PARAMETRIZED TOPOLOGICAL COMPLEXITY OF COLLISION-FREE MOTION PLANNING IN THE PLANE

DANIEL C. COHEN, MICHAEL FARBER, AND SHMUEL WEINBERGER

ABSTRACT. Parametrized motion planning algorithms have high degrees of universality and flexibility, as they are designed to work under a variety of external conditions, which are viewed as parameters and form part of the input of the underlying motion planning problem. In this paper, we analyze the parameterized motion planning problem for the motion of many distinct points in the plane, moving without collision and avoiding multiple distinct obstacles with a priori unknown positions. This complements our prior work [3], where parameterized motion planning algorithms were introduced, and the obstacle-avoiding collision-free motion planning problem in three-dimensional space was fully investigated. The planar case requires different algebraic and topological tools than its spatial analog.

1. INTRODUCTION

The goal of this paper is to give a topological measurement of the complexity that robots must confront when navigating in a two-dimensional environment, avoiding impediments.

This work is a refinement of the work of Farber [6, 7] who studied how much forking is necessary in the programming of a robotic motion planner operating in a configuration space X via a numerical invariant $\mathsf{TC}(X)$. This, in turn, was modeled on the seminal paper of Smale [19], which studied the complexity of "the fundamental theorem of algebra," that is, the amount of forking that arises in the course of computation of solutions to polynomial equations. The invariant $\mathsf{TC}(X)$ also measures the amount of instability that any motion planner must have, that

Date: October 22, 2020.

2010 *Mathematics Subject Classification.* 55S40, 55M30, 55R80, 70Q05.

Key words and phrases. parameterized topological complexity, obstacle-avoiding collision-free motion.

D. Cohen was partially supported by an LSU Faculty Travel Grant.

M. Farber was partially supported by a grant from the Leverhulme Foundation.

S. Weinberger was partially supported by National Science Foundation grant DMS 1811071.

is, the number of different overlapping sets in a hybrid motion planning system, or similarly how much forking arises in routing algorithms.

Interesting as this invariant is, it only captures part of the difficulty that a robot needs to negotiate. A more realistic theory would take into account the sensing capacity of the robot, multiple robots that maneuver autonomously, energy, timing, and communication. We hope to investigate such issues in future work. In this paper and the previous one in this series [3], we focus on the problem of the computational complexity of flexibly solving motion planning in a potentially changing environment.

A (point) robot[†] moving around a convex room has a simple task. It can go from any point to any other along the straight line connecting them. If there is a single obstacle then any algorithm must fork - the one described would require a decision about whether to go around the obstacle to the left or the right. It turns out that two obstacles are harder than one, but then it gets no harder.

Similarly, the complexity of motion in a graph can only have three values - trivial for a tree, complexity 1 for a graph with a single cycle, but only getting bigger one more time when the number of cycles is larger than 1. The reason for this is that any connected graph can be described as a union of two trees, so if there is a specific graph that needs to be navigated, one can make use of such a decomposition. (A similar statement can be made regarding the part of a room that is complementary to any union of a finite number of convex subsets.)

Here we shall see that if the robot each day needs to move around the room where the obstacles have also been moved around, the complexity of the problem to be solved indeed grows. More generally, our main result provides a solution to the analogous problem for an arbitrary finite number of robots that are centrally controlled. The predecessor paper [3] studies the three-dimensional version of this problem, for example, for submarines navigating a mined part of the ocean. Interestingly, the mathematics is somewhat more difficult in this two-dimensional situation than the three-dimensional case.

In both cases, however, the general formalism is the same. We consider a parameter space that describes the possible location of obstacles, and therefore study a parametrzed form of topological complexity. In our situation we have the mathematical structure of a fibration describing the set of motion planning problems, which enables the application of the powerful apparatus of algebraic topology. Some

[†] Of course, the idea of a point robot is an idealization. The difficulties confronted by a physical robot will only be greater. Dealing with larger robots is related to the issue of dealing with families of problems that need not form fibrations, and will not be addressed in this paper.

of the other problems mentioned above require a weakening of this hypothesis, and cannot be directly approached by the methodology of this paper.

Parameterized motion planning. An autonomously functioning system in robotics typically includes a motion planning algorithm which takes as input the initial and terminal states of the system, and produces as output a motion of the system from the initial state to the terminal state. The theory of robot motion planning algorithms is an active area in the field of robotics, see [14, 15] and the references therein. A topological approach to the robot motion planning problem was developed in [6, 7], where topological techniques clarify relationships between instabilities occurring in robot motion planning algorithms and topological features of the configuration spaces of the relevant autonomous systems.

In a recent article [3], we developed a new approach to the theory of motion planning algorithms. In this "parameterized" approach, algorithms are required to be *universal*, so that they are able to function under a variety of situations, involving different external conditions which are viewed as parameters and are part of the input of the underlying motion planning problem. Typical situations of this kind arise when one is dealing with the collision-free motion of many objects (robots) moving in two- or three-dimensional space avoiding a set of obstacles, and the positions of the obstacles are a priori unknown.

In the current paper, we continue our investigation of the problem of collision-free motion of many particles avoiding multiple moving obstacles, focusing primarily on the planar case. A team of robots moving in an obstacle-filled room is one example. As another illustration, consider a spymaster coordinating the motion of a team of spies in a planar theatre of operations each day. Spies must avoid opposition checkpoints, which may be repositioned daily, and may not meet so as to avoid potentially compromising one another. The analogous problem in three-dimensional space, for instance, maneuvering a submarine fleet in waters infested with repositionable mines, was analyzed in [3].

In each of these motion planning problems, one is faced with a space of allowable configurations of the robots/spies/submarines which depends on parameters, the daily positions of the obstacles/checkpoints/mines. A motion planning algorithm should then be flexible enough to deal with changes in the parameters. The algebraic and topological tools used to analyze the complexity of such algorithms in the planar and spatial cases are essentially different. These differences are reflected by a numerical invariant, the *parameterized topological complexity*, which differs in the planar and spatial cases.

Parameterized topological complexity. We reformulate these considerations mathematically, using the language of algebraic topology.

Let X be a path-connected topological space. Viewing X as the space of all states of a mechanical system, the motion planning problem from robotics takes as input an initial state and a terminal state of the system, and requests as output a continuous motion of the system from the initial state to the terminal state. That is, given $(x_0, x_1) \in X \times X$, one would like to produce a continuous path $\gamma\colon I \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, where $I = [0, 1]$ is the unit interval.

Let X^I be the space of all continuous paths in X, equipped with the compact-open topology. The map $\pi\colon X^I \to X \times X$, $\pi(\gamma) = (\gamma(0), \gamma(1))$, is a fibration, with fiber ΩX, the based loop space of X. A solution of the motion planning problem, a motion planning algorithm, is then a section of this fibration, a map $s\colon X \times X \to X^I$ with $\pi \circ s = \mathrm{id}_{X\times X}$. If X is not contractible, the section s cannot be globally continuous, see [7].

The topological complexity of X is defined to be the sectional category, or Schwarz genus, of the fibration $\pi\colon X^I \to X\times X$, $\mathsf{TC}(X) = \mathsf{secat}(\pi)$. That is, $\mathsf{TC}(X)$ is the smallest number k for which there is an open cover $X \times X = U_0 \cup U_1 \cup \cdots \cup U_k$ and the map π admits a continuous section $s_j\colon U_j \to X^I$ satisfying $\pi \circ s_j = \mathrm{id}_{U_j}$ for each j. The numerical homotopy type invariant $\mathsf{TC}(X)$ provides a measure of the navigational complexity in X. Significant recent advances in the subject include work of Dranishnikov [4] on the topological complexity of spaces modeling hyperbolic groups, and work of Grant and Mescher [12] on the topological complexity of symplectic manifolds. We refer to the surveys [2, 8] and recent work of Ipanaque Zapata and González [13] for discussions of topological complexity and motion planning algorithms in the context of collision-free motion.

A parameterized approach to the motion planning problem was recently put forward in [3]. In the parameterized setting, constraints are imposed by external conditions encoded by an auxiliary topological space B, and the initial and terminal states of the system, as well as the motion between them, must satisfy the same external conditions.

This is modeled by a fibration $p\colon E \to B$, with nonempty path-connected fibers. For $b \in B$, the fiber $X_b = p^{-1}(b)$ is viewed as the space of achievable configurations of the system given the constraints imposed by b. Here, a motion planning algorithm takes as input initial and terminal (achievable given b) states of the system, and produces a continuous (achievable given b) path between them. That is, the initial and terminal points, as well as the path between them, all lie within the same fiber X_b. The parameterized topological complexity of the fibration $p\colon E \to B$ is then defined to be the sectional category of the associated fibration $\Pi\colon E^I_B \to E \times_B E$, where $E \times_B E$ is the space of all pairs of configurations lying in the same fiber of

p, E^I_B is the space of paths in E lying in the same fiber of p, and the map Π sends a path to its endpoints.

Obstacle-avoiding, collision-free motion. Investigating the collision-free motion of n distinct ordered particles in a topological space Y leads one to study the standard (unparameterized) topological complexity of the classical configuration space

$$\mathrm{Conf}(Y,n)=\{(y_1,y_2,\dots,y_n)\in Y^n \mid y_i\neq y_j \text{ for } i\neq j\}$$

of n distinct ordered points in Y. Similarly, investigating the collision-free motion of n distinct particles in a manifold Y in the presence of m distinct obstacles, with a priori not known positions, leads one to study the parameterized topological complexity of the classical Fadell-Neuwirth bundle, the locally trivial fibration

$$p\colon \mathrm{Conf}(Y,m+n)\to\mathrm{Conf}(Y,m),\quad p(y_1,\dots,y_m,y_{m+1},\dots,y_{m+n})=(y_1,\dots,y_m),$$

with fiber $p^{-1}(y_1,\dots,y_m)=\mathrm{Conf}(Y\smallsetminus\{y_1,\dots,y_m\},n)$.

In this paper, we complete the determination of the parameterized topological complexity of the Fadell-Neuwirth bundles of Euclidean configuration spaces begun in [3]. Our main result, Theorem 4.1, includes the following as a special case.

Theorem. *For positive integers m and n, the parameterized topological complexity of the motion of n non-colliding particles in the plane $\mathbb{R}^2$, in the presence of m non-colliding point obstacles with a priori unknown positions is equal to $2n+m-2$.*

The case $m=1$ of this result reduces to the previously known determination of the (standard) topological complexity of $\mathrm{Conf}(\mathbb{R}^2\smallsetminus\{0\}),n)$, see Remark 4.2.

Different techniques yield the same parameterized topological complexity for obstacle-avoiding collision-free motion in $\mathbb{R}^d$ for any $d\geq 4$ even, as discussed in Section 4. The analogous motion planning problem in $\mathbb{R}^d$, for $d\geq 3$ odd, was analyzed in [3, Thm. 9.1], where it was shown that the parameterized topological complexity is $2n+m-1$. These results provide examples of fibrations for which the parameterized topological complexity exceeds the (standard) topological complexity of the fiber, since $\mathsf{TC}(\mathrm{Conf}(\mathbb{R}^d\smallsetminus\{y_1,\dots,y_m\},n))=2n$ as shown in [10].

Our main result also illustrates that parameterized topological complexity may differ significantly from other notions of the topological complexity of a map which appear in the literature. If $p\colon E\to B$ is a fibration which admits a (homotopy) section, as is the case for many Fadell-Neuwirth bundles, then the topological complexity of p, as defined in either [16] or [17], is equal to $\mathsf{TC}(B)$. For the Fadell-Neuwirth bundle $p\colon \mathrm{Conf}(\mathbb{R}^d,m+n)\to\mathrm{Conf}(\mathbb{R}^d,m)$ with $d\geq 2$ even, we have $\mathsf{TC}(B)=\mathsf{TC}(\mathrm{Conf}(\mathbb{R}^d,m))=2m-2$ (see, for instance, [8]), which differs from the parameterized topological complexity of the bundle unless the number of obstacles is twice the number of robots.

2. Parametrized topological complexity

In this brief section, we recall requisite material from [3]. Recall the broad framework: We wish to analyze the complexity of a motion planning algorithm in an environment which may change under the influence of external conditions. These conditions, parameters treated as part of the input of the algorithm, are encoded by a topological space B. Associated to each choice of conditions, that is, to each point $b \in B$, one has a configuration space X_b of achievable configurations in which motion planning must take place. The motion planning algorithim must thus be sufficiently flexible so as to adapt to different external conditions, that is, different points in the parameter space B.

Let $p\colon E \to B$ be a fibration, with nonempty, path-connected fiber X. Let E^I_B denote the space of all continuous paths $\gamma\colon I \to E$ which lie in a single fiber of p, so that $p \circ \gamma$ is the constant path in B. Let

$$E \times_B E = \{(e, e') \in E \times E \mid p(e) = p(e')\}$$

be the space of pairs of points in E which lie in the same fiber. The map

$$\Pi\colon E^I_B \to E \times_B E, \qquad \gamma \mapsto (\gamma(0), \gamma(1))$$

given by sending a path to its endpoints is a fibration, with fiber ΩX, the space of based loops in X.

Definition 2.1. The parameterized topological complexity $\mathsf{TC}[p\colon E \to B]$ of the fibration $p\colon E \to B$ is the sectional category of the fibration $\Pi\colon E^I_B \to E \times_B E$,

$$\mathsf{TC}[p\colon E \to B] := \mathsf{secat}(\Pi\colon E^I_B \to E \times_B E).$$

That is, $\mathsf{TC}[p\colon E \to B]$ is equal to the smallest nonnegative integer k for which the space $E \times_B E$ admits an open cover

$$E \times_B E = U_0 \cup U_1 \cup \cdots \cup U_k,$$

and the map $\Pi\colon E^I_B \to E \times_B E$ admits a continuous section $s_i\colon U_i \to E^I_B$ for each i, $0 \leq i \leq k$.

If the fibration p is clear from the context, we sometimes use the abbreviated notation $\mathsf{TC}[p\colon E \to B] = \mathsf{TC}_B(X)$, to emphasize the role of the fiber X.

As shown in [3, Prop. 5.1], parameterized topological complexity is an invariant of fiberwise homotopy equivalence.

For a topological space Y, let $\dim(Y)$ denote the covering dimension of Y, and let $\mathrm{hdim}(Y)$ denote the homotopy dimension of Y, the minimal dimension of a space Z homotopy equivalent to Y. Since the parameterized topological complexity of $p\colon E \to B$ is defined to be the sectional category of the associated fibration $\Pi\colon E^I_B \to E \times_B E$, we have

$$\mathsf{TC}[p\colon E \to B] \leq \mathsf{cat}(E \times_B E) \leq \operatorname{hdim}(E \times_B E),$$

where $\mathsf{cat}(Y)$ is the Lusternik-Schnirelmann category of Y (cf. [18]). We also have the following.

Proposition 2.2 ([3, Prop. 7.1]). *Let $p\colon E \to B$ be a locally trivial fibration of metrizable topological spaces, with path-connected fiber X. Then,*

$$\mathsf{TC}_B(X) = \mathsf{TC}[p\colon E \to B] \leq 2\dim(X) + \dim(B).$$

Parameterized topological complexity admits a cohomological lower bound. For a graded ring A, let $\mathsf{cl}(A)$ denote the cup length of A, the largest integer q for which there are homogeneous elements $a_1, \dots, a_q$ of positive degree in A such that $a_1 \cdots a_q \neq 0$.

Proposition 2.3 ([3, Prop. 7.3]). *Let $p\colon E \to B$ be a fibration with path-connected fiber, and let $\Delta\colon E \to E \times_B E$ be the diagonal map, $\Delta(e) = (e, e)$. Then the parameterized topological complexity of $p\colon E \to B$ is greater than or equal to the cup length of the kernel of the map in cohomology induced by Δ,*

$$\mathsf{TC}[p\colon E \to B] \geq \mathsf{cl}\left(\ker\left[\Delta^*\colon H^*(E \times_B E; R) \to H^*(E; \Delta^* R)\right]\right),$$

for any system of coefficients R on $E \times_B E$.

We conclude this section by recording the following product inequality for parameterized topological complexity, which we will make use of in Section 4 below.

Proposition 2.4 ([3, Prop. 6.1]). *Let $p'\colon E' \to B'$ and $p''\colon E'' \to B''$ be fibrations with path-connected fibers X' and X'' respectively. Let $B = B' \times B''$, $E = E' \times E''$, $X = X' \times X''$, and $p = p' \times p''$. Then the product fibration $p\colon E \to B$ satisfies*

$$\mathsf{TC}[p\colon E \to B] \leq \mathsf{TC}[p'\colon E' \to B'] + \mathsf{TC}[p''\colon E'' \to B''].$$

Equivalently, in abbreviated notation,

$$\mathsf{TC}_{B' \times B''}(X' \times X'') \leq \mathsf{TC}_{B'}(X') + \mathsf{TC}_{B''}(X'').$$

3. Cohomology of the obstacle-avoiding configuration space

In this section, we study the structure of the cohomology rings of configuration spaces arising in the context of our main theorem. Let $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$ and $B = \mathrm{Conf}(\mathbb{R}^d, m)$. Then, the Fadell-Neuwirth bundle of configuration spaces is $p\colon E \to B$, with fiber $X = \mathrm{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_m, n)$, where $\mathcal{O}_m$ is a set of m distinct points in $\mathbb{R}^d$. In order to utilize Proposition 2.3 subsequently, we analyze the cohomology ring of the "obstacle-avoiding configuration space" $E \times_B E$.

We use homology and cohomology with integer coefficients, and suppress the coefficients, throughout. The principal objects of study, E, B, X, and $E \times_B E$, all have torsion free integral homology and cohomology. This is well known for the classical configuration spaces, see [5].

Proposition 3.1. *Let $p\colon E = \mathrm{Conf}(\mathbb{R}^d, m+n) \to B = \mathrm{Conf}(\mathbb{R}^d, m)$ be the Fadell-Neuwirth bundle of configuration spaces. The integral cohomology groups of the space $E \times_B E$ are torsion free. The cohomology ring $H^*(E \times_B E)$ is generated by degree $d-1$ elements $\omega_{i,j}$ and $\omega'_{i,j}$, $1 \le i < j \le m+n$, which satisfy the relations*

$$\begin{aligned} &\omega'_{i,j} = \omega_{i,j} \text{ for } 1 \le i < j \le m, && \omega_{i,j}\omega_{i,k} - \omega_{i,j}\omega_{j,k} + \omega_{i,k}\omega_{j,k} = 0 \text{ for } i < j < k, \\ &(\omega_{i,j})^2 = (\omega'_{i,j})^2 = 0 \text{ for } i < j, && \omega'_{i,j}\omega'_{i,k} - \omega'_{i,j}\omega'_{j,k} + \omega'_{i,k}\omega'_{j,k} = 0 \text{ for } i < j < k. \end{aligned}$$

Since $\omega'_{i,j} = \omega_{i,j}$ for $1 \le i < j \le m$, the last of these relations may be expressed as $\omega_{i,j}\omega'_{i,k} - \omega_{i,j}\omega'_{j,k} + \omega'_{i,k}\omega'_{j,k} = 0$ for such i and j. We refer to relations of this general form as "three term relations".

Proof. Let $\mathbf{z} = (z_1, \dots, z_{m+n})$ and $\mathbf{z}' = (z'_1, \dots, z'_{m+n})$ be points in the configuration space $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$, so that $z_i \neq z_j$ and $z'_i \neq z'_j$ for all $i < j$. Points in the space $E \times_B E$ may be expressed as pairs of such points $(\mathbf{z}, \mathbf{z}')$ which satisfy $z_i = z'_i$ for $1 \le i \le m$. That is, $E \times_B E$ may be realized as the intersection $E \times_B E = (E \times E) \cap S$, where $S = \{(\mathbf{z}, \mathbf{z}') \in \mathbb{R}^{2(m+n)} \mid z_i = z'_i \text{ for } 1 \le i \le m\}$. Let $\iota\colon E \times_B E \to E \times E$ denote the inclusion.

For $1 \le i < j \le m+n$, define maps $p_{i,j}, p'_{i,j}\colon E \times E \to \mathrm{Conf}(\mathbb{R}^d, 2)$ by $p_{i,j}(\mathbf{z}, \mathbf{z}') = (z_i, z_j)$ and $p'_{i,j}(\mathbf{z}, \mathbf{z}') = (z'_i, z'_j)$. The space $\mathrm{Conf}(\mathbb{R}^d, 2)$ is homotopy equivalent to the sphere S^{d-1}. Fix a generator $\sigma \in H^{d-1}(\mathrm{Conf}(\mathbb{R}^d, 2))$, and define $\Omega_{i,j}, \Omega'_{i,j} \in H^{d-1}(E \times E)$ by $\Omega_{i,j} = (p_{i,j})^*(\sigma)$ and $\Omega'_{i,j} = (p'_{i,j})^*(\sigma)$. From well known results [5] on the cohomology of the configuration space $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$ and the Künneth formula, the elements $\Omega_{i,j}, \Omega'_{i,j}$ generate $H^*(E \times E)$ and satisfy $(\Omega_{i,j})^2 = (\Omega'_{i,j})^2 = 0$ and the three term relations (involving $\{\Omega_{i,j}, \Omega_{i,k}, \Omega_{j,k}\}$ and $\{\Omega'_{i,j}, \Omega'_{i,k}, \Omega'_{j,k}\}$).

Now let $\omega_{i,j} = \iota^*(\Omega_{i,j})$ and $\omega'_{i,j} = \iota^*(\Omega'_{i,j})$ in $H^{d-1}(E \times_B E)$ for $1 \le i < j \le m+n$. Then, as shown in [3, Prop. 9.2], these cohomology classes satisfy the asserted relations. In particular, since $z'_i = z_i$ for $i \le m$, we have $\omega'_{i,j} = \omega_{i,j}$ for $i < j \le m$. The other relations follow immediately from naturality.

It remains to show that $H^*(E \times_B E)$ is torsion free, generated by the classes $\omega_{i,j}, \omega'_{i,j}$. The space $E \times_B E$ may also be realized (up to homeomorphism) as the total space of the bundle obtained by pulling back the product bundle $p \times p\colon E \times E \to$

$B \times B$ along the diagonal map $\Delta_B\colon B \to B \times B$. The common fiber $X \times X$ is totally non-homologous to zero in each of these bundles, both inclusion-induced maps $H^*(E \times E) \to H^*(X \times X)$ and $H^*(E \times_B E) \to H^*(X \times X)$ are surjective. Additionally, $H^*(X \times X)$ is torsion free since $H^*(X) = H^*(\mathrm{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_m, n)$ is. Consequently, the classical Leray-Hirsch theorem applies to both bundles, see [5, 3]. From this, we obtain an additive isomorphism $H^*(B) \otimes H^*(X \times X) \cong H^*(E \times_B E)$. Since the cohomology groups of $B = \mathrm{Conf}(\mathbb{R}^d, m)$ are also torsion free, so are those of $E \times_B E$.

Lastly, using the commuting diagram

$$\begin{array}{ccc} H^*(B \times B) \otimes H^*(X \times X) & \xrightarrow{\cong} & H^*(E \times E) \\ \Big\downarrow{\scriptstyle \Delta_B^* \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \iota^*} \\ H^*(B) \otimes H^*(X \times X) & \xrightarrow{\cong} & H^*(E \times_B E) \end{array}$$

and the fact that $\Delta_B^*\colon H^*(B \times B) \to H^*(B)$ is surjective, we see that the inclusion $\iota\colon E \times_B E \to E \times E$ induces a surjection in cohomology. Since the classes $\Omega_{i,j}$ and $\Omega'_{i,j}$ generate the ring $H^*(E \times E)$, their images $\omega_{i,j} = \iota^*(\Omega_{i,j})$ and $\omega'_{i,j} = \iota^*(\Omega'_{i,j})$ generate the ring $H^*(E \times_B E)$. $\square$

For a natural number q, let $[q] = \{1, 2, \ldots, q\}$. Let $I = (i_1, \ldots, i_\ell)$ and $J = (j_1, \ldots, j_\ell)$ be sequences of elements in $[m+n]$. If $i_k < j_k$ for each k, $1 \le k \le \ell$, we write $I < J$ and define cohomology classes

$$\omega_{I,J} = \omega_{i_1,j_1}\omega_{i_2,j_2}\cdots\omega_{i_\ell,j_\ell} \quad \text{and} \quad \omega'_{I,J} = \omega'_{i_1,j_1}\omega'_{i_2,j_2}\cdots\omega'_{i_\ell,j_\ell}$$

in $H^{(d-1)\ell}(E \times_B E)$. If $\ell = 0$, set $\omega_{I,J} = \omega'_{I,J} = 1$.

Call the sequence $J = (j_1, j_2, \ldots, j_\ell)$ increasing if $j_1 < j_2 < \cdots < j_\ell$.

Proposition 3.2 ([3, Prop. 9.3])**.** *A basis for $H^*(E \times_B E)$ is given by the set of cohomology classes*

$$\omega_{I_1,J_1}\omega_{I_2,J_2}\omega_{I_3,J_3},$$

where $J_1 \subset [m]$, $J_2, J_3 \subset [m+n]$ are increasing sequences, and I_1, I_2, and I_3 are sequences with $I_1 < J_1$, $I_2 < J_2$, and $I_3 < J_3$.

We conclude this section with a technical result which will be used in the proof of the main theorem. For a sequence $J = (j_1, \ldots, j_\ell)$, let $J' = (j_1, \ldots, j_{\ell-1})$.

Definition 3.3. Let $J = (j_1, \ldots, j_\ell)$ be an increasing sequence. A *J-admissible* sequence $I = (i_1, \ldots, i_\ell)$ is defined recursively as follows. If $|J| = \ell = 1$, then I is J-admissible if and only if $I = J$. If $|J| = \ell \ge 2$, then I is J-admissible if

(i) I is nondecreasing, $i_1 \le \cdots \le i_\ell$,

(ii) $I' = (i_1, \ldots, i_{\ell-1})$ is J'-admissible, and
(iii) either $i_\ell = i_{\ell-1}$ or $i_\ell = j_\ell$.

For instance, if $J = (j_1, j_2)$, the J-admissible sequences are (j_1, j_1) and J itself.

Proposition 3.4. *If $J = (j_1, \ldots, j_\ell)$ is an increasing sequence and $r > j_\ell$, then*

$$\omega_{j_1,r}\omega_{j_2,r}\cdots\omega_{j_\ell,r} = (-1)^\ell \sum_I (-1)^{d_I} \omega_{i_1,j_2}\omega_{i_2,j_3}\cdots\omega_{i_{\ell-1},j_\ell}\omega_{i_\ell,r},$$

and

$$\omega'_{j_1,r}\omega'_{j_2,r}\cdots\omega'_{j_\ell,r} = (-1)^\ell \sum_I (-1)^{d_I} \omega'_{i_1,j_2}\omega'_{i_2,j_3}\cdots\omega'_{i_{\ell-1},j_\ell}\omega'_{i_\ell,r},$$

where the sums are over all J-admissible sequences I, and d_I is the number of distinct elements in I.

Observe that the sums above are linear combinations of distinct elements of the basis for $H^*(E \times_B E)$ given in Proposition 3.2.

Proof. Let $R = (r, r, \ldots, r)$ be the constant sequence of length ℓ. The proposition asserts that

$$\omega_{J,R} = (-1)^\ell \sum_I (-1)^{d_I} \omega_{I,K} \quad \text{and} \quad \omega'_{J,R} = (-1)^\ell \sum_I (-1)^{d_I} \omega'_{I,K},$$

where $K = (j_2, \ldots, j_\ell, r)$. Clearly, it suffices to consider $\omega_{J,R}$.

The proof is by induction on $\ell = |J|$, with the case $\ell = 1$ trivial. The case $\ell = 2$ is the three term relation $\omega_{j_1,r}\omega_{j_2,r} = \omega_{j_1,j_2}\omega_{j_2,r} - \omega_{j_1,j_2}\omega_{j_1,r}$, which will be crucial subsequently.

Assume that $\ell \geq 3$. For $J = (j_1, \ldots, j_\ell)$, recall that $J' = (j_1, \ldots, j_{\ell-1})$, and let R' be the constant sequence of length $\ell - 1$. By induction, we have

$$\omega_{J',R'} = (-1)^{\ell-1} \sum_{I'} (-1)^{d_{I'}} \omega_{i_1,j_2}\cdots\omega_{i_{\ell-2},j_{\ell-1}}\omega_{i_{\ell-1},r},$$

where the sum is over all J'-admissible sequences $I' = (i_1, \ldots, i_{\ell-1})$. Since $\omega_{J,R} = \omega_{J',R'}\omega_{j_\ell,r}$, we obtain

$$\begin{aligned}\omega_{J,R} &= (-1)^{\ell-1} \sum_{I'} (-1)^{d_{I'}} \omega_{i_1,j_2}\cdots\omega_{i_{\ell-2},j_{\ell-1}}\omega_{i_{\ell-1},r}\omega_{j_\ell,r} \\ &= (-1)^{\ell-1} \sum_{I'} (-1)^{d_{I'}} \omega_{i_1,j_2}\cdots\omega_{i_{\ell-2},j_{\ell-1}}(\omega_{i_{\ell-1},j_\ell}\omega_{j_\ell,r} - \omega_{i_{\ell-1},j_\ell}\omega_{i_{\ell-1},r}),\end{aligned}$$

using the three term relations on the second line. For I' as above, let $P = (i_1, \ldots, i_{\ell-1}, j_\ell)$ and $Q = (i_1, \ldots, i_{\ell-1}, i_{\ell-1})$. Note that $d_P = d_{I'} + 1$ and $d_Q = d_{I'}$.

Further, as is clear from Definition 3.3, every J-admissible sequence I arises from a J'-admissible sequence I' by adjoining either j_ℓ or $i_{\ell-1}$. Thus,

$$\begin{aligned}\omega_{J,R} &= (-1)^{\ell-1}\sum_{I'}(-1)^{d_{I'}}\omega_{P,K} + (-1)^{\ell}\sum_{I'}(-1)^{d_{I'}}\omega_{Q,K}\\ &= (-1)^{\ell}\sum_{P}(-1)^{d_P}\omega_{P,K} + (-1)^{\ell}\sum_{Q}(-1)^{d_Q}\omega_{Q,K}\\ &= (-1)^{\ell}\sum_{I}(-1)^{d_I}\omega_{I,K},\end{aligned}$$

where the last sum is over all J-admissible sequences as required. □

4. Obstacle-avoiding collision-free motion in the plane

In this section, we state and prove our main theorem, determining the parameterized topological complexity of obstacle-avoiding collision-free motion in any Euclidean space $\mathbb{R}^d$ of positive even dimension. The case $d = 2$ of the plane was highlighted in the Introduction.

Theorem 4.1. *For positive integers n, m, and $d \geq 2$ even, the parameterized topological complexity of the motion of n non-colliding particles in $\mathbb{R}^d$ in the presence of m non-colliding point obstacles with a priori unknown positions is equal to $2n + m - 2$. In other words, the parameterized topological complexity of the Fadell-Neuwirth bundle $p\colon \operatorname{Conf}(\mathbb{R}^d, m+n) \to \operatorname{Conf}(\mathbb{R}^d, m)$ is*

$$\mathsf{TC}\big[p\colon \operatorname{Conf}(\mathbb{R}^d, m+n) \to \operatorname{Conf}(\mathbb{R}^d, m)\big] = 2n + m - 2.$$

Let $E = \operatorname{Conf}(\mathbb{R}^d, m+n)$ and $B = \operatorname{Conf}(\mathbb{R}^d, m)$, so that the Fadell-Neuwirth bundle is $p\colon E \to B$. The fiber of this bundle is $X = \operatorname{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_m, n)$, where $\mathcal{O}_m$ is a set of m distinct points (representing the obstacles). Each of the spaces E, B, X, and $E \times_B E$ has the homotopy type of a finite CW-complex of known dimension. For the configuration spaces B, E, and X, see [5]. For $E \times_B E$, this can be shown using various forms of Morse theory, cf. [1, 11]. The dimensions of these CW-complexes are

$$\begin{aligned} &\operatorname{hdim} B = (m-1)(d-1), && \operatorname{hdim} E = (m+n-1)(d-1),\\ &\operatorname{hdim} X = n(d-1), && \operatorname{hdim} E \times_B E = (2m+n-1)(d-1).\end{aligned} \tag{4.1}$$

Furthermore, each of the spaces E, B, X, and $E \times_B E$ is $(d-2)$-connected, as each is obtained removing codimension d subspaces from a Euclidean space.

Remark 4.2. If $m = 1$, the base space $B = \operatorname{Conf}(\mathbb{R}^d, m) = \mathbb{R}^d$ of the Fadell-Neuwirth bundle is contractiible, and the bundle is trivial. The parameterized topological complexity of this trivial bundle is equal to the (standard) topological complexity of the fiber $X = \mathrm{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_1, n)$, see [3, Ex. 4.2], and Theorem 4.1 is a restatement of results of [9] in this instance.

We subsequently assume that $m \geq 2$. We first show that

$$\mathsf{TC}\big[p\colon \mathrm{Conf}(\mathbb{R}^d, m+n) \to \mathrm{Conf}(\mathbb{R}^d, m)\big] \geq 2n + m - 2.$$

By Proposition 2.3, this is a consequence of the following.

Proposition 4.3. *For $d \geq 2$ even, $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$ and $B = \mathrm{Conf}(\mathbb{R}^d, m)$, the ideal*

$$\ker[\Delta^*\colon H^*(E \times_B E) \to H^*(E)]$$

in $H^*(E \times_B E)$ has cup length $\mathsf{cl}(\ker \Delta^*) \geq 2n + m - 2$.

Proof. The ideal

$$\mathcal{J} = \langle \omega_{i,j} - \omega'_{i,j} \mid 1 \leq i < j \text{ and } m < j \leq n+m \rangle \tag{4.2}$$

is generated by degree $d-1$ elements in $H^*(E \times_B E)$. One can check (cf. [3, Prop. 9.4]) that $\mathcal{J} \subseteq \ker \Delta^*$. So to prove the proposition it is enough to show that $\mathsf{cl}(\mathcal{J}) \geq 2n + m - 2$. We establish this by showing that the product

$$\Psi = \prod_{i=1}^{m}(\omega_{i,m+1} - \omega'_{i,m+1}) \prod_{j=m+2}^{m+n}(\omega_{1,j} - \omega'_{1,j}) \prod_{j=m+2}^{m+n}(\omega_{j-1,j} - \omega'_{j-1,j})$$

is nonzero in $H^*(E \times_B E)$.

If a_i, b_i, $1 \leq i \leq q$, are cohomology classes of the same degree, then

$$\prod_{i=1}^{\ell}(a_i - b_i) = \sum_{S \subset [q]} (-1)^{|S|} c_1 c_2 \cdots c_q, \text{ where } c_j = \begin{cases} a_j & \text{if } j \notin S, \\ b_j & \text{if } j \in S. \end{cases}$$

Using this, we have

$$\begin{aligned}
\prod_{i=1}^{m}(\omega_{i,m+1} - \omega'_{i,m+1}) &= \sum_{S}(-1)^{|S|}\lambda_1 \cdots \lambda_m, & \xi_i &= \begin{cases} \omega_{i,m+1} & \text{if } i \notin S, \\ \omega'_{i,m+1} & \text{if } i \in S, \end{cases} \\
\prod_{j=m+2}^{m+n}(\omega_{1,j} - \omega'_{1,j}) &= \sum_{T_1}(-1)^{|T_1|}\mu_{m+2} \cdots \mu_{m+n}, & \mu_j &= \begin{cases} \omega_{1,j} & \text{if } j \notin T_1, \\ \omega'_{1,j} & \text{if } j \in T_1, \end{cases} \\
\prod_{j=m+2}^{m+n}(\omega_{j-1,j} - \omega'_{j-1,j}) &= \sum_{T_2}(-1)^{|T_2|}\xi_{m+2} \cdots \xi_{m+n}, & \lambda_j &= \begin{cases} \omega_{j-1j} & \text{if } j \notin T_2, \\ \omega'_{j-1j} & \text{if } j \in T_2, \end{cases}
\end{aligned}$$

where, writing $[p,q] = \{p, p+1, \ldots, q\}$, $S \subset [m]$ and $T_1, T_2 \subset [m+2, m+n]$.

For $T = (j_1, \dots, j_\ell)$ a sequence in $[p, q]$, let $T^{\mathsf{c}} = (p, \dots, \widehat{j_1}, \dots, \widehat{j_\ell}, \dots, q)$ denote the complementary sequence, and let ϵ_T be the sign of the shuffle permutation taking $[p, q]$ to (T^{c}, T). Denote the constant sequence $(1, 1, \dots, 1)$ (of appropriate length) by $\mathbf{1}$, and let $T - \mathbf{1} = (j_1 - 1, \dots, j_\ell - 1)$. Then, the latter two products above may be expressed as

$$\begin{aligned}\prod_{j=m+2}^{m+n} (\omega_{1j} - \omega'_{1j}) &= \sum_{T_1} (-1)^{|T_1|} \epsilon_{T_1} \omega_{\mathbf{1},T_1^{\mathsf{c}}} \omega'_{\mathbf{1},T_1},\\ \prod_{j=m+2}^{m+n} (\omega_{j-1j} - \omega'_{j-1j}) &= \sum_{T_2} (-1)^{|T_2|} \epsilon_{T_2} \omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}} \omega'_{T_2-\mathbf{1},T_2},\end{aligned} \tag{4.3}$$

Since, for $i = 1, 2$, T_i and T_i^{c} are increasing sequences in $[m+2, m+n]$ and $\mathbf{1} < T_i$, $\mathbf{1} < T_i^{\mathsf{c}}$, and $T_i - \mathbf{1} < T_i$, the monomials $\omega_{\mathbf{1},T_1^{\mathsf{c}}} \omega'_{\mathbf{1},T_1}$ and $\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}} \omega'_{T_2-\mathbf{1},T_2}$ arising in (4.3) are elements of the basis for $H^*(E \times_B E)$ of Proposition 3.2.

Similarly, with $R = (m+1, m+1, \dots, m+1)$, the first of the three products above may be expressed as

$$\begin{aligned}\prod_{i=1}^{m} (\omega_{i,m+1} - \omega'_{i,m+1}) &= \sum_{S} (-1)^{|S|} \epsilon_S \omega_{S^{\mathsf{c}},R} \omega'_{S,R}\\ &= \sum_{\emptyset \subsetneq S \subsetneq [m]} (-1)^{|S|} \epsilon_S \omega_{S^{\mathsf{c}},R} \omega'_{S,R} + \epsilon_\emptyset \omega_{[m],R} + (-1)^m \epsilon_{[m]} \omega'_{[m],R}.\end{aligned}$$

None of the monomials $\omega_{S^{\mathsf{c}},R} \omega'_{S,R}$ is an element of the basis of Proposition 3.2. Rewriting using Proposition 3.4 and some sign simplification yields

$$\begin{aligned}\prod_{i=1}^{m} (\omega_{i,m+1} - \omega'_{i,m+1}) = &\sum_{\emptyset \subsetneq S \subsetneq [m]} \epsilon_S \left(\sum_{I_1} (-1)^{d_{I_1}} \omega_{I_1,K_1} \right) \left(\sum_{I_2} (-1)^{d_{I_2}} \omega'_{I_2,K_2} \right)\\ &+ \epsilon_\emptyset \sum_{I_1} (-1)^{d_{I_1}} \omega_{I_1,K_1} + \epsilon_{[m]} \sum_{I_2} (-1)^{d_{I_2}} \omega'_{I_2,K_2}\end{aligned} \tag{4.4}$$

where I_1 and I_2 range over all S^{c}-and S-admissible sequences respectively, and if $S^{\mathsf{c}} = (i_1, \dots, i_p)$ and $S = (j_1, , \dots, j_q)$, then $K_1 = (i_2, \dots, i_{p-1}, m+1)$ and $K_2 = (j_2, \dots, j_{q-1}, m+1)$. Note that $K_1 = [2, m+1]$ if $S = \emptyset$ and $S^{\mathsf{c}} = [m]$, while $K_2 = [2, m+1]$ if $S = [m]$ and $S^{\mathsf{c}} = \emptyset$.

The product Ψ may then be obtained by multiplying the expressions of (4.4) and (4.3). Expanding yields an expression of Ψ as a linear combination of monomials $\omega_{P_!,Q_1} \omega_{P_2,Q_2} \omega'_{P_3,Q_3}$, where $Q_1 \subset [m]$ and $Q_2, Q_3 \subset [m+1, m+n]$. Some of these monomials are elements of the basis of Proposition 3.2, others are not. One of the basis elements appearing in this expansion of Ψ is

$$(4.5)\quad \omega_{1,2}\omega_{1,3}\cdots\omega_{1,m}\omega_{1,m+1}\omega_{1,m+2}\cdots\omega_{1,m+n}\omega'_{m+1,m+2}\omega'_{m+2,m+3}\cdots\omega'_{m+n-1,m+n}.$$

This element is obtained by taking $S=\emptyset$, $S^{\mathsf{c}}=[m]$ and $I_1=\mathbf{1}$ in (4.4), so that the expansion of $\omega'_{S,R}$ is simply 1, and by taking $T_1=T_2^{\mathsf{c}}=\emptyset$, $T_1^{\mathsf{c}}=T_2=[m+2,m+n]$ in (4.3), so that $\omega'_{\mathbf{1},T_1}=\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}=1$. It may be expressed briefly as $x=\omega_{\mathbf{1},K}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega'_{T_2-\mathbf{1},T_2}$, where $K=[2,m+1]$.

We assert that the basis element $x=\omega_{\mathbf{1},K}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega'_{T_2-\mathbf{1},T_2}$ is unaffected by rewriting non-basis monomials in the expansion of Ψ using the three term relations. This will insure the non-vanishing of Ψ as needed. Let $y=\omega_{P_!,Q_1}\omega_{P_2,Q_2}\omega'_{P_3,Q_3}$ be a monomial in the expansion of Ψ. From the expansions (4.4) and (4.3), we have

$$(4.6)\qquad y=\omega_{P_!,Q_1}\omega_{P_2,Q_2}\omega'_{P_3,Q_3}=\big(\omega_{I_1,K_1}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}\big)\big(\omega'_{I_2,K_2}\omega'_{\mathbf{1},T_1}\omega'_{T_2-\mathbf{1},T_2}\big),$$

where, for $j=1,2$, K_j is either empty or is an increasing sequence in $[2,m+1]$ and I_j is K_j-admissible, and T_j and T_j^{c} are complementary increasing sequences in $[m+2,m+n]$. From Proposition 3.4, non-empty sequences K_1 and K_2 are of the form $(k_1,\dots,k_\ell,m+1)$ with $k_\ell\le m$, so K_1' and K_2' are increasing sequences in $[2,m]$ of the form $(k_1,\dots,k_\ell)$. Since $\omega'_{i,j}=\omega_{i,j}$ for $1\le i<j\le m$, up to sign, the monomial $\omega_{P_!,Q_1}\omega_{P_2,Q_2}\omega'_{P_3,Q_3}$ can be rewritten as

$$(4.7)\qquad y=\big(\omega_{I_1',K_1'}\omega_{I_2',K_2'}\big)\big(\omega_{\alpha,m+1}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}\big)\big(\omega'_{\beta,m+1}\omega'_{\mathbf{1},T_1}\omega'_{T_2-\mathbf{1},T_2}\big),$$

where $I_1=(I_1',\alpha)$ and $I_2=(I_2',\beta)$.

Suppose the monomial y of (4.6) is not an element of the basis of Proposition 3.2. First, consider the case where the subset S of $[m]$ in (4.4) is non-empty, so that $K_2\neq\emptyset$. As indicated in (4.7) above, this gives rise to a factor of $\omega'_{\beta,m+1}$ in the monomial y. Subsequent simplifications, for instance if $\omega'_{\mathbf{1},T_1}\omega'_{T_2-\mathbf{1},T_2}$ is not a basis element, either annihilate y or given rise to basis elements involving $\omega'_{\beta,m+1}$ or $\omega'_{1,m+1}$. No factor of this form appears in the monomial x of (4.5).

It remains to consider the case where the subset S of $[m]$ in (4.4) is empty. For $S=\emptyset$, we have $K_1=[2,m+1]$ and $\alpha=m$ in (4.7). In this instance,

$$(4.8)\qquad y=\big(\omega_{I_1',K_1'}\big)\big(\omega_{m,m+1}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}\big)\big(\omega'_{\mathbf{1},T_1}\omega'_{T_2-\mathbf{1},T_2}\big),$$

where $I_1=(I_1',m)$ and $K_1'=[2,m]$. We have either $T_1\neq\emptyset$ or $T_2^{\mathsf{c}}\neq\emptyset$, since the basis element x of (4.5) is obtained by taking $S=\emptyset$ and $T_1=T_2^{\mathsf{c}}=\emptyset$.

If $T_1\neq\emptyset$, then $\omega'_{1,k}$ is a factor of y, where $k\in[m+2,m+n]$ denotes the minimal element of T_1. If $k\notin T_2$, then $\omega'_{1,k}$ is a factor of the basis element $\omega'_{\mathbf{1},T_1}\omega'_{T_2-\mathbf{1},T_2}$, which survives in each term of the expansion of y arising from application of the three term relations to $\omega_{m,m+1}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}$ and resulting expressions. If, on the other hand, $k \in T_2$, then $\omega'_{1,k}\omega'_{k-1,k}$ is a factor of y. Rewriting using the three term relation $\omega'_{1,k}\omega'_{k-1,k} = \omega'_{1,k-1}(\omega'_{k-1,k} - \omega'_{1,k})$ yields expressions involving $\omega'_{1,k-1}$. Continuing as necessary yields a linear combination of basis elements, each of which contains a factor of $\omega'_{1,j}$, for some j, $m+1 \le j \le k$. No factor of this form appears in the monomial x of (4.5).

Finally, if $T_1 = \emptyset$, then $T_2^{\mathsf{c}} \neq \emptyset$. This implies that T_2 is a proper subset of $[m+2, m+n]$, and consequently that the factor $\omega'_{m+1,m+2}\omega'_{m+2,m+3}\cdots\omega'_{m+n-1,m+n}$ appearing in the monomial x of (4.5) cannot appear in y. Since $\omega'_{T_2-\mathbf{1},T_2}$ is a basis element, any necessary expansion of y involves applications of the three term relations to the factor $\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega_{T_2^{\mathsf{c}}-\mathbf{1},T_2^{\mathsf{c}}}$. Since these, and subsequent simplifications, cannot introduce any factors of the form $\omega'_{p,q}$, the factor $\omega'_{m+1,m+2}\omega'_{m+2,m+3}\cdots\omega'_{m+n-1,m+n}$ of x cannot appear in any resulting monomial.

Thus, as asserted, expressing Ψ in terms of the basis of Proposition 3.2 does not alter the summand $x = \omega_{\mathbf{1},K}\omega_{\mathbf{1},T_1^{\mathsf{c}}}\omega'_{T_2-\mathbf{1},T_2}$. Therefore, $\Psi \neq 0$ and $\mathsf{cl}(\ker\Delta^*) \geq \mathsf{cl}(\mathcal{J}) \geq 2n+m-2$ as required. ☐

Thus, for d even and $m \geq 2$, we have

$$\mathsf{TC}\big[p\colon \operatorname{Conf}(\mathbb{R}^d, m+n) \to \operatorname{Conf}(\mathbb{R}^d, m)\big] \geq \mathsf{cl}(\ker\Delta^*) \geq 2n+m-2. \tag{4.9}$$

We establish the reverse inequality for the case $d = 2$ of the plane and for the case $d \geq 4$ of higher even dimensions using different methods. Since the result in the planar case will play a role in the proof in the higher dimensional case, we begin with the former.

The plane. Consider the case $d = 2$ of the plane $\mathbb{R}^2 = \mathbb{C}$. Express the configuration space $\operatorname{Conf}(\mathbb{R}^2, \ell)$ as

$$\operatorname{Conf}(\mathbb{R}^2, \ell) = \operatorname{Conf}(\mathbb{C}, \ell) = \{(y_1, \dots, y_\ell) \in \mathbb{C}^\ell \mid y_i \neq y_j \text{ if } i \neq j\}$$

in complex coordinates.

For any $\ell \geq 3$, the map $h_\ell\colon \operatorname{Conf}(\mathbb{C}, \ell) \to \operatorname{Conf}(\mathbb{C} \smallsetminus \{0,1\}, \ell-2) \times \operatorname{Conf}(\mathbb{C}, 2)$ defined by

$$h_\ell(y_1, y_2, y_3, \dots, y_\ell) = \left(\left(\frac{y_3 - y_1}{y_2 - y_1}, \dots, \frac{y_\ell - y_1}{y_2 - y_1}\right), (y_1, y_2)\right) \tag{4.10}$$

is a homeomorphism. It follows that the bundle $p\colon \operatorname{Conf}(\mathbb{C}, m+n) \to \operatorname{Conf}(\mathbb{C}, m)$ is trivial for $m = 2$. The parameterized topological complexity is then equal to the topological complexity of the fiber $\operatorname{Conf}(\mathbb{C} \smallsetminus \{0,1\}, n)$, see [3, Ex. 4.2]. Since $\mathsf{TC}(\operatorname{Conf}(\mathbb{C} \smallsetminus \{0,1\}, n)) = 2n$ as shown in [10], for $m = 2$, we have

$$\mathsf{TC}[p\colon \operatorname{Conf}(\mathbb{C}, n+2) \to \operatorname{Conf}(\mathbb{C}, 2)] = \mathsf{TC}(\operatorname{Conf}(\mathbb{C} \smallsetminus \{0,1\}, n)) = 2n$$

as asserted.

For $m \geq 3$, the maps (4.10) give rise to an equivalence of fibrations

$$\begin{array}{ccc} \mathrm{Conf}(\mathbb{C}, m+n) & \xrightarrow{h_{m+n}} & \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m+n-2) \times \mathrm{Conf}(\mathbb{C}, 2) \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\ \mathrm{Conf}(\mathbb{C}, m) & \xrightarrow{h_m} & \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m-2) \times \mathrm{Conf}(\mathbb{C}, 2), \end{array}$$

where $q = q' \times q''$, with q' the forgetful map and $q'' = \mathrm{id}$ the identity map. Since $\mathsf{TC}[q''\colon \mathrm{Conf}(\mathbb{C}, 2) \to \mathrm{Conf}(\mathbb{C}, 2)] = 0$, the product inequality Proposition 2.4 implies that $\mathsf{TC}\,[p\colon \mathrm{Conf}(\mathbb{C}, m+n) \to \mathrm{Conf}(\mathbb{C}, m)]$ is less than or equal to

$$\mathsf{TC}\left[q'\colon \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m+n-2) \to \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m-2)\right]. \tag{4.11}$$

Let $E' = \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m+n-2)$ and $B' = \mathrm{Conf}(\mathbb{C} \smallsetminus \{0,1\}, m-2)$. The fiber of $q'\colon E' \to B'$ is the configuration space $X = \mathrm{Conf}(\mathbb{C} \smallsetminus \mathcal{O}_m, n)$, which has the homotopy type of a CW-complex of dimension n. Similarly, B' has the homotopy type of a CW-complex of dimension $m-2$. Using Proposition 2.2, we obtain the following upper bound for (4.11):

$$\mathsf{TC}[q'\colon E' \to B'] \leq 2\dim(X) + \dim(B) = 2n + m - 2.$$

Combining the above observations yields

$$\mathsf{TC}\,[p\colon \mathrm{Conf}(\mathbb{C}, m+n) \to \mathrm{Conf}(\mathbb{C}, m)] \leq 2n + m - 2.$$

Together with the lower bound (4.9), this completes the proof of Theorem 4.1 in the case $d = 2$ of the plane $\mathbb{R}^2 = \mathbb{C}$.

Theorem 4.1 for the planar case $d = 2$ informs on the structure of the cohomology ring $H^*(E \times_B E)$ for any even d. This structure will be utilized in the case $d \geq 4$ of higher even dimensions below. Recall the ideal $\mathcal{J}$ in $H^*(E \times_B E)$ from (4.2).

Corollary 4.4. *For positive integers m and d with $m \geq 2$ and $d \geq 2$ even, let $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$, and $B = \mathrm{Conf}(\mathbb{R}^d, m)$. Then the ideal*

$$\mathcal{J} = \langle \omega_{i,j} - \omega'_{i,j} \mid 1 \leq i < j \text{ and } m < j \leq n+m \rangle$$

in $H^*(E \times_B E)$ has cup length $\mathsf{cl}(\mathcal{J}) = 2n + m - 2$.

Higher even dimensions. For $d \geq 4$ even, we use obstruction theory to complete the proof of Theorem 4.1.

The Schwarz genus of a fibration $p\colon E \to B$ with fiber X is at most $r-1$ if and only if its r-fold fiberwise join admits a continuous section, cf. [18, Thm. 3]. Consequently, $\mathsf{TC}[p\colon E \to B] \leq r-1$ if and only if the r-fold fiberwise join

$$\Pi_r\colon \underset{r}{\ast}(E^I_B) \to E \times_B E$$

admits a section. Note that the fiber of Π_r is $*_r(\Omega X)$, the r-fold join of the loop space of X. In the case of the Fadell-Neuwirth bundle of configuration spaces, we have $B = \mathrm{Conf}(\mathbb{R}^d, m)$, $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$, and $X = \mathrm{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_m, n)$. As noted previously, X is $(d-2)$-connected. Since the join of p- and q-connected CW-complexes is $(p+q+2)$-connected, the fiber $*_r(\Omega X)$ of Π_r is $(rd-r-2)$-connected. Thus, to show that $\mathsf{TC}\big[p\colon \mathrm{Conf}(\mathbb{R}^d, m+n) \to \mathrm{Conf}(\mathbb{R}^d, m)\big] \leq 2n + m - 2$, it suffices to prove the following.

Proposition 4.5. *For positive integers m and d with $m \geq 2$ and $d \geq 4$ even, let $E = \mathrm{Conf}(\mathbb{R}^d, m+n)$, $B = \mathrm{Conf}(\mathbb{R}^d, m)$, and $r = 2n+m-1$. Then the fibration*

$$\Pi_r\colon \mathop{*}_{r} E^I_B \to E \times_B E$$

admits a section.

Proof. From the connectivity of $X = \mathrm{Conf}(\mathbb{R}^d \smallsetminus \mathcal{O}_m, n)$ noted previously, the primary obstruction to the existence of a section of $\Pi_r\colon *_r E^I_B \to E \times_B E$ is an element $\Theta_r \in H^{r(d-1)}(E \times_B E; \pi_{r(d-1)-1}(*_r \Omega X))$. Since $\mathrm{hdim}\big(E \times_B E\big) = r(d-1)$ as noted in (4.1), higher obstructions vanish for dimensional reasons. So Θ_r is the only obstruction. By the Hurewicz theorem, we have $\pi_{r(d-1)-1}(*_r \Omega X) = H_{r(d-1)-1}(*_r \Omega X)$. For spaces Y and Z with torsion free integral homology, the (reduced) homology of the join is given by $\widetilde{H}_{q+1}(Y * Z) = \bigoplus_{i+j=q} \widetilde{H}_i(Y) \otimes \widetilde{H}_j(Z)$. This, together with the fact that the homology groups of X (and ΩX) are free abelian, yields

$$\pi_{r(d-1)-1}(\mathop{*}_{r} \Omega X) = H_{r(d-1)-1}(\mathop{*}_{r} \Omega X) = [\widetilde{H}_{d-1}(X)]^{\otimes r} = [H_{d-1}(X)]^{\otimes r},$$

the last equality since $d \geq 4$. Thus, $\Theta_r \in H^{r(d-1)}(E \times_B E; [H_{d-1}(X)]^{\otimes r})$.

By [18, Thm. 1], the obstruction Θ_r decomposes as $\Theta_r = \theta \smile \cdots \smile \theta = \theta^r$, where $\theta \in H^{d-1}(E \times_B E; H_{d-1}(X))$ is the primary obstruction to the existence of a section of $\Pi\colon E^I_B \to E \times_B E$. Since $E \times_B E$ is simply connected, the system of coefficients $H_{d-1}(X)$ on $E \times_B E$ is trivial. As noted above, $H_{d-1}(X)$ is torsion free. By Proposition 3.1, the cohomology ring $H^*(E \times_B E)$ is also torsion free. It follows that $H^*(E \times_B E; [H_{d-1}(X)]^{\otimes q})$ is torsion free for any $q \geq 1$.

Since θ is the primary obstruction to the existence of a section the fibration $\Pi\colon E^I_B \to E \times_B E$, we have

$$\theta \in \ker[\Delta^*\colon H^*(E \times_B E; H_{d-1}(X)) \longrightarrow H^*(E; \Delta^* H_{d-1}(X))].$$

For brevity, denote the free abelian group $H_{d-1}(X)$ by A. Using a Universal Coefficient theorem (for a (co)chain complex computing $H^*(E \times_B E)$), we can identify $H^{d-1}(E \times_B E; A)$ with $H^{d-1}(E \times_B E) \otimes A$, and $H^{d-1}(E; A)$ with $H^{d-1}(E) \otimes A$.

With these identifications, we have $\Delta^*\colon H^{d-1}(E \times_B E) \otimes A \to H^{d-1}(E) \otimes A$, and $\theta \in \ker(\Delta^*)$ may be expressed as a linear combination of elements of the form $\eta_j \otimes a_j$, where the elements η_j are the degree $d-1$ generators of $\ker[\Delta^*\colon H^*(E \times_B E)) \to H^*(E;\mathbb{Z}))]$ and $a_j \in A$.

The r-fold cup product $\Theta_r = \theta^r \in H^{r(d-1)}(E \times_B E) \otimes A^{\otimes r}$ is then realized as a linear combination of elements of the form $\eta_J \otimes a_J$, where $\eta_J = \eta_{j_1} \smile \cdots \smile \eta_{j_r}$ is an r-fold cup product of degree $d-1$ generators of $\ker[\Delta^*\colon H^*(E \times_B E)) \to H^*(E))]$, and $a_J \in A^{\otimes r}$. But the degree $d-1$ generators of $\ker \Delta^*$ are the generators of the ideal $\mathcal{J}$ of (4.2). As noted in Corollary 4.4, we have $\mathsf{cl}(\mathcal{J}) = 2n+m-2$. It follows that for $r = 2n+m-1$, we have $\mathcal{J}^r = 0$, and consequently $\theta^r = 0$. Since the primary obstruction $\Theta_r = \theta^r$ vanishes, the fibration $\Pi_r\colon \ast_r E_B^I \to E \times_B E$ admits a section. $\square$

This completes the proof of Theorem 4.1 in the case where $d \geq 4$ is even.

Acknowledgments. The first author thanks Emanuele Delucci, Nick Proudfoot, and He Xiaoyi for productive conversations, and the organizers of the virtual workshop *Arrangements at Home* for facilitating several of these conversations. Portions of this work were undertaken when the first and second authors visited the University of Florida Department of Mathematics in November, 2019. We thank the department for its hospitality and for providing a productive mathematical environment.

References

1. K. Adiprasito, *Combinatorial stratifications and minimality of 2-arrangements*. J. Topol. **7** (2014), 1200–1220; MR3286901
2. D. Cohen, *Topological complexity of classical configuration spaces and related objects*, in: *Topological complexity and related topics*, pp. 41–60, Contemp. Math., vol. 702, Amer. Math. Soc., Providence, RI, 2018; MR3762831
3. D. Cohen, M. Farber, S. Weinberger, *Topology of parameterised motion planning algorithms*, preprint 2020; `arXiv:2009.06023`
4. A. Dranishnikov, *On topological complexity of hyperbolic groups*, Proc. Amer. Math. Soc. **148** (2020), 4547–4556; MR4135318
5. E. Fadell, S. Husseini, *Geometry and Topology of Configuration Spaces*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001; MR1802664.
6. M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003), 211–221; MR1957228.
7. M. Farber, *Topology of robot motion planning*, in: *Morse Theoretic Methods in Non-linear Analysis and in Symplectic Topology*, pp. 185–230, NATO Science Series II: Mathematics, Physics and Chemistry, vol. 217, Springer, 2006; MR2276952.
8. M. Farber, *Configuration spaces and robot motion planning algorithms*, in: *Combinatorial and Toric Homotopy: Introductory Lectures*, pp. 263–303, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 35, World Sci. Publ., Hackensack, NJ, 2018; MR3792464.

9. M. Farber, M. Grant, *Topological complexity of configuration spaces*, Proc. Amer. Math. Soc. **137** (2009), 1841–1847; MR2470845.
10. M. Farber, M. Grant, S. Yuzvinsky, *Topological complexity of collision free motion planning algorithms in the presence of multiple moving obstacles*, in: *Topology and Robotics*, pp. 75–83, Contemp. Math., vol. 438, Amer. Math. Soc., Providence, RI, 2007; MR2359030.
11. M. Goresky, R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 14, Springer-Verlag, Berlin, 1988; MR0932724.
12. M. Grant, S. Mescher, *Topological complexity of symplectic manifolds*, Math. Z. **295** (2020), 667–679; MR4100027.
13. C. Ipanaque Zapata, J. González, *Multitasking collision-free optimal motion planning algorithms in Euclidean spaces*, Discrete Math. Algorithms Appl. **12** (2020), 2050040, 19 pp.; MR4120432.
14. J.-C. Latombe, *Robot Motion Planning*, Springer Science+Business Media, New York, 2012.
15. S.M. LaValle, *Planning Algorithms*, Cambridge University Press, Cambridge, 2006; MR2424564.
16. A. Murillo, J. Wu, *Topological complexity of the work map*, J. Topol. Anal., to appear; `arXiv:/1706.09157`.
17. P. Pavešić, *Topological complexity of a map*, Homology Homotopy Appl. **21** (2019), 107–130; MR3921612.
18. A.S. Schwarz, *The genus of a fiber space*, Amer. Math. Sci. Transl., **55** (1966), 49–140; MR0154284.
19. S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.) **4** (1981), 1–36; MR0590817

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803
Email address: `cohen@math.lsu.edu`
URL: `www.math.lsu.edu/~cohen`

School of Mathematical Sciences, Queen Mary University of London, E1 4NS London
Email address: `M.Farber@qmul.ac.uk`

Department of Mathematics, The University of Chicago, 5734 S University Ave, Chicago, IL 60637
Email address: `shmuel@math.uchicago.edu`