COORDINATE-FREE COVERAGE IN SENSOR NETWORKS WITH CONTROLLED BOUNDARIES VIA HOMOLOGY

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Abstract. We introduce tools from computational homology to verify coverage in a sensor network. Our methods are unique in that, while they are coordinate-free and assume no localization or orientation capabilities for the nodes, there are also no probabilistic assumptions. We demonstrate the robustness of the techniques by adapting them to a variety of settings, including static planar coverage, 3-d barrier coverage, and time-dependent sweeping coverage. We also give results on hole repair, error tolerance, and variable radii.

`draft; version 8.0; simulation-free`

1. Introduction.

The *coverage problem* appears in a variety of settings: in sensor networks, broadcasting, beacon navigation, and security applications. Given a collection of nodes $\mathcal{X}$ in a bounded domain $\mathcal{D}$, assume that each node can sense, broadcast to, or otherwise cover a region of fixed *coverage radius* r_c about the node. The most basic form of coverage problem is the simple query: given the nodes, does the collection of radius r_c balls at $\mathcal{X}$ cover the domain $\mathcal{D}$?

We provide a sufficiency criterion for coverage. We do not answer the problem of how the nodes should be placed in order to maximize coverage — nodes are assumed to be allocated a priori. The coverage criterion we introduce is computable, as we demonstrate. It is also centralized. We do not here demonstrate how to reduce the homological criterion to a distributed computation.

1.1. Assumptions.

We assume a complete absence of localization capabilities. **Nodes can determine neither distance nor direction.** Only connectivity data between nodes is used. The only strong assumption we make is on the *fence nodes* set up along the boundary of the domain. This strong degree of control along the boundary is not strictly required (see, e.g., [7]), but it simplifies the statements and proofs of theorems dramatically.

- **A1:** Nodes $\mathcal{X}$ broadcast their unique ID numbers. Each node can detect the identity of any node within *broadcast* radius r_b.
- **A2:** Nodes have radially symmetric covering domains of *cover* radius $r_c \geq r_b/\sqrt{3}$.
- **A3:** Nodes $\mathcal{X}$ lie in a compact connected domain $\mathcal{D} \subset \mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is connected and piecewise-linear with vertices marked *fence nodes* $\mathcal{X}_f$.
- **A4:** Each fence node $v \in \mathcal{X}_f$ knows the identities of its neighbors on $\partial\mathcal{D}$ and these neighbors both lie within distance r_b of v.

To summarize, the sensor data for each node consists of a list of node ID numbers within signal detection range, as well as a binary flag denoting whether or not it is a marked fence node.

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1.2. Results. We claim that, surprisingly, such coarse coordinate-free data is sufficient to rigorously verify coverage in many instances. One determines the *communication graph* whose vertices are the nodes of the network and whose edges represent signal detection connectivity (at radius r_b). From this graph we build the *Rips complex* $\mathcal{R}$: the largest simplicial complex with the corresponding graph as its 1-d skeleton. By assumption **A4** the boundary $\partial\mathcal{D}$ can be represented as a 1-dimensional *fence complex* $\mathcal{F} \subset \mathcal{R}$ which is canonically identified with $\partial\mathcal{D}$.

Our results are all based on a certain algebraic-topological invariant of these simplicial complexes — homology — reviewed in Appendix A. The following is the principal criterion for coverage we derive in this paper:

Main Theorem: *The union of the covering discs contains $\mathcal{D}$ if there is a nontrivial element of the relative homology group $H_2(\mathcal{R}, \mathcal{F})$ whose boundary is nonvanishing.*

See Theorem 3.3 for details.

In §4-§10 we provide several extensions of this result. These include the following:

1. Criteria for performing 'hole repair' in systems for which the coverage criterion fails;
2. Criteria for coverage in domains with multiple boundary components;
3. A homological approach to identifying redundant nodes in a cover;
4. Coverage criteria for systems with varying communication and coverage radii, including systems with communication errors;
5. Barrier coverage for 3-d systems in a tunnel-like domain.
6. Pursuit-evasion coverage criteria for time-dependent systems.

1.3. Related work. There is a large literature on the subject of static or 'blanket' coverage; see, e.g., [10, 3, 21] and references therein. In addition, there are variants on these problems involving 'barrier' coverage to separate regions. Dynamic or 'sweeping' coverage [4] is a common and challenging task with applications ranging from security to housekeeping.

There are two primary approaches to static coverage problems in the literature. The first uses computational geometry tools applied to exact node coordinates. This typically involves computational geometry [15] and Delaunay triangulations of the domain [21, 19, 25]. Such approaches are very rigid with regards to inputs: one must know exact node coordinates and one must know the geometry of the domain precisely to determine the Delaunay complex.

To alleviate the former requirement, many authors have turned to probabilistic tools. For example, in [17], the author assumes a randomly and uniformly distributed collection of nodes in a domain with a fixed geometry and proves expected area coverage. Other approaches [20, 24, 18, 14] give probabilisticvand/or percolation results about coverage and network integrity for randomly distributed nodes. The drawback of these methods is the need for a uniform distribution of nodes.

The mathematical tools we use — homology theory for simplicial complexes — date roughly from the 1930s. The use of homology as an effective tool in scientific computation is more recent: see, e.g., the textbook of [16] and its references. Homology has recently been used is several applied contexts, from point cloud shape representation and high-dimensional data analysis [26, 6], vision [1], applied differential equations [16, 22], and hybrid controls [2]. The reader who is not familiar with homology theory can find a brief summary tailored towards the applications of this paper in the Appendix.

2. The Rips complex. Given a collection of nodes $\mathcal{X}$ in a domain, we wish to determine the global properties of $\mathcal{U}$, the union of coverage domains centered at these nodes. However, we are constrained to use only communication connectivity data between nodes. Instead of restricting attention to the graph of pairwise-connectivity data, we complete it to a higher-dimensional complex. This type of simplicial complex was introduced by Vietoris in the early history of homology theory [23], and has more recently been reinterpreted by Rips [11] and used extensively in geometric group theory.

DEFINITION 2.1. *Given a set of points $\mathcal{X} = \{x_\alpha\}$ in a metric space and a fixed $\epsilon > 0$, the* **Rips complex** *of $\mathcal{X}$, $\mathcal{R}_\epsilon(\mathcal{X})$, is the abstract simplicial complex whose k-simplices correspond to unordered $(k+1)$-tuples of points in $\mathcal{X}$ which are pairwise within distance ϵ of each other.*

Our goal is to compare the topology of the Rips complex $\mathcal{R} = \mathcal{R}_{r_b}(\mathcal{X})$ to the union of covering discs $\mathcal{U} = \mathcal{U}_{r_c}(\mathcal{X})$. The cover $\mathcal{U}$ is necessarily a subset of $\mathbb{R}^2$; the Rips complex, in contrast, may have any dimension, depending on clustering of nodes. This paper asserts that homological content in $\mathcal{R}$ is enough to conclude certain properties of $\mathcal{U}$.

The following lemma demonstrates that the choice of bound for r_c in **A2** is the appropriate one.

LEMMA 2.2. *The convex hull of any collection of nodes in $\mathcal{D}$ which form a simplex of $\mathcal{R}$ lies within $\mathcal{U}$.*

Proof: Any collection of circular disks which meet at a common point x necessarily covers the convex hull of x and the centers of the discs. So, it suffices to show that the balls of radius r_c intersect. It also suffices to prove this for a 2-simplex of $\mathcal{R}$ thanks to Helly's theorem [9], which implies that a collection of $k \geq 4$ convex sets in $\mathbb{R}^2$ has a nonempty common intersection provided only that the same is true for each subset of size 3.

Therefore, consider a triple of points $\{x_i\}_1^3$ which span a triangle with side lengths at most r_b. We must show that the three discs of radius r_c centered on $\{x_i\}_1^3$ meet at a common point. If the triangle is obtuse (or right-angled), then the midpoint of the longest side is common to all three discs; hence $r_c \geq r_b/2$ suffices. If the triangle is acute then the largest angle, say θ_1 at vertex x_1, satisfies $\pi/3 \leq \theta_1 \leq \pi/2$ and so $\sin(\theta_1) \geq \sqrt{3}/2$. We can compute the circumradius R of the triangle as $R = \|x_2 - x_3\|/2\sin\theta_1$ and hence we deduce $R \leq r_b/\sqrt{3} \leq r_c$. Thus, in this case, the three discs meet at the circumcenter. ◇

REMARK 2.3. The ratio $r_c \geq r_b/\sqrt{3}$ is optimal: consider an equilateral triangle of side length r_b.

Unfortunately, the radius-r_b Rips complex of a set of nodes in $\mathbb{R}^2$ does not always capture the topology of the union of radius-r_c balls centered on these nodes. Fig. 2.1 gives examples for which the Rips complex fails to capture the topology of the cover.

3. A homological criterion for coverage. We use the homology of $\mathcal{R}$ relative to $\mathcal{F}$ to obtain a coverage criterion. Geometrically, the fence subcomplex $\mathcal{F}$ is equal to $\partial\mathcal{D}$. The following SIMPLE algebraic lemmas complete the setup.

LEMMA 3.1. *Any nonzero cycle $\zeta \in Z_1(\mathcal{F})$ defines a nonzero element of $H_1(\partial\mathcal{D})$.*

Proof: By the definition of homology, $H_1(\mathcal{F}) = Z_1(\mathcal{F})/B_1(\mathcal{F})$. However, $B_1(\mathcal{F}) = \partial(C_2(\mathcal{F})) = 0$, since $C_2(\mathcal{F}) = 0$ in the simplicial category; hence $Z_1(\mathcal{F}) = H_1(\mathcal{F}) = H_1(\partial\mathcal{D})$. ◇

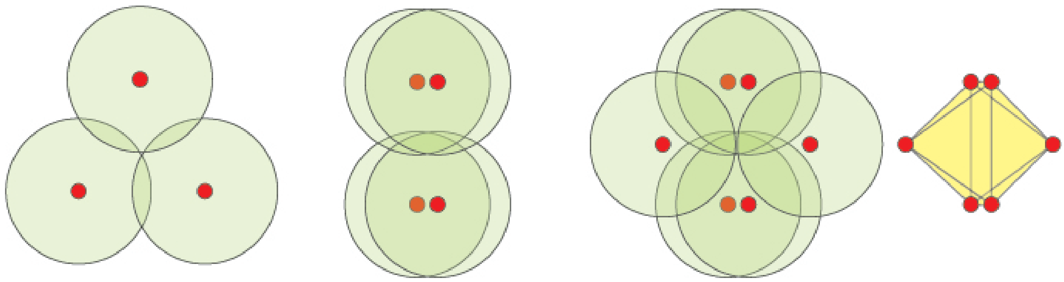

FIG. 2.1. *[left] The Rips complex has the property that all 2-simplices determine triangles in the domain which lie within the radius r_c cover. However, the Rips complex does not capture the topology of the cover. A contractible union of r_c balls can have Rips complex with nontrivial homology in dimension one [center, in which $\mathcal{R}$ is a quadrilateral], two [right, in which $\mathcal{R}$ is the boundary of a solid octahedron], or higher.*

LEMMA 3.2. *A cycle $\zeta \in Z_1(\mathcal{F})$ is nonzero if and only if it has a nonzero coefficient at every fence edge.*

Proof: If ζ is a cycle, then the coefficient of ζ at any pair of adjacent edges is the same up to a sign, because $\partial\zeta$ has coefficient zero at their common vertex. Since the boundary is connected, ζ has the same coefficient at every edge of $\mathcal{F}$ up to a sign. The lemma follows immediately. $\diamond$

The following theorem is our principal coverage criterion:

THEOREM 3.3. *For a set of nodes $\mathcal{X}$ in a domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying assumptions* **A1**-**A4**, *the sensor cover $\mathcal{U}_c$ contains $\mathcal{D}$ if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

We note (by Lemma 3.2) that the condition $\partial\alpha \neq 0$ can easily be evaluated by picking a fence edge and testing whether the coefficient of $\partial\alpha$ on that edge is nonzero.

Proof: We consider the simplicial realization map $\sigma : \mathcal{R} \to \mathbb{R}^2$ which sends vertices of the abstract complex $\mathcal{R}$ to the points $\mathcal{X} \subset \mathcal{D}$ and which sends a k-simplex of $\mathcal{R}$ to the (potentially singular) k-simplex given by the convex hull of the vertices implicated. Via **A4**, σ takes the pair $(\mathcal{R}, \mathcal{F})$ to $(\mathbb{R}^2, \partial\mathcal{D})$; we therefore construct the following diagram from the long exact sequences:

$$\begin{array}{ccc} H_2(\mathcal{R}, \mathcal{F}) & \xrightarrow{\delta_*} & H_1(\mathcal{F}) \\ \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \sigma_*} \\ H_2(\mathbb{R}^2, \partial\mathcal{D}) & \xrightarrow{\delta_*} & H_1(\partial\mathcal{D}) \end{array} \quad . \tag{3.1}$$

Here, δ_* acts on a class $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ by taking the boundary: $\delta_*[\alpha] = [\partial\alpha] \in H_1(\mathcal{F})$. It follows from the naturality of the long exact sequence that the diagram of Eqn. (3.1) is commutative: $\delta_*\sigma_* = \sigma_*\delta_*$. The homology class $\sigma_*\delta_*[\alpha]$ is the winding number of $\partial\alpha$ about $\partial\mathcal{D}$.

By assumption, $\partial\alpha \neq 0$; hence, by way of Lemma 3.1, we observe that $\sigma_*\delta_*[\alpha] = \sigma_*[\partial\alpha] \neq 0$. By commutativity of Eqn. (3.1), $\delta_*\sigma_*[\alpha] \neq 0$, and thus $\sigma_*[\alpha] \neq 0$.

Assume that $\mathcal{U}$ does not contain $\mathcal{D}$ and choose $p \in \mathcal{D} - \mathcal{U}$. Since, by Lemma 2.2, every point in $\sigma(\mathcal{R})$ lies within $\mathcal{U}$, we have that $\sigma : (\mathcal{R}, \mathcal{F}) \to (\mathbb{R}^2, \partial\mathcal{D})$ factors through the pair $(\mathbb{R}^2 - p, \partial\mathcal{D})$. However, $H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) = 0$, as the following simple computation shows. Let $A = \mathbb{R}^2 - p$ and B be a small ball about p, so that $A \cap B$ is an open annulus homotopic to S^1. Let $A' = \partial\mathcal{D}$ and $B' = \emptyset$. Using Eqn. (A.7), we have

$$\cdots \longrightarrow H_2(S^1) \xrightarrow{\phi_*} H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) \oplus 0 \xrightarrow{\psi_*} H_2(\mathbb{R}^2, \partial\mathcal{D}) \xrightarrow{\partial_*} H_1(S^1) \xrightarrow{\phi_*} \cdots \tag{3.2}$$

Since $(\mathbb{R}^2, \partial\mathcal{D})$ deformation retracts to the pair $(\mathcal{D}, \partial\mathcal{D})$ fixing $\mathcal{D}$, we have that

$$H_2(\mathbb{R}^2, \partial\mathcal{D}) \cong H_2(\mathcal{D}, \partial\mathcal{D}) \cong H_2(\mathcal{D}/\partial\mathcal{D}) \cong H_2(S^2) \cong \mathbb{R}. \tag{3.3}$$

Since $p \in \mathcal{D}$, the homomorphism ∂_* takes the generator of $H_2(\mathbb{R}^2, \partial\mathcal{D})$ to that of $H_1(S^1)$. Eqn. (3.2) therefore simplifies to

$$\cdots \longrightarrow 0 \longrightarrow H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) \longrightarrow \mathbb{R} \xrightarrow{\cong} \mathbb{R} \longrightarrow \cdots \tag{3.4}$$

By exactness, $H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) = 0$ and thus $\sigma_*[\alpha] = 0$: contradiction. $\diamond$

Remark 3.4. This is not a sharp criterion. It is clearly possible to have the criterion always fail for injudicious choice of r_c. For example, if r_c is much larger than the bound in Assumption (A3), then there will be many instances of coverage without a homological forcing. This being said, we note that even if one chooses the minimal acceptable bounds from Assumption (A3), it is still possible to arrange the points to cover $\mathcal{D} - \mathcal{C}$ without the homological criterion detecting this, as illustrated in Fig. 3.1.

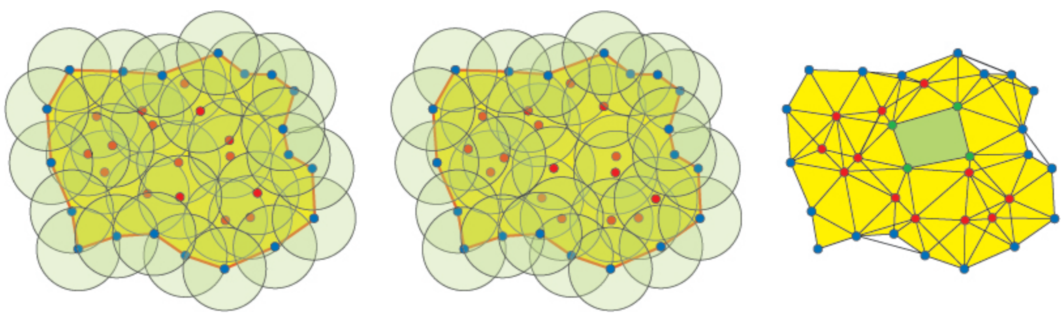

Fig. 3.1. *Examples of two covers. The homological criterion holds for one [left] but not for the other [center], because of a 1-cycle in $\mathcal{R}$ [right]. Note the fragility of the cover [center] within the 1-cycle.*

4. Generators for redundant covers. Theorem 3.3 guarantees that the covering discs in fact cover the desired area. For reasons of power conservation, one would like to know which nodes could be “turned off” without impinging upon the coverage integrity. This is an important problem with a large literature, e.g., [18, 14]. A practical approach to this problem is implicit in our homology criterion.

Corollary 4.1. *If a homology class in $H_2(\mathcal{R}, \mathcal{F})$ satisfies the criterion of Theorem 3.3, then the restriction of $\mathcal{U}$ to those nodes which make up the representative α suffice to cover $\mathcal{D}$, for any choice of α in the homology class.*

Proof. Let $\mathcal{U}^\alpha$ denote the restriction of $\mathcal{U}$ to the nodes implicated by the representative α. Assume that $\mathcal{U}^\alpha$ does not contain $\mathcal{D}$ and choose $p \in \mathcal{D} - \mathcal{U}^\alpha$. Lemma 2.2 implies that

$\sigma(\mathcal{R}) \subset \mathcal{U}^{\alpha}$. Thus, $\sigma : (\mathcal{R}, \mathcal{F}) \to (\mathbb{R}^2, \partial\mathcal{D})$ factors through the pair $(\mathbb{R}^2 - p, \partial\mathcal{D})$, which has vanishing homology in dimension two. $\diamond$

The independence of the choice of representative in the homology class is extremely important. If one chooses a "minimal" generator α — in the sense that α minimizes the number of 0-simplices within $[\alpha]$ — then Corollary 4.1 yields a small subset of nodes which is guaranteed to cover the domain. Existing software packages for computing homology classes can "shrink" generators (though without rigor in terms of being truly minimal); hence, this is an implementable strategy.

5. Hole repair. Of course, since the result of Theorem 3.3 is merely a criterion, one wishes to implement a strategy for guaranteeing coverage when the criterion fails. We present an elementary means for doing so via homology, the idea being to compute 'minimal' generators in $H_1(\mathcal{R})$ so as detect holes. We consider a sensor network in which all nodes are initially in a 'power saving' mode of low coverage radius r_c with the ability to increase the coverage radii of certain nodes. The following result is most useful in this setting, where the homological criterion fails, but just barely.

THEOREM 5.1. *Consider a set of nodes $\mathcal{X}$ satisfying assumptions* **A1**-**A4**. *Let $\Gamma = \{\gamma_i\}_1^K$ be a basis for $H_1(\mathcal{R})$ with $\|\gamma_i\| \leq N_i$ for each i, where $\|\cdot\|$ denotes length of the generator in terms of the number of nodes implicated. Let $\mathcal{U}'$ denote the set obtained from the collection $\mathcal{U}$ by enlarging all balls based at nodes in γ_i to balls of radius*

$$r'_c(i) = \frac{r_b}{2} \csc \frac{\pi}{N_i}. \tag{5.1}$$

Then $\mathcal{D} \subset \mathcal{U}'$.

Proof: The quantity $r'_c(i)$ represents the minimal radius needed to cover a regular N_i-gon. We claim that this is the limiting case.

Consider the image $\mathcal{L}_1 = \sigma(\gamma_i)$ of the loop γ_i in $\mathcal{D}$. This is a (not necessarily embedded) loop in $\mathcal{D}$. A point $x \in \mathcal{D}$ is enclosed by $\mathcal{L}_i$ if $[\mathcal{L}_i]$ is nonzero in $H_1(\mathbb{R}^2 - x) \cong \mathbb{Z}$ (this class is the *winding number* of the loop about x). We demonstrate that covering each node of γ_i with a ball of radius $r'_c(i)$ covers any such x. For such an x it follows that one or more of the N_i edges of $\mathcal{L}$ subtends an angle at x of at least $2\pi/N_i$; for otherwise there would exist rays originating at x which miss $\sigma(\gamma_i)$ entirely, making $\mathcal{L}_i$ contractible in $\mathbb{R}^2 - x$ and the winding number zero. Let ab be such an edge. Taking cosines this inequality becomes

$$\cos(\frac{2\pi}{N_i}) \geq \frac{|xa|^2 + |xb|^2 - |ab|^2}{2|xa||xb|} \geq 1 - \frac{r_b^2}{2|xa||xb|} \tag{5.2}$$

where we use the AM-GM inequality and the fact that $|ab| \leq r_b$ for the latter inequality. Since $\cos(2\pi/N_i) = 1 - 2\sin^2(\pi/N_i)$ we can rearrange to obtain $|xa||xb| \leq r'_c(i)^2$. Thus x must lie within distance $r'_c(i)$ of the nearer of the two nodes a, b, as required.

We now create a modified complex $\mathcal{R}'$ obtained from $\mathcal{R}$ in the following manner. For each i, sew in an abstract 2-d disc along the loop γ_i. (If one wishes to remain in the simplicial category, one can triangulate the disc.) Next, extend the map σ to a continuous map $\sigma' : \mathcal{R}' \to \mathbb{R}^2$.

The long exact sequence yields a commutative diagram as in Eqn. (3.1):

$$\begin{tikzcd}
H_2(\mathcal{R}',\mathcal{F}) \arrow[r, "\delta_*"] \arrow[d, "\sigma'_*"] & H_1(\mathcal{F}) \arrow[r, "i_*"] \arrow[d, "\sigma'_*"] & H_1(\mathcal{R}') \arrow[d, "\sigma'_*"] \\
H_2(\mathbb{R}^2,\partial\mathcal{D}) \arrow[r, "\delta_*"] & H_1(\partial\mathcal{D}) \arrow[r, "i_*"] & H_1(\mathbb{R}^2)
\end{tikzcd} \tag{5.3}$$

Because we have filled in all the generators of $H_1(\mathcal{R})$, we have that $H_1(\mathcal{R}') = 0$ and $\delta_* : H_2(\mathcal{R}',\mathcal{F}) \to H_1(\mathcal{F})$ is onto. Exactness implies that there exists a generator $[\alpha]$ of $H_2(\mathcal{R}')$ with $\partial\alpha = \mathcal{F}$.

Assume by way of contradiction that there exists a point $p \in \mathcal{D}-\mathcal{U}'$. If $[\mathcal{L}_i] \neq 0 \in H_1(\mathbb{R}^2-p)$ for any i, then $p \in \mathcal{U}'$ by the argument above. Therefore, assume that these homology classes vanish for all i. Since the set $\{\gamma_i\}$ forms a basis for $H_1(\mathcal{R})$, there exists a 2-chain ζ in $C_2(\mathcal{R})$ such that $\partial\zeta = \mathcal{F} - \sum_i c_i\gamma_i$ for some constants c_i. Applying σ to these 1-chains yields the equation $\partial\sigma(\zeta) = \partial\mathcal{D} - \sum_i c_i\mathcal{L}_i$. This descends to an equation in $H_1(\mathbb{R}^2 - p)$, since p is assumed to be not in $\mathcal{U}'$ and $\sigma(\zeta) \subset \mathcal{U} \subset \mathcal{U}'$ by Lemma 2.2. We know that $[\partial\mathcal{D}] \neq 0$ in $H_1(\mathbb{R}^2 - p)$ since $p \in \mathcal{D}$. By assumption that all the winding numbers of $\mathcal{L}_i$ about p vanish, we have that $[\partial\sigma(\zeta)] \neq 0 \in H_1(\mathbb{R}^2 - p)$. However, $\zeta \in C_2(\mathcal{R})$ and is an algebraic sum of 2-simplices in $\mathcal{R}$. At least one such 2-simplex ς of ζ must therefore satisfy $\sigma(\partial\varsigma) \neq 0 \in H_1(\mathbb{R}^2 - p)$, implying that $p \in \sigma(\zeta) \subset \mathcal{U} \subset \mathcal{U}'$. Contradiction. $\diamond$

It follows from this argument that, if one has the hardware constraint of a fixed coverage radius r_c which is larger that the bound $r_b/\sqrt{3}$, then one can get a better coverage criterion, as follows. Let N be the largest integer for which $r_c \leq 2r_b/\csc(\pi/N)$. Then, build a version of the Rips complex for the network which has all loops in the network of length less than or equal to N filled in by abstract 2-cells. Coverage is guaranteed if the resulting cell complex has a cycle in H_2 with nonvanishing boundary.

6. Domains with arbitrary planar topology. Assumption **A3** restricts the topology of the domain $\mathcal{D}$ in two features: connectivity of $\mathcal{D}$ and connectivity of $\partial\mathcal{D}$. It is not difficult to eliminate both of these requirements. If $\mathcal{D}$ is disconnected, then each connected component of $\mathcal{D}$ can be treated separately. If $\partial\mathcal{D}$ is disconnected, we can succeed if we have some extra information about the connected components of $\partial\mathcal{D}$.

THEOREM 6.1. *Consider a set of nodes $\mathcal{X}$ satisfying assumptions* **A1**-**A4**, *with* **A3** *modified as follows:*

- **A3′** Nodes $\mathcal{X}$ lie in a compact connected domain $\mathcal{D} \subset \mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is piecewise-linear with vertices marked *fence nodes* $\mathcal{X}_f$. There is a partition of $\mathcal{X}_f$ into $\mathcal{X}_f^+ \sqcup \mathcal{X}_f^-$ representing those on the outer and inner boundary components respectively.

The sensor cover $\mathcal{U}_c$ contains $\mathcal{D}$ if there exists $[\alpha] \in H_2(\mathcal{R},\mathcal{F})$ such that $\partial\alpha$ is nonzero on the outermost boundary component.

To evaluate the condition on α, we can pick any edge on the outermost boundary component and check whether $\partial\alpha$ has a nonzero coefficient at that edge (compare Lemma 3.2).

Proof. This is a modification of the proof of Theorem 3.3. To start with, we can write the fence subcomplex as a disjoint union $\mathcal{F} = \mathcal{F}^+ \sqcup \mathcal{F}^-$ where $\mathcal{F}^+$ is the outermost fence component, and $\mathcal{F}^-$ is the union of the inner fence components. Similarly one can write $\partial\mathcal{D} = \partial^+\mathcal{D} \sqcup \partial^-\mathcal{D}$ for the domain boundary. The condition on α is then equivalent to the assertion that $\delta_*[\alpha] \neq 0$ where $\delta_* : H_2(\mathcal{R}, \mathcal{F}) \to H_1(\mathcal{F}, \mathcal{F}^-) \cong H_1(\mathcal{F}^+)$ is the boundary map in the long exact sequence for the triple $(\mathcal{R}, \mathcal{F}, \mathcal{F}^-)$.

This time we have a simplicial realization map $\sigma : (\mathcal{R}, \mathcal{F}, \mathcal{F}^-) \to (\mathbb{R}^2, \partial\mathcal{D}, \partial^-\mathcal{D})$, which gives us the following commutative diagram:

$$\begin{tikzcd}
H_2(\mathcal{R},\mathcal{F}) \arrow[r, "\delta_*"] \arrow[d, "\sigma_*"] & H_1(\mathcal{F},\mathcal{F}^-) \arrow[r, equal] \arrow[d, "\sigma_*"] & H_1(\mathcal{F}^+) \arrow[d, "\sigma_*"] \\
H_2(\mathbb{R}^2,\partial\mathcal{D}) \arrow[r, "\delta_*"] & H_1(\partial\mathcal{D},\partial^-\mathcal{D}) \arrow[r, equal] & H_1(\partial^+\mathcal{D})
\end{tikzcd} \tag{6.1}$$

The equalities on the right of the diagram come from excision. Since $\sigma_* : H_1(\mathcal{F}^+) \to H_1\partial^+\mathcal{D}$ is an isomorphism, the same is true of $\sigma_* : H_1(\mathcal{F}, \mathcal{F}^-) \to H_1(\partial\mathcal{D}, \partial^-\mathcal{D})$.

Suppose there exists $[\alpha]$ satisfying the criterion in the theorem, so $\delta_*[\alpha] \neq 0$. By commutativity of Eqn. (3.1) and since the middle map σ_* is an isomorphism, it follows that $\delta_*\sigma_*[\alpha] = \sigma_*\delta_*[\alpha] \neq 0$.

Now assume, for a contradiction, that there is some point $p \in \mathcal{D} - \mathcal{U}$. Since it lies in $\mathcal{D}$ the point p is encircled by the outermost boundary component $\partial^+\mathcal{D}$ but not by any of the other boundary components. Since $p \notin \mathcal{U}$ the composite $\delta_*\sigma_*$ factors as

$$H_2(\mathcal{R},\mathcal{F}) \xrightarrow{\sigma_*} H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) \xrightarrow{i_*} H_2(\mathbb{R}^2, \partial\mathcal{D}) \xrightarrow{\delta_*} H_1(\partial\mathcal{D}, \partial^-\mathcal{D}) \tag{6.2}$$

We claim that $\delta_* i_* : H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) \to H_1(\partial\mathcal{D}, \partial^-\mathcal{D})$ is the zero map, which gives the required contradiction since it implies that $\delta_*\sigma_*[\alpha] = 0$.

In fact $\delta'_* = \delta_* i_*$ is the boundary map in the long exact sequence for the triple $(\mathbb{R}^2 - p, \partial\mathcal{D}, \partial^-\mathcal{D})$. Consider the following excerpt from that sequence:

$$\cdots \longrightarrow H_2(\mathbb{R}^2 - p, \partial\mathcal{D}) \xrightarrow{\delta'_*} H_1(\partial\mathcal{D}, \partial^-\mathcal{D}) \xrightarrow{j_*} H_1(\mathbb{R}^2 - p, \partial^-\mathcal{D}) \longrightarrow \cdots \tag{6.3}$$

By exactness, we can prove that $\delta'_* = 0$ by establishing instead that j_* is one-to-one. This can be read off from the following commutative diagram with exact rows, coming from the inclusion map of pairs $j : (\partial\mathcal{D}, \partial^-\mathcal{D}) \to (\mathbb{R}^2 - p, \partial^-\mathcal{D})$.

$$\begin{tikzcd}
\cdots & H_1(\partial\mathcal{D}) \arrow[r, "i_*"] \arrow[d, "i_*"] & H_1(\partial\mathcal{D},\partial^-\mathcal{D}) \arrow[d, "j_*"] & \cdots \\
H_1(\partial^-\mathcal{D}) \arrow[r, "0"] \arrow[ur] & H_1(\mathbb{R}^2 - p) \arrow[r, "k_*"] & H_1(\mathbb{R}^2 - p, \partial^-\mathcal{D}) & \cdots
\end{tikzcd} \tag{6.4}$$

The geometric content here is that the map $H_1(\partial^-\mathcal{D}) \to H_1(\mathbb{R}^2 - p)$ is zero, since the interior boundary cycles do not enclose p, whereas the map $H_1(\partial\mathcal{D}) \to H_1(\mathbb{R}^2 - p)$ is onto since the outer boundary cycle does encircle p. It follows that the two maps labeled i_* have the same kernel and are both onto. By exactness the map labeled k_* is one-to-one and therefore the same is true of j_*. This is what was required. $\diamond$

It is not enough to have $\partial\alpha \neq 0$ as before. Consider the situation of Fig. 6.1, in which a small interior boundary component is a loop of four edges. Then, one can generate a relative 2-cycle consisting of the four boundary nodes along with a single interior node which is properly situated. This, of course, does not cover the domain.

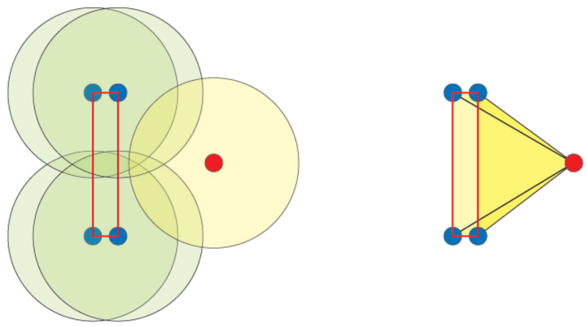

FIG. 6.1. *An example of a small internal boundary component [left] giving rise to a fake relative 2-cycle [right] in the Rips complex.*

We leave it to the reader to modify the statements of theorems in the following sections to accommodate the case of domains which for which connectivity or simple connectivity fail.

7. Opaque boundaries and communication errors. We have not carefully specified the mechanism by which nodes communicate presence over a distance. From Assumption **A1** it follows that communication signals are picked up purely as a function of distance, permeating the boundary of the domain if necessary. In certain physical situations, these communication signals may not be capable of boundary penetration (e.g., if they are visually-detected beacons). One might wish to modify the assumptions with the following *opaque* boundary condition: *Each node can detect the identity of any node connected by a straight line in $\mathcal{D}$ of length at most r_b.* One changes the Rips complex to include only those edges which communicate through unobstructed signals.

This is a particular example of the more general phenomenon of having communication errors of the form where two nodes within communication distance fail to establish a link. For the most general case, consider a system satisfying **A1-A4** with Rips complex $\mathcal{R}$. Define a *Rips complex with omissions*, $\mathcal{ER}$, to be any subcomplex of $\mathcal{R}$ containing $\mathcal{F}$ (we assume perfect control of the fence nodes). This $\mathcal{ER}$ may result as a random error in establishing communication links or, as above, as a systematic failure to establish links near certain types of boundaries.

THEOREM 7.1. *Consider a set of nodes $\mathcal{X}$ in a domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying assumptions* **A1**-**A4** *with $\mathcal{ER}$ a Rips complex with omissions. The sensor cover $\mathcal{U}_c$ contains $\mathcal{D}$ if there exists $[\alpha] \in H_2(\mathcal{ER}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

Proof: Since $\mathcal{ER} \subset \mathcal{R}$, we have

$$\begin{tikzcd} H_2(\mathcal{ER},\mathcal{F}) \arrow[r, "\delta_*"] \arrow[d, "\sigma_*"] & H_1(\mathcal{F}) \arrow[d, "\sigma_*"] \\ H_2(\mathbb{R}^2,\partial\mathcal{D}) \arrow[r, "\delta_*"] & H_1(\partial\mathcal{D}) \end{tikzcd} \quad . \tag{7.1}$$

The remainder of the proof follows exactly as in Theorem 3.3. $\diamond$

This result implies that the homological coverage criterion relies on the coarse metric data of Assumption **A1** only in the positive sense. The criterion does not use the fact that a failure to communicate implies a lower bound on the distance between nodes.

8. Variable Radii.

Assumptions **A1**-**A2** on the radial symmetry of sensors are physically unrealistic: a more accurate model would incorporate asymmetry and/or variable radii, to accommodate errors or fluctuations in signals. It is possible to apply the homological criterion to systems with asymmetric broadcast domains by using the Rips complex with omissions of §7. One chooses r_b to be an upper bound for the broadcast signal distance and $r_c \geq r_b/\sqrt{3}$. The communication network then establishes links between certain nodes, but not purely as a function of distance. While this method is applicable, there is a wastefulness in the bound on r_c in terms of the maximal broadcast distance.

We therefore consider systems whose radii r_c and r_b vary from node to node, as a next step toward dealing with asymmetry in sensor networks. Consider the case where a system of nodes $\mathcal{X} = \{x^i\}$ satisfies a modified set of assumptions:

- **V1:** Nodes $\mathcal{X} = \{x^i\}$ broadcast their unique ID numbers. The identity of each node can be detected any node within its *broadcast* radius r_b^i.
- **V2:** Nodes have radially symmetric covering domains of *cover* radius $r_c^i \geq r_b^i/\sqrt{3}$.
- **V3:** Nodes $\mathcal{X}$ lie in a compact connected domain $\mathcal{D} \subset \mathbb{R}^2$ whose boundary $\partial\mathcal{D}$ is connected and piecewise-linear with vertices marked *fence nodes* $\mathcal{X}_f$.
- **V4:** Each fence node $v \in \mathcal{X}_f$ knows the identities of its neighbors on $\partial\mathcal{D}$ and these neighbors both lie within distance r_b^i of v.

We modify the construction of the Rips complex as follows. For any pair of nodes x^i and x^j, there is an edge in $\mathcal{R}$ if and only if the distance between x^i and x^j in $\mathcal{D}$ is less than or equal to the minimum of r_b^i and r_b^j. The full complex $\mathcal{R}$ is then the maximal simplicial complex for the edge set as defined. The fence subcomplex $\mathcal{F}$ is defined in the same way as before, with vertex set $\mathcal{X}_f$ and an edge between each pair of adjacent nodes along the fence. We define the variable-radius cover $\mathcal{U}_c$ in this context to be the union of closed discs of radii r_c^i centered at node x^i.

Theorem 8.1. *For a set of nodes $\mathcal{X}$ in a domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying the variable-radius assumptions* **V1**-**V4**, *the variable-radius cover $\mathcal{U}_c$ contains $\mathcal{D}$ if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ such that $\partial\alpha \neq 0$.*

Proof. The proof of Theorem 3.3, being topological, is largely independent of the geometry of the system. The crucial geometric step is in the application of Lemma 2.2. We now verify that the variable-radius version of this lemma holds.

Consider a triple of points $\{x_1, x_2, x_3\}$ which span a triangle in $\mathcal{R}$ with side lengths ℓ_{12}, ℓ_{13}, and ℓ_{23}, where $\ell_{ij} \leq \min(r_d^i, r_d^j)$. We must show that the three discs of radius r_b^i centered on x_i meet at a common point (and hence cover the triangle spanned by x_1, x_2, x_3).

Consider the continuous function

$$f(x) = \max_{i=1,2,3} f_i(x) = \max_{i=1,2,3} \frac{\|x - x_i\|}{r_d^i}.$$

Since $f(x) \to \infty$ as $\|x\| \to \infty$ the function attains a global minimum, say $\lambda = f(x_0)$. We must show that $\lambda \le 1/\sqrt{3}$.

The minimizer x_0 must lie inside the triangle $x_1x_2x_3$, because any point x outside the triangle can be perturbed so as to decrease all three distances $\|x - x_i\|$ simultaneously. In more detail this argument shows that x_0 lies within the convex hull of its *critical vertices*, defined as those vertices x_i for which $f(x_0) = f_i(x_0)$.

There are two cases. If x_0 has two critical vertices x_i, x_j, then x_0 lies on the edge x_ix_j and $\lambda = f_i(x_0) = f_j(x_0) = \ell_{ij}/(r_d^i + r_d^j) \le 1/2$, which is less than $1/\sqrt{3}$. Otherwise all three vertices x_1, x_2, x_3 are critical. The largest of the three angles $\theta_{ij} = \angle x_ix_0x_j$ satisfies $\theta_{ij} \ge 2\pi/3$. The interior bisector of this angle meets the edge x_ix_j at a point y which divides the edge in the ratio $\|x_0 - x_i\| : \|x_0 - x_j\|$ or $r_i : r_j$. Using the sine rule for triangle x_0yx_i we then have

$$\lambda r_i = \|x_0 - x_i\| = \|y - x_i\| \cdot \frac{\sin \angle x_0yx_i}{\sin(\theta_{ij}/2)} \le \frac{\ell_{ij} r_i}{(r_i + r_j)} \cdot \frac{1}{\sin(\pi/3)} \le \frac{r_i}{\sqrt{3}}$$

giving the required bound.

The proof of the theorem now follows that of Theorem 3.3 precisely. $\diamond$

Of course, the results on minimal generators and Rips complexes with omissions still apply in this setting as well, as the reader may check.

9. Barrier coverage in 3-d. We consider the following modification of the physical workspace of the nodes. Let the nodes be points in $\mathcal{D} \times \mathbb{R}$ for $\mathcal{D} \subset \mathbb{R}^2$ as in **A3**, and let the fence nodes lie in $\mathcal{D} \times \{0\}$ and satisfy **A4**. We define $\mathcal{U} \subset \mathbb{R}^2 \times \mathbb{R}$ by placing a 3-d ball of radius r_c at each $x_i \in \mathcal{X}$. We construct a Rips complex as before, connecting nodes if they are within distance r_b in $\mathcal{D} \times \mathbb{R}$. From **A4** it follows that the fence subcomplex $\mathcal{F}$ is, as before, exactly $\partial\mathcal{D} \times \{0\}$.

The problem of **barrier coverage** is to determine whether there is a path connecting $\mathcal{D} \times \{-\infty\}$ to $\mathcal{D} \times \{+\infty\}$ avoiding $\mathcal{U}$. Our homological criterion immediately yields a criterion for barrier coverage (the nonexistence of such a path).

THEOREM 9.1. *A system of nodes in $\mathcal{D} \times \mathbb{R}$ satisfying* **A1**-**A4** *as above has barrier coverage if there exists $[\alpha] \in H_2(\mathcal{R}, \mathcal{F})$ with $\partial\alpha \neq 0$.*

Proof. We prove a stronger result in the spirit of Corollary 4.1. The proof of Lemma 3.1 holds for the 2-skeleton of the Rips complex: three points determine a plane which intersects the balls in discs of radius r_c. Hence, the simplicial realization map $\sigma : \mathcal{R} \to \mathcal{D} \times \mathbb{R}$ takes any 2-cycle α to a subset of $\mathcal{U}^\alpha$, the cover restricted to the nodes of α.

Let $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ denote projection to the second factor. Assume that $p : \mathbb{R} \to \mathcal{D} \times \mathbb{R} - \mathcal{U}^\alpha$ is a curve with $\lim_{x\to\pm\infty} \pi \circ p(x) = \pm\infty$. Since every point in $\sigma(\alpha)$ lies within $\mathcal{U}^\alpha$, we have that $\sigma : (\alpha, \partial\alpha) \to (\mathbb{R}^2 \times \mathbb{R}, \partial\mathcal{D} \times \{0\})$ factors through the pair $(\mathbb{R}^2 \times \mathbb{R} - p, \partial\mathcal{D} \times \{0\})$. However, let $A = (\mathbb{R}^2 \times \mathbb{R}) - p$ and B be a neighborhood of p, so that $A \cap B$ is an annular tube homotopic to S^1. Let $A' = \partial\mathcal{D} \times \{0\}$ and $B' = \emptyset$. Using Eqn. (A.7), we have

$$\longrightarrow H_2(S^1) \xrightarrow{\phi_*} H_2((\mathbb{R}^2 \times \mathbb{R}) - p, \partial\mathcal{D} \times \{0\}) \oplus 0 \xrightarrow{\psi_*} H_2((\mathbb{R}^2 \times \mathbb{R}), \partial\mathcal{D} \times \{0\}) \xrightarrow{\partial_*} H_1(S^1) \longrightarrow \qquad (9.1)$$

Since $H_2((\mathbb{R}^2 \times \mathbb{R}), \partial\mathcal{D} \times \{0\}) \cong H_2(\mathcal{D}, \partial\mathcal{D}) \cong \mathbb{R}$ and ∂_* is an isomorphism, we obtain

$$\cdots \longrightarrow 0 \longrightarrow H_2((\mathcal{D} \times \mathbb{R}) - p, \partial\mathcal{D} \times \{0\}) \longrightarrow \mathbb{R} \xrightarrow{\cong} \mathbb{R} \longrightarrow \cdots \tag{9.2}$$

By exactness, $H_2((\mathbb{R}^2 \times \mathbb{R}) - p, \partial\mathcal{D} \times \{0\}) = 0$ and thus, $\sigma_*[\alpha] = 0$: contradiction. $\diamond$

10. Pursuit-evasion and mobile nodes. Consider a situation in which the node positions are a continuous function of time: $\mathcal{X} = \mathcal{X}_t \subset \mathcal{D}$ for $t = 0...1$. Assume that the network is sampled to give a finite sequence of connectivity graphs $\{\Gamma_i\}_0^N$ at times $0 = t_0 < \cdots < t_N = 1$, as in Fig. 10.1. We assume the following:

T1 If two nodes are connected at time steps t_i and t_{i+1}, then they remain within the broadcast radius r_b for all $t_i \leq t \leq t_{i+1}$.
T2 Nodes may go off-line or come on-line, represented by deleting the nodes in the appropriate graph Γ_i.
T3 Fence nodes always remain fixed and on-line.

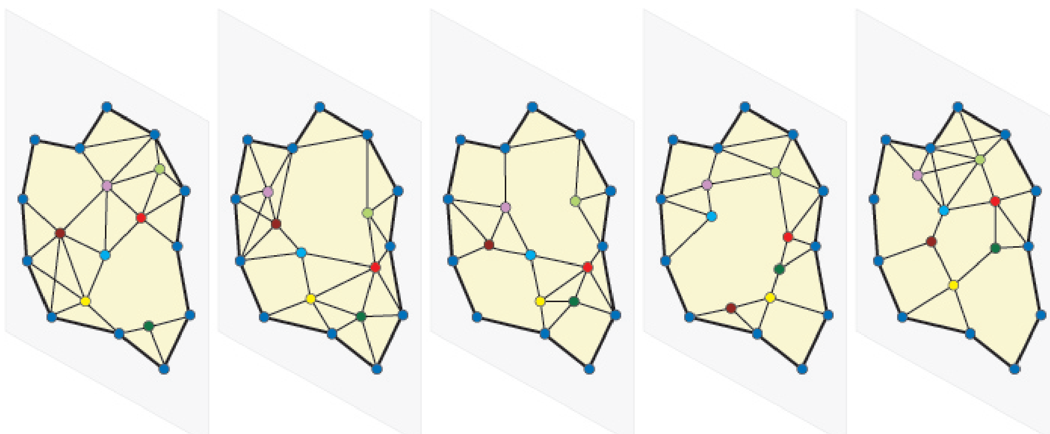

FIG. 10.1. *A mobile network with fixed fence nodes sampled at five time segments: can an evader avoid the cover?*

We now address the question of whether there can be a "wandering" loss of coverage. It may be the case that at no time $t \in [0, 1]$ does there exist a complete sensor coverage of the domain; however, the changes may obstruct any sequence of points from 'jumping' from one hole to the next, avoiding the coverage domain. Verifying the lack of wandering holes is a particular type of pursuit-evasion problem with relevance to problems in security and defense. Note that this problem is distinct from the "sweeping" coverage problem, in which one wants to know whether the union of the cover sets $\cup_t \mathcal{U}(t)$ contains $\mathcal{D}$.

10.1. A prism complex. We present a homological criterion for guaranteeing no wandering holes via computing the homology of a certain space derived from the sequence of Rips complexes $\mathcal{R}_i$.

DEFINITION 10.1. *Given a sequence $\{\Gamma_i\}$ of vertex-labeled communication graphs as above, define the* **stacked Rips complex** $\mathcal{SR}$ *to be the cell complex obtained from the disjoint union $\coprod_i \mathcal{R}_i$ of the Rips complexes $\mathcal{R}_i$ of Γ_i by the following operation:*

For each k-simplex $[v_{\alpha_1}, \ldots, v_{\alpha_{k+1}}]$ of $\mathcal{R}_i$ which is also a k-simplex on the same vertices in $\mathcal{R}_{i+1}$, connect these k-simplices by a prism $\Delta^k \times [0,1]$ with $\Delta^k \times \{0\}$ glued to $\mathcal{R}_i$ and $\Delta^k \times \{1\}$ glued to $\mathcal{R}_{i+1}$.

We treat the time variable $t \in [0,1]$ as an extra dimension and consider the problem of evasive coverage in $\mathcal{D} \times [0,1]$. The complex $\mathcal{SR}$ has a natural 'prism' structure: $\mathcal{SR}$ is a 1-parameter family of simplicial Rips complexes indexed by $t \in [0,1]$, these 'slices' being equal to $\mathcal{R}_i$ at t_i. See Fig. 10.2. We likewise consider the moving covers as a 1-parameter family in a 3-dimensional setting. If $\mathcal{U}_t$ denotes the radius r_c cover of nodes $\mathcal{X}_t$ at time t, embed the time-varying covers into $\mathcal{D} \times [0,1]$ via $\mathcal{U}_t \subset \mathcal{D} \times \{t\}$. The problem of wandering loss of coverage now becomes the question of whether the union $\cup_t \mathcal{U}_t$ in $\mathcal{D} \times [0,1]$ has a 'tunnel' running from bottom ($t = 0$) to top ($t = 1$).

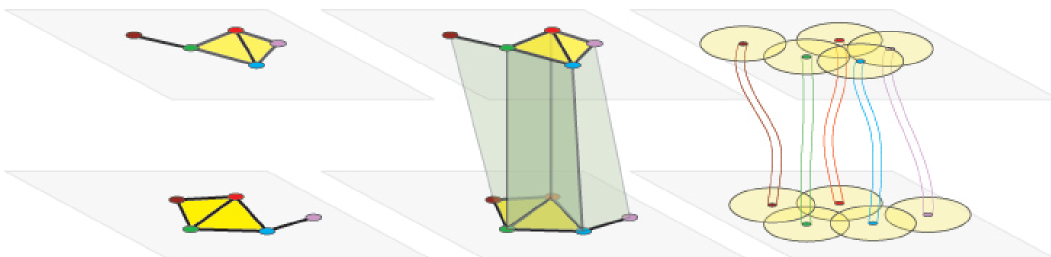

FIG. 10.2. *Subsequent Rips complexes [left] are attached via prisms between matching simplices [center] to capture the topology of the mobile cover [right].*

THEOREM 10.2. *Consider a time-varying set of nodes $\mathcal{X}_t$ in a domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying assumptions* **A1**-**A4** *and* **T1**-**T3**. *Then, for any continuous curve $p : [0,1] \to \mathcal{D}$, $p(t)$ must lie in $\mathcal{U}_t$ for some $0 \le t \le 1$ if there exists $[\alpha] \in H_2(\mathcal{SR}, \mathcal{F} \times [0,1])$ such that $\pi_*(\partial\alpha) \neq 0$, where $\pi : \mathcal{F} \times [0,1] \to \mathcal{F}$ is the projection map.*

Proof. As in the proof of Theorem 3.3, we consider a simplicial realization map $\overline{\sigma} : \mathcal{SR} \to \mathbb{R}^2 \times [0,1]$. Define $\overline{\sigma}$ as follows. Given the structure of $\mathcal{SR}$ as a family of Rips complexes $\mathcal{R}_t$ indexed by $t \in [0,1]$, let $\overline{\sigma}$ send each slice to $\sigma(\mathcal{R}_t) \subset \mathcal{D} \times \{t\}$, where σ is the realization map from the proof of Theorem 3.3 and the vertices are sent to $\mathcal{X}_t$.

The map $\overline{\sigma}$ takes the pair $(\mathcal{SR}, \mathcal{F} \times [0,1])$ to $(\mathbb{R}^2 \times [0,1], \partial\mathcal{D} \times [0,1])$, yielding the following diagram:

$$\begin{tikzcd} H_2(\mathcal{SR}, \mathcal{F} \times [0,1]) \arrow[r, "\delta_*"] \arrow[d, "\overline{\sigma}_*"] & H_1(\mathcal{F} \times [0,1]) \arrow[d, "\overline{\sigma}_*"] \\ H_2(\mathbb{R}^2 \times [0,1], \partial\mathcal{D} \times [0,1]) \arrow[r, "\delta_*"] & H_1(\partial\mathcal{D} \times [0,1]) \end{tikzcd} \quad . \tag{10.1}$$

It follows from assumption **T3** and Lemma 3.1 that $\pi_*\overline{\sigma}_*\delta_*[\alpha] \neq 0$. By commutativity of Eqn. (3.1), $\overline{\sigma}_*[\alpha] \neq 0$.

Assume that there exists a continuous curve $p : [0,1] \to \mathcal{D} \times [0,1]$ of points $p(t) \in \{\mathcal{D} \times \{t\} - \mathcal{U}_t\}$. We claim that $\overline{\sigma}(\mathcal{SR}) \subset \cup_t \mathcal{U}_t$. Assume that the nodes $\{x_i(t)\}_{i=1}^{k+1}$ span a k-simplex of $\mathcal{R}_t \subset \mathcal{SR}$ at some fixed time t. Then $\overline{\sigma}$ sends this to the convex hull of these nodes in $\mathbb{R}^2 \times \{t\}$. From Definition 10.1 and assumption **T1**, any edge in $\mathcal{R}_t$ implies that the node points implicated by this edge are within distance r_b at time t. An application of Lemma 2.2 then guarantees that the convex hull of these nodes lies within $\mathcal{U}_t$.

We conclude from this and the existence of the wandering curve p that $\overline{\sigma} : (\mathcal{SR}, \mathcal{F} \times [0,1]) \to (\mathbb{R}^2 \times [0,1], \partial\mathcal{D} \times [0,1])$ factors through the pair $(\mathbb{R}^2 \times [0,1] - p, \partial\mathcal{D} \times [0,1])$. However, this has vanishing H_2, using the same argument as in Theorem 9.1. Thus, $\overline{\sigma}_*[\alpha] = 0$: contradiction. $\diamond$

10.2. A simplicial model. In practice, computing with the stacked Rips complex is inconvenient. The software we use is meant for simplicial complexes, not the more general prism complex $\mathcal{SR}$. We therefore provide a simple means of reducing the stacked Rips complex to a simplicial object which is much smaller and simpler to encode.

Definition 10.3. *Given a collection of network graphs $\{\Gamma_i\}$ as in Definition 10.1, define the* **amalgamated Rips complex** *to be the space obtained from the disjoint union $\coprod_i \mathcal{R}_i$ of the Rips complexes $\mathcal{R}_i$ of Γ_i by the following operation:*

> *For each k-simplex $[v_{\alpha_1}, \ldots, v_{\alpha_{k+1}}]$ of $\mathcal{R}_i$ which is also a k-simplex on the same vertices in $\mathcal{R}_{i+1}$, identify these simplices.*

A few observations are in order. First, the amalgamated Rips complex $\mathcal{AR}$ is a cell complex built from simplices. It is not, properly speaking, a [combinatorial] simplicial complex since there may be, e.g., more than one 1-simplex connecting two vertices; hence, cells in this complex are not uniquely defined by their faces. Second, since the fence nodes are assumed stationary, the fence subcomplex $\mathcal{F}$ is fixed in each $\mathcal{R}_i$ and thus is identified to yield a well-defined subcomplex $\mathcal{F} \subset \mathcal{AR}$.

Proposition 10.4. *The pair $(\mathcal{SR}, \mathcal{F} \times [0,1])$ is homotopy equivalent to $(\mathcal{AR}, \mathcal{F})$.*

Proof: For each i, consider the maximal subcomplex $S_i \subset \mathcal{R}_i$ which is also a subcomplex of $\mathcal{R}_{i+1}$. The prism subcomplex $S_i \times [0,1] \subset \mathcal{SR}$ is a properly embedded subcomplex; hence the collapse of $S_i \times [0,1]$ to the simplicial subcomplex S_i in $\mathcal{AR}$ is a homotopy equivalence. The amalgamated complex $\mathcal{AR}$ is the result of applying the sequence of collapses to $\mathcal{SR}$, and the subcomplex $\mathcal{F} \times [0,1] \subset \mathcal{SR}$ is collapsed via projection of the second factor. $\diamond$

This immediately implies the following:

Corollary 10.5. *The homological condition of Theorem 10.2 is satisfied if and only if $H_2(\mathcal{AR}, \mathcal{F})$ has a generator $[\alpha]$ with $\partial\alpha \neq 0$.*

These hypotheses are preferable to those of Theorem 10.2 in that the spaces involved are smaller, simplicial, and there is no condition involving the projection of the boundary of the generator. For a software package that can handle only true combinatorial simplicial complexes, there is a simple modification of $\mathcal{AR}$ available. Since the homological criterion resides in H_2, one can identify all k-simplices with the same boundary for $k \geq 2$. Only the multiple 1-simplices need be distinguished, and these may be handled by inserting additional vertices and refining the cell structure.

11. Conclusions. The applicability of homology theory to sensor networks initiated in this paper is not as surprising as might at first appear. Indeed, the two fields share several features. Problems in both homology and sensor networks have as inputs a large collection of local objects (simplices, sensors) with local interaction rules (faces, communication). From this collection (chain complex, sensor network), one seeks to determine global properties of the system. The primary point of departure is that chain complexes carry with them a rich algebraic structure which can be exploited to great effect. We have demonstrated that certain features of this algebraic structure carry over to answer important questions in coverage, power conservation, and evasion-detection.

11.1. Remarks.

1. We have not specified communication protocols on the level of hardware, having concerned ourselves in this paper with the mathematical tools. We claim, however, that the Rips complex can be built in a distributed fashion on the hardware level. We expect the signal complexity of this operation to be reasonable, since the Rips complex is completely determined by its 1-skeleton.
2. In this paper, we have focused on the case where there is complete control over the fence nodes. In practice, such control may not be available. By endowing nodes with the capability of detecting the boundary of the domain, it is possible to reconstruct a fence subcomplex $\mathcal{F}$ composed of nodes near the boundary. Since these are not assumed to be well spaced (as in A4) the proofs of all the results here are invalid. We demonstrate in [8, 7] how to recover some of the results of this paper in that more general case via *persistent homology*.
3. We stress that the coverage criterion is not if-and-only-if. It is a rigorous test to guarantee coverage, and, thus, any system which is "just barely" covered will likely fail that test.
4. The test as given in this paper is centralized: a distributed coverage algorithm is greatly desirable.

11.2. Questions. This paper represents merely the first step in applications of algebraic topology to sensor networks. We comment on possible and probable extensions below.

1. What is the computational complexity of the homological criterion as a function of number of nodes? The standard algorithm for computing homology (using Smith normal form) is quintic in the number of simplices. More recent algorithms are much faster, but the subquadratic algorithm of [5] relies on duality for Euclidean spaces, and is not applicable for arbitrary Rips complexes.
2. Can one construct a homological coverage criterion which is distributed, allowing nodes with limited computational capabilities to compute local homology?
3. Can the mobile-network coverage criterion for wandering holes be made asynchronous? Rather than sampling the entire network at once, subsets of nodes should sample their connectivity and register their network graph with a central processor. Does a homological criterion holds for such systems?
4. By changing the bound in **A2** to $r_c \geq r_b$, the homological criterion verifies 3-coverage in a planar network [a simple exercise]. Is it possible to verify k-coverage for any k via homology? One wants to impose as few restrictions on r_c as possible.
5. In practice, coverage and communication domains are not radially symmetric: elliptical or conical shapes are closer to the reality in many cases. Is it possible to construct a homological coverage criterion for sensors whose communication and/or coverage domains are not radially symmetric? What additional capabilities do the sensors require in order to handle such asymmetry?
6. With the exception of the work in §10, we are working in a setting for which it is desired that there are more than enough sensors necessary to cover the domain. In

such a sensor-rich environment, it is possible for the Rips complex to attain a very high dimension. This is highly undesirable for computational reasons. Is there a way to compress the Rips complex in a preprocessing step without changing the appropriate homology group? This seems reasonable: a 20-dimensional simplex implies a cluster of nodes, most of which should be redundant.

7. If we endow the nodes with additional capabilities, such as the ability to measure some angular data about neighboring nodes, what global problems can be solved? We believe that problems involving degree computation and target isolation are solvable with only a very weak form of angular data at the nodes.

REFERENCES

[1] M. Allili, K. Mischaikow, and A. Tannenbaum, "Cubical homology and the topological classification of 2D and 3D imagery," in *IEEE Intl. Conf. Image Proc.*, pp. 173–176, 2001.

[2] A. Ames, "A homology theory for hybrid systems: hybrid homology," *Lect. Notes in Computer Science* 3414, pp. 86–102, 2005.

[3] M. A. Batalin and G. S. Sukhatme, "Spreading out: A local approach to multi-robot coverage," in *Proc. of 6th International Symposium on Distributed Autonomous Robotic Systems*, (Fukuoka, Japan), 2002.

[4] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Trans. Robotics. and Automation*, 20:2, pp. 243-255, 2004.

[5] C. Delfinado and H. Edelsbrunner, "An incremental algorithms for Betti numbers of simplicial complexes on the 3-spheres," *Comp. Aided Geom. Design*, 12:7, pp. 771-784, 1995.

[6] V. de Silva and G. Carlsson, "Topological estimation using witness complexes," in *Symp. Point-Based Graphics*, ETH Zurich, 2004.

[7] V. de Silva and R. Ghrist, "Coverage in sensor networks via persistent homology," preprint.

[8] V. de Silva, R. Ghrist, and A. Muhammad, "Blind swarms for coverage in 2-d," in *Proc. Robotics: Systems and Science*, 2005.

[9] J. Eckhoff, "Helly, Radon, and Carathéodory Type Theorems." Ch. 2.1 in *Handbook of Convex Geometry* (Ed. P. M. Gruber and J. M. Wills). Amsterdam, Netherlands: North-Holland, pp. 389-448, 1993.

[10] D. W. Gage, "Command control for many-robot systems," in *Nineteenth Annual AUVS Technical Symposium*, pp. 22-24, (Huntsville, Alabama, USA), 1992.

[11] M. Gromov, *Hyperbolic groups*, in Essays in Group Theory, MSRI Publ. **8**, Springer-Verlag, 1987.

[12] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[13] J.-C. Hausmann, "On the Vietoris-Rips complexes and a cohomology theory for metric spaces," in *Ann. Math. Studies 138*, Princeton Univ. Press, pp. 175–188, 1995.

[14] C.-F. Hsin and M. Liu, "Network coverage using low duty-cycled sensors: random & coordinated sleep algorithms," in *Proc. IPSN*, 2004.

[15] C.-F. Huang and Y.-C. Tseng, "The coverage problem in a wireless sensor network," in *ACM Intl. Workshop on Wireless Sensor Networks and Applications*, pp. 115121, 2003.

[16] T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational Homology,* Applied Mathematical Sciences 157, Springer-Verlag, 2004.

[17] H. Koskinen, "On the coverage of a random sensor network in a bounded domain," in *Proceedings of 16th ITC Specialist Seminar*, pp. 11-18, 2004.

[18] S. Kumar, T. H. Lai, and J. Balogh, "On k-coverage in a mostly sleeping sensor network," in *Proc. 10th Intl. Conf. on Mobile Computing and Networking*, 2004.

[19] X.-Y. Li, P.-J. Wan, and O. Frieder, "Coverage in wireless ad-hoc sensor networks" IEEE Transaction on Computers, Vol. 52, No. 6, pp. 753-763, 2003.

[20] B. Liu and D. Towsley, "A study of the coverage of large-scale sensor networks," in *IEEE International Conference on Mobile Ad-hoc and Sensor Systems*, 2004.

[21] S. Meguerdichian, F. Koushanfar, M. Potkonjak, and M. Srivastava, "Coverage problems in wireless ad-hoc sensor network," in *IEEE INFOCOM*, pp. 13801387, 2001.

[22] K. Mischaikow, M. Mrozek, J. Reiss, and A. Szymczak, "Construction of symbolic dynamics from experimental time series," *Phys. Rev. Lett.* 82(6), p. 1144, 1999.

[23] L. Vietoris, "Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen," *Math. Ann.* 97 (1927), 454–472.

[24] F. Xue and P. R. Kumar, "The number of neighbors needed for connectivity of wireless networks,"

Wireless Networks, pp. 169-181, vol. 10, no. 2, March 2004.

[25] H. Zhang and J. Hou, "Maintaining Coverage and Connectivity in Large Sensor Networks," in *International Workshop on Theoretical and Algorithmic Aspects of Sensor, Ad hoc Wireless and Peer-to-Peer Networks*, Florida, Feb. 2004

[26] A. Zomorodian and G. Carlsson, "Computing persistent homology," in *Proc. 20th ACM Sympos. Comput. Geom.* pp. 346-356, 2004.

[27] Plex home page, http://math.stanford.edu/comptop/programs/plex/

Appendix A. Homology basics.

The mathematical tools we use are by no means novel: with the exception of the simulations, this paper could have been written in the middle of the prior century. However, as these tools are not in the repertoire of researchers in sensor networks, we give a primer. Those wanting a more complete treatment can find it in the excellent text of Hatcher [12].

A.1. Simplicial homology. Homology is an algebraic procedure for counting 'holes' in topological spaces. There are numerous variants of homology: we use **simplicial homology with real coefficients**, a theory adapted to **simplicial complexes**.

Given a set of points V, a k-simplex is an unordered subset $\{v_0, v_1, \ldots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$. The **faces** of this k-simplex consist of all $(k-1)$-simplices of the form $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$ for some $0 \leq i \leq k$. A **simplicial complex** is a collection of simplices which is closed with respect inclusion of faces. Triangulated surfaces form a concrete example, where the vertices of the triangulation correspond to V. The orderings of the vertices correspond to an orientation. Any abstract simplicial complex on a (finite) set of points V has a geometric realization in some $\mathbb{R}^n$.

Let X denote a simplicial complex. Roughly speaking, the homology of X, denoted $H_*(X)$, is a sequence of vector spaces $\{H_k(X) : k = 0, 1, 2, 3 \ldots\}$, where $H_k(X)$ is called the **k-dimensional homology of** X. The dimension of $H_k(X)$, called the k^{th} **Betti number of** X, is a coarse measurement of the number of different holes in the space X that can be sensed by using subcomplexes of dimension k.

For example, the dimension of $H_0(X)$ is equal to the number of connected components of X. These are the types of 'holes' in X that points can detect — are two points connected by a sequence of edges or not? The simplest basis for $H_0(X)$ consists of a choice of vertices in X, one in each path-component of X. Likewise, the simplest basis for $H_1(X)$ consists of loops in X, each of which surrounds a different 'hole' in X. For example, if X is a graph, then $H_1(X)$ is a measure of the number and types of cycles in the graph.

Let X denote a simplicial complex. Define for each $k \geq 0$, the vector space $C_k(X)$ to be the vector space whose basis is the set of **oriented** k-simplices of X; that is, a k-simplex $\{v_0, \ldots, v_k\}$ together with an order type denoted $[v_0, \ldots, v_k]$ where a change in orientation corresponds to a change in the sign of the coefficient:

$$[v_0, \ldots, v_i, \ldots, v_j, \ldots, v_k] = -[v_0, \ldots, v_j, \ldots, v_i, \ldots, v_k].$$

For k larger than the dimension of X, we set $C_k(X) = 0$. The **boundary map** is defined to be the linear transformation $\partial : C_k \to C_{k-1}$ which acts on basis elements $[v_0, \ldots, v_k]$ via

$$\partial[v_0, \ldots, v_k] := \sum_{i=0}^{k} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k]. \tag{A.1}$$

This gives rise to a **chain complex**: a sequence of vector spaces and linear transformations

$$\cdots \xrightarrow{\partial} C_{k+1} \xrightarrow{\partial} C_k \xrightarrow{\partial} C_{k-1} \cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

Consider the following two subspaces of C_k: the **cycles** (those subcomplexes without boundary) and the **boundaries** (those subcomplexes which are themselves boundaries).

$$\begin{array}{ccccc} k\text{-cycles} & : & Z_k(X) & = & \ker(\partial : C_k \to C_{k-1}) \\ k\text{-boundaries} & : & B_k(X) & = & \operatorname{im}(\partial : C_{k+1} \to C_k) \end{array} \tag{A.2}$$

A simple lemma demonstrates that $\partial \circ \partial = 0$; that is, the boundary of a complex has empty boundary. It follows that B_k is a subspace of Z_k. This has great implications. The k-cycles in X are the basic objects which count the presence of a 'hole of dimension k' in X. But, certainly, many of the k-cycles in X are measuring the same hole; still other cycles do not really detect a hole at all — they bound a subcomplex of dimension $k+1$ in X.

We say that two cycles ξ and η in $Z_k(X)$ are **homologous** if their difference is a boundary:

$$[\xi] = [\eta] \quad \leftrightarrow \quad \xi - \eta \in B_k(X).$$

The k-dimensional **homology** of X, denoted $H_k(X)$ is the quotient vector space,

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}. \tag{A.3}$$

Specifically, an element of $H_k(X)$ is an equivalence class of homologous k-cycles. This inherits the structure of a vector space in the natural way: $[\xi] + [\eta] = [\xi + \eta]$ and $c[\xi] = [c\xi]$ for $c \in \mathbb{R}$.

By arguments utilizing barycentric subdivision, one may show that the homology $H_*(X)$ is a topological invariant of X: it is indeed an invariant of homotopy type. Readers familiar with the Euler characteristic of a triangulated surface will not find it odd that intelligent counting of simplicies yields an invariant.

A.2. Relative homology. The precise version of homology used in our theorems is a 'relative' homology. Often, one wishes to compute holes modulo some region of the space.

Let $Y \subset X$ be a subcomplex of X. We define the **relative** chains as follows: $C_k(X, Y)$ is the quotient space obtained from $C_k(X)$ by collapsing the subspace generated by k-simplices in Y. One verifies that this quotient is respected by ∂ and that the subspaces defined by the kernel and image are well-defined and satisfy

$$B_k(X, Y) \subset Z_k(X, Y) \subset C_k(X, Y).$$

It then follows that the **relative homology**

$$H_k(X, Y) = \frac{Z_k(X, Y)}{B_k(X, Y)} \tag{A.4}$$

is well-defined. This homology $H_*(X, Y)$ measures holes detected by chains whose boundaries lie in Y.

It follows from the Excision Theorem that the relative homology is equal to the regular homology of the quotient space X/Y obtained by identifying all simplices in Y to a single abstract vertex.

$$H_k(X, Y) \cong H_k(X/Y) \quad k > 0.$$

A.3. Induced homomorphisms. Is it often remarked that *homology is functorial*, by which it is meant that things behave the way they ought. A simple example of this which is crucial to our applications arises as follows.

Consider two simplicial complexes X and X'. Let $f : X \to X'$ be a continuous simplicial map: f takes each k-simplex of X to a k'-simplex of X', where $k' \leq k$. Then, the map f induces a linear transformation $f_\# : C_k(X) \to C_k(X')$. It is a simple lemma to show that $f_\#$ takes cycles to cycles and boundaries to boundaries; hence there is a well-defined linear transformation on the quotient spaces

$$f_* : H_k(X) \to H_k(X') \quad : \quad f_* : [\xi] \mapsto [f_\#(\xi)].$$

This is called the **induced homomorphism** of f on H_*. Functoriality means that (1) the identity map $Id : X \to X$ induced the identity map on homology; and (2) the composition of two maps $g \circ f$ induces the composition of the linear transformation: $(g \circ f)_* = g_* \circ f_*$.

A.4. Exact sequences. Computing algebraic topological invariants is greatly simplified by the use of exact sequences. A sequence of vector spaces $\{V_i\}$ connected by linear transformations $\varphi_i : V_i \to V_{i-1}$ is said to be **exact** if the kernel of φ_i is equal to the image of φ_{i+1}.

Given a simplicial complex X with subcomplex $Y \subset X$, the **long exact sequence** of the pair (X, Y) is

$$\cdots \longrightarrow H_k(Y) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,Y) \xrightarrow{\delta_*} H_{k-1}(Y) \xrightarrow{i_*} \cdots \tag{A.5}$$

Here, i_* is the map induced by inclusion $i : Y \hookrightarrow X$, j_* is induced by the quotient $X \to X/Y$, and δ_* is the map which takes a relative k-cycle α in $H_k(X,Y)$ and returns the boundary, $\partial\alpha$, a $(k-1)$-cycle in Y.

This sequence is exact and is an effective means of computing relative homology groups. Of equal importance is the **Mayer-Vietoris** sequence of a space $X = A \cup B$:

$$\cdots \longrightarrow H_k(A \cap B) \xrightarrow{\phi_*} H_k(A) \oplus H_k(B) \xrightarrow{\psi_*} H_k(A \cup B) \xrightarrow{\partial_*} H_{k-1}(A \cap B) \xrightarrow{\phi_*} \cdots \tag{A.6}$$

Here $\phi(c) = (c, -c)$ and $\psi(c, c') = c + c'$, with ∂_* of a cycle $\zeta = c \cup c'$ being $[\partial c] = [-\partial c']$. Also of relevance to the proofs of this paper is a relative version of the Mayer-Vietoris sequence:

$$\begin{aligned} &\cdots \longrightarrow H_k(A \cap B, A' \cap B') \xrightarrow{\phi_*} H_k(A, A') \oplus H_k(B, B') \xrightarrow{\psi_*} H_k(A \cup B, A' \cup B') \\ &\xrightarrow{\partial_*} H_{k-1}(A \cap B, A' \cap B') \xrightarrow{\phi_*} \cdots \end{aligned} \tag{A.7}$$

Here $(X, Y) = (A \cup B, A' \cup B')$.