomology group $\mathbb{H}^{k-n}(X;IC_X)$. In general, for a l-dimensional stratified pseudomanifold L with only even codimension strata (e.g., L can be the link of a stratum), the intersection cohomology groups are defined by $IH^k(L;\mathbb{Q}) := \mathbb{H}^{k-2l}(L;IC_L^{top})$.

Since complex algebraic varieties are *compactifiable*, their rational intersection cohomology groups (with either compact or closed support) are finite dimensional (cf. [3], V.10.13), therefore the intersection homology Euler characteristics of complex algebraic varieties are well-defined. We let $I\chi(X)$, resp. $I\chi_c(X)$, denote the intersection homology Euler characteristic of X with closed, resp. compact support.

For simplicity, we first consider proper algebraic morphisms with a non-singular domain. We will also work under the trivial monodromy assumption [2], e.g. assume that $\pi_1(V) = 0$, for all $V \in \mathcal{V}$.

The first main result of this note is the following:

Theorem 3.1. *Let $f : X^n \to Y^m$ be a proper surjective map of algebraic varieties, and assume X is non-singular and Y is irreducible. Let $\mathcal{V}$ be the set of components of strata of Y in a stratification of f, and assume $\pi_1(V) = 0$ for all $V \in \mathcal{V}$. For each $V \in \mathcal{V}$, define inductively*

$$\widehat{I\chi}(\bar{V}) = I\chi(\bar{V}) - \sum_{W<V} \widehat{I\chi}(\bar{W}) \cdot I\chi(c^\circ L_{W,V}),$$

where the sum is over all $W \in \mathcal{V}$ with $\bar{W} \subset \bar{V} \setminus V$ and $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar{V}$. Then:

$$\chi(X) = I\chi(Y) \cdot \chi(F) + \sum_{V\in\mathcal{V}, \dim V < \dim Y} \widehat{I\chi}(\bar{V}) \cdot [\chi(F_V) - \chi(F) I\chi(c^\circ L_{V,Y})], \tag{3.1}$$

where F is the generic fiber, F_V is the fiber of f above the stratum V and $L_{V,Y}$ is the link of V in Y.

and to continue…

Theorem 3.4. *Let $f : X^n \to Y^m$ be a proper surjective map of irreducible complex algebraic varieties. Under the trivial monodromy assumption, and with the notations from Theorem 3.1, the following holds:*

(3.5)
$$I\chi(X) = I\chi(Y) \cdot I\chi(F) + \sum_{V\in\mathcal{V}, \dim V < \dim Y} \widehat{I\chi}(\bar{V}) \cdot \left[I\chi(f^{-1}(c^\circ L_{V,Y})) - I\chi(F) I\chi(c^\circ L_{V,Y})\right].$$

where the sum is over all $W \in \mathcal{V}$ with $\bar{W} \subset \bar{V} \setminus V$, $c^\circ L_{W,V}$ denotes the open cone on the link of W in $\bar{V}$, and i_* is used universally to denote the appropriate map induced by inclusion. Then:

(3.7)

$$f_* IT_y(X) = IT_y(Y) \cdot I\chi_y(F) + \sum_{V \in \mathcal{V}_0} i_* \widehat{IT}_y(\bar{V}) \cdot \left[I\chi_y(f^{-1}(c^\circ L_{V,Y})) - I\chi_y(F) I\chi_y(c^\circ L_{V,Y}) \right],$$

where $\mathcal{V}_0 = \{V \in \mathcal{V}, dimV < dimY\}$, and $L_{V,Y}$ is the link of V in Y.

This last therem is from a different paper. It takes place is a category of Mixed Hodge Modules and cannot be explicated here.

All of this work had application to the theory of toric varieties and indirectly to counting lattice points and summing functions over lattice points in polytopes in Euclidean space.

For this, the formulae are complicated and involve a lot of notation, so with your indulgence, I will just wave my hands, and explain the ideas orally.

Some final examples and gems

A perturbative $SU(3)$ Casson invariant

S. E. Cappell, R. Lee and E. Y. Miller

Abstract. A perturbative $SU(3)$ Casson invariant $\Lambda_{SU(3)}(X)$ for integral homology 3-sphere is defined. Besides being fully perturbative, it has the nice properties: (1) $4 \cdot \Lambda_{SU(3)}$ is an integer. (2) It is preserved under orientation change. (3) A connect sum formula. Explicit calculations of the invariant for $1/k$ surgery of $(2, q)$ torus knot are presented and compared with Boden–Herald's different $SU(3)$ generalization of Casson's invariant. For those cases computed, the invariant defined here is a quadratic polynomial in k for $k > 0$ and a different quadratic polynomial for $k < 0$.

Mathematics Subject Classification (2000). 57M25, 57M05, 58G25.

Keywords. Gauge theory, Malsov index, Floer homology, spectral flow, Chern–Simons, Heegard decomposition, three manifolds, index theory, eta invariant.

I should point out that this work required a detailed analysis of how small eigenvalues decompose when one stretches a neck separating two pieces of a manifold.

There is a Maslov index piece. Cappell, Lee, and Miller gave a very useful axiomatization of this and used it to clarify a number of its many appearances in analysis and symplectic geometry.

Cappell, S. E. (1-NY-X); **Lee, R. [Lee, Ronnie]** (1-YALE); **Miller, E. Y.** (1-PINY)
The action of the Torelli group on the homology of representation spaces is nontrivial. (English summary)
Topology **39** (2000), *no. 4,* 851–871.

Let M be a compact Riemann surface of genus $g \geq 2$, $R_{\mathrm{SU}(2)}(M)$ the space of conjugacy classes of representations of $\pi_1(M)$ into the Lie group $\mathrm{SU}(2)$ and $R_{\mathrm{SU}(2)}(M)_{\mathrm{irred}}$ the subspace of $R_{\mathrm{SU}(2)}(M)$ consisting of conjugacy classes of irreducible representations. Then the mapping class group of M and its subgroup, the Torelli group, act naturally on these representation spaces. The main result of this paper is that the action of the Torelli group on the rational homology of $R_{\mathrm{SU}(2)}(M)$ and $R_{\mathrm{SU}(2)}(M)_{\mathrm{irred}}$ is nontrivial for $g > 3$. The authors also obtain formulas for the Poincaré polynomial of $R_{\mathrm{SU}(2)}(M)$ and other related spaces. This nontriviality result contrasts with previous results of M. F. Atiyah and R. H. Bott [Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615; MR0702806 (85k:14006)] and F. C. Kirwan [Proc. London Math. Soc. (3) **53** (1986), no. 2, 237–266; MR0850220 (88e:14012)] where they studied closely related spaces and showed in particular that the action of the mapping class group on the homology of those spaces factors through the symplectic group $\mathrm{Sp}(2g, \mathbf{Z})$ so that the action of the Torelli group is trivial.

Cohomology of Harmonic Forms on Riemannian Manifolds With Boundary

Sylvain Cappell, Dennis DeTurck, Herman Gluck, and Edward Y. Miller

To Julius Shaneson on the occasion of his 60th birthday

Theorem 1. *Let M be a compact, connected, oriented, smooth Riemannian n-dimensional manifold with non-empty boundary. Then the cohomology of the complex $(\mathrm{Harm}^*(M), d)$ of harmonic forms on M is given by the direct sum:*

$$H^p(\mathrm{Harm}^*(M), d) \cong H^p(M; \mathbb{R}) + H^{p-1}(M; \mathbb{R})$$

for $p = 0, 1, \ldots, n$.

I have to apologize that I skipped so much.

And, that what I covered, I covered so superficially.

However, I hope that I showed some of the breadth and hinted at the depth:

- topics from high and low dimensional topology.
- analytic aspects, pure geometrical constructions.
- algebraic-geometric side
- both smooth and singular.
- connections to arithmetic.