Before proving that Nash equilibria in mixed strategies exist, we need a theorem that a fundamental component of many equilibrium existence proofs.

1. Brouwer Fixed Point Theorem

BROUWER FIXED POINT THEOREM. *Let $S \subset \mathbb{R}^n$ be convex and compact. If $T\colon S \to S$ is continuous, then there exists a fixed point. I.e., there exists $x^* \in S$ such that $T(x^*) = x^*$.*

ONE-DIMENSIONAL CASE. I won't prove the general case. However, the one-dimensional case is much easier. We saw a diagrammatic argument in class. Here is the proof. When $n = 1$, the only compact and convex sets are closed intervals $[a, b]$. Let $T\colon [a, b] \to [a, b]$. If $T(a) = a$ or $T(b) = b$, we are done. Now suppose $T(a) > a$ and $T(b) < b$. Consider $g(x) = T(x) - x$. Then $g(a) > 0$ and $g(b) < 0$. The function g is continuous because T is continuous. The Intermediate Value Theorem (from calculus) tells us that there is some x^*, $a < x^* < b$, where $g(x^*) = 0$. This x^* is the required fixed point. □

2. Existence of Nash Equilibrium

Consider a two-person finite game in strategic form. Label the strategies of player 1 by $1, 2, \ldots, m$ and label player two's strategies by $1, 2, \ldots, n$. Define the k-simplex by $\Delta_k = \{x \in \mathbb{R}^{k+1}_+ : \sum_{i=1}^{k+1} x_i = 1\}$. We can regard any mixed strategy of player one as a point Δ_{m-1} and any mixed strategy of player two as a point in Δ_{n-1}. The payoff functions are defined on $S = \Delta_{m-1} \times \Delta_{n-1}$, which is the product of the strategy sets.

Given strategies $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$, the expected payoff are $Eu_\ell(p, q) = \sum_{i=1}^m \sum_{j=1}^n p_i q_j u_\ell(i, j)$. We define matrices A and B by $a_{ij} = u_1(i, j)$ and $b_{ij} = u_2(i, j)$. The expected payoffs can then be written $Eu_1(p, q) = p'Aq$ and $Eu_2(p, q) = p'Bq$ where p and q are regarded as column vectors and the prime denotes transpose.

We will denote the ith column of A by A_i and the jth row of B by B_j. Thus $A_i q$ gives the expected payoff to player one when playing the pure strategy i against player two's mixed strategy q. Similarly, $p'B_j$ gives the expected payoff to player two when playing the pure strategy j against player one's mixed strategy p.

THEOREM 1. *Every two-person finite game has a Nash equilibrium in mixed strategies.*

PROOF. Define $c_i(p, q) = \max\{A_i q - p'Aq, 0\}$ for $i = 1, \ldots, m$ and $d_j(p, q) = \max\{p'B_j - p'Bq, 0\}$ for $j = 1, \ldots, n$. Thus c_i and d_j represent the gain (if any) from switching to the pure strategy i (j) from the mixed strategy p (q).

Now define functions

$$P_i(p, q) = \frac{p_i + c_i(p, q)}{1 + \sum_{k=1}^m c_k(p, q)}$$
$$Q_j(p, q) = \frac{q_j + d_j(p, q)}{1 + \sum_{k=1}^n d_j(p, q)}$$

Since $A_i q - p'Aq$ and $p'B_j - p'Bq$ are linear functions, they are continuous. Moreover, the maximum of two continuous functions is continuous, which implies both c_i and d_j are continuous. Finally, both numerator and denominator of P_i and Q_i are continuous and the denominator is strictly positive. It follows that P_i and Q_i are continuous.

Furthermore,

$$\sum_{i=1}^{m} P_i(p,q) = \frac{\sum_{i=1}^{m} p_i + \sum_{i=1}^{m} c_i(p,q)}{1+\sum_{k=1}^{m} c_k(p,q)} = \frac{1+\sum_{k=1}^{m} c_k(p,q)}{1+\sum_{k=1}^{m} c_k(p,q)} = 1$$

which means $P \in \Delta_{m-1}$. Similarly, $\sum_{j=1}^{n} Q_j(p,q) = 1$ and so $Q \in \Delta_{n-1}$. Define the function T, $T\colon \Delta_{m-1} \times \Delta_{n-1} \to \Delta_{m-1} \times \Delta_{n-1}$ by $T(p,q) = (P(p,q), Q(p,q))$. Note that T is continuous.

Moreover, $S = \Delta_{m-1} \times \Delta_{n-1}$ is compact and convex. The Brouwer Fixed Point Theorem yields $(p^*, q^*) \in \Delta_{m-1} \times \Delta_{n-1}$ such that $T(p^*, q^*) = (p^*, q^*)$. Let $u_1^* = p^{*\prime} A q^*$ be player one's expected utility at (p^*, q^*).

I claim that: $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$ Suppose the claim is false. Then $\sum_{k=1}^{m} c_k(p^*, q^*) > 0$.

Because (p^*, q^*) is a fixed point,

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^{m} c_k(p^*, q^*)}$$

for every i. Clearing the fraction and cancelling p_i^* yields $p_i^*[\sum_k c_k(p^*, q^*)] = c_i(p^*, q^*)$. Thus $p_i^* = 0$ whenever $c_i(p^*, q^*) = 0$.

Let $I = \{i : p_i^* > 0\}$. Note that $I \neq \emptyset$ because $\sum_i p_i^* = 1$. By the previous paragraph, $I \subset \{i : c_i(p^*, q^*) > 0\}$. Finally, the definition of I implies $\sum_{i=1}^{m} p_i^* = \sum_{i \in I} p_i^* = 1$.

For $i \in I$, $c_i(p^*, q^*) > 0$ which means $A_i q^* > u_1^*$. Multiplying by $p_i^* > 0$ yields $p_i^* A_i q^* > p_i^* u_1^*$. Summing over $i \in I$ yields

$$u_1^* = \sum_{i=1}^{m} p_i^* A_i q^* \geq \sum_{i \in I} p_i^* A_i q^* > (\sum_{i \in I} p_i^*) u_1^* = u_1^*.$$

But this is a contradiction, which means our supposition that $\sum_{k=1}^{m} c_k(p^*, q^*) > 0$ must be false. This establishes our claim.

Since $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$, $c_i(p^*, q^*) = 0$ for every $i = 1, \ldots, m$. Expanding c_i yields $A_i q^* \leq u_1^*$. Let $p \in \Delta_{m-1}$. Then $p' A q^* = \sum_i p_i A_i q^* \leq (\sum_i p_i) u_1^* = u_1^*$. In other words, p^* is a best response to q^*. A similar argument shows that q^* is a best response to p^*. Since p^* and q^* are mutual best responses, we have a Nash equilibrium. $\square$