Dynamics in General Equilibrium Theory

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I find myself something of an outsider in writing a paper like this; and for a mathematician to write a paper with no technical mathematics is a real (but very constructive) confrontation with the problem of communication. What I hope to achieve here is to convey some of my ideas on the relation between dynamics and the traditionally static, economic equilibrium theory.

Since I have been working in mathematical economics, I have been struck by the number of attacks on general equilibrium theory, on mathematical economics and even economic theory in general. Coming from a radical background, I have much sympathy for some of the arguments brought forth.

I first want to say a few words on the subject of mathematics in economics and economic theory in general. The role of theory per se hardly requires defense; theory can give a deeper understanding of any subject, subtle relations are seen, inconsistent ideas are exposed, new horizons are revealed.

A criticism commonly made of economic theory is its failure to make predictions of crises in the country or to anticipate correctly unemployment or inflation. One must be cautious in the social sciences about looking toward physics for answers. However, some comparisons with the physical sciences seem profitable in connection with the above criticism. In those sciences, where theory itself is in a far more advanced state, limitations can be seen in a similar way. For example a given individual human body functions according to physical principles; however no physical scientist would predict a heart attack. The physical theory gives understanding of aspects of what goes on in the human body only under very idealized conditions. The physical theories eventually play some role in the education of medical doctors, who can then say some things, some times about a patient's susceptibility to a heart attack, preventive measures, and cures.

The economy of the world or even a nation is a very complex phenomenon, like a human body, involving a number of factors, both economic and political. It is no more reasonable to expect economic theorists to predict a nation's economic future than for a theoretical scientist to predict the future health of an individual.

Questions about the need for mathematics in economic theory have been raised. Indeed, the successes of mathematics in economics have not been nearly as impressive as in physics. Yet the notion of money and prices already introduces mathematics into economics; and the mathematics becomes deeper with the equation of equilibrium, supply equals demand. When one considers the equation, supply equals demand for several interdependent markets, the mathematical problem already takes on some sophistication.

What is special about "general equilibrium theory" as opposed to economic theory in general? For me the importance

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of general equilibrium theory lies in its traditions which are deeper than any other part of economic theory. These traditions, which of course derive from actual economic history, explain why equilibrium theory has played such a central role and possesses such depth in content. Many of the procedures and mathematical methods used in other parts of economics grew out of developments in general equilibrium theory. Also equilibrium theory plays an important role in communication within the economics profession. Since most economists are knowledgeable in equilibrium theory, they can understand new ideas more readily when presented in that context. Also, since equilibrium theory has been studied so much, new ideas introduced there show weaknesses and also strengths most quickly.

After all of this is said, equilibrium theory will eventually stand or fall, depending on its truth as an important idealization of actual economic systems or as a model with values of justice, of efficient distribution and of utilization of resources. As a normative theory, I find great merit in its decentralization features (Schumacher's popular book, "Small is Beautiful" expresses some of my sentiment on decentralization). There are also, without doubt, basic failings in the theory which are well-presented in the economic literature, and there are some weaknesses which I wish to discuss presently. In fact these problems can be seen and understood especially clearly due to the well-developed structure of the theory; and one can use the body of general equilibrium theory as a tool in developing alternate models. To me it would seem overly difficult to construct and communicate successfully any alternate economic theory without having first studied very thoroughly equilibrium theory.

I would like to give some reasons why I feel equilibrium theory is far from satisfactory. For one thing the theory has not successfully confronted the question, "How is equilibrium reached?" Dynamic considerations would seem necessary to resolve this problem. Another weakness is the reliance of the theory on long range optimization.

In the main model of equilibrium theory, say as presented in Gerard Debreu's *Theory of Value,* economic agents make one life-long decision, optimizing some value. With future dating of commodities, time has almost an artificial role. The model is reminiscent of John von Neumann's game theory. I like to make an analogy between "Theory of Value" and the game theoretic approach to chess. The possible strategies are laid out to each player in advance, paths in a game tree, or a set of moves, one move to each position that could possibly occur. Each player makes a single choice of strategy. The strategies are compared and the game is over. Of course, chess isn't played like this. And in a situation more complicated than chess, where life-long consumption plans replace strategies, I don't believe economic agents act that way either.

In fact even the very best chess players don't analyze very many moves and certainly don't make future commitments. Their experience together with the environment at the moment (the position), some rules of thumb and some other considerations lead to decisions on the playing board.

My personal economic decisions are of a similar nature, from buying a book to buying a house; from a decision to travel to decisions about my job. Between one economic decision and another, there has been a real passage of time, circumstances have changed, and the new decision takes place in this new environment.

Long-run optimization would be impractical, even if it were emotionally acceptable, because of barriers of complexity. Complexity keeps us from analyzing very far ahead. The amount of time involved in making a decision is an important factor, for a chess player or a purchaser. Dynamic models based on some kind of behavioral strategy could meet these objections.

Sometimes static theories pose paradoxes whose resolution lies in a dynamic perspective. Let me give an example from the theory of duopoly of the Cournot (or Nash) equilibrium solution. Under classical hypotheses on the profit functions, consider such an equilibrium (r_1, r_2) where r_i is the rate of production of duopolist number i. Then agent 1's rate of production maximizes profit among all such rates with r_2 fixed. On the other hand the solution is unsatisfactory because there is another state, say, (r'_1, r'_2), where each duopolist with reduced production is taking a higher profit. In the actual dynamic world it is unlikely that the duopolists would stay at the Cournot solution, knowing that they would both be better off at nearby states.

It doesn't require explicit cooperation for these agents to move off the Cournot solution. In fact with flow of information and implicit threats in the context of a passage of time, one can argue that the duopolists will move to an optimal state from a Cournot state. But this resolution requires a real passage of time, that after each market move, the opposing duopolist has another opportunity to move. One can readily think of examples of duopoly where an increase of advertising is withheld by one agent knowing the other agent would match an increase and both would be worse off. James Friedman has written on this topic.

I feel that dynamics could also play a role in the resolution of Kenneth Arrow's paradox in the politics of social choice. Politics and elections in particular are actually part of a dynamic process, balloting being just a stage. The process looked at as a whole involves a number of moves such as a candidate's speech, a political ad, revising a position on some issue, etc. After each action of a candidate, other candidates have the option of taking an action of their own; voters' opinions evolve. For this reason, I would think that a dynamical model of the political process would give much better perspective than a static model of simple voting. In relation to this, the work of G. Kramer comes to mind.

We return to the subject of equilibrium theory. The existence theory of the static approach is deeply rooted to the use of the mathematics of fixed point theory. Thus one step in the liberation from the static point of view would be to use a mathematics of a different kind. Furthermore, proofs of fixed point theorems traditionally use difficult ideas of algebraic topology, and this has obscured the economic phenomena underlying the existence of equilibria. Also the economic equilibrium problem presents itself most directly and with the most tradition not as a fixed point problem, but as an *equation,* supply equals demand. Mathematical economists have translated the problem of solving this equation into a fixed point problem.

I think it is fair to say that for the main existence problems in the theory of economic equilibria, one can now bypass the fixed point approach and attack the equations directly to give existence of solutions, with a simpler kind of mathematics and even mathematics with dynamic and algorithmic overtones. In the last part of the paper we elaborate on this point.

Behind my own work on the questions of dynamics in economics, lies certain foundational work in the equilibrium theory in terms of calculus. The early development of mathematical economics, including the 19th century and even up to World War II, was largely in terms of calculus; it was no doubt the influence of game theory, and associated fixed point theorems that gradually reduced the dependence on calculus. In *Theory of Value* in 1959, Debreu wrote of the work of von Neumann and Oskar Morgenstern which freed mathematical economics from its traditions of differential calculus and compromises with logic. He wrote of the radical change of mathematical tools from calculus to convexity and topology.

But then in his paper on a finite number of equilibria, Debreu returned to calculus tools; my own work, "Global Analysis and Economics," has been to try to systematize the use of calculus in equilibrium theory. This can be justified on several grounds. First, the theory is brought closer to the practice. With calculus, one has in the derivative a linear approximation. It is these linear conditions that are so basic to practical economic studies. Comparative statics depend on derivatives; the same is usually true for stability conditions; dynamic questions are more accessible via calculus. When general equilibrium theory is developed on calculus mathematics, not only is theory brought closer to practice, but greater unity is achieved. Furthermore, recent work on approximation by differentiable functions in economics gives further justification to the use of calculus.

Finally before moving to the constructive side of the question of dynamics in equilibrium theory, it is worth making a remark on the nature of goods. It seems from our experience that it is important especially in modeling dynamics to put goods into two ideal classes, either completely perishable or completely durable. The theory seems different for the two kinds. For example, Walras equilibrium seems suited to the perishable, continually endowed class of goods, while for durable goods, the kind of equilibrium found in the fundamental theorem of welfare economics seems more appropriate. We hope that the rest of the paper makes this point clearer.

We discuss now the results of our paper, "Exchange Processes with Price Adjustment." This is a model of a market of durable goods; a particular example from personal experience of such a market is a weekend "mineral bourse" where agents with minerals and/or money meet to exchange, buy, and sell fine mineral specimens for collectors. Here one sees a truly dynamic process of exchange and price adjustments which converges to an equilibrium toward the end of the weekend of the "bourse."

An early work in some of the same spirit was done in 1962 by Frank H. Hahn, Hirofumi Uzawa, and Takashi Negishi. This approach is called a "non-tatonment," because in place of the Walrasian Tatonment, a sequence of actual trades takes place over time. On the other hand, our model differs from the 1962 work in that it is nondeterministic, the dynamics proceeds without an ordinary differential equation, long-run optimization is dropped, and a large body of examples is constructed.

The main result of our paper can be stated as follows:

"In a pure exchange economy, an exchange price adjustment process, responsive to transaction costs, and which doesn't stop unless forced to by market conditions, converges to a price equilibrium. There exists such processes starting from any state of any (pure exchange) economy."

In the paper, mathematical content is given to all of the phrases used here, and the result is proved. Here we give a brief explanation of some of the terms used.

A "state" of an economy means a set of

data characterizing the economy at a given time. Our use of the word "state" is akin to its use in physics. For example in a pure exchange economy, a state will consist of an allocation of the resources, or equivalently the set of goods of each agent and a price system.

A state in general will change over time, e.g., by exchange and price adjustments. Thus a "process" as used in the main result, means a passage in time of a state. Or equivalently, a process is a path (over time) in the space of states of an economy.

This process, to qualify as an *exchange* process, must satisfy economically justifiable conditions for exchange which are embodied in the following axiom. The exchange axiom for the process asserts that: (a) the total resources of the economy are constant (there is no production); (b) exchange takes place at current prices; (c) an exchange increases satisfaction of the participating agents; and (d) some exchange will take place provided that it is possible consistent with (a), (b), and (c).

For the process to qualify as a price adjustment process (as in the quoted main result), we demand that it satisfy a price adjustment axiom defined in terms of a short run version of demand. A usual excess demand approach requires long-run optimization for the agents while our spirit is closer to that of behavioral strategies. At given prices and goods possessed one defines the infinitesimal demand of an agent to be the direction his preferences take him when restricted to his budget set.

The price adjustment axiom asserts that prices adjust in the direction of some weighting of the infinitesimal demands of all the agents.

A Walrasian price equilibrium depends on the traders' endowments. Thus if one allows a real passage of time, say an actual exchange to take place, and several such, this initial endowment becomes forgotten. Thus if one allows a "nontatonment" kind of time passage, one must replace a Walrasian price equilibrium by a different notion of price equilibrium.

As stated in our main result above, a price equilibrium is a feasible allocation and price system where, for each agent, satisfaction is maximized on his budget set defined relative to his wealth at equilibrium. Equivalently, a price equilibrium is an optimal allocation together with a supporting price system, as studied in the correspondence of the fundamental theorem of welfare economics.

A detailed mechanism of price setting and transactions is not developed, but it seems likely that the model is consistent with doing this explication.

Next we discuss some problems and results on the classical Walrasian model from the point of view of dynamics and algorithms. We prefer an alternate, well-known interpretation of the Walras model to that given in Debreu's *Theory of Value*.

Suppose that the goods are perishable, with labor a main example. One might envision a situation where each day an agent starts his economic activity with a fixed endowment of labor, or fish which won't keep. The next day he will have a new endowment of the same, but none left from the day before.

The consumption variables are the amounts of commodities consumed each day, of an agent. Thus both the endowment and consumption bundles in commodity space will be interpreted as the *rates* of endowment (fixed over time) and consumption respectively. A completely satisfactory dynamics (which isn't available) for this problem would construct and analyze paths over time in the space of states, that is commodity vectors for each agent and price systems (or sets of price systems). These paths should obey economically justifiable axioms of exchange and price adjustment, and probably should lead, at least under some economic conditions to a Walrasian equilibrium, starting from an endowment allocation and any price system. At the most satisfactory level, these paths should be given interpretations in terms of individual agent's actions in price offerings and purchases. In my view, a behavioral strategy for agents would be more desirable than decisions based on long run optimization.

Martin Shubik and I have worked on the problem in this setting without any definite success. On the other hand, it seems as if we might eventually obtain such a model with convergence provided a condition such as gross substitutes is satisfied. I should emphasize that I am not talking about any "tatonment" in this kind of dynamic, but rather an actual process, where agents are adjusting their goals, consumptions, prices over time to arrive at balanced budgets where the value of the rate of consumption equals the value of the rate of the fixed endowment for each agent.

I would like to turn now to some work carried out in my article "A Convergent Process of Price Adjustment and Global Newton Methods," which has more success on the mathematical side of the above problem.

One way of looking at this work is to first alter the Scarf Algorithm from finding fixed points to solving a system of equations, especially the system, supply equals demand in many variables. We define an ordinary differential equation, called a "global Newton," which is a version of (the altered) Scarf's Algorithm. Under rather general hypotheses (comparable to those needed in the execution of Scarf's Algorithm), solutions of the global Newton converge to the set of solutions of the original system (e.g., supply equals demand). Combining this fact with methods of numerical analysis, one obtains a different but analogous algorithm to that of Scarf. Morris Hirsch and I have implemented this effectively on a computer and are developing the algorithm from a numerical analysis point of view. It applies to systems of n nonlinear equations in n variables, without hypotheses on the system of nonzero Jacobian, convexity, or monotonicity.

Let $z(p)$ be the excess demand as a function of prices $p=(p_1 \ldots p_l)$ so that p^* is an equilibrium if $z(p^*)=0$. Then the global Newton takes the form

$$Dz(p)\frac{dp}{dt} = -\lambda z(p), \tag{1}$$

$$\text{sign } \lambda = \text{sign Determinant } Dz(p)$$

Here $Dz(p)$ is the matrix of first partial derivatives of z. If one takes $\lambda=1$ and uses Euler's method of discrete approximation to (1) then one obtains Newton's method for solving $z(p)=0$. Using equation (1), one can obtain a proof of the existence of economic equilibrium without using fixed point theorems or algebraic topology.

Consider the problem of representing a process of price adjustments by (1). Recall that the classical "tatonment" process has an embodiment in the equation

$$\frac{dp}{dt} = z(p) \tag{2}$$

Arrow, Leonid Hurwicz and Herbert Block have shown that solutions of (2) converge to economic equilibrium under hypotheses on z such as gross substitutes. These hypotheses are substantial and are strong enough to imply the existence of a unique equilibrium. On the other hand Scarf subsequently showed that under classical properties on preference relations, almost all solutions of (2) could oscillate for all time.

Now (1) can be considered as a modification of (2), which involves more subtle intermarket relations and which will converge when (2) doesn't. In particular (1) will converge in the Scarf example.

Generally speaking, (1) converges starting from almost any initial price system on the boundary of the price simplex. One could also formulate the equation in quantity space and look for an interpretation of the process in terms of budget balancing actions on the part of the agents. The interested reader could pursue these topics further in the papers cited.

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