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Journal of Economic Theory ▮ (▮▮▮▮) ▮▮▮–▮▮▮

JOURNAL OF Economic Theory

http://www.elsevier.com/locate/jet

From evolutionary to strategic stability

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Received 28 February 2001; final version received 16 September 2002

Abstract

A connected component of Nash equilibria is (dynamically) *potentially stable* if there exits an evolutionary selection dynamics from a broad class for which the component is asymptotically stable. A necessary condition for potential stability is that the component's index agrees with its Euler characteristic. Second, if the latter is nonzero, the component contains a *strategically stable set*. If the Euler characteristic would be zero, the dynamics (that justifies potential stability) could be slightly perturbed so as to remove all zeros close to the component. Hence, any *robustly potentially stable* component contains equilibria that satisfy the strongest rationalistic refinement criteria.

JEL classification: C72

Keywords: Dynamic stability; Evolutionary game theory; Index theory; Strategic stability

1. Introduction

Nash equilibrium rests on two assumptions. One is that players *maximize* utility at no cost, with absolute precision, and under complete information about the game. The other is that expectations about the opponents are *consistent*, i.e., they are correct in equilibrium. In the light of both everyday experience and experimental evidence these assumptions appear controversial.

Therefore, theorists have turned to justifications for noncooperative solutions that mitigate these assumptions. One such approach has already been suggested by Nash

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0022-0531/03/$ - see front matter
doi:10.1016/S0022-0531(03)00078-4

[42, pp. 21–23]. It has become known as the "mass-action" interpretation of Nash equilibrium and refers to a quasi-biological setup (as initiated in biology by Maynard Smith and Price [36], Maynard Smith [35], and Taylor and Jonker [50]). Players are replaced by large populations, one for each player position, of boundedly rational individuals with little or no information about the game. Period after period agents are randomly drawn to interact. Individual agents come "programmed" to use a particular strategy, but occasionally they "wake up" and revise their routines. Strategy revisions may be guided by imitation, myopic best replies, learning, or experimentation. In the aggregate this yields a dynamic process on the population distributions over strategies available to the various player positions. (The literature on such models is too large to be reviewed here; see e.g. the survey by Mailath [34], or the textbooks by Hofbauer and Sigmund [27], Weibull [56], Vega-Redondo [54], Samuelson [47], and Fudenberg and Levine [19].)

Evolutionary models combine two processes. A *selection process* favors some strategies over others. The variety of strategies on which selection operates is created by a *mutation process*. Modelling approaches can be distinguished by which of these two processes they emphasize. Stochastic models of finite, but large populations focus on the role of mutations (an approach pioneered by Foster and Young [16], Fudenberg and Harris [18], Young [59,60], and Kandori et al. [30]). They generate predictions in terms of stationary long run distributions (on appropriately specified states). Deterministic continuous time dynamics emphasize the selection process and take account of mutations by way of stability analysis (see e.g. [6,17,27,41,46,48,50]). Predictions are obtained in terms of dynamically stable (mixed) strategy combinations or sets thereof.

Evolutionary predictions often, but not always, support the rationalistic paradigm of Nash equilibrium. In many cases evolutionary dynamics even select among Nash equilibria. This has raised the issue under which conditions the predictions from evolutionary dynamics will agree or disagree with noncooperative solutions.

Focussing on (deterministic continuous time) selection processes, it has been shown for the "replicator dynamics" that either convergence (of an interior trajectory) or (Lyapunov) stability imply Nash equilibrium ([6,41])—a result that generalizes to a larger class of selection dynamics (see [17,41,46,48]). Swinkels [49] shows that dynamic asymptotic stability of a set of Nash equilibria, in a selection dynamics from a wide class, implies (together with a topological condition) that this set meets certain refinement criteria. Ritzberger and Weibull [46] give a necessary and sufficient condition in terms of the data of the game, "closure under better replies", for (the face spanned by) a product of pure strategy sets to be asymptotically stable in a large class of selection dynamics. Balkenborg and Schlag [1] show that every (connected) asymptotically stable set of rest points, that contains a pure strategy combination, is a "strict equilibrium set", viz. a set of Nash equilibria such that every player's strategy can (unilaterally) be replaced by an alternative best reply without leaving the set. For two-player games and for convex strict equilibrium sets they also show the converse.

Many of these results concern *sets* of strategy combinations rather than points. In a dynamic model set-valuedness is easier to interpret than in a rationalistic approach.

Dynamic stability simply predicts that, once in the set, the population state will remain in the set, with no particular prescription about which of its elements will obtain at any point in time.

This suggests that the appropriate objects for dynamic considerations may be *connected components* of Nash equilibria. The set of all Nash equilibria for any finite normal form game consists of finitely many such connected and closed components (see [32]). For generic extensive form games the outcomes are even *constant* across equilibria in the same component (see [33, Appendix A3]).

But that an evolutionary process supports Nash equilibrium requires conditions on the dynamics. And this represents a major difficulty, because often the precise form of the dynamics is unknown or only given by certain coarse properties. A goal of research, therefore, is to find conditions, that apply to a class of selection dynamics as broad as possible, and still allow conclusions on dynamic stability properties. Ideally, one would like to infer directly from the data of the game whether certain components of Nash equilibria can be dynamically stable for at least some from a wide class of selection dynamics.

1.1. Results

This paper combines a focus on Nash equilibrium components with minimal conditions on the selection dynamics. The latter is modelled by a deterministic evolutionary selection dynamics in continuous time. In the light of recent approximation results [2,4,7,8,44], however, the conclusions obtained also have implication for some stochastic models of mutation processes.

More precisely, selection dynamics on finite normal form games are considered for which all Nash equilibria are rest points, that do not point outwards at the boundary of the strategy space, that are Lipschitz continuous in mixed strategy combinations, and that satisfy a mild payoff consistency condition. These are truly minimal conditions. All classes of (deterministic continuous time) selection dynamics studied so far meet these criteria.

The first result of this paper gives a necessary condition for a Nash equilibrium component (or, more generally, a connected set of rest points) to be asymptotically stable in some dynamics from this class. And the second identifies a strong rationalistic implication of dynamic stability.

The necessary condition is as follows. If there exists such a selection dynamics for which a given Nash equilibrium component is asymptotically stable, the component is called *potentially stable* and we show that its *index* (see [45]) equals its *Euler characteristic*. Thus, for a component to be asymptotically stable in *some* selection dynamics it is *necessary* that the topological structure of the component, as described by the Euler characteristic, agrees with the index.

This condition depends purely on the data of the game. Indeed it can be shown (see [13,14]) that the index of a Nash equilibrium component is independent of the particular vector field used to compute it. It can be expressed as the local degree of the projection mapping from the graph of the Nash equilibrium correspondence to the space of games. By ‘‘local degree’’ we mean that the relevant projection is from a neighborhood of the component in the graph to a neighborhood of the game.

The first result is then used to identify a connection to the strongest rationalistic solution concept. Under the mild condition that the Euler characteristic of the Nash equilibrium component is nonzero, potential stability implies that the component contains a *strategically stable set* in the sense of Mertens ([39,40], henceforth an *M-stable* set).

This generalizes the result by Swinkels [49] in two respects. First, the topological hypothesis of a nonzero Euler characteristic is weaker than the existence of a neighborhood (contained in the component's basin of attraction) that is homeomorphic to the strategy space—which is what Swinkels assumes. Second, Swinkels deduces that the component contains a *hyperstable* set (see [32]), while we obtain strategic stability in its strongest form. (There are, of course, other notions of strategic stability, for instance by Hillas [24], Vermeulen [55], or ‘‘homotopy-stability’’ by Mertens [39]. But it is known (see [25]) that M-stability is the strongest notion, i.e., it implies all the others.)

The added condition of a nonzero Euler characteristic in our second result also has evolutionary significance. Call a Nash equilibrium component *robustly potentially stable* if (it is potentially stable and) sufficiently small perturbations of the dynamics (that justifies potential stability) still have zeros nearby. It can be shown that a component with zero Euler characteristic cannot be robustly potentially stable. Hence, the second result says that any robustly potentially stable component contains an M-stable set.

1.2. Applications

There are a number of applications of these results. First, consider generic normal form games for which all equilibria are *regular* (see [21,45,52]). The index of a regular equilibrium can only be $+1$ or -1. Moreover, if there are m equilibria with index $+1$, there must be at least $m-1$ equilibria with index -1 (see [20,45]). Since the Euler characteristic of a point is $+1$, no equilibrium with index -1 can be asymptotically stable in any selection dynamics. Hence, in many games quite a number of equilibria are ruled out by evolutionary considerations. This is despite the fact that regular equilibria meet all known refinement criteria.

Second, if for general games attention is restricted to *convex* (or merely contractible) Nash equilibrium components, then the present result identifies those (contractible) components that can be asymptotically stable: They must have index $+1$, because a contractible set has Euler characteristic $+1$. This represents a generalization of the stability criterion for regular rest points.

Consider, for instance, the widely studied class of ‘‘signaling games’’. Those are two-player games, where first player 1 learns her type and sends a message; then player 2 learns the message, but not the type, and chooses an action. For generic such games every agent's part of an equilibrium component consists either of a singleton (if the agent is reached) or of a simplex (if the agent is not reached) (see [52, p. 272]). Since the mapping from behavior to mixed strategies is an embedding, every component is contractible in mixed strategy combinations. Therefore, in such games it suffices to determine the index of a component to test for potential stability: only components with index $+1$ qualify. In the well-known ''beer-quiche game'' (see [10]), for example, only the component, that is generally viewed as more intuitive (and has index $+1$), can be asymptotically stable in some selection dynamics (because the other component has index 0).

Third, the present approach reveals that there are games in which dynamic stability will not support Nash equilibrium. Examples can be constructed (see Examples 1 and 3 in Section 4) of games, that do not have any Nash equilibrium component for which the Euler characteristic agrees with the index. For such games *no* component will be asymptotically stable in *any* selection dynamics.

Fourth, the present result yields a straightforward evolutionary analysis of ''forward induction'' à la van Damme [53] and Hauk and Hurkens [23] (see Example 1 in Section 4). Consider a two-player game, where player 1 first chooses between an outside option and a simultaneous move subgame, with a single equilibrium that yields 1 more than the outside option. If the forward induction equilibrium in the subgame has index $+1$, the component, where player 1 takes the outside option, fails potential stability. If it has index -1, the game has no potentially stable component. In the first case evolution unambiguously supports forward induction, in the second it remains agnostic. This reproduces results by Hauk and Hurkens [23].

Fifth, the role of assumptions explicitly or implicitly used in other models is clarified. As mentioned, Swinkels [49] uses the topological condition that the asymptotically stable set has a basin of attraction which contains a neighborhood homeomorphic to the space of (mixed) strategy combinations. This is assuming a version of contractibility. Ritzberger and Weibull [46] consider faces, that are convex by construction, and, thus, have Euler characteristic $+1$. Balkenborg and Schlag [1] need convexity of the strict equilibrium set to establish asymptotic stability. (A convex strict equilibrium set is either a strict equilibrium or a face that consists entirely of Nash equilibria.) All these assumptions restrict the topological structure of the set under scrutiny. In view of the present result this is *necessary* to enable asymptotic stability.

The result can also be used to clarify an ambiguity in Corollary 4 of Ritzberger and Weibull [46]. They conclude from asymptotic stability of a face (in a ''sign-preserving'' dynamics) that it contains an essential component of Nash equilibria and a hyperstable set. But an essential component need *not* contain a hyperstable set (see [23] for an example). Still, their statement is correct for the following reason.

Given a selection dynamics in which a face is asymptotically stable, modify the vector field such that the face consists entirely of zeros, by multiplying with the (product of the weights of all) strategies that are not used in the face. Since (on the interior) this is a reparameterization of time, it only changes the velocity, but not the orbits implied by the vector field. Hence, the face remains asymptotically stable. Since a face is convex by construction, our first result implies that its index must equal the Euler characteristic $+1$. Since the index sum of Nash equilibrium components contained in the face is $+1$, the asymptotically stable face must contain a Nash equilibrium component with nonzero index. But a Nash equilibrium component with nonzero index does contain a hyperstable set (and, in fact, an M-stable set).

Furthermore, our second result lends evolutionary support to "backwards induction", by the same reasoning as in Swinkels [49]. Consider any (finite) extensive form game. By the second result, no component, that fails to contain a *sequential equilibrium* (see [33]), can be robustly potentially stable. Hence, if at all, evolution favors backwards induction.

In particular, for generic *perfect information* extensive form games Kuhn's algorithm identifies a unique subgame perfect equilibrium. Since generically outcomes are constant across each component, such a game has a *unique* Nash equilibrium component that can be robustly potentially stable: The backwards induction component. This observation parallels results by Cressman and Schlag [11] and Hart [22].

Yet, the second result not only shows that robust asymptotic stability in some dynamics supports strategic stability. It can also be used in a purely static framework. In some games it is computationally easier to construct a dynamics for which a component is asymptotically stable than to verify M-stability. Since the Euler characteristic is also easy to compute, robust potential stability can be used to identify M-stable sets.

The rest of the paper is organized as follows. Section 2 describes evolutionary selection dynamics on games and briefly reviews index theory. Section 3 contains the two main results. Section 4 considers examples, and Section 5 concludes.

2. The model

The analysis will focus on two objects. First, f will be a vector field on a smooth orientable K-dimensional manifold Θ, i.e., a Lipschitz continuous function from Θ to the tangent space of Θ. Second, $C \subseteq \Theta$ will be a compact connected component of zeros for f. To avoid pathologies, it is assumed that C is a semi-algebraic set and, therefore, homeomorphic to a polyhedron.

The results apply to this general set-up. That the objects from evolutionary game dynamics fit this general framework, is shown next.

2.1. Game dynamics

Consider a finite n-player normal form game $\Gamma = (S, u)$, where $S = \times_{i=1}^{n} S_i$ is the product of the n sets S_i of pure strategies for players $i = 1, \ldots, n$, each with a finite number $K_i = |S_i| > 1$ of elements, and $u = (u_1, \ldots, u_n) : S \to \mathbb{R}^n$ is the payoff function. For each player i number pure strategies from 1 to K_i, so $S_i = \{s_i^j\}_{j=1}^{K_i}$. Let

$$\Sigma_i = \left\{ \sigma_i \in \mathbb{R}_+^{K_i - 1} \,\middle|\, 1 \geqslant \sum_{j=1}^{K_i - 1} \sigma_i^j \right\} \tag{1}$$

denote player i's mixed strategy set, where $\sigma_i^j = \sigma_i(s_i^j) \geqslant 0$ for all $j = 1, \dots, K_i - 1$ and $1 - \sum_{j=1}^{K_i-1} \sigma_i^j = \sigma_i(s_i^{K_i}) \geqslant 0$ for all $i = 1, \dots, n$. And let $\Sigma = \times_{i=1}^n \Sigma_i$ be the space of mixed strategy combinations.

An evolutionary game dynamics is given by a vector field on the space of mixed strategy combinations. Two minimal technical assumptions, invariably adopted on such dynamics, are Lipschitz continuity in mixed strategy combinations and that the vector field does not point outwards at the boundary of the strategy space. The first ensures existence of a unique solution to the associated (system of) differential equation(s) for any initial condition by the Picard–Lindelöf theorem. The second guarantees that the strategy space is forward invariant.

A minor technical difficulty is that the space of mixed strategy combinations is not a smooth manifold. It is locally diffeomorphic to an orthant of a Euclidean space, but not to a half-space. Yet, it will now be shown that any such vector field can be continuously extended to a manifold with boundary that properly contains the strategy space such that no zeros appear outside the space of mixed strategy combinations.

Choose a convex manifold Θ with boundary $\partial\Theta$ such that $\Sigma \subseteq (\Theta \backslash \partial\Theta)$. This can always be done.[1] For each player i and each $\sigma \in \Theta$ define $x_i(\sigma_i) = \arg\min_{x \in \Sigma_i} ||x - \sigma_i||$. This function is the identity on Σ_i, satisfies $x_i(\sigma_i) \in \partial\Sigma_i$ for all $\sigma_i \notin \Sigma_i$, and is such that $x_i(\sigma_i) - \sigma_i$ is perpendicular to the boundary $\partial\Sigma_i$ of Σ_i wherever possible. Define $x : \Theta \to \Sigma$ by $x(\sigma) = (x_1(\sigma_1), \dots, x_n(\sigma_n))$.

Given a vector field $\hat{f}$ on Σ, it is extended to Θ by

$$f(\sigma) = \hat{f}(x(\sigma)) + x(\sigma) - \sigma. \tag{2}$$

If there would be some $\sigma \in \Theta \backslash \Sigma$ such that $f(\sigma) = 0$, then

$$\hat{f}(x(\sigma)) \cdot (x(\sigma) - \sigma) + ||x(\sigma) - \sigma||^2 = 0$$

$$\text{would imply that } \hat{f}(x(\sigma)) \cdot (\sigma - x(\sigma)) > 0$$

because $\sigma \notin \Sigma$ implies $x(\sigma) \neq \sigma$. But the latter contradicts the assumption that $\hat{f}$ does not point outwards at the boundary $\partial\Sigma$ of Σ. Hence, the extension f has no zeros outside of Σ. Moreover, since at the boundary $\partial\Theta$ of Θ the extended vector field f is the sum of a vector field that does not point outwards and one that points inwards, it is inward pointing at $\partial\Theta$.

The technical assumptions of Lipschitz continuity and that the vector field does not point outwards at the boundary of Σ do not provide any link to the payoffs for the game. The latter requires an extra condition. For the present purpose, the weakest form of *payoff consistency* will do.

Definition 1. A *payoff consistent selection dynamics* is a Lipschitz continuous vector field $f = (f_1, \dots, f_n)$ on Σ, that does not point outwards along the boundary of Σ,

[1] Let L be the smallest affine space containing Σ and B_ε the closed ε-ball around the origin. Then $\Theta = (\Sigma + B_\varepsilon) \cap L$ gives such a manifold. We are grateful to an anonymous referee for this simple argument.

and satisfies for all $i = 1, \ldots, n$ and all $\sigma \in \Sigma$

$$f_i(\sigma) \cdot \nabla_{\sigma_i} U_i(\sigma) \geqslant 0, \tag{3}$$

where $\nabla_{\sigma_i} U_i$ denotes the gradient of the extension U_i of the payoff function u_i to mixed strategy combinations with respect to player i's strategy $\sigma_i \in \Sigma_i$.

Condition (3) is a mild condition, satisfied by all classes of evolutionary selection dynamics studied so far in the literature. It, roughly, says that unilaterally for each player position/population i the vector field points in the direction of nondecreasing average population payoffs. Swinkels [49] calls such vector fields "myopic adjustment dynamics" if every Nash equilibrium is a zero.

The literature has focussed on two subclasses thereof, "payoff monotonic" and "payoff positive" (or "sign-preserving") selection dynamics. A *regular* selection dynamics is a vector field f on Σ such that growth rate functions g_i^j satisfying $f_i^j(\sigma) = g_i^j(\sigma)\sigma_i^j$ for all $\sigma \in \Sigma$, all $j = 1, \ldots, K_i$, and all $i = 1, \ldots, n$, can be chosen satisfying Lipschitz continuity, where $f_i^{K_i}(\sigma) = -\sum_{j=1}^{K_i-1} f_i^j(\sigma)$ and $\sigma_i^{K_i} = 1 - \sum_{j=1}^{K_i-1} \sigma_i^j$ for all $i = 1, \ldots, n$. A regular selection dynamics is *payoff monotonic* if for all $\sigma \in \Sigma$, all $i = 1, \ldots, n$, and all $s_i^j, s_i^h \in S_i$

$$g_i^j(\sigma) > g_i^h(\sigma) \Leftrightarrow U_i(\sigma_{-i}, s_i^j) > U_i(\sigma_{-i}, s_i^h) \tag{4}$$

(see [56, Definition 5.5]; the same property appears under different names in [17,41,48]). A regular selection dynamics is *payoff positive* if for all $\sigma \in \Sigma$, all $i = 1, \ldots, n$, and all $s_i^j \in S_i$

$$\operatorname{sign}(g_i^j(\sigma)) = \operatorname{sign}(U_i(\sigma_{-i}, s_i^j) - U_i(\sigma)) \tag{5}$$

(see [56, Definition 5.6]; see also [41,46]). These two classes are distinct, but overlap. Their intersection contains the "payoff-linear" [56, Definition 5.7] or "aggregate monotonic" [48] selection dynamics, of which the replicator dynamics is the most prominent example.

Remark 1. It is easily seen that every payoff monotonic or payoff positive selection dynamics is payoff consistent. For, let f be the vector field of a payoff monotonic selection dynamics and g the associated growth rate function. Choose a player position i and a strategy combination $\sigma \in \Sigma$ and assume without loss of generality that

$$g_i^1(\sigma) \geqslant g_i^2(\sigma) \geqslant \cdots \geqslant g_i^{K_i}(\sigma).$$

There are two possibilities: Either $f_i^j(\sigma) = 0$ for all $j = 1, \ldots, K_i$ or there is $j < K_i$ such that $f_i^j(\sigma) > 0$. In the first case $f_i(\sigma) \cdot \nabla_{\sigma_i} U_i(\sigma) = 0$ verifies payoff consistency (3). In the second case there exists l with $K_i - 1 \geqslant l$ such that $f_i^j(\sigma) > 0 \geqslant f_i^h(\sigma)$ for all $h \geqslant l > j$. Hence,

$$
\begin{aligned}
f_i(\sigma) \cdot \nabla_{\sigma_i} U_i(\sigma) &= \sum_{j=1}^{K_i - 1} f_i^j(\sigma)[U_i(\sigma_{-i}, s_i^j) - U_i(\sigma_{-i}, s_i^{K_i})] \\
&= \sum_{j=1}^{l-1} g_i^j(\sigma)\sigma_i^j[U_i(\sigma_{-i}, s_i^j) - U_i(\sigma_{-i}, s_i^l)] \\
&\quad + \sum_{h=l}^{K_i} g_i^h(\sigma)\sigma_i^h[U_i(\sigma_{-i}, s_i^h) - U_i(\sigma_{-i}, s_i^l)] \geqslant 0,
\end{aligned}
$$

verifies (3).

Second, let f be the vector field associated with a payoff positive selection dynamics. Then, for all $i = 1, \ldots, n$

$$
\begin{aligned}
f_i(\sigma) \cdot \nabla_{\sigma_i} U_i(\sigma) &= \sum_{j=1}^{K_i - 1} f_i^j(\sigma)[U_i(\sigma_{-i}, s_i^j) - U_i(\sigma_{-i}, s_i^{K_i})] \\
&= \sum_{j=1}^{K_i} g_i^j(\sigma)\sigma_i^j[U_i(\sigma_{-i}, s_i^j) - U_i(\sigma)] \geqslant 0
\end{aligned}
$$

verifies payoff consistency (3).

Since Remark 1 shows that the two most prominent classes of dynamics are covered, adopting payoff consistency (3) represents a sufficiently broad class of dynamics.

When dynamics on games are considered, the component $C \subset \Theta$ of zeros will be assumed to be a connected component of Nash equilibria. This is done to exhibit the relation between the noncooperative concept of the *index of a Nash equilibrium component* with evolutionary stability properties.

Definition 2. A *Nash dynamics* is a payoff consistent selection dynamics such that $f(\sigma) = 0$ if and only if $\sigma \in \Sigma$ is a Nash equilibrium. The set of all Nash dynamics is denoted $\mathscr{F}$.

Many selection dynamics on games, like the replicator dynamics, for instance, allow rest points, that are not Nash equilibria. There are two reactions to this. One is to let C simply be a connected set of zeros for the vector field, assign an index to C the same way it would be done for Nash equilibrium components, and interpret the first result as relating this index to the Euler characteristic of C. The other approach is to slightly perturb the vector field such that its zeros coincide with the Nash equilibria of the game. This has the advantage that the noncooperative concept of Nash equilibrium gets one-to-one associated with the dynamic property of being a rest point.

Remark 2. Effectively, every payoff consistent selection dynamics is homotopic to a Nash dynamics, i.e., for every payoff consistent selection dynamics f for which all Nash equilibria are zeros there is a continuous function $G : \Sigma \times [0,1] \to \mathbb{R}^K$ such that $g_\lambda \equiv G(.,\lambda)$ is a Nash dynamics for all $\lambda > 0$ and $g_0 = f$.

To see this, let f be a payoff consistent vector field on Θ such that $f^{-1}(0) \subseteq \Sigma$ and all Nash equilibria of the game belong to $f^{-1}(0)$ (but there may be more zeros). Let $b = (b_1, \dots, b_n) : \Sigma \to \mathbb{R}^K$ be a smoothed version of the dynamics introduced by Brown and von Neumann [9], i.e., for all $i = 1, \dots, n$ and $\sigma \in \Sigma$ define

$$b_i^j(\sigma) = \frac{\varphi(U_i(\sigma_{-i}, s_i^j) - U_i(\sigma)) - \sigma_i^j \sum_{h=1}^{K_i} \varphi(U_i(\sigma_{-i}, s_i^h) - U_i(\sigma))}{1 + \sum_{h=1}^{K_i} \varphi(U_i(\sigma_{-i}, s_i^h) - U_i(\sigma))},$$

for all $j = 1, \dots, K_i - 1$, where $\varphi(y) = 0$ for all $y \leqslant 0$ and $\varphi(y) = e^{-1/y}$ for all $y > 0$. Since b is based on the Nash mapping [43], $b(\sigma) = 0$ if and only if σ is a Nash equilibrium, hence, $b \in \mathscr{F}$ and $b^{-1}(0) \subseteq f^{-1}(0)$. Therefore, a family of vector fields g_λ for $\lambda \in [0,1]$ can be defined by $g_\lambda = f + \lambda b$ (with g_λ extended to Θ as in (2)), such that $g_0 = f$ and all zeros of g_λ for $\lambda > 0$ are Nash equilibria.

To see the latter, first note that, by (2), $g_\lambda^{-1}(0) \subseteq \Sigma$, for all $\lambda \in [0,1]$. Suppose there is $\bar{\sigma} \in g_\lambda^{-1}(0)$ that is not a Nash equilibrium for $\lambda > 0$. Observe that b is payoff consistent, (3), and $b_i(\sigma) \cdot \nabla_{\sigma_i} U_i(\sigma) = 0$ for all i if and only if σ is a Nash equilibrium. Then, $g_\lambda(\bar{\sigma}) = 0$ implies for all i

$$f_i(\bar{\sigma}) \cdot \nabla_{\sigma_i} U_i(\bar{\sigma}) + \lambda b_i(\bar{\sigma}) \cdot \nabla_{\sigma_i} U_i(\bar{\sigma}) = 0$$

so that $f_i(\bar{\sigma}) \cdot \nabla_{\sigma_i} U_i(\bar{\sigma}) < 0$, in contradiction to the assumption that f is payoff consistent, (3). Hence, all zeros for g_λ with $\lambda > 0$ must be Nash equilibria.

Moreover, the Jacobian $D_\sigma b$ of the vector field b at any zero (i.e. at any Nash equilibrium) is identically zero. Therefore, in first-order approximation the behavior of g_λ around equilibria is locally the same as the behavior of f, irrespective of the homotopy parameter λ, i.e., $D_\sigma g_\lambda(\bar{\sigma}) = D_\sigma f(\bar{\sigma})$ whenever $\bar{\sigma} \in \Sigma$ is a Nash equilibrium, for all $\lambda \in [0,1]$.[2]

While focussing on Nash dynamics narrows the allowed class of dynamics, Remark 2 shows that the loss of generality is small. Therefore, the component C of zeros will henceforth be taken to be a connected component of Nash equilibria, i.e., we focus on Nash dynamics $f \in \mathscr{F}$. Still, the two core results do not depend on this. Theorem 1 applies to any suitable vector field and Theorem 2 applies to arbitrary payoff consistent dynamics (see Corollary 1).

For a vector field $f \in \mathscr{F}$ on Θ, its associated flow is denoted $F_t(\sigma)$ for all $t \in \mathbb{R}$, where $\sigma = F_o(\sigma)$ is the initial condition. A set $B \subseteq \Theta$ is *invariant* (resp. forward invariant) if $F_t(\sigma) \in B$ for all $\sigma \in B$ and all $t \in \mathbb{R}$ (resp. all $t \geqslant 0$). A closed invariant set $B \subseteq \Theta$ is (Lyapunov) *stable* if for every neighborhood V'_B of B there is a

[2] First-order approximations are not used in this paper, except at this point. They may be useful in applications, e.g. when linear stability is being considered.

neighborhood V_B'' of B such that $F_t(\sigma) \in V_B'$ for all $\sigma \in V_B'' \cap \Theta$ and all $t \geqslant 0$. It is an *attractor* if there is a neighborhood V_B of B such that $\min_{\sigma \in B} ||\sigma - F_t(\sigma^o)|| \rightarrow_{t \rightarrow +\infty} 0$ for all $\sigma^o \in V_B \cap \Theta$. Finally, it is *asymptotically stable* if it is stable and an attractor.

An invariant, stable, or asymptotically stable set need *not* be a set of rest points. A set of rest points, on the other hand, is always closed and invariant. Here we focus on stability properties of sets of rest points or, more precisely, of connected components of Nash equilibria. The key concepts are as follows.

Definition 3. (a) A connected component C of Nash equilibria for Γ is (dynamically) *potentially stable* if there exists a Nash dynamics $f \in \mathscr{F}$ such that C is asymptotically stable for f.

(b) It is *robustly potentially stable* if it is potentially stable and any sufficiently small (in the $\mathscr{C}^\infty$ norm) perturbation $\tilde{f}$ of f has zeros close to C in Θ (not necessarily in Σ).[3]

The strengthening of potential to *robust* potential stability is again motivated by the desire to eliminate a dependence on the precise specification of the dynamics. After all, perturbations in the form of "drift" have been a concern in the literature on evolution (see [3]).

2.2. Index theory

Next, we turn to a classification of Nash equilibrium components. If $C \subset \Theta$ is a Nash equilibrium component, there exists a relatively compact neighborhood V of C that isolates C, i.e. such that $V \cap f^{-1}(0) = C$ for any $f \in \mathscr{F}$. This isolating neighborhood can be used to assign an *index* to the component (see [45]).

If C happens to be a *regular* zero, i.e. a point $C = \{\bar{\sigma}\}$, where the Jacobian $D_\sigma f(\bar{\sigma})$ is nonsingular, its index is given by

$$\mathrm{ind}_f\,(\bar{\sigma}) = \mathrm{sign}(|-D_\sigma f(\bar{\sigma})|), \tag{6}$$

where $|-D_\sigma f|$ denotes the determinant of (-1 times) the Jacobian $D_\sigma f$ evaluated at $\bar{\sigma}$. For an arbitrary component C, if $\tilde{f}$ is a (sufficiently small) perturbation of f that is equal to f outside of V and such that all of its zeros in V are regular, then

$$\mathrm{Ind}(C) = \sum_{\sigma \in \tilde{f}^{-1}(0) \cap V} \mathrm{ind}_{\tilde{f}}(\sigma) \tag{7}$$

provides an elementary definition of the index. Such "regular" perturbations $\tilde{f}$ of f can be shown to exist by Sard's theorem.

The subscript on $\mathrm{Ind}(C)$ has been dropped in (7), because it can be shown that it does not matter which particular vector field f is used to compute the index, as long as f is Lipschitz continuous in strategies, continuous in payoffs, does not point outwards at the boundary, and all Nash equilibria are zeros (see [13,14]). Moreover,

[3] If in a prisoners' dilemma the vector field is perturbed by adding a small positive constant to all coordinates, the zero approximating the dominant strategy equilibrium will indeed lie outside of Σ.

due to the Poincaré–Hopf theorem, the index sum across all Nash equilibrium components is a constant, the Euler characteristic $+1$ of Σ.

It can be shown that the index of a component agrees with the local degree of the projection mapping from the graph of the Nash equilibrium correspondence to the space of games (see [12,13]). Hence, the index provides a classification of Nash equilibrium components, that depends only on the (local) geometry of the equilibrium correspondence. Still, it will be shown that the index—combined with information on the topological structure of the component—also carries potential information about dynamic stability.

The required extra information, the Euler characteristic, is also easy to compute. Since every component C of Nash equilibria is a semi-algebraic set (see [5]), it admits a finite triangulation. (That is, there exists a homeomorphism from C to a polyhedron.) If r_k denotes the number of its "faces" (i.e. the simplices of the polyhedron) with dimension k, the Euler characteristic is given by the alternating sum

$$\chi(C) = \sum_{k=0}^{K} (-1)^k r_k. \tag{8}$$

The Euler characteristic is a topological invariant and, therefore, does not depend on the choice of the triangulation.

3. Potential stability

The first result says that for a component to be potentially stable its index must agree with its Euler characteristic. To develop some intuition, consider first the normal form generic case of a regular equilibrium. Its index is either $+1$ or -1. If it is asymptotically stable for some $f \in \mathscr{F}$, then all eigenvalues of the Jacobian at the equilibrium must have negative real parts. Hence, its index must be $+1$, which is the Euler characteristic of a point.

Yet, it is well known that regular equilibria are the exception rather than the rule. Since any nontrivial extensive form gives rise to payoff ties, components of Nash equilibria tend to be higher-dimensional in general.

In the general case the theorem would be more transparent to prove, if it could be assumed that C admits an *invariant* neighborhood V_C that is a manifold with boundary and deformation retracts onto C. The logic of the proof would then, very intuitively, run as follows. Since V_C would be invariant, an appropriate perturbation of f would give a vector field on V_C that points inwards at the boundary ∂V_C and has only finitely many regular zeros. By the Poincaré–Hopf theorem, the index sum across these zeros would then equal the Euler characteristic of V_C which must agree with $\chi(C)$, because C is a deformation retract of V_C. By definition, this would then agree with $\mathrm{Ind}(C)$.

For the case, where C is a smooth manifold, the existence of such an invariant *open* (tubular) neighborhood is shown by Wilson (Theorem 3.4 of [58], using techniques developed in [57]).[4] We believe that it is possible to adapt his argument to a rigorous proof in the general case, where C is not a manifold, and to modify the neighborhood such that it becomes a manifold with boundary. But, because this would require a lengthy technical digression, we instead prefer to give two self-contained proofs in the next section.

For the moment, the statement of the Theorem is presented.[5] Note that the Theorem does *not* require the vector field f to be a Nash dynamics, nor a payoff consistent dynamics. Its assumptions are much more general indeed: Let f be a Lipschitz continuous vector field on a smooth orientable K-dimensional manifold Θ and $C \subseteq f^{-1}(0) \subseteq \Theta$ a compact connected semi-algebraic set of zeros for f. Denote by F_t the flow associated with f for all $t \in \mathbb{R}$ and by $\chi(C)$ the Euler characteristic of C.

Theorem 1. *If C is asymptotically stable for F_t, then* $\chi(C) = \mathrm{Ind}(C)$.

For evolutionary selection dynamics on games Theorem 1 means that, if a component of Nash equilibria is *potentially stable*, then its index equals its Euler characteristic. The converse of Theorem 1 is not true. Example 2 in Section 4 illustrates this.

In the following subsection Theorem 1 is proved. Readers, who are not primarily interested in the details of the proof, may skip the subsection and return to it later.

3.1. Proof of Theorem 1

Two conceptually equivalent proofs are provided. The first assumes familiarity with an axiomatic approach to index theory, as described by McLennan [37].[6] The second requires some knowledge of elementary homology theory. But the two proofs differ only in the last step. Therefore, we start with their common part, showing that the vector field around C can be homotopically deformed into a retraction onto C.

Let C be the component of zeros for f and denote by $F_t(\sigma)$ the associated flow at time t for the initial condition $\sigma \in \Theta$. Choose open neighborhoods $V_1 \supseteq V_2 \supseteq V_3$ of C with compact closures such that

(a) C is a deformation retract of V_3, i.e., there is a family of maps $r_\lambda : V_3 \to V_3$ for $\lambda \in [0, 1]$ such that r_0 is the identity, $r_1 : V_3 \to C$, and the restriction $r_\lambda | C$ is the identity for all $\lambda \in [0, 1]$;
(b) the closure $\bar{V}_3$ of V_3 is a smooth manifold with boundary;
(c) $F_t(\sigma) \in V_1$ for all $\sigma \in V_2$ and all $t \geqslant 0$;
(d) there is $T > 0$ such that $F_t(\sigma) \in V_3$ for all σ in the closure $\bar{V}_2$ of V_2 and all $t \geqslant T$;
(e) there is a constant $\delta > 0$ such that $||f(\sigma)|| \geqslant \delta$ for all $\sigma \in V_1 \setminus V_3$.

[4] We are grateful to an anonymous referee for pointing out this reference.

[5] The following Theorem is the result quoted in the CORE Working Paper version of Demichelis and Germano [12] (CORE DP 2000/17, Corollary 2).

[6] We are grateful to an anonymous referee for pointing out this reference and suggesting a simplification in the proof.

That V_1 and V_2 can be chosen such that (c) holds, follows from Lyapunov stability of C. That V_3 can be chosen such that (a) and (b) hold, follows from the assumption that f is semi-algebraic. That (d) holds, follows from C being an attractor (as defined at the end of Section 2.1): for every $\sigma \in \bar{V}_2$ there is $t_\sigma \geqslant 0$ such that $F_t(\sigma) \in V_3$ for all $t > t_\sigma$; because $\bar{V}_2$ is compact, $\sup_{\sigma \in \bar{V}_2} t_\sigma$ is finite and yields T. That V_1 and V_3 can be chosen such that (e) holds, follows from the fact that V_1 can be chosen inside an isolating compact neighborhood of C that contains no other zeros than those in C.

Let $V = \cup_{t \geqslant 0} F_t(V_2) = \cup_{t \geqslant 0} \cup_{\sigma \in V_2} F_t(\sigma)$, $\bar{V}$ its closure, and $\partial V = \bar{V} \backslash V$. By construction, $V_1 \supseteq \bar{V} \supseteq V_3 \supset C$, $F_t(\bar{V}) \subseteq \bar{V}$ for all $t \geqslant 0$, and $F_t(\bar{V}) \subseteq V_3$ for all $t \geqslant T$. (Note that $\bar{V}$ is not necessarily a smooth manifold with boundary.) Because $\partial V \cap V_3 = \emptyset$, we have $f|\partial V \neq 0$. Moreover, F_t on $\bar{V} \backslash V_3$ has no periodic orbits (i.e., $F_t(\sigma) \neq \sigma$ if $t \neq 0$ for all $\sigma \in \bar{V} \backslash V_3$), because otherwise $F_T(\bar{V})$ could not be contained in V_3, as implied by (d). Now, let $R : \bar{V} \to C$ be defined by $R(\sigma) = r_1(F_T(\sigma))$ for all $\sigma \in \bar{V}$.

Lemma 1. *There is a homotopy* $h : \bar{V} \times [0, 1] \to C$ *between* $h(\sigma, 0) = f(\sigma)$ *and* $h(\sigma, 1) = R(\sigma) - \sigma$ *such that* $h(\sigma, \lambda) \neq 0$ *for all* $\sigma \in \bar{V} \backslash V_3$ *and all* $\lambda \in [0, 1]$.

Proof. First, f is clearly homotopic to εf for $\varepsilon > 0$. Second, εf is homotopic to $(F_\varepsilon - \mathrm{id})$ for sufficiently small $\varepsilon > 0$: by Lipschitz continuity, there is a constant γ such that $\gamma\varepsilon^2 \geqslant ||\varepsilon f(\sigma) - (F_\varepsilon(\sigma) - \sigma)||$; choosing ε small enough such that $\gamma\varepsilon^2 < \delta\varepsilon/2$ implies that the segment between $\varepsilon f(\sigma)$ and $F_\varepsilon(\sigma) - \sigma$ does not contain a zero as long as $\sigma \in \bar{V} \backslash V_3$, by (e).

Third, $(F_\varepsilon - \mathrm{id})$ is homotopic to $(F_T - \mathrm{id})$ via $(F_t - \mathrm{id})$. Because there are no periodic orbits in $\bar{V} \backslash V_3$, we conclude $F_t(\sigma) - \sigma \neq 0$ for all $\sigma \in \bar{V} \backslash V_3$. Fourth, $(F_T - \mathrm{id})$ is homotopic to $(R - \mathrm{id})$ via $r_\lambda \circ F_T - \mathrm{id}$. Because $r_\lambda(F_T(\sigma)) \in V_3$ for all $\sigma \in \bar{V} \backslash V_3$, it follows that $r_\lambda(F_T(\sigma)) - \sigma \neq 0$ for all $\sigma \in \bar{V} \backslash V_3$ and all $\lambda \in [0, 1]$. Glueing these homotopies together verifies the statement of the lemma. □

This first step shows that the vector field around C can be homotopically deformed into a retraction onto C. It remains to show in the second step that the index of this retraction equals the Euler characteristic of C. It is for this second and last step that two versions are being offered.

We begin with the approach by axiomatic index theory. The homotopy identified in Lemma 1 gives a homotopy from $f + \mathrm{id}$ to R with no fixed points on $\partial V = \bar{V} \backslash V \subseteq \bar{V} \backslash V_3$. Now, given an open subset V of a topological space X and a map $V \to X$ or an upper hemi-continuous correspondence $M : \bar{V} \to X$ with no fixed points in ∂V, a Lefschetz fixed point index $\Lambda(M, V)$ is defined in McLennan ([37, p. 7]). There it is shown in Section 2.3 that, if V is as here, $X = \mathbb{R}^K$, and $M = f + \mathrm{id}$, then this definition agrees with the elementary definition of the index given in (7), i.e. $\Lambda(f + \mathrm{id}, V) = \mathrm{Ind}(C)$.

Now apply the six axioms given by McLennan [37, p. 17]. First, by the homotopy between f and $(R - \mathrm{id})$ identified in Lemma 1, axiom I3 implies $\Lambda(f + \mathrm{id}, V) = \Lambda(R, V)$. Second, if $\imath$ denotes the inclusion $\imath : C \hookrightarrow V$, one has $R \circ \imath = \mathrm{id}_C$ and $\imath \circ R = R$;

therefore, by commutativity (axiom I4),

$$\Lambda(R, V) = \Lambda(\imath \circ R, V) = \Lambda(R \circ \imath, C) = \Lambda(\mathrm{id}, C).$$

Now, $\Lambda(\mathrm{id}, C)$ is McLennan's definition of the Euler characteristic of C (see [37, Definition 3.2, p. 32]). On p. 31 of McLennan [37] it is claimed that his definition of a Lefschetz number agrees with the standard one, viz. $L(f) = \sum_i (-1)^i \operatorname{trace} f_* | H_i$ (where f_* denotes the homomorphism on homology groups H_i induced by f); so, the Euler characteristic defined by McLennan [37, p. 32] will be $\sum_i (-1)^i \dim H_i(C)$. That the latter agrees with the elementary definition given in (8) is well known. It follows that $\Lambda(f + \mathrm{id}, V) = \chi(C)$ which completes the first variant of the proof.

Next, we give a direct proof in terms of homology. It is well known that the index can be defined as the integer $\mathrm{Ind}_f\,(C)$ that solves $f_*(\alpha) = \mathrm{Ind}_f\,(C)\beta$, where $f_* : Z \simeq H_K(\Theta, \Theta \backslash C) \to H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}) \simeq Z$ is the homomorphism induced on relative homology by f, and $\alpha \in H_K(\Theta, \Theta \backslash C)$ and $\beta \in H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\})$ are the respective orientation classes (and Z denotes the integers). The inclusion $(\Theta, \Theta \backslash V_3) \hookrightarrow (\Theta, \Theta \backslash C)$ is a homotopy equivalence, so the groups $H_K(\Theta, \Theta \backslash V_3)$ and $H_K(\Theta, \Theta \backslash C)$ are isomorphic. Therefore, one can think of α as an element in $H_K(\Theta, \Theta \backslash V_3)$. Now, the homomorphism f_* factorizes through

$$H_K(\Theta, \Theta \backslash V_3) \xrightarrow{e} H_K(\bar{V}, \bar{V} \backslash V_3) \xrightarrow{f_*} H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}),$$

where e is induced by excision. Let $e(\alpha) = \mu \in H_K(\bar{V}, \bar{V} \backslash V_3)$; since f is homotopic to $(R - \mathrm{id})$ by Lemma 1, this composition is the same as

$$H_K(\Theta, \Theta \backslash V_3) \xrightarrow{e} H_K(\bar{V}, \bar{V} \backslash V_3) \xrightarrow{(R-\mathrm{id})_*} H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}),$$

i.e. $(R - \mathrm{id})_*(\mu) = \mathrm{Ind}_f\,(C)\beta$. Since $\bar{V}$ may not be a manifold, we aim at replacing it by V_3: If $\imath$ denotes the inclusion of pairs $\imath$: $(\bar{V}, \bar{V} \backslash V_3) \hookrightarrow (\bar{V}, \bar{V} \backslash C)$ and $\imath_*(\mu) = \mu'$, $(R - \mathrm{id})_*(\mu') = (R - \mathrm{id})_*(\mu) = \mathrm{Ind}_f\,(C)\beta$, because of the commutativity of the diagram

$$\begin{array}{ccc} \mu \in H_K(\bar{V}, \bar{V} \backslash V_3) & \xrightarrow{(R-\mathrm{id})_*} & H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}) \\ \downarrow \imath_* & & \downarrow \mathrm{id} \\ \mu' \in H_K(\bar{V}, \bar{V} \backslash C) & \xrightarrow{(R-\mathrm{id})_*} & H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}) \end{array}$$

Next, if $e' : H_K(\bar{V}, \bar{V} \backslash C) \to H_K(\bar{V}_3, \bar{V}_3 \backslash C)$ is the isomorphism induced by excision and $e'(\mu') = \mu''$, one has $(R - \mathrm{id})_*(\mu'') = (R - \mathrm{id})_*(\mu') = \mathrm{Ind}_f\,(C)\beta$. Because the diagram

$$\begin{array}{ccc} \mu' \in H_K(\bar{V}, \bar{V} \backslash C) & \xrightarrow{(R-\mathrm{id})_*} & H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}) \\ \downarrow e' & & \downarrow \mathrm{id} \\ \mu'' \in H_K(\bar{V}_3, \bar{V}_3 \backslash C) & \xrightarrow{(R-\mathrm{id})_*} & H_K(\mathbb{R}^K, \mathbb{R}^K \backslash \{0\}) \end{array}$$

commutes, $(R-\mathrm{id})_*(\mu')$ can be computed as the Lefschetz index of the map R on the smooth manifold with boundary $\bar{V}_3$ (rather than on $\bar{V}$).

By the Lefschetz fixed point theorem for manifolds with boundary (see [31, Proposition 8.4.6]), $(R-\mathrm{id})_*(\mu'') = L\beta$, where $L = \sum_{i=0}^{K}(-1)^i \mathrm{trace}\, R_*|H_i(\bar{V}_3)$ is the associated Lefschetz number. But V_3 deformation retracts onto C, so $H_i(\bar{V}_3) \simeq H_i(C)$ for all $i = 0, \ldots, K$, and R_* is the identity on C, so $L = \sum_{i=0}^{K}(-1)^i \dim H_i(V_3)$ which is $\chi(C)$. This completes the second variant of the proof. □

3.2. *Rationalistic implications*

That a component is potentially stable may be a knife-edge phenomenon. Consider, for instance, a potentially stable component with *zero* index. The definition of the index, (7), suggests that the vector field (for which the component is asymptotically stable) may be slightly perturbed so as to remove all zeros close to the component. For example, consider a component that is homeomorphic to a circle. (The circle has Euler characteristic zero, so this is compatible with equality of Euler characteristic and index.) The vector field could be slightly modified such as to induce a slow motion along the circle, removing all rest points.

For evolutionary predictions this would cause a problem. If a component *can* be asymptotically stable, but only in exceptional cases, then the prediction requires deep trust in the precise specification of the dynamics. Unless the application justifies such confidence, strengthening the criterion to *robust potential stability* seems natural. Such robustness yields a first connection to rationalistic criteria.

An equilibrium component is *essential*, roughly, if every nearby game has an equilibrium close to the component (see [29]).[7] Obviously, a robustly potentially stable component must be essential. For, if it would be potentially stable, but not essential, then around the component the dynamics could be modified towards the dynamics of a nearby game with no equilibrium (rest point) close to the component. Hence, it would not be robustly potentially stable. The next result makes this precise.

Proposition 1. *Any robustly potentially stable component has nonzero Euler characteristic and is essential.*[8]

Proof. Assuming that the equilibrium component C is asymptotically stable for $f \in \mathscr{F}$, we show that if $\chi(C) = 0$, then there is an arbitrary small perturbation $\tilde{f}$ of f such that $\tilde{f}^{-1}(0) \cap V$ is empty, where V is the isolating neighborhood of C.

[7] This is a somewhat different meaning of "essential" than in the theory of fixed points. In the latter not only payoffs of the game are perturbed, but the whole map (see [37]).

[8] A partial converse of the first part of Proposition 1 is obvious. If C is potentially stable and $\chi(C) \neq 0$, then Ind $(C) \neq 0$ by Theorem 1 and any perturbation of f must have zeros in V by the definition of the index, (7).

By the hypothesis and Theorem 1, Ind $(C)=0$. Let $\hat{f}$ be a small perturbation of f such that $\hat{f}=f$ outside $\mathrm{int}(V)$ and

$$\hat{f}^{-1}(0)\cap V=\{\sigma^1,\ldots,\sigma^m\}\subseteq \mathrm{int}(V),$$

where the σ^h's are regular zeros for all $h=1,\ldots,m$. By (7) $\sum_{h=1}^m \mathrm{ind}_{\hat{f}}(\sigma^h)=0$ must hold. By Lemma 2.9 of Hirsch [26, Chapter 5, p. 137] there is a function $g:V\to\mathbb{R}^K\backslash\{0\}$ such that $g=f$ on ∂V. Finally, glue g and f on ∂V and smooth the result by a small perturbation around ∂V such that no zeros are introduced. The resulting map is $\tilde{f}$.

Hence, if C is robustly potentially stable, then $\chi(C)\neq 0$. Thus, by Theorem 1, Ind $(C)\neq 0$ and, therefore, C is essential by Theorem 4 of Ritzberger [45]. □

That robust potential stability has rationalistic implications is to be expected. Many noncooperative refinement concepts, including strategic stability, are defined by robustness criteria, albeit mostly in strategy perturbations. What is less obvious is that robust potential stability is *more* than what is needed for even the strongest rationalistic criterion.

The next theorem adds to potential stability only the condition that the Euler characteristic of the component is nonzero, and continuity in payoffs. Proposition 1 shows that this is a weaker hypothesis than robust potential stability. Yet, it is sufficient. By Theorem 1 potential stability and a nonzero Euler characteristic imply that the index of the component is nonzero. And this is sufficient for the component to contain an M-stable set.

Theorem 2. *If C is asymptotically stable in a Nash dynamics $f\in\mathscr{F}$, that is continuous in payoffs,*[9] *and $\chi(C)\neq 0$, then C contains an M-stable set.*

Proof. By potential stability and Theorem 1 the index of C agrees with $\chi(C)$ which is nonzero by hypothesis. Since, by continuity in payoffs, the index equals the local degree (see [12,13]), the local degree is also nonzero. Then the proof of existence for strategically stable sets [39, Theorem 1] can be adapted as follows.

Let D be a sufficiently small ball around $u=((u_i(s))_{s\in S})_{i=1}^n$, the payoff vector for the game Γ, and $\mathscr{G}$ the graph of the Nash equilibrium correspondence. Let N be a neighborhood of $C\times\{u\}$ in the ambient space such that $N_D=\mathscr{G}\cap N$ constitutes a neighborhood of C in $\mathscr{G}$ which projects onto D. By the definition of the local degree this projection is homologically nontrivial. Let W be the space of sufficiently small strategy perturbations for the game Γ and $\mathscr{G}_W$ the graph of the Nash equilibrium correspondence on W. Since strategy trembles are particular payoff perturbations, $W\subseteq D$ and $N'_W=\mathscr{G}_W\cap N_D$ is the part of $\mathscr{G}_W$ that is close to C. The projection of N'_W to W is not homologous to zero, due to the basic result of Mertens ([38], explained in Remark 2 of Section 2 and the discussion of Theorem 1 in Mertens [39]).

[9] Up to this point only (Lipschitz) continuity in strategies has been assumed. The added assumption, therefore, concerns continuity in payoff parameters.

Replacing $\mathscr{G}_W$ by N'_W, the existence part of the proof for Theorem 1 of Mertens [39] can now be applied.

Briefly, the argument works as follows. Consider the projection map $P:(N'_W, \partial N'_W) \to (W, \partial W)$ which is nontrivial in homology. Let $N_1, N_2, \ldots, N_k$ be the connected components of $N'_W \backslash \partial N'_W$. Once perturbations are small enough, the number of components is constant, because everything is semi-algebraic. Let $\bar{N}_j$ be the closure of N_j in N'_W and $\partial N_j = \bar{N}_j \backslash N_j$ for all j. By the excision axiom $H_*(N'_W, \partial N'_W) \simeq \oplus_j H_*(\bar{N}_j, \partial N_j)$. So, at least one of the maps $P:(\bar{N}_j, \partial N_j) \to (W, \partial W)$ is homologically nontrivial. Choosing this particular component, $\bar{N}_j$ is a closed connected set of equilibria (for the perturbed games) which projects nontrivially on perturbation space. Hence, its (Hausdorff) limit is the set required in the definition of M-stable sets. □

A converse of Theorem 2 is not true. A component can contain an M-stable set and have nonzero Euler characteristic, but may still not be potentially stable. Example 2 in Section 4 illustrates this possibility.

Theorem 2 constitutes strong evolutionary support for the rationalistic paradigm, when robustly potentially stable components exist. In this case the evolutionary prediction agrees with the strongest known noncooperative criterion. Unless a component contains an M-stable set, it cannot be robustly potentially stable.

Indeed, Theorem 2 extends easily to all payoff consistent dynamics (that may have rest points outside the set of Nash equilibria). Formally:

Corollary 1. *If C is a connected component of rest points, that is asymptotically stable in some payoff consistent dynamics for which all Nash equilibria are zeros, and that is continuous in payoffs, and if C satisfies $\chi(C) \neq 0$, then it contains an M-stable set.*

Proof. By Theorem 1 $\text{Ind}_f\,(C) = \chi(C) \neq 0$, where $\text{Ind}_f\,(C)$ is computed with respect to the payoff consistent dynamics f. Let $\tilde{f}$ be a perturbation of f such that $\tilde{f}$ is a Nash dynamics (see Remark 2). Let $C_1, \ldots, C_k$ be the components of Nash equilibria contained in C. By the invariance of the index

$$\sum_{j=1}^{k} \text{Ind}\,(C_j) = \text{Ind}_f\,(C)$$

holds. Therefore, for at least one j we have $\text{Ind}\,(C_j) \neq 0$. Since Theorem 2 effectively shows that every Nash equilibrium component with nonzero index contains an M-stable set, it can now be applied directly. □

When Γ is the normal form of some extensive form game, this provides an evolutionary foundation for backwards induction, because any M-stable set contains a proper equilibrium, and any proper equilibrium induces a sequential equilibrium in any compatible extensive form (see [51,32, Proposition 0]).

Corollary 2. *Any robustly potentially stable component, that is asymptotically stable for a Nash dynamics, that is continuous in payoffs, contains a proper (and, hence, sequential) equilibrium.*

For the normal form of a generic perfect information extensive form game there is a *unique* component, that induces the backwards induction outcome. All other equilibrium components must have index zero, because otherwise they would contain an M-stable set (this is what the proof of Theorem 2 shows) and, hence, a proper (sequential) equilibrium. Therefore, the unique backwards induction component has index +1. Moreover, all components for such games are contractible (see [15]) and, thus, have Euler characteristic +1. Since the backwards induction component is the unique one with index +1, it always meets the necessary condition for potential stability. So, in these games either evolution unambiguously supports backwards induction or there is no equilibrium outcome supported by potential stability. This simple observation reproduces exactly, for all payoff consistent selection dynamics, the result that Cressman and Schlag [11] obtain for the replicator dynamics.[10]

Of course, by Corollary 1, in the statement of Corollary 2 "robustly potentially stable" can be replaced by "asymptotically stable in some payoff consistent dynamics and nonzero Euler characteristic".

4. Examples

Example 1. The first (class of) example(s) illustrates both the cutting power of potential stability and its possible failure of existence. Consider "outside option games" with unique forward induction equilibria. These are two-player games where player 1 can first choose either an outside option, or to move into a (finite) simultaneous move subgame. The subgame is assumed to have only one equilibrium, that yields player 1 more than the outside option (see [23,53]). Such a game has two components of equilibria, one where player 1 moves into the subgame and his preferred equilibrium is played—the "forward induction" equilibrium—and a higher-dimensional component where player 1 takes the outside option.

The main argument about such games is that, if the "forward induction" equilibrium is "viable" [53], then player 2 should conclude from the fact, that she gets to move, that player 1 intends to play her preferred equilibrium. Such a forward induction argument, of course, depends on what "viable" means. In the present context a straightforward interpretation is suggested.

Consider the generic case where the "forward induction" equilibrium is regular. Then it has either index +1 or −1. If it has index +1 (e.g. because it is strict), then the

[10] Hart [22] constructs a finite-population stochastic discrete-time dynamics under which the backwards induction component of a perfect information game is the unique stochastically stable outcome. The continuous-time and large-population limit of this dynamics, however, is not necessarily continuous.

outside option component must have index zero. If it has index -1, the outside option component has index $+2$ (for an example see [23, Fig. 8]). But the outside option component is convex, because it consists of all strategies for player 2, that do not induce player 1 to enter. Therefore, the outside option component has Euler characteristic $+1$. By Theorem 1 the outside component cannot be potentially stable in either case. In the first case potential stability uniquely selects the "forward induction" equilibrium. In the second case, where the equilibrium of the subgame is mixed, there is no potentially stable component.

These simple calculations mimic the results by Hauk and Hurkens [23]. And the suggested interpretation of a "viable" forward induction equilibrium is that it be potentially stable.

Example 2. The second example (due to Hofbauer and Swinkels [28]) also serves double purpose. Consider the three-player game in Table 1, parameterized by $q \in [0, 1]$. (Player 1's payoff is in the upper left, 2's in the middle, and 3's in the lower right corner.)

First, let $q = 0$. Then the game has two components of equilibria. The first, C_1, is a singleton where all players use all their strategies with probability $\frac{1}{2}$. The second, C_2, is homeomorphic to a circle and connects, by the corresponding edges, the pure strategy combinations (s_1^2, s_2^2, s_3^2), (s_1^2, s_2^1, s_3^2), (s_1^1, s_2^1, s_3^2), (s_1^1, s_2^1, s_3^1), (s_1^1, s_2^2, s_3^1), (s_1^2, s_2^2, s_3^1), back to (s_1^2, s_2^2, s_3^2). (Hence, C_2 does *not* satisfy the topological condition used by Swinkels [49].)

We claim that the singleton component $C_1 = \{\bar{\sigma}\}$ cannot be potentially stable, but that C_2 is. At $q = 0$ all three players have the same payoff function, that serves as a Lyapunov function. By payoff consistency, (3),

$$\frac{dU_i(\sigma)}{dt} = \sum_{j=1}^{3} f_j(\sigma) \cdot \nabla_{\sigma_j} U_i(\sigma) = \sum_{j=1}^{3} f_j(\sigma) \cdot \nabla_{\sigma_j} U_j(\sigma) \geqslant 0$$

for all i. Choose $\sigma \in \operatorname{int} \Sigma$ arbitrary close to $\bar{\sigma}$ such that $U_i(\sigma) > -\frac{1}{4} = U_i(\bar{\sigma})$. From such an initial condition the trajectory can never converge to $\bar{\sigma}$, because the payoff cannot decrease. Hence, $C_1 = \{\bar{\sigma}\}$ is *not* potentially stable. On the other hand, since C_2 constitutes the unique set of payoff maximizing strategy combinations where

Table 1

s_3^1:

	s_2^1	s_2^2
s_1^1	$0,\ -q,\ 0$	$-q,\ 0,\ 0$
s_1^2	$-1,\ -1,\ -1$	$0,\ 0,\ -q$

s_3^2:

	s_2^1	s_2^2
s_1^1	$0,\ 0,\ -q$	$-1,\ -1,\ -1$
s_1^2	$-q,\ 0,\ 0$	$0,\ -q,\ 0$

$U_i(\sigma) = 0$ for all i, the component C_2 is asymptotically stable for any Nash dynamics for which (3) holds with strict inequality outside the set of Nash equilibria.

It follows from Theorem 1, $\chi(C_2) = 0$, and the additivity of the index that Ind $(C_2) = 0$ and Ind $(C_1) = +1$. This shows that the converse of Theorem 1 is false. For $C_1 = \{\bar{\sigma}\}$ the index agrees with the Euler characteristic, $\chi(C_1) = +1$, but C_1 cannot be potentially stable.

This example also shows that a converse of Theorem 2 is false. The component $C_1 = \{\bar{\sigma}\}$ is a singleton M-stable set (because it is completely mixed) with nonzero index and Euler characteristic (both equal to $+1$), but it is not potentially stable.

Whether a potentially stable component with zero Euler characteristic may occasionally contain an M-stable set we do not know. But we see no reason why there should not be such cases.[11]

Certainly, a potentially stable component with zero Euler characteristic *need not* contain an M-stable set. The easiest way to see this is to let $q \in [0, 1]$ be the (mixed) strategy of a fourth player (with two pure strategies) for whom $q = 0$ is strictly dominant. By payoff consistency, (3), $dq/dt < 0$ must hold for any Nash dynamics, so C_2 remains asymptotically stable and equilibria and the index calculations remain as before.

But now C_2 does clearly *not* contain an M-stable set. To test for strategic stability *strategy trembles* have to be considered. Yet, if $q > 0$ then there is no Nash equilibrium (of the game among players $i = 1, 2, 3$) close to C_2. (The only equilibrium for $q > 0$ is $\bar{\sigma}$.) Note that all equilibria in C_2 are perfect and, indeed, proper (because each player has only two pure strategies). Still, C_2 fails the test for M-stability.

This shows that in Theorem 2 the hypothesis of a nonzero Euler characteristic is *necessary*. With four players the component C_2 is potentially stable, but has zero Euler characteristic and does *not* contain an M-stable set. Indeed, by slightly modifying the vector field the movement along the circle for $q > 0$ can be extended to $q = 0$ so as to remove all zeros at C_2. So, C_2 is *not* robustly potentially stable.

Example 3. The last example illustrates the limitations, that arise from using *asymptotic* stability as the relevant dynamic stability criterion. In the two-player game in Table 2 (due to Kohlberg and Mertens [32, p. 1034]) the set of all Nash equilibria is a single connected component, that is again homeomorphic to a circle.

It consists of the edges connecting the pure strategy combinations (s_1^1, s_2^1) and (s_1^1, s_2^3), (s_1^1, s_2^3) and (s_1^2, s_2^3) , (s_1^2, s_2^3) and (s_1^2, s_2^2), (s_1^2, s_2^2) and (s_1^3, s_2^2), (s_1^3, s_2^2) and (s_1^3, s_2^1), and back again from (s_1^3, s_2^1) to (s_1^1, s_2^1). Hence, the only component of equilibria has index $+1$ and Euler characteristic zero. By Theorem 1 it cannot be potentially stable.

Yet, both the second and third strategy for each player is weakly dominated (by the first). It is known (see [56, Proposition 5.8]) that, if a weakly dominated strategy

[11] For the present example (with $q = 0$) the component C_2 contains a strategically stable set in the sense of Kohlberg and Mertens (1986). The argument demonstrating this claim is available from the authors upon request.

Table 2

	s_2^1	s_2^2	s_2^3
s_1^1	1, 1	0, −1	−1, 1
s_1^2	−1, 0	0, 0	−1, 0
s_1^3	1, −1	0, −1	−2, −2

does not vanish along an interior solution path to a payoff-linear selection dynamics, then the opponent's strategy against which it does worse (than the dominating strategy) must vanish along that path. Hence, if σ_i^2 does not converge to zero, then σ_{3-i}^1 must converge to zero, for $i = 1, 2$ and any interior trajectory. Likewise, if $\sigma_i^1 + \sigma_i^2$ does not converge to 1, then $\sigma_{3-i}^1 + \sigma_{3-i}^2$ must converge to 1, for $i = 1, 2$ along any interior trajectory.

If σ_i^2 converges to zero along an interior path, then in a payoff-linear dynamics σ_i^1 must converge to 1. But this implies that σ_{3-i}^2 converges to zero, so that $\lim_{t\to\infty} \sigma_i = (1, 0)$ and $\lim_{t\to\infty} \sigma_{3-i} = (y, 0)$ for $y \in [0, 1]$, for $i = 1, 2$. Hence, the limit point is a Nash equilibrium for any interior trajectory where σ_i^2 converges to zero, for $i = 1, 2$. If σ_i^2 does not converge to zero along an interior path, then σ_{3-i}^1 must converge to zero, for $i = 1, 2$. But if σ_{3-i}^1 converges to zero, then in a payoff-linear dynamics the growth rates of σ_i^1 and σ_i^2 become identical and nonnegative. Thus, $\sigma_i^1 + \sigma_i^2$ converges to 1 along an interior path. But then either $\sigma_{3-i}^2 \to 0$ or $\sigma_i^1 \to 0$. In the fist case we are back to the previous argument and conclude that the limit point is a Nash equilibrium. If $\sigma_i^1 \to 0$, then all limit points sit on the edges connecting (s_1^2, s_2^2) with (s_1^2, s_2^3) or with (s_1^3, s_2^2). Those are again all Nash equilibria.

The conclusion is that all interior paths will converge to a Nash equilibrium. Yet, Theorem 1 asserts that the set of Nash equilibria cannot be asymptotically stable. This is due to a failure of Lyapunov stability.

Consider the face where $\sigma_i^1 + \sigma_i^2 = 1$ for $i = 1, 2$ and, say, the replicator dynamics. The latter becomes $\dot{\sigma}_i^1 = 2\sigma_i^1(1 - \sigma_i^1)\sigma_{3-i}^1$, so $\sigma_i^1 = 0$ implies $\dot{\sigma}_i^1 = \dot{\sigma}_{3-i}^1 = 0$ for $i = 1, 2$. This means that the edges connecting (s_1^2, s_2^2) with (s_1^1, s_2^2) and with (s_1^2, s_2^1) consist entirely of zeros. Therefore, trajectories starting on these edges do not converge to (s_1^2, s_2^2) and the set of Nash equilibria is *not* asymptotically stable in the replicator dynamics.

If the replicator dynamics would be slightly modified, as in Remark 2, then the movement along the edges connecting (s_1^2, s_2^2) with (s_1^1, s_2^2) and with (s_1^2, s_2^1) would be away from (s_1^2, s_2^2). Hence, a trajectory starting close to (s_1^2, s_2^2) would leave a neighborhood of the component, cross the interior of the face, and eventually return to a Nash equilibrium "at the other end", close to (an edge containing) (s_1^1, s_2^1).

While in this example the only component of Nash equilibria cannot be potential stable, there is still a sense in which evolutionary (payoff-linear) dynamics lend support to Nash equilibrium. The unique Nash equilibrium component is an attractor for all *interior* trajectories. Theorem 1 only points out that it fails Lyapunov stability (for a related example see [56, Example 3.6, p. 90]).

This example also shows that Lyapunov stability is necessary for Theorem 1. The slight perturbation of the replicator dynamics pointed out above would make the unique equilibrium component an attractor. Yet, its index (+1) does not agree with its Euler characteristic (0).

5. Conclusions

This paper first identifies a necessary condition for dynamic evolutionary stability of a component of Nash equilibria. If there exists a dynamics for which a given component is asymptotically stable, then the component's index must agree with its Euler characteristic. Second, if moreover the component's Euler characteristic is nonzero, then it will contain a strategically stable set in the sense of Mertens ([39,40]). This is the weakest hypothesis on dynamic evolutionary stability so far identified which implies the strongest known rationalistic refinement criterion.

If evolutionary dynamics are meant to be a selection criterion among Nash equilibria, the present results provide strong cutting power. In generic normal form games roughly half of the equilibria fail to be potentially stable, despite meeting all rationalistic refinement criteria. Moreover, potential stability selects among equilibria in classes of games, that have motivated certain refinement concepts. In two-player outside option games either the "forward induction" equilibrium is potentially stable, or no component is. Likewise, in generic perfect information games either the subgame perfect equilibrium outcome is potentially stable, or no component is. In signaling games it suffices to find equilibrium components with index +1 to identify the potentially stable outcomes.

Yet, the results also highlight that evolutionary stability may be overly selective. There are games, that do not have any potentially stable Nash equilibrium component.

Acknowledgments

We thank Jean-François Mertens, Josef Hofbauer, and two anonymous referees for helpful comments, and CORE and the Institute for Advanced Studies, Vienna, for their hospitality and support. The usual disclaimer applies.

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