*Chapter 8*

# GLOBAL ANALYSIS AND ECONOMICS

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One main goal of this work is to show how the existence proof for equilibria can be based on Sard's theorem and calculus foundations. At the same time, equations such as "supply equals demand", are used rather than fixed points methods. The existence proofs given here are constructive in some reasonable and practical sense. These equilibria can be found on a machine using numerical analysis methods.

Our motivation for providing a proof of the Arrow–Debreu theorem (Appendix A) is to show that calculus can be used for the foundations of equilibrium theory.

Also in the paper optimization and the fundamental theorems of welfare economics are developed via the calculus. Abstract optimization theorems are proved in Section 3 and applied in Section 4 to pure exchange economies. Debreu's finiteness of equilibria theorem is proved in Section 5. In this section a manifold structure is put on the set of optima and on a certain set of equilibria as well.

The reader can see Smale (1976b) for a general motivation for a calculus approach to equilibrium theory (as well as references to other topics in Global Analysis and Economics). Furthermore some justification is given in this reference for the continued study of classical equilibrium theory in spite of its deep inadequacies for analyzing the problems of our day.

The account here could be used as a basis for a short course and in fact it was written when giving such a course at Berkeley in the winter of 1977. Much of the background for this exposition is to be found in our papers in the Journal of Mathematical Economics.

## 1. The existence of equilibria

The basic idea of equilibrium theory is to study solutions of the equation; supply equals demand or $S(p)=D(p)$. For the simple case of one market, where prices are measured in terms of some extra market standard, the familiar diagram below gives some justification for existence of the equilibrium price $p^*$.

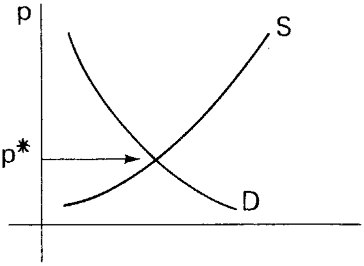


Figure 1.1

General equilibrium theory treats this problem for several markets. Let us be more precise: Suppose an economy with $\ell$ commodities is given. Then the space $R^\ell_+ = \{(x^1,\ldots,x^\ell) \in R^\ell;\ x^i \geq 0,\text{ each } i\}$ will play two roles for us: The first is as *commodity space*; so $x \in R^\ell_+$ will be interpreted as a *commodity bundle*. Thus $x$ is the $\ell$-tuple $(x^1,\ldots,x^\ell)$ with the first coordinate measuring the units of good number one, etc. But also $R^\ell_+$ with the origin 0 removed will be the space of *price systems*; thus if $p \in R^\ell_+ - 0$, $p = (p^1,\ldots,p^\ell)$ represents a set of *prices* of the $\ell$ goods, $p^1$ being the price of one unit of the first good, etc.

We suppose that the economy under study presents us (axiomatically) with *demand and supply functions* $D, S: R^\ell_+ - 0 \to R^\ell_+$, from the set of price systems to commodity space. Thus $D(p)$ will be the commodity bundle demanded by the economy (or its agents in sum) at prices $p$. In other words, at prices $(p^1,\ldots,p^\ell)$, the vector of goods that would be purchased is $D(p)$. The *equilibrium problem* is to find (and study) under suitable conditions on $D, S$ a price system $p^* \in R^\ell_+ - 0$ such that $D(p^*) = S(p^*)$ (equality as vectors).

Let us write $Z(p) = D(p) - S(p)$ so that the *excess demand* is a map $Z: R^\ell_+ - 0 \to R^\ell$, and we look for solution $p^* \in R^\ell_+ - 0$ of

$$Z(p^*) = 0. \tag{1.1}$$

The goal of this section is to put conditions on $Z$ which are reasonable from economics and then to show the existence of solutions of (1.1) by a constructive method through the differential calculus. This will be done without passing to the micro-foundations of the excess demand. Then in Section 2 we will give a classical micro-foundational development for the excess demand via aggregation of demand functions of individual economic agents for a pure exchange economy. In Appendix A, we prove the full Arrow–Debreu theorem this way.

Also in this exposition, existence will at first be shown under strong hypotheses, so that one can see the methods in their simplest form. Later the hypotheses will be relaxed.

The conditions on the excess demand $Z$ are

$$Z: R^{\ell}_{+} - 0 \to R^{\ell} \quad \text{is continuous,} \tag{1.2}$$

$$Z(\lambda p) = Z(p) \quad \text{for all} \quad \lambda > 0. \tag{1.3}$$

Thus $Z$ is homogeneous; if the price of each good is raised or lowered by the same factor, the excess demand is not changed. This supposes we are in a complete or self-contained economy so that the prices of the commodities are not based on a commodity lying outside the system,

$$p \cdot Z(p) = 0 \quad \left( \text{using the dot product,} \quad \sum_{i=1}^{\ell} p^i Z^i(p) = 0 \right). \tag{1.4}$$

This expression states that the value of the excess demand is zero and (1.4) is called *Walras Law*. One can think of this as asserting that the demand in an economy is consistent with the assets of that economy. It is a budget constraint. The total value demanded is equal to the total value of the supply of the agents. Walras Law is no doubt the most subtle of the conditions we impose on $Z$ here, and a micro-foundational justification will be given subsequently.

Before we state our final condition on the excess demand we give a geometric interpretation of the preceding conditions. Let $S^{\ell-1}_{+} = \{p \in R^{\ell}_{+} \mid \|p\|^2 = \sum (p^i)^2 = 1\}$ be the space of normalized price systems. By homogeneity, it is sufficient to study the restriction $Z: S^{\ell-1}_{+} \in R$. By Walras Law $Z$ is *tangent* to $S^{\ell-1}_{+}$ at each point; $p \cdot Z(p) = 0$ says that the vector $Z(p)$ is perpendicular to $p$. Thus one can interpret $Z$ as a field of tangent vectors on $S^{\ell-1}_{+}$.

The final condition on the excess demand $Z$ is the boundary condition

$$Z^i(p) \geqslant 0 \quad \text{if} \quad p^i = 0. \tag{1.5}$$

Here $Z(p) = (Z^1(p), \ldots, Z^{\ell}(p)) \in R^{\ell}$ and $p = (p^1, \ldots, p^{\ell})$. Condition (1.5) can be interpreted simply as: if the $i$th good is free then there will be a positive (or at least non-negative) excess demand for it. Goods have a positive value in our model.

*Theorem 1.1*

If an excess demand $Z: R^{\ell}_{+} - 0 \to R^{\ell}$ is continuous, homogeneous, and satisfies Walras Law and the boundary condition [i.e., (1.2), (1.3), (1.4) and (1.5)], then there is a price system $p^* \in R^{\ell}_{+} - 0$ such that $Z(p^*) = 0$. This price system $p^*$ is given constructively.

The last sentence will be elucidated in the proof.

The proof of Theorem 1.1 is proved via Theorems 1.2 and 1.3. These theorems are general, purely mathematical theorems about solutions of equations systems.

*Theorem 1.2*

Let $f: D^\ell \to R^\ell$ be a continuous map satisfying the boundary condition:

($B_D$) if $x \in \partial D^\ell$ then $f(x)$ is not of the form $\mu x$ for any $\mu > 0$.

Then there is $x^* \in D^\ell$ with $f(x^*) = 0$.

Here $D^\ell = \{x \in R^\ell \mid \|x\| \leqslant 1\}$ and $\partial D^\ell = \{x \in D^\ell \mid \|x\| = 1\}$.

We use for the proof of this theorem two results that have been central to global analysis and its applications to economics, the inverse mapping theorem (or implicit function theorem) and Sard's theorem. To state these results, one uses the idea of a singular point (a critical point) of a differentiable map, $f: U \to R^n$ where $U$ is some open set of a Cartesian space, say $R^k$. We will say that $f$ is $C^r$ if its $r$th derivatives exist and are continuous. For $x$ in $U$, the derivative $Df(x)$ (i.e., matrix of partial derivatives) is a linear map from $R^k$ to $R^n$. Then $x$ is called a *singular point* if this derivative is not surjective ("onto"). Note that if $k < n$ all points are singular. The *singular values* are simply the images under $f$ of all of the singular points; and $y$ in $R^n$ is a *regular value* if it is not singular.

*Inverse Mapping Theorem*

If $y \in R^n$ is a regular value of a $C^1$ map $f: U \to R^n$, $U$ open in $R^k$, then either $f^{-1}(y)$ is empty or it is a submanifold $V$ of $U$ of dimension $k-n$.

Here $V$ is a *submanifold* of $U$ of dimension $m = k-n$ if given $x \in V$, one can find a differentiable map $h: N(x) \to \mathcal{O}$ with the following properties:

(a) $h$ has a differentiable inverse.
(b) $N(x)$ is an open neighborhood of $x$ in $U$.
(c) $\mathcal{O}$ is an open set containing 0 in $R^k$.
(d) $h(N(x) \cap V) = \mathcal{O} \cap C$ where $C$ is a coordinate subspace of $R^k$ of dimension $m$.

*Sard Theorem*

If $f: U \to R^n$, $U \subset R^k$ is sufficiently differentiable (of class $C^r$, $r > 0$ and $r > k-n$), then the set of singular values has measure zero.

For a proof see, for example, Abraham and Robbin (1967); general background material can be found here. We say in this case that the set of regular values has *full measure*. Both of these theorems apply directly to the case of maps $f: U \to C$ where $U$ is a submanifold of dimension $k$ of Cartesian space of some dimension and $V$ is a submanifold of dimension $n$ (perhaps of some other Cartesian space). In that case the derivative $\mathrm{D}f(x): T_x(U) \to T_{f(x)}(V)$ is a linear map on the tangent spaces.

The above summarizes the basic mathematics that one uses in the application of global analysis to economics.

Toward the proof of Theorem 1.2 consider the following problem of finding a zero of a system of equations. Suppose $f: D^\ell \to R^\ell$ is a $C^2$ map satisfying the very strong boundary condition:

(SB) $f(x) = -x$ for all $x \in \partial D^\ell$.

The problem is to find $x^* \in D^\ell$ with $f(x^*) = 0$. We are following Smale (1976a), influenced by a modification of Varian (1977); for history see the paper by Smale.

To solve this problem define an auxiliary map $g: D^\ell - E \to S^{\ell-1}$ by $g(x) = f(x)/\|f(x)\|$ where $E = \{x \in D^\ell | f(x) = 0\}$ is the solution set. Since $g$ is $C^2$, Sard's theorem yields that the set of regular values of $g$ is of full measure in $S^{\ell-1}$ (using a natural measure on $S^{\ell-1}$). Let $y$ be such a value. Then by the inverse function theorem $g^{-1}(y)$ is a 1-dimensional submanifold which must contain $-y$ by the boundary condition. Let $V$ be the component of $g^{-1}(y)$ starting from $-y$. So $V$ must be a non-singular arc starting from $-y$ and open at the opposite end. Also $V$ does not meet $\partial D^\ell$ at any point other than $-y$ by the boundary condition and meets $-y$ only at its initial point, since it is non-singular at $-y$. Now $V$ is a closed subset of $D^\ell - E$ and so all its limit points lie in $E$. In particular $E$ is not empty and by following along $V$ starting from $-y$, one must eventually converge to $E$. This gives a geometrically constructive proof of the existence of $x^* \in D^\ell$ with $f(x^*) = 0$.

We remark that to further explicate the constructive nature of this solution, one can show that $V$ is a solution curve of the "Global Newton" ordinary differential equation $\mathrm{D}f(x)(\mathrm{d}x/\mathrm{d}t) = -\lambda f(x)$ where $\lambda = \pm 1$ is chosen according to the sign of the determinant of $\mathrm{D}f(x)$ and changes with $x$. If $\mathrm{D}f(x)$ is non-singular, then Eulers method of discrete approximation yields

$$x_n = x_{n-1} \mp \mathrm{D}f(x_{n-1})^{-1} f(x_{n-1}),$$

which, with fixed sign, is Newton's method for solving $f(x) = 0$. Thus the "Global Newton" indeed is a global version of Newton's method in some reasonable sense. M. Hirsch and I have had some success with the computer using the Global Newton as a tool of numerical analysis in solving systems of equations.

Now suppose only that $f: D^\ell \to R^\ell$ is only continuous and still satisfies $f(x) = -x$ for $x \in \partial D^\ell$. Define a new continuous map $f_0: D_2^\ell \to R^\ell$ by

$$f_0(x) = f(x) \quad \text{for} \quad \|x\| \leq 1,$$
$$f_0(x) = -x \quad \text{for} \quad \|x\| \geq 1.$$

Take a sequence of $\varepsilon_i \to 0$. For each $i$ we construct a $C^\infty$ approximation $f_i$ of $f_0$, so $\|f_i(x) - f_0(x)\| < \varepsilon_i$, all $x \in D_2$. One can use "convolution" here. See Lang (1969) for details. Let $\varphi_r$ be a $C^\infty$ function on $R^\ell$ such that $\int \varphi_r = 1$ and the support of $\varphi_r$ is contained in the disk $D_r$ of radius $r$.

Then define $f_i(y) = \int f_0(y-x)\varphi_r(x)\,dx = \int f_0(x)\varphi_r(y-x)\,dx$ with $r$ small enough relative to $\varepsilon_i$, and always $r < \frac{1}{2}$. Then $f_i$ approximates $f_0$ and $f_i(x) = -x$ for $x \in \partial D_2$. We can apply the result proved above to obtain $x_i \in D_2^\ell$ with $f_i(x_i) = 0$. Clearly $x_i \in D^\ell$ and also $x_i \to \{x \in D^\ell \mid f(x) = 0\}$ as $i \to \infty$. This proves Theorem 1.2 in case of the strong boundary condition (SB). Finally, suppose only $f: D^\ell \to R^\ell$ is continuous and satisfies $(\mathrm{B_D})$ as in the Theorem 1.2.

We will define a continuous map $\hat{f}: D_2^\ell \to R^\ell$ such that $\hat{f}(x) = -x$ for $x \in \partial D_2^\ell$, as follows:

$$\hat{f}(x) = f(x) \quad \text{for} \quad \|x\| \leq 1,$$
$$f(x) = (2 - \|x\|) f(x/\|x\|) + (\|x\| - 1)(-x) \quad \text{for} \quad \|x\| \geq 1.$$

Now by the preceding result there is $x^* \in D_2^\ell$ with $\hat{f}(x^*) = 0$. But $\|x^*\| \leq 1$, for otherwise the boundary condition $(\mathrm{B_D})$ would be violated. Thus $f(x^*) = 0$ and the proof of Theorem 1.2 is finished.

For the main result on the existence of equilibria, we need to modify Theorem 1.2 from disks to simplices. Define

$$\Delta_1 = \left\{ p \in R_+^\ell \mid \textstyle\sum p^i = 1 \right\}, \quad \partial\Delta_1 = \{ p \in \Delta_1 \mid \text{some } p_i = 0 \},$$
$$\Delta_0 = \left\{ z \in R^\ell \mid \textstyle\sum z^i = 0 \right\},$$

and

$$p_c = (1/\ell, \ldots, 1/\ell) \in \Delta_1, p_c \text{ being the center of } \Delta_1.$$

We will deal with continuous maps $\phi: \Delta_1 \to \Delta_0$ which satisfy the boundary condition:

(B) $\phi(p)$ is not of the form $\mu(p - p_c)$, $\mu > 0$ for $p \in \partial\Delta_1$.

If one thinks of $\phi(p)$ as a vector based at $p$ in $\partial D_1$, then $\phi(p)$ does not point radially outward in $\Delta_1$ according to condition (B).

*Theorem 1.3*

Let $\phi: \Delta_1 \to \Delta_0$ be a continuous map satisfying the boundary condition (B). Then there is $p^* \in \Delta_1$ with $\phi(p^*)=0$.

For the proof of Theorem 1.3. we will construct a "ray" preserving homeomorphism into the situation of Theorem 1.2 and apply that theorem. Define $h: \Delta_1 \to \Delta_0$ by $h(p)=p-p_c$; let $\lambda: \Delta_0 - 0 \to R^+$ be the map $\lambda(p)= -(1/\ell)(1/\min_i p_i)$. Then let $D=D^\ell \cap \Delta_0$; $\psi: D \to h(\Delta_1)$ defined by $\psi(p)= \lambda(p/\|p\|)_p$ is a ray preserving homeomorphism.

Consider the composition $\alpha: D \to \Delta_0$,

$$D \xrightarrow{\psi} h(\Delta_1) \xrightarrow{h^{-1}} \Delta_1 \xrightarrow{\phi} \Delta_0.$$

We assert that $\alpha$ satisfies the boundary condition ($B_D$) of Theorem 1.2. To that end, consider $q \in \partial D$ and let $p=\psi(q)+p_c=h^{-1}\psi(q)$. Now by (B) there is no $\mu>0$ with $\phi(p)=\mu(p-p_c)$ or with $\mu(p-p_c)=\alpha(q)$. Equivalently there is no $\mu>0$ with $\alpha(q)=\mu\psi(q)$, and since $\psi$ is ray preserving that means $\alpha(q)\neq\mu q$, $\mu>0$. This proves our assertion.

We conclude from Theorem 1.2 that there is $q^* \in D$ with $\alpha(q^*)=0$; or if $p^*=\psi(q^*)+p_c$ then $\phi(p^*)=0$. This proves Theorem 1.3.

To obtain Theorem 1.1, define from $Z: R^\ell_+ - 0 \to R^\ell$ of that theorem, a new map $\phi: \Delta_1 \to \Delta_0$ by $\phi(p)=Z(p)-(\sum Z^i(p))\,p$. Note $\sum\phi^i(p)=\sum Z^i(p)- \sum Z^i(p)\sum p^i=0$, so that $\phi$ is well-defined; $\phi$ is clearly continuous. Also if $p\in\partial\Delta_1$, $p^i=0$ for some $i$ and so $\phi^i(p)=Z^i(p)\geqslant 0$. Thus (B) of Theorem 1.3 is satisfied for $\phi$. Thus by Theorem 1.3 there is $p^* \in \Delta_1$ with $\phi(p^*)=0$ or $Z(p^*)=\sum Z^i(p^*)p^*$. Take the dot product of both sides with $Z(p^*)$ to obtain, using Walras Law, that $\|Z(p^*)\|^2=0$ or that $Z(p^*)=0$. This proves Theorem 1.1.

There can be natural equilibrium situations where $D(p^*)\neq S(p^*)$ as in the following one-market example for $p=0$.

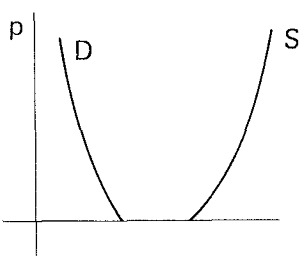


Figure 1.2

Thus for an excess demand $Z\colon R^{\ell}_{+} - 0 \to R^{\ell}$, any $p^*$ in $R^{\ell}_{+} - 0$ with $Z(p^*) \leqslant 0$, i.e., $Z^i(p^*) \leqslant 0$ all $i$, is sometimes called an equilibrium, e.g. as in Arrow–Hahn (1971). One might also think of such a $p^*$ as a *free disposal equilibrium* for after destroying excess supplies, one has an equilibrium with $Z(p) = 0$.

*Proposition*

If $Z\colon R^{\ell}_{+} - 0 \to R^{+}$ satisfies Walras law, $p \cdot Z(p) = 0$, and $Z(p^*) \leqslant 0$, then for each $i$, either $Z^i(p^*) = 0$ or $p^{*i} = 0$.

Otherwise for some $i$, $Z^i(p^*) < 0$ and $p^{*i} > 0$; and for all $i$, $p^{*i} Z^i(p^*) \leqslant 0$ which contradicts Walras Law.

With weaker hypotheses than those of Theorem 1.1 one can obtain a free disposal equilibrium.

*Theorem 1.4* (Debreu–Gale–Nikaidô)

Let $Z\colon R^{\ell}_{+} - 0 \to R^{\ell}$ be continuous and satisfy this weak form of Walras Law, namely, $p \cdot Z(p) \leqslant 0$. Then there is $p^* \in R^{\ell}_{+} - 0$ with $Z(p^*) \leqslant 0$. See Debreu (1959).

Note first that Theorem 1.4 implies Theorem 1.1. For let $Z$ satisfy the hypotheses of Theorem 1.1, then by Theorem 1.4 there is $p^*$ with $Z(p^*) \leqslant 0$. By the above proposition, for each $i$, $Z^i(p^*) = 0$ or $p^{*i} = 0$. But by the boundary condition of Theorem 1.1, if $p^{*i} = 0$ then $Z^i(p^*) \geqslant 0$, so in fact $Z^i(p^*) = 0$ and thus $Z(p) = 0$.

For the proof of Theorem 1.4, let $\beta\colon R \to R$ be the function $\beta(t) = 0$ for $t \leqslant 0$, and $\beta(t) = t$ for $t \geqslant 0$. Define $\bar{Z}\colon R^{\ell}_{+} - 0 \to R^{\ell}_{+}$ by $\bar{Z}_i(p) = \beta(Z^i(p))$ for all $i, p$. Now just as in the proof of Theorem 1.1 above, define $\phi\colon \Delta_1 \to \Delta_0$ by $\phi(p) = \bar{Z}(p) - \left(\sum \bar{Z}^i(p)\right) p$. This $\phi$ satisfies the hypotheses of Theorem 1.3 and so there is $p^* \in \Delta_1$ with $\phi(p^*) = 0$ or $\bar{Z}(p^*) = \sum Z^i(p^*) p^*$. Take the inner product of both sides by $Z(p)$ and use the weak Walras to obtain $\sum Z^i(p^*) \beta(Z^i(p^*)) \leqslant 0$. But $t\beta(t) > 0$ unless $t \leqslant 0$ in which case $t\beta(t) = 0$. Therefore $Z^i(p^*) \leqslant 0$ all $i$. This proves Theorem 1.4.

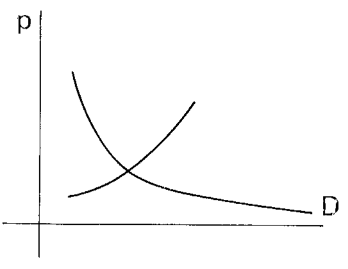


Figure 1.3

Another generalization of Theorem 1.1, and Theorem 1.4 as well, will be proved to account for $Z^i(p)\to\infty$ as $p^i\to 0$, e.g. the phenomenon illustrated in Figure 1.3. This theorem, Theorem 1.5 below, is a slight generalization of a theorem in Arrow–Hahn (1971, ch. 2, theorem 3).

Suppose now that the excess demand $Z$ is defined only on a subset $\mathfrak{D}$ of $R^\ell_+ - 0$ where $\mathfrak{D}$ contains all of the interior of $R^\ell_+$ and if $p\in\mathfrak{D}$, so does $\lambda p$ for each $\lambda>0$. Consider

$$Z:\mathfrak{D}\to R^l \quad \text{is continuous,} \tag{1.2'}$$

$$Z(\lambda p)=Z(p), \quad \text{all} \quad p\in\mathfrak{D}, \quad \lambda>0, \tag{1.3'}$$

$$p\cdot Z(p)\leqslant 0, \quad \text{all} \quad p\in\mathfrak{D}, \tag{1.4'}$$

$$\sum Z^i(p_k)\to\infty \quad \text{if} \quad p_k\to\bar{p}\notin\mathfrak{D}. \tag{1.5'}$$

*Theorem 1.5*

Let $Z:\mathfrak{D}\in R^\ell$ satisfy (1.2′), (1.3′), (1.4′) and (1.5′). Then there is a $p^*\in\mathfrak{D}$ with $Z(p^*)\leqslant 0$.

Let $\beta: R\to R$ be as in the previous proof and define $\alpha: R\to R$ by fixing $c>0$ and letting

$$\begin{aligned}\alpha(t)&=0 && \text{for} \quad t\leqslant 0,\\ &=1 && \text{for} \quad t\geqslant c,\\ &=t/c && \text{otherwise.}\end{aligned}$$

Define $\bar{Z}: R^\ell_+ - 0\to R^\ell_+$ by

$$\begin{aligned}\bar{Z}^i(p)&=1 \quad \text{if} \quad p\notin\mathfrak{D},\\ &=\Big(1-\alpha\big(\textstyle\sum Z^i(p)\big)\Big)\beta\big(Z^i(p)\big)+\alpha\big(\textstyle\sum Z^i(p)\big) \quad \text{otherwise.}\end{aligned}$$

Then $\bar{Z}$ is continuous.

Just as in the proof of Theorems 1.1 and 1.4 above, define $\phi:\Delta_1\to\Delta_0$ by $\phi(p)=\bar{Z}(p)-\sum\bar{Z}^i(p)p$. Then $\phi$ satisfies the hypotheses of Theorem 1.3, and so there is $p^*\in\Delta_1$ with $\phi(p^*)=0$ or

$$\bar{Z}(p^*)=\sum\bar{Z}^i(p^*)p^*.$$

First suppose that $p^*\in\mathfrak{D}$. Take the inner product of both sides with $Z(p^*)$ to obtain $Z(p^*)\cdot\bar{Z}(p^*)\leqslant 0$ (using the weak Walras Law). Then

$$\sum_i\left(1-\alpha\left(\sum_i Z^i(p^*)\right)\right)Z^i(p^*)\beta(Z^i(p^*))+\alpha\left(\sum Z^i(p^*)\right)\sum Z^i(p^*)\leqslant 0.$$

Since for any $t$, $t\alpha(t)\geqslant 0$, we have as a consequence that

$$\left(1-\alpha\left(\sum Z^i(p^*)\right)\right)\sum Z^i(p^*)\beta(Z^i(p^*))\leqslant 0,$$

and even

$$\sum Z^i(p^*)\beta(Z^i(p^*))\leqslant 0.$$

But $t\beta(t)$ is strictly positive unless $t\leqslant 0$. Therefore $Z^i(p^*)\leqslant 0$ all $i$.

On the other hand if $p^*\notin\mathfrak{D}$, it follows from the above equation on $\bar{Z}$ that $p^*$ is $(1,\dots,1)1/\ell$ which is in $\mathfrak{D}$. So in fact $p^*$ can't be outside $\mathfrak{D}$. This proves Theorem 1.5.

## 2. Pure exchange economy: Existence of equilibria

This section has two parts; in the first we make stronger hypotheses and emphasize differentiability, while the second is more general. The two are pretty much independent. The existence theorems are special cases of the Arrow–Debreu theorem; see Debreu (1959) and Appendix A.

To start with, consider a single trader with commodity space $P=\{x\in R^\ell\,|\,x=(x^1,\dots,x^\ell),\ x^i>0\}$. Thus $x$ in $P$ will represent a commodity bundle associated with this economic agent. It will be supposed that a preference relation on $P$ is represented by a "utility function" $u: P\to R$ so that the trader prefers $x$ to $y$ in P exactly when $u(x)>u(y)$. The sets $u^{-1}(c)$ in $P$ for $c$ in $R$ are called the *indifference surfaces*. Strong hypotheses of classical type are postulated:

$$u: P\to R \text{ is } C^2. \tag{2.1}$$

Now let $g(x)$ be the oriented unit normal vector to the indifference surface $u^{-1}(c)$ at $x$, $c=u(x)$. One can express $g(x)$ as $\operatorname{grad} u(x)/\|\operatorname{grad} u(x)\|$ where $\operatorname{grad} u=(\partial u/\partial x^1,\dots,\partial u/\partial x^n)$. Then $g$ is a $C^1$ map from $P$ to $S^{\ell-1}$, $S^{\ell-1}=\{p\in R^\ell\,|\,\|p\|=1\}$. It plays a basic role in the analysis of consumer preferences and demand theory.

Our second hypothesis is a strong differentiable version of free disposal, "more is better", or monotonicity,

$$g(x)\in P\cap S^{\ell-1}=\operatorname{int} S_+^{\ell-1}\quad\text{for each}\quad x\in P. \tag{2.2}$$

The word interior is shortened to int. So (2.2) means that all of the partial derivatives $\partial u/\partial x^i$ are positive.

Our third hypothesis is one of convexity, again in a strong and differentiable form. For $x \in P$, the derivative $Dg(x)$ is a linear map from $R^\ell$ to the perpendicular hyperplane $g(x)^\perp$ of $g(x)$. One may think of $g(x)^\perp$ as either the tangent space $T_{g(x)}(S^{\ell-1})$ or as the tangent plane of the indifference surface at $x$. The restriction of $Dg(x)$ to $g(x)^\perp$ is a symmetric linear map of $g(x)^\perp$ into itself,

$Dg(x)$ restricted to $g(x)^\perp$ has strictly negative eigenvalues. (2.3)

We have sometimes called this condition (2.3) "differentiably convex". One can restate (2.3) equivalently as

The second derivative $D^2u(x)$ as a symmetric bilinear form restricted to the tangent hyperplane $g(x)^\perp$ of the indifference surface at $x$ is negative definite. (2.3′)

We can see the equivalence of (2.3) and (2.3′) as follows: Let $Du(x)\colon R^\ell \to R$ be the first derivative of $u$ at $x$ with kernel denoted by $\operatorname{Ker} Du(x)$. Then since $v \cdot g(x) = Du(x)(v)/\|\operatorname{grad} u(x)\|$, $v \in \operatorname{Ker} Du(x)$ is the same condition as $v \cdot \operatorname{grad} u(x) = 0$ or $v \cdot g(x) = 0$ or yet $v \in g(x)^\perp$. Let $v_1, v_2 \in \operatorname{Ker} Du(x)$. Then $v_1 \cdot g(x) = Du(x)(v_1)/\|\operatorname{grad} u(x)\|$ and $v_1 \cdot Dg(x)(v_2) = D^2u(x)(v_1, v_2)/\|\operatorname{grad} u(x)\|$. This implies that (2.3) and (2.3′) are equivalent.

Next we show:

*Proposition 2.1*

If $u\colon P \to R$ satisfies (2.3) then $u^{-1}[c, \infty)$ is strictly convex for each $c$.

*Proof*

We show that the minimum of $u$ on any segment can not be in the interior of that segment. More precisely let $x, x' \in P$ with $u(x) \geqslant c$, $u(x') \geqslant c$. Let $S$ be the segment $\{\lambda x + (1-\lambda)x' | 0 < \lambda < 1\}$. Let $x^* = \lambda^* x + (1-\lambda^*)x'$ be a minimum for $u$ on $S$. Then $Du(x^*)(v) = 0$ where $v = x' - x$; since $x^*$ is a minimum, $D^2u(x^*)(v, v) \geqslant 0$. This contradicts our hypothesis (2.3′) that $D^2u(x^*) < 0$ on $\operatorname{Ker} Du(x^*)$. Therefore $u$ is greater than $c$ on $S$.

The final condition on $u$ is a boundary condition and has the effect of avoiding problems associated with the boundary of $R^\ell_+$:

The indifference surface $u^{-1}(c)$ is closed in $R^\ell$ for each $c$. (2.4)

This may be interpreted as the condition that the agent desires to keep at least a little of each good. It is used in Debreu (1959).

We derive now the *demand* function from the utility function of the trader. For this suppose given a *price* system $p \in \operatorname{int} R^{\ell}_{+}$ (of course $\operatorname{int} R^{\ell}_{+} = P$) and a *wealth* $w \in R_{+} = \{w \in R \mid w > 0\}$. This definition of $R_{+}$ is convenient though maybe not consistent. Consider the *budget set* $B_{p,w} = \{x \in P \mid p \cdot x = w\}$. One thinks of $B_{p,w}$ as the set of goods attainable at prices $p$ with wealth $w$. The demand $f(p,w)$ is the commodity bundle maximizing satisfaction (or utility) on $B_{p,w}$. Note that $B_{p,w}$ is bounded and non-empty, and that $u$ restricted to $B_{p,w}$ has compact level surfaces. Therefore $u$ has a maximum $x$ on $B_{p,w}$ which is unique by our convexity hypothesis (2.3) (Proposition 2.1).

Then $x = f(p,w)$ is the *demand* of our agent at prices $p$ with wealth $w$. It can be seen that the demand is a continuous map $f: \operatorname{int} R^{\ell}_{+} \times R_{+} \to P$. Since $x = f(p,w)$ is a maximum for $u$ on $B_{p,w}$, the derivative $\mathrm{D}u(x)$ restricted to $B_{p,w}$ is zero or $g(x) = p/\|p\|$. From the definition $p \cdot f(p,w) = w$ and $f(\lambda p, \lambda w) = f(p,w)$ for all $\lambda > 0$. Thus:

*Proposition 2.2*

The individual demand $f: \operatorname{int} R^{\ell}_{+} \times R_{+} \to P$ is continuous and satisfies

(a) $g(f(p,w)) = p/\|p\|$,
(b) $p \cdot f(p,w) = w$,
(c) $f(\lambda p, \lambda w) = f(p,w)$ if $\lambda > 0$.

Furthermore we will show the following classical fact with a modern version in Debreu (1972).

*Proposition 2.3*

The demand is $C^1$ (and will have the class of differentiability of $g$ in general).

For the proof, note that from Proposition 2.2, we can obtain

$$\varphi: P \to (\operatorname{int} S^{\ell-1}_{+}) \times R_{+}, \qquad \varphi(x) = (g(x), x \cdot g(x)),$$

which is an *inverse* to the restriction of $f$ to $(\operatorname{int} S^{\ell-1}_{+}) \times R_{+}$. Since $\varphi$ is $C^1$, by a version of the inverse function theorem, $f$ will be $C^1$ if the derivative $\mathrm{D}\varphi(x)$, of $\varphi$ at an arbitrary $x \in P$ is non-singular. To show that $\mathrm{D}\varphi(x)$ is non-singular, it is sufficient to prove, $\mathrm{D}\varphi(x)(\eta) = 0$ implies $\eta = 0$. For $\eta \in R^{\ell}$, we may write

$$\mathrm{D}\varphi(x)(\eta) = (\mathrm{D}g(x)(\eta), \eta \cdot g(x) + x \cdot \mathrm{D}g(x)(\eta)).$$

So if $\mathrm{D}\varphi(x)(\eta) = 0$, by this expression surely $\mathrm{D}g(x)(\eta) = 0$, so $\eta \in \operatorname{Ker} \mathrm{D}g(x)$.

But also $\eta \cdot g(x)=0$, $\eta \in g(x)^{\perp}$ and we know (3) that $Dg(x)$ restricted to $g(x)^{\perp}$ is non-singular. In other words $g(x)^{\perp} \cap \operatorname{Ker} Dg(x)=0$. This proves Proposition 2.3.

Let us elucidate this a bit. From what we have just said we may write $R^{\ell}$ as a direct sum $R^{\ell}=g(x)^{\perp} \oplus \operatorname{Ker} Dg(x)$ or write $\eta \in R^{\ell}$ uniquely as $\eta=\eta_1+\eta_2$ with $\eta_1 \cdot g(x)=0$, $Dg(x)(\eta_2)=0$. See Figure 2.1.

Here we are basing vectors at $x$. We may orient the line $\operatorname{Ker} Dg(x)$ by saying $\eta \in \operatorname{Ker} Dg(x)$ is positive if $\eta \cdot g(x)>0$. The following interpretation can be given to this line: Since $Dg(x)$ is always non-singular, the curve $g^{-1}(p)$ with $p=g(x)$, $p$ fixed in $S_{+}^{\ell-1}$ is non-singular. It is called the *income expansion path*. At $x \in P$, the tangent line to $g^{-1}(p)$ is exactly $\operatorname{Ker} Dg(x)$ (from the definition). This curve may be interpreted as the path of demand increasing with wealth as long as prices are fixed. One may consider wealth as a function $w: P \to R$ defined by $w(x)=x \cdot g(x)$. Then $w$ is strictly increasing along each income expansion path, and in fact $g^{-1}(p)$ can be differentiably parameterized by $w$.

Suppose now that the trader's wealth comes from an endowment $e$ in $P$, and is the function $w=p \cdot e$ of $p$. Then the last property of the demand is given by:

*Proposition 2.4*

Let $p_i$ be a sequence of price vectors in int $R_{+}^{\ell}$ tending to $p^*$ in $\partial R_{+}^{\ell}$ as $i \to \infty$. Then $\| f(p_i, p_i \cdot e)\| \to \infty$ as $i \to \infty$.

*Proof*

If the conclusion were false, by taking a subsequence and re-indexing we have $f(p_i, p_i e) \to x^*$. Since $u(f(p_i, p_i \cdot e)) \geqslant u(e)$ all $i$, by use of (2.4), $x^*$ is in $P$. Therefore $g(x^*)$ is defined and equals $p^*$. But since $p^* \in \partial R_{+}^{\ell}$, we have a contradiction with our monotonicity hypothesis (2.2). This proves Proposition 2.4.

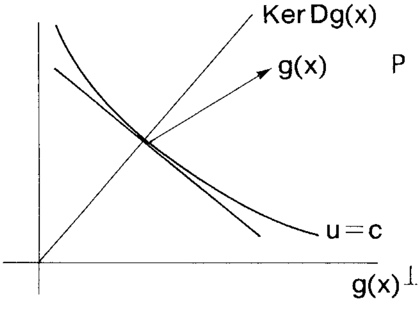


Figure 2.1

A *pure exchange economy* consists of the following: there are $m$ agents, who are traders, and to each is associated the same commodity space $P$. Agent number $i$ for $i=1,\dots,m$ has a preference represented by a utility function $u_i: P\to R$ satisfying the conditions (2.1)–(2.4). We suppose also that to the $i$th agent is associated an endowment $e_i\in P$. Thus at a price system, $p\in R^\ell_+ - 0$, the income or wealth of the $i$th agent is $p\cdot e_i$.

One may interpret this model as a trading economy where each agent would like to trade his endowed goods for a commodity bundle which would improve or even maximize his/her satisfaction (constrained by the budget). The notion of economy may be posed as follows:

A *state* consists of an *allocation* $x\in(P)^m$, $x=(x_1,\dots,x_m)$, $x_i\in P$ together with a price system $p\in S^{\ell-1}_+$. An allocation is called *feasible* if $\sum x_i=\sum e_i$. Thus the total resources of the economy impose a limit on allocations; there is no production. The state $(x,p)\in(P)^m\times S^{\ell-1}_+$ will be called a competitive or Walras *equilibrium* if it satisfies conditions (A) and (B):

(A) $\sum x_i=\sum e_i$.

This is the feasibility condition mentioned above.

(B) For each $i$, $x_i$ maximizes $u_i$ on the *budget set* $B=\{y\in P\mid p\cdot y=p\cdot e_i\}$.

Note that by the monotonicity condition (2.2) above, (B) does not change if in the definition of the budget set $p\cdot y=p\cdot e_i$ is replaced by $p\cdot y\leqslant p\cdot e_i$.

Note that (B) can be replaced by conditions ($B_1$) and ($B_2$):

($B_1$) $p\cdot x_i=p\cdot e_i$ for each $i$.

($B_2$) $g_i(x_i)=p$ for each $i$.

With (A), ($B_1$), and ($B_2$), equilibrium is given explicitly as the solution of a system of equations. We will show:

*Theorem 2.5*

Suppose given a pure exchange economy. More precisely let there be $m$ traders with endowments $e_i\in P$, $i=1,\dots,m$, and preferences represented by utilities $u_i: P\to R$, each satisfying conditions (2.1)–(2.4). Then there is an equilibrium; i.e., there are $x_i\in P$, $i=1,\dots,m$, and $p\in S^{\ell-1}_+$ satisfying (A) and (B).

We may translate the equilibrium conditions (A) and (B) into a problem of supply and demand. Let $S: R^\ell_+ - 0\to R^\ell_+$ be the constant map, $S(p)=\sum e_i$. Let $D:\operatorname{int} R^\ell_+\to R^\ell_+$ be defined by $D(p)=\sum f_i(p,p\cdot e_i)$ where $f_i(p,p\cdot e_i)$ is the demand generated by $u_i$ (Proposition 2.2). Define the excess demand $Z:\operatorname{int} R^\ell_+\to R^\ell$ by $Z(p)=D(p)-S(p)$. We note that the equilibrium conditions (A) and (B) are satisfied for $(x,p)$ if and only if $Z(p)=0$ and $x_i=f_i(p,p\cdot e)$. So if we can find a solution of $Z(p)=0$ by Section 1, we will have shown the existence of an economic equilibrium in the setting of a pure exchange economy.

Walras Law for $Z$ [(1.4)] is verified directly; if $p\in \operatorname{int} R^{\ell}_{+}$,

$$p\cdot Z(p)=p\cdot D(p)-p\cdot S(p)=\sum p\cdot f_i(p,p\cdot e_i)-p\cdot\sum e_i=0.$$

Homogeneity, that $Z(\lambda p)=Z(p)$ for $\lambda>0$ is checked as easily.

To apply the existence theorem, Theorem 1.6, we take $\mathfrak{D}$ to be $\operatorname{int} R^{\ell}_{+}$. It remains only to verify the boundary condition (2.5′), that if $p$ tends to a point in the boundary of $R^{\ell}_{+}-0$, the $\sum Z^i(p)\to\infty$. But that is a consequence of Proposition 2.4, using the fact that $Z$ is bounded below. Thus we have shown the existence of $p^*\in\mathfrak{D}$ with $Z(p^*)\leqslant 0$. But by the proposition preceding Theorem 1.4, it must be that $Z(p^*)=0$ since Walras Law is satisfied. This proves Theorem 2.5.

We give another setting for a pure exchange economy where we use only continuous preferences.

For this consider a preference relation on the full $R^{\ell}_{+}$ as commodity space (rather than its interior $P$) represented by a continuous utility function $u: R^{\ell}_{+}\to R$. We replace conditions (2.1) to (2.4) simply by:

$$u: R^{\ell}_{+}\to R \text{ is continuous,} \tag{2.1′}$$

and

$$u(\lambda x+(1-\lambda)x')>c \quad\text{if}\quad u(x)\geqslant c,\quad u(x')\geqslant c \quad\text{and}\quad 0<\lambda<1. \tag{2.2′}$$

The latter is a strict convexity condition on the preference relation.

Suppose that to each trader, in addition to a preference of the above type, is associated an endowment $e_i$ in $P$. Thus each agent has a positive amount of each commodity.

*Theorem 2.6*

Given a utility $u_i: R^{\ell}_{+}\to R$ for agents $i=1,\dots,m$ satisfying (2.1′), (2.2′) above and endowments $e_i\in P$, $i=1,\dots,m$, there is a ("free disposal") equilibrium $(x^*,p^*)$. Thus:

(a) $\sum x_i^*\leqslant\sum e_i$, and

(b) $x_i^*$ maximizes $u_i$ on the budget set $\{x_i\in R^{\ell}_{+}\mid p^*\cdot x_i\leqslant p^*\cdot e_i\}$ at $x_i^*$ for each $i$.

*Proof*

Before constructing a demand, we cut off commodity space near $\infty$ to avoid problems with unboundedness. We are able to get away with this because of the feasibility condition. More precisely choose $c > \|\sum e_i\|$ and let $D_c$ be the ball of radius $c$ or $D_c = \{p \in R^\ell \mid \|p\| \leqslant c\}$. Define an associated *false demand* function $\hat{f}_i : (R^\ell_+ - 0) \times R_+ \to X_c$, $X_c = D_c \cap R^\ell_+$, by taking $\hat{f}_i$ at $(p, w)$ to be the maximum of $u_i$ on $\hat{B}_{p,w} = \{x \in X_c \mid p \cdot x \leqslant w\}$. Then since $\hat{B}_{p,w}$ is compact, convex, and non-empty, by the strict convexity property of $u_i$, $\hat{f}_i(p, w)$ is well-defined.

*Proposition 2.7*

The false demand $\hat{f}_i : (R^\ell_+ - 0) \times R_+ \to X_c$ is continuous, $\hat{f}_i(\lambda p, \lambda w) = \hat{f}_i(p, w)$ for $\lambda > 0$, and $p \cdot \hat{f}_i(p, w) \leqslant w$. Also if $\|\hat{f}_i(p, w)\| < c$, then the maximum, $f_i(p, w)$, of $u_i$ on $B_{p,w} = \{x \in R^\ell_+ \mid p \cdot x \leqslant w\}$ exists (the true demand!) and $f_i(p, w) = \hat{f}_i(p, w)$.

*Proof*

This is straightforward except perhaps for the last. Let $\hat{x}_i = \hat{f}_i(p, w)$ with $\|\hat{x}_i\| < c$ and consider $x_i \in B_{p,w}$ with $u_i(x_i) \geqslant u_i(\hat{x}_i)$. Let $S$ be the segment between $\hat{x}_i$ and $x_i$ in $R^\ell_+$. For any $x_i' \neq \hat{x}_i$ on $S \cap X_c$, $u(x_i') > u_i(\hat{x}_i)$ by strict convexity (2.2′), contradicting the choice of $\hat{x}_i$. This proves Proposition 2.7.

Next define $\hat{D}(p) = \sum \hat{f}_i(p, p \cdot e_i)$, $S(p) = \sum e_i$, and $\hat{Z}: R^\ell_+ - 0 \to R^\ell$ by $\hat{Z} = \hat{D} - S$. Then $\hat{Z}$ satisfies the weak Walras Law, so by Theorem 1.4, there exists $p$ with $\hat{Z}(p) = 0$. Thus if $\hat{f}_i(p, p \cdot e_i) = \hat{x}_i$, $\sum \hat{x}_i = \sum e_i$ and $\|\hat{x}_i\| < c$. Therefore by Proposition 2.7, $\hat{x}_i = x_i = f_i(p, p \cdot e_i)$ and $(x_1, \ldots, x_m, p)$ is an equilibrium; Theorem 2.6 is proved.

Suppose $u_i : R^\ell_+ \to R$ satisfies:

*No Satiation Condition*: $u_i : R^\ell_+ \to R$ has no maximum.

Then we claim that the commodity vector $x_i = f_i(p, w)$ at the end of the proof of Theorem 2.6 satisfies $p \cdot f_i(p, w) = w$ (rather than inequality). Otherwise choose $x_i^*$ in $R^\ell_+$ outside $B_{p,w}$ with $u_i(x_i^*) \geqslant u_i(f_i(p, w))$ by the No Satiation Condition. By strict convexity, as in the proof of Proposition 2.7, we get a contradiction. Thus in this case we have that for the excess demand $Z(p) = \sum f_i(p, p \cdot e_i) - S(p)$, the usual Walras Law is satisfied at equilibrium and we obtain a more satisfactory interpretation of the free disposal equilibrium (see the proposition preceding Theorem 1.4.).

The question of relaxing strict convexity in Theorem 2.6, as well as questions of production, we defer to Appendix A.

## 3. Pareto optimality

Towards the problems of Pareto optimality in equilibrium theory and the "fundamental theorem of welfare economics", we consider abstract optimization problems in this section.

Our setting is an open set $W$ in $R^n$ ($W$ could be a smooth manifold or submanifold in what follows) together with $C^2$ functions $u_i: W \to R$, $i=1,\ldots,m$. One might think of $W$ as the space of states of society and the members of that society have preferences represented by the $u_i$. A point $x \in W$ is called *Pareto optimal* (or just optimal) if there is no $y \in W$ with $u_i(y) \geqslant u_i(x)$ all $i$ and strict inequality for some $i$. Such a $y$ could be called *Pareto superior* to $x$. If $m=1$, an optimum is the same thing as an ordinary maximum. The point $x \in W$ is a *local optimum* if there is a neighborhood $N$ of $x$ and $x$ is an optimum for $u_1,\ldots,u_m$ restricted to $N$. A point $x \in W$ is a *strict optimum* if whenever $y \in W$ satisfies $u_i(y) \geqslant u_i(x)$, all $i$, then $y=x$ (like a strict maximum). Finally a *local strict optimum* is defined similarly. Note that these definitions apply generally, e.g. to non-open $W$ in $R^n$. The goal of this section is to give calculus conditions for local optima. The following theorem is proved in Smale (1975) and Wan (1975); we follow the Smale paper especially, which one can see for more history.

*Theorem 3.1*

Let $u_1,\ldots,u_m: W \to R$ be $C^2$ functions where $W$ is an open set in $R^n$. If $x \in W$ is a local optimum, then there exist $\lambda_1,\cdots,\lambda_m \geqslant 0$, not all zero and

$$\sum \lambda_i \mathrm{D}u_i(x)=0. \tag{3.1}$$

Further suppose $\lambda_1,\ldots,\lambda_m, x$ are as above and

$$\begin{aligned}&\textstyle\sum \lambda_i \mathrm{D}^2 u_i(x) \text{ is negative definite on the space}\\ &\{v \in R^n \mid \lambda_i \mathrm{D}u_i(x)(v)=0,\ i=1,\ldots,m\}.\end{aligned} \tag{3.2}$$

Then $x$ is a local strict optimum.

Here $\mathrm{D}u_i(x)$ is the derivative of $u_i$ at $x$ as a real valued linear function on $R^n$, and $\mathrm{D}^2 u_i(x)$ is the second derivative as a quadratic form on $R^n$ [one could think of $\mathrm{D}^2 u_i(x)$ as the square matrix of second partial derivatives]. $\sum \lambda_i \mathrm{D}^2 u_i(x)$ is then also a quadratic form.

Note that if one takes $m=1$ and $n=1$, the theorem becomes the basic beginning calculus theorem on maxima. For $m=1$, and $n$ arbitrary, the theorem might be in an advanced calculus course. It has been pointed out to me by several people that one can reduce the proof of Theorem 3.1 to this case of $m=1$. However the direct proof we will give has some advantages with the geometry and symmetry in the $u_i$'s. In the following Im stands for image.

*Proof of Theorem 3.1*

Let $Pos=\{v \in R^m \mid v=(v_1,\ldots,v_m),\ v_i>0\}$ and $\overline{Pos}$ its closure. Then the first condition of the theorem may be stated as there is $\lambda \in \overline{Pos}-0$ with $\lambda \cdot \mathrm{D}u(x)=0$

(dot product). Here $u=(u_1,\ldots,u_m)$ maps $W$ into $R^m$. Let $x$ be a local optimum and suppose $\operatorname{Im} \mathrm{D}u(x) \cap Pos \neq \phi$. Then choose $v \in R^n$ with $\mathrm{D}u(x)(v) \in Pos$, and $\alpha(t)$ a curve through $x$ in $W$ with $\alpha(0)=x$ and the $\alpha'(0)=v$. Clearly for small values of $t$, $u_i(\alpha(t)) > u_i(\alpha(0)) = u_i(x)$ so that $x$ is no local optimum. Thus we know that $\operatorname{Im} \mathrm{D}u(x) \cap Pos = \phi$.

From this it follows from an exercise in linear algebra that there is some $\lambda \in Pos - 0$ with $\lambda$ orthogonal to $\operatorname{Im} \mathrm{D}u(x)$. Thus $\lambda \cdot \mathrm{D}u(x) = 0$, and the first part of the theorem is proved.

Suppose that the theorem (second part) is true in case $\lambda_i > 0$, all $i$, and consider the general case. Let the indices be such that $\lambda_1,\ldots,\lambda_k > 0$, $\lambda_{k+1} = \cdots = \lambda_m = 0$. Then conditions (3.1) and (3.2) are the same for optimizing $u_1,\ldots,u_m$ at $x$ and optimizing $u_1,\ldots,u_k$ at $x$. So (3.1) and (3.2) are satisfied for $u_1,\ldots,u_k$ also; and since by assumption the theorem is true in this case, $x$ is a strict local optimum for the $u_1,\ldots,u_k$. But then it is also a strict local optimum for $u_1,\ldots,u_m$. From this it is sufficient to prove the theorem in the case all the $\lambda_i$ are strictly positive.

We may suppose that $x$ is the origin of $R^n$ and $u(x)=0$ in $\boldsymbol{R}^m$, so that the symbol $x$ will remain free to denote any point in $W$. Then the condition that $0 \in W$ is a local <u>strict</u> optimum is that there is some neighborhood $N$ of 0 in $W$ with $(u(N)-0) \cap \overline{Pos} = \phi$. We will show that under the conditions of Theorem 3.1, indeed there is such an $N$.

Denote by $K$ or $\operatorname{Ker} \mathrm{D}u(0)$ the kernel of $\mathrm{D}u(0)$ as a linear subspace of $R^n$ and by $K^{\perp}$ its orthogonal complement.

*Lemma 3.2*

There exist $r, \delta > 0$ with the property that when $\|x\| < r$, $x=(x_1,x_2)$, $x_1 \in K$, $x_2 \in K^{\perp}$ and $\delta\|x_1\| \geqslant \|x_2\|$ then $\lambda \cdot u(x) < 0$ if $x \neq 0$.

*Proof*

Let $H = \sum \lambda_i \mathrm{D}^2 u_i(0)$. By (3.2) there is some $\sigma > 0$ so that $H(x,x) \leqslant -\sigma\|x\|^2$ for $x \in K$. For $x \in R^n$, $x=(x_1,x_2)$, $x_1 \in K$, $x_2 \in K^{\perp}$, we may write $H(x,x) = H(x_1,x_1) + 2H(x_1,x_2) + H(x_2,x_2)$. Since $|H(x_1,x_2)| \leqslant C\|x_1\|\,\|x_2\|$, $|H(x_2,x_2)| \leqslant C_1\|x_2\|^2$, we choose $\eta, \delta > 0$ so that if $\delta\|x_1\| \geqslant \|x_2\|$ then $H(x,x) \leqslant -\eta\|x\|^2$. Write by Taylor's theorem for $\|x\| < r$, $u(x) = \mathrm{D}u(0)(x) + \mathrm{D}^2 u(0)(x,x) + R_3(x)$ where $\|\lambda \cdot R_3(x)\| < \eta/2\|x\|^2$. Taking the dot product with $\lambda$ yields the lemma.

Now write $J = \operatorname{Im} \mathrm{D}u(0)$ and write $u$ in $R^m$ as $u=(u_a,u_b)$, $u_a \in J$, $u_b \in J^{\perp}$.

*Lemma 3.3*

Given $\alpha > 0$ and $\delta > 0$ there is $s > 0$ so that if $\|x\| < s$, $x=(x_1,x_2)$, $x_1 \in K$, $x_2 \in K^{\perp}$ with $\|x_2\| \geqslant \delta\|x_1\|$, then $\|u_b(x)\| \leqslant \alpha\|u_a(x)\|$.

*Proof*

The restriction

$$\mathrm{D}u(0)_{K^\perp}: K^\perp \to \mathrm{Im}\,\mathrm{D}u(0)$$

is a linear isomorphism so there are positive constants $c_1, c$ with

$$\begin{aligned} \|\mathrm{D}u(0)(x)\| = \|\mathrm{D}u(0)(x_2)\| &\geqslant c_1\|x_2\| \quad \text{all} \quad x=(x_1,x_2), \\ &\geqslant c\|x\| \quad \text{if} \quad \|x_2\| \geqslant \delta\|x_1\|. \end{aligned}$$

By the Taylor's series

$$u_a(x)+u_b(x)=u(x)=Du(0)(x)+R(x),$$

so that given $\beta>0$, we may assume $\|R(x)\| \leqslant \beta\|x\|$ for $\|x\|<$some number $s$. With $R=(R_a, R_b)$ we have

$$\|u_a(x)\| = \|\mathrm{D}u(0)(x)+R_a(x)\| \geqslant (c-\beta)\|x\|,$$

and

$$\|u_b(x)\| = \|R_b(x)\| \leqslant \beta\|x\|,$$

say with $\beta$ small enough and $\beta/(c-\beta)<\alpha$. Then $\|u_b(x)\| \leqslant \alpha\|u_b(x)\|$, finishing the proof of the lemma.

To finish the proof of Theorem 3.1, choose $\alpha$ of Lemma 3.3 so that if $\|u_a(x)\| \leqslant \alpha\|u_a(x)\|$ then $u(x) \notin \overline{Pos}-0$. This can be done since $\mathrm{Im}\,\mathrm{D}u(0) \cap \overline{Pos} = 0$, all the $\lambda_i$s being strictly positive. Choose a disk around 0 of radius $r_0$, $r_0<r$ of Lemma 3.2 and $r_0<s$ of Lemma 3.3. Let the $\delta$ of Lemma 3.3 be given by Lemma 3.2. Now from the two lemmas we have that $u(x) \notin \overline{Pos}$ if $x \neq 0$, $\|x\|<r_0$, proving $x$ to be a local strict optimum and Theorem 3.1.

We pass now to an extension of Theorem 3.1 to the setting of constrained optimization. Thus let $C^2$ functions $u_1, \ldots, u_m$ be defined on an open set $W \subset R^\ell$, subject to constraints given by conditions of the form $g_\beta(x) \geqslant 0$, $\beta=1,\ldots,k$, with $g: W \to R$ of class $C^2$. One may express the problem by defining $W_0=\{x \in W \mid g_\beta(x) \geqslant 0,\ \beta=1,\ldots,k\}$ and seeking conditions for optima of the restrictions $u_1,\ldots,u_m$ to $W_0$.

*Theorem 3.4*

Suppose $x \in W_0$ is a local optimum for the functions $u_1, \dots, u_m$ on $W_0$, $W_0$ as above. Then there exist non-negative numbers $\lambda_i$, $\mu_\beta$, not all zero such that

$$\sum_{i=1}^{m} \lambda_i \mathrm{D}u_i(x) + \sum \mu_\beta \mathrm{D}g_\beta(x) = 0, \tag{3.1'}$$

where

$$\mu_\beta = 0 \quad \text{if} \quad g_\beta(x) \neq 0.$$

Furthermore suppose $x \in W_0$, $\lambda_0 \geqslant 0$, $\mu_\beta \geqslant 0$ with not all the $\lambda_i$, $\mu_\beta$ zero, are given so that (3.1′) is true. If the bilinear symmetric form

$$\sum_{i=1}^{m} \lambda_i \mathrm{D}^2 u_i(x) + \sum \mu_\alpha \mathrm{D}^2 g_\alpha(x) \tag{3.2'}$$

is negative definite on the linear space

$$\{v \in R^\ell \mid v \cdot \lambda_i \operatorname{grad} u_i(x) = 0, \text{ all } i, \text{ and}$$
$$v \cdot \mu_\alpha \operatorname{grad} g_\alpha(x) = 0, \text{ all } \alpha\}$$

then $x$ is a local strict optimum for $u_1, \dots, u_m$ restricted to $W_0$.

For the first part let us suppose that $g_\beta(x) = 0$ (by renumbering if necessary) precisely for all $\beta = 1, \dots, k$, and define $\phi : W \to R^{m+k}$ by $\phi = (u_1, \dots, u_m, g_1, \dots, g_k)$. Then we claim that $\operatorname{Im} \mathrm{D}\phi(x) \cap Pos = \phi$. Otherwise let $\mathrm{D}\phi(x)(v) \in Pos$ and let $\alpha(t)$ be a curve in $W$ satisfying $\alpha(0) = x$, $\alpha'(0) = v$. For small enough $\varepsilon$, $\alpha(\varepsilon)$ is in $W_0$ and a Pareto improvement over $\alpha(0) = x$. So $x$ could not be locally optimal. So $\operatorname{Im} \mathrm{D}\phi(x) \cap Pos = \phi$ and there is a vector $(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k) \in \overline{Pos} - 0$ normal to $\operatorname{Im} \mathrm{D}\phi(x)$, as in Theorem 3.1. This proves the first part of Theorem 3.4.

For the proof of the last part we first note, with $\phi : W \to R^{m+k}$ as above, that if $x \in W_0$ is a local strict optimum for $\phi$ on $W$, then it is also a local strict optimum for $u_1, \dots, u_m$ on $W_0$. This follows from the definitions. But the hypotheses on $x$ in the second part of Theorem 3.4 imply that $x$ is a local strict optimum of $\phi$ as a consequence of Theorem 3.1. Thus Theorem 3.4 is proved.

We end this section with some final remarks:

(1) Note Theorem 3.1 is the special case of Theorem 3.4 when $k = 0$.
(2) Suppose the $g_\alpha$ satisfies the *Non-Degeneracy Condition* at $x \in W_0$. The set

$Dg_\beta(x)$ for $\beta$ with $g_\beta(x)=0$ is linearly independent. If this condition is satisfied then in (1) at least one of the $\lambda_i$ is not zero.

(3) If in Theorem 3.4 $m=1$, the first part is related to the Kuhn–Tucker theorem, and if the Non-Degeneracy Condition is met, one has $\lambda_1=1$.

Theorem 3.4 is in Smale (1974–76, V) and Wan (1975). See also Simon (forthcoming) for further information on this.

## 4. Fundamental theorem of welfare economics

We return to a pure exchange economy as in Section 2, with traders preferences represented by $C^2$ utility functions $u_i: P \to R$, $P = \operatorname{int} R^\ell_+$, $i=1,\dots,m$, satisfying the differentiable convexity, monotonicity and strong boundary conditions (2.2), (2.3), and (2.4). Also as in Section 2, the maps $g_i: P \to S^{\ell-1}_+$ defined by $g_i(x) = \operatorname{grad} u_i(x)/\|\operatorname{grad} u_i(x)\|$ will be used in our approach. While we do not presume that each agent is given an endowment, it will be supposed that the total resources $r$ of the economy are a fixed vector in $P$.

Thus the set $W$ of attainable allocations or states has the form

$$W = \left\{ x \in (P)^m \mid x = (x_1,\dots,x_m),\ x_i \in P,\ \sum x_i = r \right\}.$$

The individual utility $u_i: P \to R$ of the $i$th agent induces a map $v_i: W \to R$, $v_i(x) = u_i(x_i)$. After Section 3 it is natural to ask, what the optimal states in $W$ for the functions $v_i$, $i=1,\dots,m$, are. The answer is in:

*Theorem 4.1*

The following three conditions on an allocation $x \in W$ (relative to the induced utilities $v_i: W \to R$) are equivalent:

(1) $x$ is a local Pareto optimum.
(2) $x$ is a strict Pareto optimum.
(3) $g_i(x_i)$ is a vector in $S^{\ell-1}_+$, independent of $i$.

Let $\theta$ be the set of $x \in W$ satisfying one of these conditions. Then $\theta$ is a submanifold of $W$ of dimension $m-1$.

In this theorem as in this whole section, we are following Smale (1974–76).

*Proof*

Note (2) implies (1). We will show that (1) implies (3). For this we do not use any conditions on $u_i: P \to R$ except that the $u_i$ are $C^1$.

Thus suppose that $x \in W$ is a local optimum. We apply the first part of Theorem 3.1 to obtain $\lambda_1, \ldots, \lambda_m \geqslant 0$, not all zero, such that $\sum \lambda_i \mathrm{D}v_i(x) = 0$ or $\sum \lambda_i \mathrm{D}u_i(x_i) = 0$. We may suppose that $\lambda_1 \neq 0$ by a change of notation. Apply the sum to the vector $\bar{x} \in (R^\ell)^m$ with $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$, $\sum \bar{x}_i = 0$ (a tangent vector to $W$). If $\bar{x} = (\bar{x}_1, 0, \ldots, 0, -\bar{x}_1, 0, \ldots, 0)$ with $-\bar{x}_1$ in the $k$th place we have $\sum \lambda_i \mathrm{D}u_i(x_i)(\bar{x}_i) = \lambda_1 \mathrm{D}u_1(x_1)(\bar{x}_1) - \lambda_k \mathrm{D}u_k(x_k)(\bar{x}_1) = 0$ for all $\bar{x}_1 \in R^\ell$. Thus $\lambda_k \mathrm{D}u_k(x_k)$ is not zero all $k$ and equal to $\lambda_1 \mathrm{D}u_1(x_1)$. This yields condition (3).

For the equivalence of the three conditions, it remains to prove that if $x$ satisfies (3) then (2), $x$ is a strict optimum. So let $x$ satisfy (3) and let $y \in W$ with $v_i(y) \geqslant v_i(x)$, all $i$, or equivalently, $u_i(y_i) \geqslant u_i(x_i)$, all $i$.

We use now:

*Lemma 4.2*

Let $u: P \to R$ satisfy differentiable convexity (2.3). If $y \in P$, $u(y) \geqslant u(x)$ and $y \neq x$, then $\mathrm{D}u(x)(y-x) > 0$. Thus also in this case, $y \cdot g(x) > x \cdot g(x)$.

*Proof*

For $t \geqslant 0$ and $t \leqslant 1$, Proposition 2.1 (strict convexity) implies that $u(t(y-x)+x) \geqslant u(x)$, and so $(\mathrm{d}/\mathrm{d}t)u(t(y-x)+x)|_{t=0} \geqslant 0$. Therefore by the chain rule $\mathrm{D}u(x)(y-x) \geqslant 0$. On the other hand by Taylor's series if $\mathrm{D}u(x)(y-x) = 0$, $u(x+t(y-x)) = u(x) + \mathrm{D}^2u(x)((t(y-x))^2) + R_3$ which yields by differentiable convexity [(2.3′)] $u(x+t(y-x)) < u(x)$ for small $t$. This lies in contradiction with the convexity. The lemma is proved.

By the lemma, for each $i$, $y_i \cdot g_i(x_i) \geqslant x_i \cdot g_i(x_i)$ with inequality in case $y_i \neq x_i$. Then let $p = g_i(x_i)$ using (2.8), so $\sum p \cdot y_i \geqslant \sum p \cdot x_i$ with inequality if $y_i \neq x_i$ any $i$. But since $y \in W$, $\sum y_i = r = \sum x_i$ and $\sum p \cdot y_i = \sum p \cdot x_i$. Thus $y_i = x_i$, each $i$, $y = x$ and $x$ is a strict optimum.

For Theorem 4.1 it remains to prove that $\theta$ is an $(m-1)$ dimensional submanifold. For this we use the inverse function theorem in the form of the transversality theorem of Thom which goes as follows:

Let $W, V$ be submanifolds of some Cartesian space (or abstract manifolds) and let $\Delta$ be a submanifold of $V$. Thus given $y \in \Delta$, there is a diffeomorphism $h$ (differentiable map with a differentiable inverse) of a neighborhood $U$ of $Y$ in $V$ onto a neighborhood $N$ of 0 in $R^k$, $k = \dim V$, and $h(\Delta \cap U) = N \cap C$ where $C$ is a coordinate subspace of $R^k$. Then $\alpha: W \to V$ is *transversal* to $\Delta$ if whenever $x \in W$ with $\alpha(x) = y \in \Delta$, $T_y(V) = \operatorname{Im} \mathrm{D}\alpha(x) + T_y(\Delta)$. In other words, the image of the derivative $\mathrm{D}\alpha(x): T_x(W) \to T_y(V)$ together with tangent vectors to $\Delta$ at $y$ spans the tangent space of $V$ at $Y$. Also one can think of $\mathrm{D}\alpha(x)$ mapping surjectively onto the complement of the tangent space of $\Delta$ in $T_y(V)$.

Then the inverse function theorem implies:

*Transversality Theorem*

Let $\alpha: W \to V$ be transversal to the closed submanifold $\Delta$ of $V$. Then $\alpha^{-1}(\Delta)$ is a submanifold of $W$ with either $\alpha^{-1}(\Delta)$ empty or $\dim W - \dim \alpha^{-1}(\Delta) = \dim V - \dim \Delta$ (codimension is preserved).

Here, the dimension is shortened to dim. References with details are Abraham and Robbin (1967) and Golubitsky and Guilemin (1973).

For the proof let $\alpha(x) = y \in \Delta$ and apply the usual inverse function theorem to the composition $\pi \circ h \circ \alpha: W \to C^{\perp}$ with $h$ as above, $C^{\perp}$ is the orthogonal complement of $C$ above and $\pi: R^k \to C^{\perp}$ is the projection.

Now take the $W$ of the Transversality Theorem as the $W$ in Theorem 4.1 and let $V$ be the Cartesian product of $m$ spheres, $V = (S^{\ell-1})^m$ and $\Delta$ to be the diagonal in $V$,

$$\Delta = \left\{ y \in (S^{\ell-1})^m \mid y = (y_1, \dots, y_m), y_i \in S^{\ell-1}, y_1 = y_2 = \cdots = y_m \right\}.$$

Define $g: W \to (S^{\ell-1})^m$ by $g(x)$ having $i$th coordinate given by $g_i(x_i)$ where $g_i: P \to S^{\ell-1}$ is the normalized gradient of the utility of the $i$th trader. By definition [first part of Theorem 4.1, condition (3)], $g^{-1}(\Delta) = \theta$. We will show that $g$ is transversal to $\Delta$ as follows:

Let $K_x = \operatorname{Ker} Du(x)$ where $u: W \to R^m$ is the map with the $i$th coordinate of $u(x)$ given by $u_i(x_i)$. Then

$$K_x = \left\{ \bar{x} \in (R^{\ell})^m \mid \bar{x}_i \in R^{\ell}, \sum \bar{x}_i = 0, \bar{x}_i \cdot g_i(x_i) = 0 \right\}.$$

Let $L_x$ for $x \in \theta$ be the set of $\bar{x} \in T_x(W)$ with $Dg(x)(\bar{x}) \in T_{g(x)}(\Delta)$ or

$$L_x = \left\{ \bar{x} \in (R^{\ell})^m \mid \sum \bar{x}_i = 0, Dg_i(x_i)(\bar{x}_i) \text{ is independent of } i \right\}.$$

[Eventually we will see that $L_x = T_x(\theta)$ is the tangent space to $\theta$ at $x$.]

*Lemma 4.3*

$L_x \cap K_x = 0$ for all $x \in \theta$. Moreover $\dim K_x = m\ell - \ell - m + 1$.

*Proof*

Let $p = g_i(x_i)$ and $\gamma_i: p^{\perp} \to p^{\perp}$ be the restriction of $Dg_i(x_i)$ to $p^{\perp}$. Then $\gamma_i$ is symmetric with negative eigenvalues [see condition (2.3)]. Also $\sum \gamma_i^{-1}$ is an isomorphism since $\gamma_i^{-1}$ is symmetric with negative eigenvalues and the sum of negative definite symmetric linear maps is negative definite (from linear algebra, or look at the corresponding bilinear symmetric forms).

Let $\bar{x} \in L_x \cap K_x$ and $Dg_i(x_i)(\bar{x}_i) = \bar{p}$. Then $\gamma_i^{-1}(\bar{p}) = \bar{x}_i$ since $\bar{x}_i \cdot g_i(x_i) = 0$ and $\sum \bar{x}_i = \sum \gamma_i^{-1}(p) = 0$ so $\bar{p} = 0$. Thus also $\bar{x}_i = 0$ each $i$, proving the first part of the lemma. The dimension of $K_x$ is easily counted.

To finish the proof of Theorem 4.1, let us count more dimensions. It is easy to see that $\dim W = m\ell - \ell$, $\dim(S^{\ell-1}) = m\ell - m$, $\dim \Delta = \ell - 1$. From these dimensions and the lemma, $Dg(x)$ restricted to $K_x$ maps $K_x$ injectively into the complement of $T_y(\Delta)$ in $T_y((S^{\ell-1})^m)$, $y = (p, \ldots, p)$. This proves that $g$ is transversal to $\Delta$ and therefore by the transversality theorem, $g^{-1}(\Delta)$ is empty or a submanifold of dimension $m-1$. However, it cannot be empty by Theorem 2.5. using any endowments $e_i$ which sums to $r$. This finishes the proof of Theorem 4.1.

*Remark*

By the definitions, $L_x = T_x(\theta)$ and so $\dim L_x = m - 1$, and so

$$T_x(W) = T_x(\theta) \oplus K_x \quad \text{(direct sum)}.$$

We give some consequences of Theorem 4.1:

*Corollary 4.4*

Let $W$ be the space of attainable states of a pure exchange economy with fixed total resources $r$ as above. Consider the map $u: W \to R^m$ defined by: $u(x)$ has $i$th coordinate $u_i(x_i)$, $i = 1, \ldots, m$, where $u_i: P \to R$ is the utility of agent $i$. Let $\theta$ be the submanifold of Pareto optimal points. Then $u/\theta$, the restriction of $u$ to $\theta$ is an imbedding of $\theta$ into $R^m$.

Here an *imbedding* means that the derivative is injective as a linear map from $T_x(\theta) \to R^m$, and the map is injective.

In fact, the corollary is an immediate consequence of the remark that $\operatorname{Ker} Du(x) \cap T_x(\theta) = 0$.

Then since $u(\theta)$ has codimension 1 in $R^m$, one may define the *Gauss map* $G: \theta \to S^{m-1}$ by letting $G(x)$ be the unit normal to $u(\theta)$ at $u(x)$, oriented so that it lies in $R_+^\ell$. By definition $G(x)$ is perpendicular to the image $Du/\theta(x)$ or $G(x) \cdot Du(x)(\bar{x}) = 0$ for all $\bar{x} \in T_x(\theta)$. Since $T_x(\theta) \cap \operatorname{Ker} Du(x) = 0$, this is the same as $G(x) \cdot Du(x)(\bar{x}) = 0$ for all $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ with $\sum \bar{x}_i = 0$. Thus if we take $\lambda = (\lambda_1, \ldots, \lambda_m) = \lambda_x$ as in Theorem 4.1 and normalized as well, so that $\|\lambda_x\| = 1$, then $\lambda_x = G(x)$. In a certain way the Gauss map $G$ is the curvature of the imbedded manifold $u(\theta)$, so that the $\lambda$ of Theorem 4.1 may be thought of as a curvature. Note that the previous discussion, in contrast to the rest of this article, depends on the utility representations $u_i$, not just the underlying preference.

*Remark*

In connection with Corollary 4.4, it is worth noting that if $x \notin \theta$, then it can be shown that $\mathrm{D}u(x): T_x(W) \to R^m$ is surjective. If $x \in \theta$, then the image (see above) of $\mathrm{D}u(x): T_x(W) \to R^m$ has dimension $m-1$ and it can be shown that the map $u$ is a *fold* at $x$ in the sense of singularities of maps. See Smale (1974–76); this aspect of the subject is developed in work of de Melo, Saari, Simon, Titus, and Wan [see Simon (forthcoming) for some references].

*Corollary 4.5*

Given $e \in W$, there is some $x$ in $\theta$ so that $e - x \in K_x$. Furthermore there is a neighborhood $N(\theta)$ of $\theta$ in $W$ so that for each $e \in N(\theta)$, there is a *unique* $x$ in $\theta$ with $e - x \in K_x$. For an endowment vector $e$ in $N(\theta)$, $e = (e_1, \dots, e_m)$ there is a corresponding unique Walras equilibrium, $(x, p)$, with $x \in \theta$, $p = g_i(x_i)$, all $i$, and the budget condition $p \cdot e_i = p \cdot x_i$, all $i$.

For the proof note that for every $x \in W$ the attainability condition of equilibrium is satisfied. If $x \in \theta$, then the satisfaction condition defining $p = g_i(x_i)$ for some $i$ (hence all $i$) is also satisfied. Finally the budget condition $p \cdot e_i = p \cdot x_i$ all $i$ may be restated as $g_i(x_i) \cdot (e_i - x_i)$, all $i$, or simply as $e - x \in K_x (= \mathrm{Ker}\, \mathrm{D}u(x))$. Then the first sentence of Corollary 4.5 just re-expresses the existence Theorem 2.5. The uniqueness theorem, second or third sentence of the corollary, follows from the tubular neighborhood theorem of differential topology [see Golubitsky and Guilemin (1973, ch. 2, sect. 7)]. While we are following Smale (1974–76, VI), this is also close to work of Balasko (1975).

Towards the final corollary of Theorem 4.1 we give the concept of welfare equilibrium. We say that a state $(x, p) \in W \times S_+^{\ell-1}$ is a *welfare equilibrium* if $x_i$ is a (in this case the) maximum of $u_i$ on the budget set $B_{p, p \cdot x_i} = \{x \in P \mid p \cdot x = p \cdot x_i\}$. The subset of welfare equilibria in $W \times S_+^{\ell-1}$ will be called $\Lambda$. From this definition it follows that $(x, p)$, $x = (x_1, \dots, x_m)$, $x_i \in P$, $p \in S_+^{\ell-1}$ is in $\Lambda$ provided $(1_{\mathrm{E}})$, $(2_{\mathrm{E}})$ hold:

$(1_{\mathrm{E}})$ $\sum x_i = r$.

$(2_{\mathrm{E}})$ $g_i(x_i) = p$, each $i = 1, \dots, m$ (from the maximization condition on $u_i$).

If one has the further data of individual initial endowments, $e_i \in P$, $i = 1, \dots, m$, summing to $r$, then a third condition $(3_{\mathrm{E}})$, with $(1_{\mathrm{E}})$ and $(2_{\mathrm{E}})$, defines the equilibria of Section 2 or the Walras equilibria:

$(3_{\mathrm{E}})$ $p \cdot e_i = p \cdot x_i, \quad i = 1, \dots, m.$

The welfare equilibria are called "equilibria relative to a price system" in

Debreu (1959). They play a central role in theorems of welfare economics as well as non-tatonment dynamics. It is important to distinguish these two kinds of related concepts of equilibria. When there is a danger of confusion, we use the words *Walras equilibria* with emphasis on the budget condition ($3_E$).

A very sharp, though perhaps not general, version of the *fundamental theorem* of *welfare economics* is the following:

*Corollary 4.6*

All as above, $\theta, \Lambda$ are $(m-1)$-dimensional submanifolds, closed as subsets of $W, W \times S_+^{\ell-1}$, respectively, and the map $\beta: \Lambda \to W$ defined by $(x, p) \to x$ is a diffeomorphism of $\Lambda$ onto $\theta \subset W$.

We recall that a *diffeomorphism* is a differentiable map with differentiable inverse so that it is bijective (one to one and onto).

The usual form [compare Debreu (1959), Arrow–Hahn (1971)] states that $\Lambda \to \theta$ is well-defined and surjective, i.e., every optimal allocation is supported by a price system and the allocation part of a welfare equilibrium is optimal.

The proof of Corollary 4.6 goes as follows: Define an imbedding $\alpha: W \to W \times S_+^{\ell-1}$ by $\alpha(x) = (x, g_1(x_1))$. Then $\alpha(\theta) = \Lambda$ using Theorem 4.1; $\alpha/\theta$ and $\beta/\Lambda$ are inverse to each other with $\alpha/\theta$ an imbedding of the submanifold $\theta$. Then $\Lambda$ is a submanifold and the corollary follows.

We now indicate how some of this goes without assuming any properties on the utilities $u_i: P \to R$ besides differentiability, i.e., $C^2$. Let $\theta_s$ be the subset of the space $W$ of attainable allocations which consists of local strict optima. Emphasizing no hypotheses on the $u_i$, we still have:

*Proposition 4.7*

If $x \in W$ is a local optimum for the utility induced functions on $W$, then

(a) there exists $\lambda_i \geqslant 0$ not all 0 with $\sum \lambda_i Du_i(x_i) = 0$ (which implies that $g_i(x_i)$ is independent of $i$).

Further let $x$ satisfy (a) and also

(b) $\sum \lambda_i D^2 u_i(x_i)((\bar{x}_i)^2)$ is negative whenever $\sum \bar{x}_i = 0$, $\bar{x}_i \cdot g_i(x_i) = 0$, all $i$, and $\bar{x}_i \neq 0$, some $i$.

Then $x \in \theta_s$.

For the proof note that the first part is done (Theorem 4.1). The last part just goes by applying the second part of Theorem 3.1; the situation is similar to the proof of Theorem 4.1.

The condition (b) is considerably weaker than differentiable convexity at $x_i$, each $i$. In general one may hope to circumvent convexity hypotheses by using the second-order conditions (as in Theorem 3.1). On the other hand, $x$ may be a strict optimum with no supporting price equilibrium. In that case there is only an "extended price equilibrium" [see e.g. Smale (1974–76, III)].

We now consider the situation of Theorem 4.1 for commodity space with boundary. Up to now in this section the analysis has been interior. Thus suppose that trader $i$, for $i=1,\dots,m$, has a $C^2$ utility representation $u_i: R^\ell_+ \to R$ of his/her preference (so $u_i$ is defined on the full $R^\ell_+$, not just the interior). The conditions of differentiable monotonicity and differentiable convexity of Section 2 will be assumed for the rest of this section. We suppose that each $u_i$: $R^\ell_+ \to R$ is the restriction of a $C^2$ function defined on some open set of $R^\ell$ containing $R^\ell_+$. Then $u_i$ off $R^\ell_+$ will never be used. In this way the derivatives $Du_i(x), D^2u_i(x)$ still make sense for $x \in \partial R^\ell_+$ and so the conditions (2.2) and (2.3) make sense on the boundary as well.

Fix a vector $r \in \operatorname{int} R^\ell_+$ of total resources and let $W_0 = \{x \in (R^\ell_+)^m \mid \sum x_i = r\}$. Then $W_0$ is the space of attainable states of our pure exchange economy. Let $W$ be a neighborhood of $W_0$ in $\{x \in (R^\ell)^m \mid \sum x_i = r\}$ on which the functions $v_i: W \to R$, can be defined by $v_i(x) = u_i(x_i)$, $i=1,\dots,m$. Let $g_i^k: W \to R$ be given by $g_i^k(x) = x_i^k$. Then we are in the situation of optimizing several functions subject to constraints, or Theorem 3.4. These $g_i^k$ are constraints as above and bear no relation to the normalized gradients of utility functions. The problem of optima in $W_0$ relative to the $v_i: W_0 \to R$ is equivalent to optimizing the $v_i: W \to R$ subject to $g_i^k(x) \geqslant 0$.

*Theorem 4.8*

For $i=1,\dots,m$, let $u_i$: $R^\ell_+ \to R$ satisfy

$$\frac{\operatorname{grad} u_i(x_i)}{\|\operatorname{grad} u_i(x_i)\|} = g_i(x_i) \in S^{\ell-1}_+, \quad \text{each } x_i, \tag{4.1}$$

and

$$D^2 u_i(x_i) \text{ on } g_i(x_i)^\perp \quad \text{is negative definite.} \tag{4.2}$$

Suppose $W_0 = \{x \in (R_+)^m \mid \sum x_i = r\}$ with $v_i: W_0 \to R$ defined by $v_i(x) = u_i(x_i)$. If $x \in W_0$ is a local optimum for the $v_i$:

(a) there exists $p \in S^{\ell-1}_+$ and $\lambda_1,\dots,\lambda_m \geqslant 0$, not all 0, with $p \geqslant \lambda_i Du_i(x_i)$ each $i$, where one has equality in the $k$th coordinate if $x_i^k \neq 0$.

Conversely let $p$, $x_1,\dots,x_m$, $\lambda_1,\dots,\lambda_m$ be as in (a) with $p \cdot x_i \neq 0$ each $i$. Then $x$ is a strict optimum.

For the proof let $g_i^j: W \to R$ be defined as above so that $g_i^j(x) = x_i^j$ are constraints for $v_i$ on $W$. Then the derivatives satisfy $Dg_i^j(x)(\bar{x}) = \bar{x}_i^j$ where $\bar{x} \in (R^\ell)^m$ with $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$ and $\sum \bar{x}_i = 0$. Also $\bar{x}_i = (\bar{x}_i^1, \ldots, \bar{x}_i^\ell)$. If $x$ in $W_0$ is a local optimum for the $v_i$, then Theorem 3.4 applies to yield the existence of $\lambda_i \geqslant 0$, $\mu_i^j \geqslant 0$, $i = 1, \ldots, m$, $j = 1, \ldots, \ell$, not all zero with $\mu_i^j = 0$ if $x_i^j \neq 0$ and

$$\sum \lambda_i Du_i(x_i)(\bar{x}_i) + \sum \mu_i^j \bar{x}_i^j = 0, \quad \text{all } \bar{x}_i \text{ as above.}$$

Take $\bar{x}_i^j = 1$, $\bar{x}^j = -1$, all other components of $\bar{x}$ zero to obtain

$$\lambda_i Du_i(x_i)^j + \mu_i^j = \lambda_k Du_k(x_k)^j + \mu_k^j,$$

where $Du_i(x_i)^j$ denotes the $j$th coordinate of $Du_i(x_i)$.

Alternately we see that $q = \lambda_i Du_i(x_i) + \mu_i$ is independent of $i$ where $\mu_i = (\mu_i^1, \ldots, \mu_i^\ell)$, $\mu_i \geqslant 0$ and $\mu_i \cdot x_i = 0$. Note that $q \neq 0$, for otherwise all the $\lambda_i$ and $\mu_i$ would be zero [recall $Du_i(x_i) \neq 0$]. Let $p = q/\|q\|$ and multiply through $q = \lambda_i Du_i(x_i) + \mu_i$ by $1/\|q\|$. By renaming the $\lambda_i$, $\mu_i$ we have now

$$p = \lambda_i Du_i(x_i) + \mu_i, \qquad \mu_i \geqslant 0, \quad \lambda_i \geqslant 0, \quad \mu_i \cdot x_i = 0.$$

This yields the first part of Theorem 4.8.

For the converse let $y \in W_0$, $u_i(y_i) \geqslant u_i(x_i)$, $i = 1, \ldots, m$, $x_i, y_i \in R_+^\ell$. We must show that $y_i = x_i$ for each $i$. By the first lemma in the proof of Theorem 4.1, $Du_i(x_i)(y_i - x_i) \geqslant 0$ with equality only if $y_i = x_i$. By our main condition above $p \cdot x_i = \lambda_i Du_i(x_i)(x_i)$ and so $\lambda_i \neq 0$ since $p \cdot x_i \neq 0$. Then by this same condition $p \cdot (y_i - x_i) \geqslant \mu_i \cdot y_i$ or $p \cdot y_i \geqslant p \cdot x_i$, with equality only if $y_i = x_i$, each $i$. On the other hand $\sum y_i = \sum x_i = r$; putting this together indeed yields $y_i = x_i$ each $i$. This finishes the proof.

*Remark*

Note that if $u_i$ satisfies the stronger monotonicity condition, that $Du_i(x_i) \in \operatorname{int} S_+^{\ell-1}$, then $p \cdot x_i \neq 0$ in Theorem 4.8 can be omitted.

Say that $(x, p)$ is a *welfare equilibrium* (as before), or $(x, p) \in \Lambda \subset W_0 \times S_+^{\ell-1}$ if $x_i$ is a maximum of $u_i$ on the budget set $B_{p, p \cdot x_i} = \{x \in R_+^\ell \mid p \cdot x \leqslant p \cdot x_i\}$, each $i$. Thus for $(x, p) \in \Lambda$, $\sum x_i = r$, since $x \in W_0$.

*Proposition 4.9*

If $(x, p) \in \Lambda$, then there exist numbers $\lambda_i \geqslant 0$, $i = 1, \ldots, m$, and $\mu_i \in R^\ell$, $\mu_i \geqslant 0$ with $x_i \cdot \mu_i = 0$ and $p = \lambda_i \cdot Du_i(x_i) + \mu_i$. Conversely, given $(x, p) \in W_0 \times S_+^{\ell-1}$, with $p \cdot x_i \neq 0$, all $i$, and $\lambda_i, \mu_i$ as above with $p = \lambda_i \cdot Du_i(x_i) + \mu_i$, then $(x, p) \in \Lambda$.

*Proof*

Since $x_i$ is a maximum of $u_i$ on $B_{p,p\cdot x_i}$, for each $i$, there exist $\lambda_i \geqslant 0$, $\mu_i \in R^\ell_+$, $\sigma_i \geqslant 0$ not all zero, with

$$\lambda_i \mathrm{D}u_i(x_i)(\bar{x}_i) + \sum \mu_i^j \mathrm{D}g_i^j(x_i)(\bar{x}_i) - \sigma_i p \cdot \bar{x}_i = 0, \quad \text{all} \quad \bar{x}_i \in R^\ell,$$

or

$$\sigma_i p = \lambda_i \mathrm{D}u_i(x_i) + \mu_i, \qquad \mu_i \cdot x_i = 0.$$

If the $\sigma_i$ were 0, then so would be $\lambda_i$, $\mu_i$. Thus we may rescale by dividing by $\sigma_i$ to obtain $p = \lambda_i \mathrm{D}u_i(x_i) + \mu_i$, $\mu_i \cdot x_i = 0$. This proves the first part. For the second let $y_i \in B_{p,p\cdot x_i}$ with $u(y_i) > u(x_i)$. Then by Lemma 4.2 in the proof of Theorem 4.1, $\mathrm{D}u_i(x_i)(y_i - x_i) > 0$, and $p \cdot y_i \geqslant y_i \cdot \lambda_i \mathrm{D}u_i(x_i) > p \cdot x_i$, $\lambda_i \neq 0$, as in an earlier argument. Then $y_i \in B_{p,p\cdot x_i}$, contrary to hypothesis. Thus $(x, p) \in \Lambda$. This proves the proposition.

For the rest of this section, let us assume for simplicity the strong monotonicity hypothesis, that $\mathrm{D}u_i(x_i) \in \operatorname{int} S_+^{\ell-1}$. The projection map $W_0 \times S_+^{\ell-1} \to W_0$, $(x, p) \to x$, induces a map $\alpha: \Lambda \to \theta$, from welfare equilibria to Pareto optima. By the proposition above and Theorem 4.8, $\alpha$ is well-defined and it is surjective. While these results have an extensive literature under the topic of "fundamental theorems of welfare economics", the question of uniqueness of a supporting price system seems not so standard. Is $\alpha$ injective?

The answer is affirmative under the further mild hypothesis of "no isolated communities" [Smale (1974–76, V)]. For $x \in W_0$, an *isolated community* is a non-empty proper subset $S \subseteq \{1, \ldots, m\}$ with the property that wherever $i \in S$ and $x_i^j \neq 0$, then $x_k^j = 0$ for all $k \notin S$.

*Theorem 4.10*

If $x$ is an optimum in $W_0$ with no isolated communities, then there is a *unique* supporting price system.

Here we are supposing $W_0$ is the space of attainable states; the utility functions $u_i: R_+^\ell \to R$ are $C^2$ with $\mathrm{D}u_i(x_i) \in \operatorname{int} S_+^{\ell-1}$ and $\mathrm{D}^2 u_i(x_i) < 0$ on $\operatorname{Ker} \mathrm{D}u_i(x_i)$.

*Lemma 4.11*

Suppose $x \in W_0$ has no isolated communities and $i, q \in \{1, \ldots, m\}$ are two agents. Then there is a sequence $i_1, \ldots, i_n$ of agents with $i_1 = i$, $i_n = q$, and a sequence of goods $j_1, \ldots, j_n$ such that $x_{i_k}^{j_k} \neq 0$, all $k$ and for any $k$, either $j_{k+1} = j_k$ or $i_{k+1} = i_k$.

*Proof*

Otherwise take any agent, say agent number 1 for convenience, and consider all above such sequences $(i_1,\dots,i_n)$, $(j_1,\dots,j_n)$ with $i_1=1$. Let $S$ be the subset of $\{1,\dots,m\}$ of all possible $i_n$ reached in this way. If $S$ is proper, then it is an isolated community. This proves the lemma.

To prove Theorem 4.10, first obtain $p, \lambda_i, \mu_i$ as in Theorem 4.8, with $p=\lambda_i Du_i(x_i)+\mu_i$, $\lambda_i \geqslant 0$, $\mu_i \in R^\ell_+$ and $\mu_i \cdot x_i=0$. The problem has to do with the ambiguity of the $\lambda_i, \mu_i$. Suppose by renumbering, that agent 1 has some of the first good so $x_1^1 \neq 0$. Normalize $p$ by taking $p^1=1$ (and not $\|p\|=1$). Then $1=p^1=\lambda_1 Du_1(x_1)^1$ since $\mu_1^1=0$, and $\lambda_1$ is thus determined. Let $q$ be any other agent; choose a sequence $(i_1,\dots,i_n)$, $i_1=1$, $i_n=q$, $(j_1,\dots,j_n)$ as in the lemma. We claim that $\lambda_{i_k}$ is determined for each $i_k$. Suppose inductively that $\lambda_{i_{k-1}}$ is determined, and $i_k \neq i_{k-1}$. Then $j_k=j_{k-1}$, both agents $i_k$ and $i_{k-1}$ have some of good $j_k$. Therefore $p^{j_k}=\lambda_{i_{k-1}} Du_{i_{k-1}}(x_{i_{k-1}})^{j_k}$ determines $p^{j_k}$ and $p^{j_k}=\lambda_{i_k} Du_{i_k}(x_{i_k})^{j_k}$ determines $\lambda_{i_k}$. Here we used the fact that the corresponding $\mu_i^j$'s are 0. Once all the $\lambda_i$'s are determined uniquely, let $k$ be any good. Choose $i$ so that $x_i^k \neq 0$. Then $p^k=\lambda_i Du_i(x_i)^k$ determines $p^k$. This proves Theorem 4.10.

## 5. Finiteness and stability of equilibria

The first goal is to give a proof that the pure exchange economy described in the first part of Section 2 has only a finite number of Walras equilibria, at least for almost all endowment allocations. At the same time we show that these equilibria are stable (better "robust") in the sense that they persist under perturbations of the endowment allocation. These results are due to Debreu (1970). Our approach to this result is to define an "equilibrium manifold" without passing to the demand functions. The hypotheses, framework (pure exchange economy), and notation will be the same as in the first part of Section 2.

Thus define the *equilibrium "manifold"* $\Sigma$ as follows: The space $(P)^m \times (P)^m$ consists of $(e,x)$, $e=(e_1,\dots,e_m)$, $x=(x_1,\dots,x_m)$ with $e_i, x_i \in P$. Here $e$ will be thought of as an *endowment allocation* parameterizing an economy. Then $\Sigma$ will be the subset of $(P)^m \times (P)^m$ of $(e,x)$ satisfying:

$\sum e_i = \sum x_i$ (a total resource or attainability condition), (5.1)

$g_i(x_i)$ is independent of $i$ (5.2)

(the first-order condition; $g_i(x_i)=\operatorname{grad} u_i(x_i)/\|\operatorname{grad} u_i(x_i)\|$),

$p\cdot(e_i-x_i)=0$ (budget condition). (5.3)

Thus if $e$ is fixed, $(e, x)\in\Sigma$, then $(x, p)$ where $p=g_i(x_i)$, is a Walras equilibrium and conversely [see Section 2, (A), ($B_1$), ($B_2$)].

*Theorem 5.1*

$\Sigma$ is a submanifold of $(P)^m\times(P)^m$ of dimension $m\ell$.

*Proof*

Define a map

$$\phi:(P)^m\times(P)^m\to R^\ell\times R^{m-1}\times(S^{\ell-1})^m,$$

by sending

$$(e,x)\to\left(\sum e_i-\sum x_i, p\cdot(e_1-x_1),\dots,p\cdot(e_{m-1}-x_{m-1}),g_1(x_1),\dots,g_m(x_m)\right).$$

Then from the definition of $\Sigma$ we may write $\Sigma=\phi^{-1}(0\times0\times\Delta)$ where $\Delta=\{(p,\dots,p)\in(S^{\ell-1})^m\}$, and we have used the fact that conditions $\sum e_i=\sum x_i$ and $p\cdot(e_i-x_i)=0$, $i=1,\dots,m-1$, imply $p(e_m-x_m)=0$.

As in Section 4, Theorem 5.1 would be a consequence of $\phi$ being transversal to $0\times0\times\Delta$, using a simple counting of equations. Following the line of proof of Theorem 4.1; if $\phi(e,x)\in\Delta$ define

$$L_{e,x}=\left\{(\bar e,\bar x)\in(R^\ell)^m\times(R^\ell)^m|\mathrm{D}\phi(e,x)(\bar e,\bar x)\in0\times0\times T(\Delta)\right\},$$

or, equivalently, from differentiating (1), (2) and (3),

$$L_{e,x}=\Big\{(\bar e,\bar x)\in(R^\ell)^m\times(R^\ell)^m|\sum\bar e_i=\sum\bar x_i,\mathrm{D}g_i(x_i)(\bar x_i)=\bar p\in p^\perp,$$
$$\bar p\cdot(e_i-x_i)+p\cdot(\bar e_i-\bar x_i)=0\Big\}.$$

Here we take $p=g_1(x_1)$ and $\bar p=\mathrm{D}g_1(x_1)(\bar x_1)$.

Now we define a second linear subspace $K_{e,x}$ of $(R^\ell)^m\times(R^\ell)^m$ by

$$K_{e,x}=(\bar e,\bar x)|\sum\bar e_i=0,\bar x_i\cdot p=0,i\leqslant m-1,\pi_p\bar e_i=0,i\leqslant m-1\Big\}.$$

Here $\pi_p: R^\ell\to P^\perp$ is the orthogonal projection so that $\bar e_i=\pi_p\bar e_i+p\cdot\bar e_i$, each $i$. This space $K_{e,x}$ is motivated only by the proof of Theorem 5.1. Clearly $\dim K_{e,x}=m\ell$, and one also can see that $\dim R^\ell\times R^{m-1}\times(S^{\ell-1})^m-\dim\Delta=m\ell$. Thus if $L_{e,x}\cap K_{e,x}=0$, we have that $\phi$ is transversal to $0\times0\times\Delta$, just as the situation was in Section 4.

*Lemma 5.2*

$L_{e,x} \cap K_{e,x} = 0.$

For the lemma let $(\bar{e}, \bar{x})$ belong to the intersection. As in Section 4, $\gamma_i: p^{\perp} \to P^{\perp}$ denotes the restriction of $Dg_i(x_i)$. Then $\sum \bar{x}_i = 0$ since $\sum \bar{e}_i = 0$ and $\sum \bar{e}_i = \sum \bar{x}_i$. So $\bar{x}_i \cdot p = 0$, all $i$, and $\gamma_i^{-1}(\bar{p}) = \bar{x}_i$, each $i$. Also $\sum \gamma_i^{-1}(\bar{p}) = \sum \bar{x}_i = 0$ and $\bar{p} = 0$, therefore $\bar{x}_i = 0$. Finally one sees that $\bar{e}_i = 0$ proving the lemma and hence the theorem.

We emphasize that we are taking $u_i: P \to R$, $i = 1, \ldots, m$, as in the first part of Section 2.

*Theorem 5.3*

There is a closed set $F \subset (P)^m$ of measure 0 so that if $e \notin F$ then there exist a finite (positive) number of Walras equilibria relative to the endowment $e = (e_1, \ldots, e_m)$. This finite set varies continuously in $e$ as long as $e$ does not meet $F$. Let $\pi: (P)^m \times (P)^m \to (P)^m$ be the projection defined by $\pi(e, x) = e$. Let $\pi_0: \Sigma \to (P)^m$ be the restriction of $\pi$.

*Lemma 5.4*

The map $\pi_0: \Sigma \to (P)^m$ is closed. The image of a closed set is closed.

*Proof*

Consider a sequence $(e^{(j)}, x^{(j)})$ in $(P)^m \times (P)^m$, $j = 1, 2, 3, \ldots$, so that $e^{(j)}$ converges to $e \in (P)^m$. Then by the equilibrium conditions defining $\Sigma$, and the boundary condition on $u_i$, the $x^{(j)}$ have a subsequence converging to some $x \in (P)^m$. This is enough to show that $\pi_0$ is closed.

Let $C \subset \Sigma$ be the closed set of critical points of $\pi_0$ and $F = \pi_0(C)$. Then $F$ is closed by Lemma and has measure 0 by Sard's theorem. Theorem now is a consequence of the inverse function theorem applied to the map $\pi_0$.

A study of comparative statics of equilibria can now be done using these theorems.

While the above approach comes from Smale (1974–76) a closely related way of proving Debreu's theorem is in Balasko (1975).

## Appendix A. Existence of economic equilibrium with production

We prove the theorem of Arrow–Debreu on the existence of economic equilibrium with production as treated in Debreu (1959). The reason we include the proof is to show that calculus can indeed be the starting point of equilibrium theory with proofs at least as short and natural as those emphasizing Kakutani's fixed point theorem. On the other hand, our approach has much in common with that of Debreu; we owe much to his exposition as well as to conversations with him.

Here the treatment is brief. One can see Debreu (1959) for economic interpretations. The proof here is based on Theorem 1.5, and it is somewhat similar to the proofs in Section 2.

An *economy* consists, first, of a production side. We suppose $\ell$ commodities including labor. To each of $n$ producers, $j=1,\ldots,n$, is associated a "technology" $Y_j \subset R^\ell$ with the conditions:

(T) (a) $0 \in Y_j$, each $j$ (possibility of no production).

Let $Y = \sum Y_j$,

(b) $Y \cap (-Y) = \{0\}$ (an irreversibility condition).

(c) $Y$ is closed and convex.

(d) $Y - R^\ell_+ \subset Y$ (free disposal).

It can be shown that (d) is a consequence of $Y \supset -R^\ell_+$ in the presence of (c); see Debreu (1959). Here $Y_j$ may be thought of as the set of productions that are available to firm $j$. We suppose that the firm is driven by profit maximization. Thus if a price system $p$ is operative, the production $y \in Y$, is sought so that the profit $p \cdot y$ is a maximum.

Pass now to the consumer side of the economy. To each of $m$ consumers, $i=1,\ldots,m$, is associated a "consumption set" $X_i \subset R^\ell$ and a utility function $u_i: X_i \to R$ which represents his/her preference. The following is assumed:

(C) (a) $X_i$ is a closed convex set.

(b) $X_i$ is bounded below.

That is, there exist $d_1,\ldots,d_n \in R^\ell$ with $X_i \subset \{x \in R^\ell \mid x \geqslant d_i\}$ or $X_i \geqslant d_i$. (Here $x \geqslant d_i$ means that each component of $x$ is $\geqslant$ the corresponding component of $d_i$.)

(c) $u_i$ satisfies the convexity condition: if $x, x' \in X_i$ with $u_i(x) > u_i(x')$, then $u_i(tx + (1-t)x') > u(x')$ for each $t \in (0,1)$.

(d) $u_i$ has no maximum (no satiation condition)

*Remark*

One could have used directly a preference relation here, as in Debreu (1959), rather than utility function. No generality is gained as one can see in Debreu's paper.

Furthermore, to each consumer is associated an endowment $e_i \in X_i$ with $e_i$ having all coordinates strictly larger than some element of $X_i$. As in Debreu (1959), this is an unhappy hypothesis. Finally (private ownership economy) let $\theta_i$, be the share of agent $i$ in firm $j$. Then it is assumed that $0 \leqslant \theta_{ij} \leqslant 1$ and $\sum_{i=1}^m \theta_{ij} = 1$. If a price system $p$ prevails, then the wealth of agent $i$ is given by $w_i = p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j$.

An *equilibrium* for an economy above is a "state" $(x, y, p)$ with $x \in \prod_{i=1}^m X_i$, $y \in \prod_{j=1}^n Y_j$, $p \in S_+^{\ell-1}$ which satisfies

A) Attainability, or $\sum x_i = \sum y_j + \sum e_i$.

B) Each consumer maximizes satisfaction or:
$x_i$ is a maximum of $u_i$ on the budget set

$$B = \left\{ \bar{x} \in X_i \,|\, p \cdot \bar{x} \leqslant p \cdot e_i + \sum_j \theta_{ij} p \cdot y_j \right\}.$$

C) Each producer maximizes profit or:
$y_j$ is a maximum of $\Pi_p$ on $Y_j$, where

$$\Pi_p : Y_j \to R \text{ is } \Pi_p(\bar{y}) = p \cdot \bar{y}.$$

*Arrow–Debreu Theorem*

For an economy above there is always an equilibrium.

We first give a proof under additional restrictions; then we extend that proof to the General Arrow–Debreu Theorem.

*Theorem A.1*

Suppose that the economy described above satisfies the further conditions:

(1) Each $Y_j$ is closed and strictly convex.
(2) Each $u_i$ has the strict convexity property of Section 2 or, more precisely, if $u_i(x) \geqslant c$, $u_i(x') \geqslant c$ and $0 < t < 1$, then $u(tx + (1-t)x') > c$.

Then there is an equilibrium.

Toward proving Theorem A.1 we use the following basic lemma for which Bowen gave me this analytic version of my more geometric account:

*Lemma A.2* (basic estimate)

Let $Y$ be a closed convex subset of $R^\ell$ with $Y \cap (-Y) = \{0\}$ and $Y \supset -R_+^\ell$. Then given $b \in R^\ell$ and $n > 0$ there is a constant $c$ so that if $y_1, \ldots, y_n \in Y$ and $\sum y_j \geqslant b$ then $\|y_j\| < c$ each $j$.

For the proof let $K=\{y\in Y \mid \|y\|=1\}$. We prove three assertions:

*Assertion 1*

The origin 0 of $R^\ell$ is not in the convex hull of $K$.

If $\alpha_1 x_1+\cdots+\alpha_r x_r=0$ with $0<\alpha_i<1$, $\alpha_1+\cdots+\alpha_r=1$, $x_i\in K$, then

$$-\alpha_1 x_1=\alpha_1\cdot 0+\alpha_2 x_2+\cdots+\alpha_r x_r\in Y,$$

and $\alpha_1 x_1$ clearly is in $Y$. Thus $\alpha_1 x_1 \in Y\cap(-Y)$ reaching a contradiction.

*Assertion 2*

There is a $q=(q_1,\ldots,q_\ell)\in R^\ell$, each $q_i>0$, such that $q\cdot x<0$ for every $x\in K$.

As $K$ is compact, so is its convex hull. By Assertion 1 there is a $q$ in $R^\ell$ with $q\cdot K<0$. If $e_i$ is a coordinate basis vector then $-e_i\in K$ and $-q_i=q\cdot(-e_i)<0$.

*Assertion 3*

There are constants $\varepsilon>0$, $\beta>0$, so that if $x\in Y$ then $q\cdot x\leqslant\beta+\varepsilon-\varepsilon\|x\|$.

Let $-\varepsilon=\max\{q\cdot x \mid x\in K\}$ and $\beta=\max\{q\cdot x \mid \|x\|\leqslant 1\}$. The inequality is clear if $\|x\|\leqslant 1$. For $\|x\|>1$, $x\in Y$, and one has $x/\|x\|\in K$ since $Y$ is convex and contains 0. Then $-\varepsilon\geqslant q\cdot x/\|x\|$ or $q\cdot x\leqslant -\varepsilon\|x\|$.

We finish the proof of Lemma 1 as follows: Suppose $\sum y_j\geqslant b$ with $y_j\in Y$. Then $q\cdot b\leqslant\sum q\cdot y_j\leqslant n(\beta+\varepsilon)-\varepsilon\sum\|y_j\|$, so $\sum\|y_j\|\leqslant(n(\beta+\varepsilon)-q\cdot b)/\varepsilon$.

An analogous lemma for the consumption side is:

*Lemma A.3*

Given $c_1\in R^\ell$, there is $a>0$ such that if $x_i\in X_i$, $X_i\geqslant d_i$ [as in (C) above] for $i=1,\ldots,m$, and $\sum x_i\leqslant c_1$, then $\|x_i\|\leqslant a$, each $i$.

We omit the very easy proof.

Now let $b=\sum d_i-\sum e_i$ and choose $c$ as in Lemma A.2, so that if $\sum y_j\geqslant b$, then $\|y_j\|<c$, each $j$. Let $\hat{Y}_j=Y_j\cap D_c$ where $D_r=\{y\in D^\ell \mid \|y\|\leqslant r\}$. For $p\in R^\ell_+-0$, let $\hat{S}_j(p)=$ the maximum of $\Pi_p: Y_j\to R$ where $\Pi_p(y)=p\cdot y$. Then $\hat{S}_j$ is the "false supply function" of firm $j$.

*Lemma A.4*

$\hat{S}_j: R^\ell_+-0\to\hat{Y}_j$ is well-defined, continuous, $\hat{S}_j(\lambda p)=\hat{S}_j(p)$ for $\lambda>0$, and if $\|\hat{S}_j(p)\|<c$, then $\hat{S}_j(p)$ is the maximum of $\Pi_p$ on $Y_j$ (the true supply).

This is clear from the definitions, recalling that we are in the situation of Theorem A.1, so that $\hat{Y}_j$ is strictly convex.

*Remark*

If $Y_j$ is merely assumed closed and convex (not necessarily strictly convex), one still has $\hat{S}_j$ defined as a correspondence; i.e., $\hat{S}_j: R^\ell_+ - 0 \to S(\hat{Y}_j)$ is a map with values, convex subsets of $\hat{Y}_j$. It is homogeneous and when restricted to $S^{\ell-1}_+$ has a compact graph

$$\Gamma = \left\{(p, y) \in S^{\ell-1}_+ \times \hat{Y}_j \mid y \in \hat{S}_j(p)\right\}.$$

Furthermore in this case if $y \in \hat{S}_j(p)$ has norm $\|y\| < c$, then $y$ is a maximum of $\Pi_p$ on $Y_j$. Note that $\Pi_p(y)$ is independent of $y \in \hat{S}_j(p)$.

Define $\hat{w}_i: R^\ell_+ - 0 \to R$, the "false income" of consumer $i$, by $\hat{w}_i(p) = p \cdot e_i + \sum_j \theta_{ij} p \cdot \hat{S}_j(p)$. Then $\hat{w}_i$ is continuous. Let $b, c, e$, be as above and choose $c_1 \in R^\ell$ such that $\sum y_j + e \leqslant c_1$ if $\|y_j\| < c$ each $j$. Choose $a$ by Lemma A.3 and let $\hat{X}_i = X_i \cap D_a$.

Define a "false demand" $\hat{D}_i: R^\ell_+ - 0 \to \hat{X}_i$ for each $i$ by $\hat{D}_i(p) =$ the maximum of $u_i$ on $\hat{B}_p = \{x \in \hat{X}_i \mid p \cdot x \leqslant \hat{w}_i(p)\}$ (compare Proposition 2.7).

*Lemma A.5*

The false demand $\hat{D}_i: R^\ell_+ - 0 \to \hat{X}_i$ is well-defined, continuous, $\hat{D}_i(\lambda p) = D_i(p)$ for $\lambda > 0$, and $p \cdot \hat{D}_i(p) = w_i(p)$. Also if $\|\hat{D}_i(p)\| < a$ then $\hat{D}_i(p)$ is the maximum of $u_i$ on the budget set $B_p = \{x \in X_i \mid p \cdot x \leqslant \hat{w}_i(p)\}$ and $p \cdot \hat{D}_i(p) = \hat{w}_i(p)$.

The proof uses the same arguments as that at the end of Section 2, uses the No Satiation Condition, and the convexity of $X_i$. The continuity uses the fact that $e_i$ dominates some element of $X_i$ (the basic hypothesis on $e_i$). We leave the detailed proof, which is not difficult, to the reader.

*Remark*

In case $u_i$ satisfies the convexity condition (c) of (C) rather than strict convexity of Theorem A.1, then $\hat{D}_i$ is defined as a correspondence with values, convex subsets of $X_i$. It is homogeneous, and the restriction $\hat{D}_i: S^{\ell-1}_+ \to X_i$ has a compact graph. Also if $x \in D_i$ satisfies $\|x\| < a$, then $x$ is a maximum of $u_i$ on $\{\bar{x} \in X_i \mid p \cdot \bar{x} \leqslant \hat{w}_i(p)\}$ and $p \cdot x = \hat{w}_i(p)$.

Now define these aggregate functions from $R^\ell_+ - 0$ to $R^\ell$: $\hat{S} = \sum \hat{S}_j + \sum e_i$, $\hat{D} = \sum \hat{D}_i$, and $\hat{Z} = \hat{D} - \hat{S}$. From Lemmas A.4 and A.5, $\hat{Z}$ satisfies homogeneity and weak Walras, so Theorem 1.5 applies to produce $p^* \in S^{\ell-1}_+$ with $\hat{Z}(p^*) \leqslant 0$. Let $y_j^* = S_j^*(p^*)$, $x_j^* = \hat{D}_j(p^*)$, so then $\sum x_i^* \leqslant \sum y_j^* + \sum e_i$. Since each $x_i^* \in \hat{X}_i \subset X_i$, this implies $b \leqslant \sum y_j^*$ (definition of $b$). Thus $\|y_j^*\| < c$ (Lemma A.2), and by

Lemma A.4, $y_j^*$ is the maximum of $\Pi_p$ on $Y_j$. By the choices of $c_1$ and $a$, via Lemma A.3, $\|x_i^*\|<a$, each $i$. By Lemma A.5, $x_i^*$ is the maximum of $u_i$ on $\{\bar{x}\in X_i \mid p^*\cdot\bar{x}\leqslant \hat{w}_i(p^*)\}$, with $\hat{w}_i(p^*)=p^*\cdot e_i+\sum_j\theta_{ij}p^*\cdot y_j^*$.

We may choose $z\in R_+^\ell$ so that $\sum x_i^*=\sum y_j^*+\sum e_i-z$. Apply $p^*$ to this to see (using Lemma A.5 again) $p^*\cdot z=0$. Then $\sum y_j^*-z$ is in $Y=\sum Y_j$ by $(T)$ so we have $y_j\in Y_j$ with $\sum y_j=\sum y_j^*-z$. Then $p\cdot\sum y_j=p\cdot\sum y_j^*$, which implies that $y_j$ also (as well as $y_j^*$) maximizes $\Pi_p$ on $Y_j$, and $(x_i^*, y_j, p^*)$ is an equilibrium, proving Theorem A.1. Note in fact $y_j=y_j^*$ by the strict convexity, but our argument covers the more general case of Theorem A.6.

We next weaken the convexity hypotheses of Theorem A.1 by using the approximation theorem of Appendix B:

*Theorem A.6*

Theorem A.1 remains true if each $Y_j$ is closed and convex (rather than strictly convex), and instead of the strict convexity hypothesis on each $u_i$, we only assume (C) as in the Arrow–Debreu Theorem.

*Proof*

Proceed as in the proof of Theorem A.1. As in the remark after Lemma A.4, we can consider $\hat{S}_j: S_+^{\ell-1}\to\hat{Y}_j$, $j=1,\dots,n$, as correspondences.

Suppose $\varepsilon>0$ is given. Apply the theorem of Appendix B to obtain continuous functions $\hat{S}_{j\varepsilon}: S_+^{\ell-1}\to\hat{Y}_j$ for each $j=1,\dots,n$, with $\Gamma_{\hat{S}_{j\varepsilon}}\subset B_\varepsilon(\Gamma_{\hat{S}_j})$. Next note that $\hat{w}_i: S_+^{\ell-1}\to R$, defined by $\hat{w}_i(p)=p\cdot e_i+\sum_j\theta_{ij}p\cdot\hat{S}_j(p)$, is a well-defined continuous function, even with $\hat{S}_j$ a correspondence. As in the remark after Lemma A.5, we can consider $\hat{D}_i: S_+^{\ell-1}\to\hat{X}_i$ defined as a correspondence. Apply the theorem of Appendix B to obtain functions $\hat{D}_{i\varepsilon}: S_+^{\ell-1}\to\hat{X}_i$ such that $\Gamma_{\hat{D}_{i\varepsilon}}\subset B_\varepsilon(\Gamma_{\hat{D}_i})$ and $|p\cdot\hat{D}_{i\varepsilon}(p)-\hat{w}_i(p)|<\varepsilon$, all $p\in S_+^{\ell-1}$.

Define $Z_\varepsilon: S_+^{\ell-1}\to R^\ell$ by $Z_\varepsilon(p)=\sum\hat{D}_{i\varepsilon}(p)-\sum\hat{S}_{j\varepsilon}(p)-\sum e_i$, and $\hat{Z}_\varepsilon(p)=Z_\varepsilon(p)-(p\cdot Z_\varepsilon(p))p$. Then $p\cdot\hat{Z}_\varepsilon(p)=0$ and $p\cdot Z_\varepsilon(p)\to 0$ as $\varepsilon\to 0$. Apply Theorem 1.5 to obtain $p_\varepsilon$ such that $\hat{Z}_\varepsilon(p_\varepsilon)=0$.

Let $y_{j\varepsilon}=\hat{S}_{j\varepsilon}(p_\varepsilon)$, $x_{i\varepsilon}=\hat{D}_{i\varepsilon}(p_\varepsilon)$. Now take a sequence of $\varepsilon_k$ tending to 0. By taking subsequences we obtain $y_{j_{\varepsilon k}}\to y_j$, $x_{i\varepsilon_k}\to x_i$, $p_{\varepsilon_k}\to p$ to obtain an equilibrium. This finishes the proof of Theorem A.6 as in Theorem A.1.

Now we give the proof of the General Arrow–Debreu Theorem. We need:

*Lemma A.7*

Let $\widehat{Z}$ denote the convex hull of a subset Z of Euclidean space. Then

$$\sum\widehat{Y_j}=\widehat{\sum Y_j}.$$

*Proof*

Since $\sum \widehat{Y_j}$ is convex it contains $\widehat{\sum Y_j}$. We will show $\hat{A}+\hat{B}\subset\widehat{A+B}$. Let $a_i \geqslant 0$ with $\sum a_i = 1$. Then $\hat{A}+y\subset\widehat{A+y}$ since $\sum a_i x_i + y = \sum a_i(x_i+y)$. Therefore $\hat{A}+B\subset\widehat{A+B}$. Finally $\hat{B}+\hat{A}\subset\widehat{B+\hat{A}}\subset\widehat{A+B}$, showing indeed that $\hat{A}+\hat{B}\subset\widehat{A+B}$. By induction the proof of Lemma A.7 is finished.

With the hypotheses and notation of the beginning of Appendix A, let $Y_j^*$ be the closure of the convex hull of $Y_j$. Recalling $Y=\sum Y_j$, we have:

*Lemma A.8*

$\sum Y_j^* = Y$.

*Proof*

Since $Y_j \subset Y_j^*$, $\sum Y_j^* \supset \sum Y_j$. On the other hand, since the sum of the closure of sets is contained in the closure of the sum, it follows from Lemma A.7 that $\sum Y_j^* \subset Y$ (recall $Y$ is closed and convex). This proves Lemma A.8.

Apply Theorem A.6 to obtain an equilibrium $(x_i^*, y_j^*, p)$ for the economy above with $Y_j^*$ replacing $Y_j$. Now $\sum y_j^* \in Y$ (Lemma A.8) and so $\sum y_j^* = \sum y_j = y$ with $y_j \in Y_j$.

Furthermore $p \cdot y_j = p \cdot y_j^*$. This is so since $y_j^*$ is a maximum of $\Pi_p$ on $Y_j^*$ and therefore $y$ is a maximum of $\Pi_p$ on $Y$. This implies (since $y=\sum y_j$) that $\Pi_p(y_j)$ is at least as much as $\Pi_p(y_j^*)$ and hence equal. The rest follows and the Arrow–Debreu Theorem is proved.

## Appendix B. A theorem on the approximation of multi-valued mappings

We prove the following theorem of Cellina (1969), using extensively an unpublished exposition of W. Hildenbrand:

*Theorem B.1*

Let $K$ be a compact set (say in some Euclidean space), $T$ a compact convex set of $R^\ell$, and $\varphi: K \to S(T)$ a correspondence with values convex subsets of $T$ such that the graph $\Gamma_\varphi = \{(x, y) \in K \times T \mid y \in \varphi(x)\}$ is compact. Then given $\varepsilon > 0$ there is a continuous function $f: K \to T$ such that $\Gamma_f \subset B_{2\varepsilon}(\Gamma_\varphi)$.

Here $\Gamma_f$ is the graph of $f$ in $K \times T$ and $B_{2\varepsilon}$ is the open set of all points of $K \times T$ within $2\varepsilon$ of $\Gamma_\varphi$.

For the proof define $\varphi^\delta: K \to S(T)$ by $\varphi^\delta(x) = $ convex hull of $\bigcup_{y \in B_\delta(x)} \varphi(y)$.

*Lemma B.2*

Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that $\Gamma_{\varphi^\delta} \subset B_\varepsilon(\Gamma_\varphi)$.

*Proof*

If the lemma were false, one could take $\delta = 1/n$ and obtain a sequence $(x_n, y_n)$ in $K \times T$, with $(x_n, y_n) \notin B_\varepsilon(\Gamma_\varphi)$, all $n$, and $y_n = \sum \lambda^i_n y^i_n$, $\sum \lambda^i_n = 1$, $\lambda^i_n > 0$, $y^i_n \in \varphi(z^i_n)$, $d(z^i_n, x_n) \leqslant 1/n$. By taking subsequences, we get $x_n \to x$, $y^i_n \to y^i$, $\lambda^i_n \to \lambda_n$, $z^i_n \to z^j = x$. So $y = \sum \lambda^i y^i$, $\lambda^i \geqslant 0$, $\sum \lambda^i = 1$ and $(x, y^i)$ is in the closure of $\Gamma_\varphi$. Since $\varphi(x)$ is convex, $(x, y)$ is in the closure of $\Gamma_\varphi$, contradicting $(x_n, y_n) \notin B_\varepsilon(\Gamma)$. The lemma is proved.

Next let $\delta$ be as in the lemma and

$$U_y = \{x \in K \mid y \in B(\varphi^\delta(x))\} \quad \text{for each} \quad y \in T,$$

and then choose $U_{y_1}, \ldots, U_{y_k}$ a finite covering of $K$. Let $\beta_i$ be a corresponding partition of unity so $\beta_i: K \to [0, 1]$, $i = 1, \ldots, k$, are continuous functions, $\beta_i(x) = 0$ exactly if $x \notin U_i$ and $\sum \beta_i \equiv 1$. For example, one could take

$$\beta_i(x) = \frac{\alpha_i(x)}{\sum_{j=1}^{k} \alpha_j(x)} \quad \text{where} \quad \alpha_j(x) = \inf_{x' \notin U_j} d(x, x').$$

Define $f(x) = \sum \beta_i(x) y_i$. Then $f$ is clearly a continuous function, $f: K \to R^\ell$, such that for $x \in K$, $f(x)$ is a convex combination of those points $y_i$ such that $x \in U_{y_i}$ or $y_i \in B_\varepsilon(\varphi^\delta(x))$.

Since an $\varepsilon$-neighborhood of convex sets is convex, $B_\varepsilon(\varphi^\delta(x))$ is convex and $f(x)$ is in it. Therefore $(x, f(x)) \in B_\varepsilon(\Gamma_{\varphi^\delta})$ and by the lemma $(x, f(x)) \in B_{2\varepsilon}(\Gamma_\varphi)$ proving the approximation theorem.

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