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A FIXED POINT THEOREM FOR PERIODIC MAPS ON LOCALLY SYMMETRIC MANIFOLDS

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To Yuri Burago,
a small token of appreciation
for your beautiful works

Abstract. Some fixed point results indicated in the title are established.

§1. Introduction

A central theme in the topology of aspherical manifolds is the attempt to understand all geometric and topological properties in terms of the fundamental group. This theme includes various kinds of rigidity theorems and the Borel conjecture, but it is broader than that[1]. In this paper, I shall consider an aspect of this philosophy in the setting of group actions.

If H is a finite group acting on a closed aspherical manifold M, which we assume has fundamental group π — a group whose center is trivial (e.g. a hyperbolic manifold, not a flat one) — then a theorem of Borel asserts that the group H is effectively represented by the homomorphism $H \to \mathrm{Out}(\pi)$ induced by the action on M. The question that I am most interested in in this paper is: *When does H have a fixed point? Can we tell just from this homomorphism?*

Rather than keep the reader in suspense, we will see below in §5 that the answer is no in general. However, for locally symmetric manifolds (and their cousins), some positive results can be deduced from technology associated with the Borel conjecture as suggested above.

For simplicity this paper can be thought of in the PL category (to avoid any analytic difficulties). Of course, if actions are constructed by, e.g. a compactification of an action on a noncompact manifold, the resulting action will only be C^0.

The obvious necessary condition is that there is a lift

$$\begin{array}{ccc} & & \mathrm{Aut}(\pi) \\ & & \downarrow \\ H & \longrightarrow & \mathrm{Out}(\pi). \end{array}$$

Of course this condition requires nothing at all about M. It can be noncompact, nonaspherical, etc. It only needs to have fundamental group π.

Let us not yet loosen asphericity. If we allow M not to be closed, then we can even allow π to be trivial. In that case, we are asking which H have PL actions on contractible manifolds with no fixed points. In that case, the classical results are that (1) for the disk (or compact contractible manifolds) the necessary and sufficient condition to guarantee

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[1] For a long essay on the Borel heuristic and some of its broad ramifications, see [We1].

that there is a fixed point is that H has "Oliver number $\neq 1$"[2] (see [O][3]) and (2) in the noncompact situation of R^n if and only if H is a p-group for some p (see [EL]).

In case (1), these conditions are a result of the interaction between Smith theory (see [Bo1]), a refinement of Lefshetz fixed point theory for periodic maps, and the obvious combinatorial observation that, when a p-group acts on a finite complex (simplicially), then the fixed set has the same Euler characteristic mod p as the original complex.

Case (2) shows strongly that methods based on finiteness and on Lefshetz theory completely fail in the noncompact setting.

Should we allow **Q**-acylic manifolds, then for case (1) the condition would be that no normal cyclic subgroup of H has prime power index $[O]$ (which is the Lefshetz condition), and there would be no condition for case (2) (as follows from the arguments below).

Intuitively this is the result of Smith theory failing, so that all we have available are the Lefshetz conditions.

In this paper, we shall discuss some situations where we cannot get Lefshetz information directly, but ideas related to the Novikov conjecture (and hence the Borel conjecture) substitute. Unfortunately, the results are weaker than we want, and are perhaps indicative of the possibility of some new phenomena.

§2. Discussion of results

Now let us return to the original question we asked about closed aspherical manifolds, and also consider their extension to $\mathbb{Q}$-aspherical ones.

Proposition 1. *If H is a p-group, then the lifting condition is the necessary and sufficient condition for having a fixed point on M.*

Addendum. The conclusion of the proposition holds even for arbitrary aspherical manifolds (closed, finite volume, or not).

Indeed, if H is a p-group and M is p-aspherical, i.e., the universal cover of M is $\mathbb{Z}/p\mathbb{Z}$ acyclic, then this conclusion holds. We shall see below that this is a simple consequence of P. A. Smith theory (see [Bo2]).

If one leaves the setting of p-aspherical manifolds, then we have the following result, which is an analog of theorems of [BW] and [ChW] and is based on an idea of [We2].

Proposition 2. *For $M = K \setminus G/\pi$ a nonuniform lattice, and $H \to \mathrm{Out}(\pi)$ the trivial map from a p-group, there is an action of H on a manifold N which is $\mathbf{Z}[1/p]\pi$-proper homology equivalent to M if and only if $\mathbb{Q}$-$\mathbf{rank}(\pi) > 2$. (For any finite group, one can do this if one inverts all the primes in the order of the group.)*

Let me dispense with a wrong heuristic argument that, for the case of M itself would suggest that the lifting property suffices for a global fixed point, at least for cyclic groups.

If H acts on M with fundamental group π, there is an associate short exact sequence

$$1 \to \pi \to \Gamma \to H \to 1,$$

where Γ is the orbifold fundamental group of the quotient, which is the same as the group of all lifts of elements of H to the universal cover of M.

If the Aut condition holds, then this sequence splits, and H acts on the universal cover of M. In our situation that would be an acton on $K \setminus G$. If we compactify this by attaching the sphere at infinity, the action extends (because it approximately preserves fundamental domains).

[2]I do not know if this condition is necessary for C^0 actions. In particular, I do not know if $\mathbf{Z}_n/\mathbf{Z}_n$ for an integer n that is divisible by 2 primes has an action on some disk with no fixed points.

[3]The first example of a fixed point free finite group action on a disk was constructed by Floyd and Richardson [FR].

Suppose, for simplicity, that H is cyclic, then the Brouwer fixed point theorem gives a fixed point. At this point, one goes heuristic. If the fixed point is in the interior, then we are done, but one "knows" that a periodic map cannot just have a single fixed point on the boundary of a disk.

That knowledge, however, is wrong. Smith theory guarantees it (the fixed set is a Z_p homology manifold properly embedded in the disk) for p-groups, and it is an elementary argument for *smooth* group actions (for an invariant metric, the exponential of a fixed normal direction gives a geodesic that will be fixed by the action). However, if one considers a fixed point free Z_{pq} action on $\mathbb{R}^n$, that after crossing with a trivial action on $[0,\infty)$, one obtains a fixed point free action on the upper halfspace. The one point compactification of this action is an action on the disk with a single fixed point, just on the boundary.[4]

Theorem 3. *Suppose that f is a periodic map with odd period on M, a rationally aspherical manifold whose fundamental group is a uniform lattice. Then if f defines an element of* $\mathrm{Aut}(\pi)$*, it has a fixed point.*

Addendum. Moreover, the dimension of the fixed set is at least as large as that of the isometric action constructed via Mostow rigidity.

Indeed, Mostow rigidity shows that $\mathrm{Out}(\pi)$ is the isometry group of $K \setminus G/\pi$ (aside from some case related to SL_2 and the mapping class group).

In some cases one can make the same conclusion if f has even period. Under the Aut condition, one obtains from the lift of H an subgroup of K' the isometry group of $K \setminus G$ that fixes a point. This is a linear representation, and our enemy it transpires, is just the eigenvalue-1.

§3. $\underline{\mathrm{E}\Gamma}$ and generalities of Smith theory

Recall, that for a countable group Γ there is a space $\underline{\mathrm{E}\Gamma}$ which is contractible and has a proper discontinuous Γ action, with the property a subgroup of Γ has nonempty fixed set, which is then contractible, if and only if the subgroup is finite. This space has the universal property that it admits an equivariant map from any proper Γ-space, and that this map is unique up to equivariant homotopy. This universal property guarantees that the equivariant homotopy type of $\underline{\mathrm{E}\Gamma}$ is well defined. (See, e.g., [BCH] for this.)

A basic example is a discrete subgroup of a real Lie group Γ, where $\underline{\mathrm{E}\Gamma}$ can be taken to be $K \setminus G$. If Γ is cocompact and G is semisimple and one is dealing with a smooth Γ-Riemannian-manifold, then there is a canonical equivariant map, produced by harmonic map theory [SY]. Here homotopy theory replaces this unique map with a contractible space of choices and dispenses with smoothness assumptions.

The most basic theorem of Smith theory asserts that if H is a p-group acting on a mod p acyclic space is mod p acyclic.

We immediately deduce (by taking mapping cones) that if H is a p-group acting on a mod p-aspherical space X, then the equivariant map from the universal cover of X to $\underline{\mathrm{E}\Gamma}$ is a mod p homology isomorphism on all fixed sets of H. Thus, in the quotient X the fixed set is $\mathbb{F}_p[\pi]$ homology equivalent to the corresponding fixed point set in $\underline{\mathrm{E}\Gamma}/\pi$.

Since we are in a situation where H is acting smoothly on a smooth manifold $\underline{\mathrm{E}\Gamma}/\pi$, the target has a top homology class in its dimension, and therefore the same holds for the domain, i.e., that the fixed set of H must be at least as high-dimensional as that of the isometric action (and if N is closed, it will be of the same dimension).

[4]There is no PL action like this, because the link of the fixed set would be preserved by the action. The link of a fixed point on the boundary is a disk, so such an action would contradict Brouwer's theorem.

We have proved Proposition 1.

§4. Equivariant Novikov ideas and localization

The technical work in this section is all from [RW1],[RW2][5], which we shall apply to prove Theorem 1. We shall need a slight improvement of Theorem 3.3 of that paper.

Suppose M is a Lipschitz G-manifold, then (based on the work of Teleman [T], see [RW2]) it is possible to define an equivaiant signature operator on M, whose symbol defines (by the usual methods of operator theory) an element of $K_m^G(M)$. In [RW1] we investigated the equivariant homotopy invariance properties of this element.

Recall that unequivariantly, the Novikov conjecture is the assertion that the push forward of this element in $K_m(B\pi)$ is rationally a homotopy invariant (and that if $B\pi$ is finite-dimensional, then perhaps even integrally a homotopy invariant). Equivariantly, one can ask the same thing regarding the push forward into $K_m^G(B\pi)$, if $B\pi$ is an "equivariant Eilenberg MacLane space"[6], but, e.g., for the circle acting freely on the sphere, this is not true (due to the classification of manifolds homotopy equivalent to CP^n using surgery).

[RW1] makes two observations about this state of affairs, both of which play an important role for us. The first is that when $B\pi$ has a finite (dimensional) model with G action, then such equivariant homotopy invariance has some plausibility. And, indeed, the work of Kasparov [K] verifies it[7] for isometric actions on locally symmetric manifolds (even of infinite volume).

The other observation, that is very important for actions on aspherical manifolds, is that the invariance property that equivariant signature operator has is somewhat stronger: it is a pseudoequivalence invariant. We say that M and N are *pseudoequivalent* if there is a G-map $M \to N$ which is an unequivariant homotopy equivalence; moreover, since this is *not* an equivalence relation as it stands, we consider the equivalence relation that this generates. This relation is exactly equivalent to $M \times \mathrm{EG}$ being equivariantly homotopy equivalent to $N \times \mathrm{EG}$, which perhaps makes the notion seem more natural.

Finally, although not explicit in [RW1], we only need a *rational pseudoequivalence* between M and N to be able to identify their equivariant higher signatures. This is because the identification goes by way of an equivariant Kaminker–Miller theorem, and we use a rational symmetric signature.[8] (If the equivalence is not degree one, one needs to correct for this, since Poincare duality requires a fundamental class. We shall see below that this does not arise in our examples.) This discussion leads to the following statement.

Theorem 4. *Suppose M is a manifold with fundamental group π, and H is a finite group acting on M, with orbifold fundamental group Γ. Assume that the Baum–Connes assembly map for $\underline{\mathrm{E}\Gamma}$ is injective, then if M and N are rationally pseudoequivalent, then the images of their signature operators in $K^H(\underline{\mathrm{E}\Gamma}/\pi)$ are equal.*

[5]The "well-known" Corollary 4.2 in that paper is nonsense. It is well known to be false. (The proof, using affine maps, does not produce the isometry.) Happily that momentary lapse impacts nothing else in the paper.

[6]The appendix to [RW1] by Peter May explains this notion; essentially it means that all fixed sets of all subgroups of G are themselves disjoint unions of Eilenberg MacLane spaces. In [We1] it is pointed out that one should require in addition that there is an injectivity condition on fundamental groupoids of fixed sets in one another, for the relevant equivariant Novikov conjecture to be plausible.

[7]Even integrally, although this is not necessary for our current paper.

[8]Indeed, the rationals were forced on us, since $\mathbb{Q}[G/H]$ (which arises in the the chain complex of a G-space with H fixed points) is a projective $\mathbb{Q}[G]$ module, but not a projective $\mathbb{Z}[G]$ module.

Notice that if M is rationally aspherical with $\pi_1 M$ a lattice in a semisimple group G (without SL_2 factors[9]), Mostow rigidity provides an equivariant rational pseudoequivalence $M \to K \setminus G/\pi$.

At this point, we can prove Theorem 4. Recall Segal's [Se] localization theorem in equivariant K-theory. $K^H(\underline{\mathrm{E}\Gamma}/\pi)$ is a module over the representation ring $R(H)$. If f is an element of H, then consider the prime ideal $I(f)$ of characters that vanish on the conjugacy class of (f). The localization theorem asserts that the inclusion of the subset of points that are fixed by an element conjugate to (f), $\underline{\mathrm{E}\Gamma}/\pi^{(f)}$ in $\underline{\mathrm{E}\Gamma}/\pi$ induces an isomorphism on localized K-theory $K^H_{I(f)}$.

Combining this with the theorem, we see that if $\varphi\colon M \to K\setminus G/\pi$ is the rational pseudoequivalence, then for a suitable characteristic class AS of the equivariant neighborhood of the fixed set we have

$$\mathrm{ch}\,\Phi * (AS(M^f)) = \mathrm{ch}(AS(\nu)) \cap L(K \setminus G/\pi^f) \in \oplus H_{n-4i}(K \setminus G/\pi^f; \mathbb{Q}),$$

where $n = \dim K \setminus G/\pi^f = \dim K \setminus G^f$. Since the Atiyah–Singer formula (see [AS]) for the contribution of the fundamental class is nontrivial for all odd order rotations (and many even order ones), M^f must have dimension at least n.

Remark. This argument has some similarities with the theorem of Fowler [Fo] about the nonexistence of rationally aspherical manifolds whose fundamental groups are lattices with odd torsion, with the major differences that he uses controlled topological ideas in place of the Atiyah–Singer formula. His argument eliminates free actions from existing, but does not directly imply the nonexistence of fixed point free actions.

It is also worth noting that the difference of odd periods from $p = 2$ that arises here because the "local p contribution" from the representation is also the phenomenon that arises in allowing nonlinear conjugacies between even order groups [CS] but not between odd order ones [HP], [MR].

§5. Construction of some actions

Now we shall construct some actions, which will prove both Proposition 2 and the statement asserted in the Introduction that there are aspherical manifolds (whose fundamental groups are centerless) with H actions, where the action of H on π does not determine whether or not there is an H-fixed point.

Proof of Proposition 2. We follow the idea of [We2] to view $X = EH/H$ as a $\mathbb{Z}[1/|H|][H]$ finitely dominated Poincare complex of dimension 0. It, of course, has nontrivial finiteness obstruction (since its rational Euler characteristic is not equivalent to a free module in $K_0(\mathbb{Q}H)$) and it has a nontrivial total surgery obstruction (in the relevant algebraic version of $BS(X)$), as the H-signature of X is not a multiple of the regular representation. □

In order to construct actions consider the map $M \to M \times_H EH$. If the $\mathbb{Q}$-rank$(\pi) > 2$, then the arguments in [BW], [ChW] show that the finiteness obstruction and the surgery obstruction lie in the trivial group, and then surgery implies that one can produce the relevant manifold. The case of $\mathbb{Q}$-rank $= 0$ is dealt with in the previous section. If the $\mathbb{Q}$-rank(π) is 1 or 2, then there is a boundary map to $L_{\dim \partial}\pi_1(\partial) \times H$, which the Novikov conjecture with coefficients [We1] or the argument from the previous section, can be used

[9] Although, using Nielsen realization [Ke] for the reducible case and Margulis superrigidity for the irreducible, this is ok even if there are SL_2 factors (and one is willing to perhaps deform the embeddings of π in G).

to show the nonvanishing. (A slightly different argument for this nonvanishing will be given in the next section.)

Now we give an example of an aspherical manifold with two H actions of a complicated group H, which are pseudoequivalent, and therefore induce the same action in Out, but one has a fixed point and the other does not. Start with a PL action of H on the disk with no fixed points. (See [O] for the class of groups for which such actions exist.) Call this disk with H action D.

There is another disk with H action, namely the cone, $c\partial D$. It has an H action with a single fixed point (the cone point). Note that there is a PL pseudoequivalence $D \to c\partial D$ which is a PL homeomorphism on the boundary.

At this point, one can do a Davis construction [D] associated with an equivariant triangulation of ∂D. If we extend to the fundamental domain using the Oliver action, we obtain an aspherical H-manifold which is fixed point free, but if we extend using the cone, it has a discrete nonempty set of H-fixed points.

Remark. It seems likely that there are such examples for any non-p-group[10], but to adapt this method to produce it would require a nontrivial extension of Oliver's theory to other aspherical manifolds with boundary in place of the disk.

§6. Degree and homology manifold rigidity

CP^2 has self maps of arbitrary square degree. Moreover, if W is a manifold of the rational homotopy type of CP^2k for any k, the degree of any map $W \to CP^{2k}$ needs to be a square, because the quadratic form on W has to be, by Poincare duality, the 1×1 form (1) and we would have a rational equivalence to (d) from a degree d map. However, if one allows rational homology manifolds, such as the Thom space of the line bundle over CP^1 with $c_1 = k$, one gets degree k.

When one discusses the rational symmetric signature of a space, one needs to specify the duality map. A rational equivalence will have some degree, which twists the Poincare duality. Multiplying by a square, however, gives an isomorphic quadratic form.

On the other hand, this change in the symmetric signature is not that terrible. The quadratic form $(d) \oplus (-1)$ has order at most 4 in the Witt group $L_0(\mathbb{Q})$ if d is positive, and therefore (by tensoring), in general, the change of a fundamental class can only change the symmetric signature by an element of exponent 4.

Now we get the following rigidity statement for rationally essential manifolds.

Corollary. *Suppose M is a rationally essential manifold, i.e., the map $M \to B\pi$ gives $[M]$ as a nontrivial rational cycle in $H_m(B\pi;\mathbb{Q})$, and the Novikov conjecture holds for π, then any orientation preserving*[11] *rational equivalence (i.e. $\mathbb{Q}\pi$ homology equivalence) from a manifold $W \to M$ has degree 1.*

This is because the higher signatures assemble to the symmetric signature, so the part of the $L(M) \cap [M]$ coming from $L_0 \cap [M]$ must be preserved. Note that if M is aspherical, then we obtain the same conclusion even if W is a rational homology manifold.

This relates to the theme of this paper, because it applies to the special case dealt with in Proposition 1 when the action of H is trivial. The orbit map would be (for the closed case) in contradiction to the above statement.

The same reasoning implies the following statement, implicit in [BFMW].

[10] Recall that in §3 we showed that there are no such examples for p-groups.

[11] We define orientation preservingness for maps between oriented manifold in terms of the sign of the determinant on top homology — not of preserving fundamental classes.

Corollary. *If M is a rationally essential manifold, and $f\colon W \to M$ is a degree one rational equivalence from an ANR homology manifold, then W is resolvable* (*i.e. a cell-like image of a topological manifold*).

In contrast, if M is simply connected and of dimension > 5, [BFMW] shows that M is *always* homotopy equivalent to a nonresolvable homology manifold.

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