NASH AND WALRAS EQUILIBRIUM VIA BROUWER

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Abstract

The existence of Nash and Walras equilibrium is proved via Brouwer's Fixed Point Theorem, without recourse to Kakutani's Fixed Point Theorem for correspondences. The domain of the Walras fixed point map is confined to the price simplex, even when there is production and weakly quasi-convex preferences. The key idea is to replace optimization with "satisficing improvement," i.e., to replace the Maximum Principle with the "Satisficing Principle."

The standard proofs of the existence of Nash and Walras equilibrium (including the original proofs by Nash [17], Arrow and Debreu [2], and McKenzie [15]) rely on Kakutani's Fixed Point Theorem for correspondences. I show that a slight perturbation of the standard arguments enables one to work entirely with Brouwer's Fixed Point Theorem for continuous functions.

Nash himself [18] gave a Brouwer fixed point proof of Nash equilibrium for matrix games. McKenzie [16] derived the existence of Walras equilibrium with production from Brouwer's Fixed Point Theorem. The only advantage of the maps I propose is that some readers may think they are simpler. For example, in my Walras existence proof the domain of the fixed point map is the price simplex. There is no need to enlarge the domain to include excess demands, as done by Gale [9] and Debreu [6], [7], or the demands of each consumer, as done in the generalized game proofs of Debreu [5] and Arrow and Debreu [2], or to add the auxiliary commodities introduced by McKenzie [16].

In Section 1, the existence of Nash equilibrium in concave games is proved. Let a game $G = (u_n, \Sigma_n)_{n \in N}$ be described by its payoffs u_n and strategy spaces Σ_n, for agents $n \in N$. The original proof by Nash relied on the best response correspondence $B_n(\overline{\sigma}_n, \overline{\sigma}_{-n}) = \operatorname{argmax}_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \overline{\sigma}_n)$. My proof simply replaces B_n with a

*I wish to thank Ken Arrow, Don Brown, and Andreu Mas-Colell for helpful comments. I first thought about using Brouwer's theorem without Kakutani's extension when I heard Herb Scarf's lectures on mathematical economics as an undergraduate in 1974, and then again when I read Tim Kehoe's 1980 Ph.D dissertation under Herb Scarf, but I did not resolve my confusion until I had to discuss Kehoe's presentation at the celebration for Herb Scarf's 65th birthday in September, 1995.

satisficing improvement *function*

$$\beta_n(\overline{\sigma}_n, \overline{\sigma}_{-n}) = \arg\max_{\sigma_n \in \Sigma_n} [u_n(\sigma_n, \overline{\sigma}_{-n}) - \|\sigma_n - \overline{\sigma}_n\|^2]$$

in which agent n moves part of the way to his optimal response. Moving all the way to a best response is irrelevant to demonstrating that a fixed point is an equilibrium. Section I also includes a discussion of earlier demonstrations of Nash equilibrium based on Brouwer's FPT for *matrix* games.

In Section 2 the existence of Walras equilibrium is proved for economies $E = ((u^h, e^h)_{h\in H},\ (Y_f)_{f\in F},\ (\theta^h_f)^{h\in H}_{f\in F})$ with quasi-concave utilities u^h and convex technologies Y_f. Let $M^h(p,\overline{p})$ be the minimum net expenditure household h must make at prices p beyond its income $I^h(p)$ in order to achieve the same utility it would obtain if it faced prices $\overline{p}$ and income $I^h(\overline{p})$.[1] It is well-known that M^h is continuous in $(p,\overline{p})$ and concave in p for any fixed $\overline{p}$. Let $M(p,\overline{p})$ be the sum of the $M^h(p,\overline{p})$ over all households h. Let S be the price simplex. In Section II it is shown that the *function* $\varphi : S \to S$ defined for each $\overline{p}$ in S by

$$\varphi(\overline{p}) = \arg\max_{p\in S} [M(p,\overline{p}) - \|p - \overline{p}\|^2]$$

is continuous and has Walras equilibrium as its fixed points.[2]

The minimum expenditure function and its properties have been very closely studied since Hicks showed that the so-called Hicksian demand is more regular than the Marshallian demand. Intermediate textbooks often emphasize the duality between utility maximization and expenditure minimization, which guarantees (through the Maxmin theorem) that a fixed point of the function φ must be a Walras equilibrium. Nevertheless, though there are many closely related ideas to be found in the literature, to the best of my knowledge nobody has used the function M to demonstrate the existence of equilibrium.

To get a piccture of the function M, let $D^h_+(\bar{p})$ be the set of all consumption bundles (budget feasible and not) that make agent h at least as well off as his Walrasian demands $D^h(\bar{p})$, and define the "better than excess demand correspondence" by $Z_+(\bar{p}) = \sum_{h\in H}(D^h_+(\bar{p}) - e^h) - \sum_{f\in F} Y_f$ in which the firms choose anything feasible. Similarly, let $Z(\bar{p}) = \sum_{h\in H}(D^h(\bar{p}) - e^h) - \sum_{f\in F} Y_f(\bar{p})$ be the usual excess demand correspondence. We shall see that

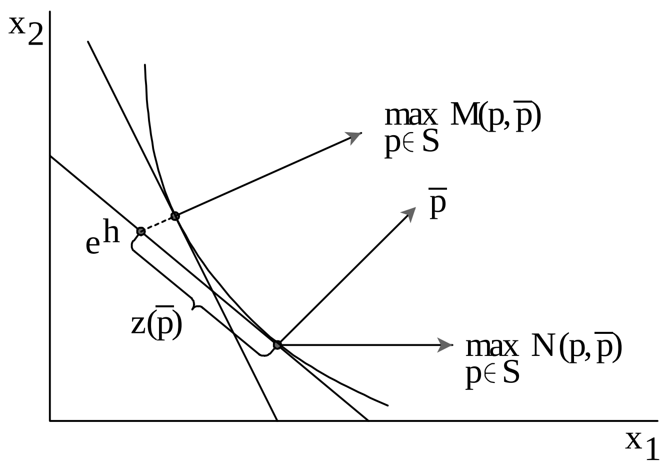
$$M(p,\bar{p}) = \min_{z\in Z_+(\bar{p})} p\cdot z.$$

Note that $M(p,\bar{p}) \leq N(p,\bar{p}) \equiv \min_{z\in Z(\bar{p})} p\cdot z$; usually $M(p,\bar{p}) < N(p,\bar{p})$. Indeed when excess demand Z is a correspondence, as will typically be the case without

[1] Income is defined by $I^h(p) = p\cdot e^h + \Sigma_{f\in F}\theta^h_f \max_{y_f\in Y_f} p\cdot y_f$.

[2] This Walrasian existence proof is evidently idential to our Nash equilibrium existence proof for a "psychological game" (not a generalized game) with only one player, the price player, whose payoff $M(p,\overline{p})$ depends on the $\overline{p}$ he is expected to play as well as the p he chooses.

further assumptions, $N(p,\bar{p})$ is not continuous.[3] Even when $Z(\bar{p})$ is a function, and N is continuous, $M(p,\bar{p}) \neq N(p,\bar{p})$. A diagram illustrates that $\max_{p\in S} M(p,\bar{p}) \neq \max_{p\in S} N(p,\bar{p})$ in an economy consisting of one consumer with a strictly concave utility.



In Section 3 we examine several special cases (e.g., the u^h strictly quasi-concave and the Y_f strictly convex) in which excess demand $Z(\bar{p})$ is a *function* $z(\bar{p})$. In these special cases there are already standard proofs of Walras equilibrium based on Brouwer's FPT. In order to facilitate comparisons with these standard proofs, in Section 3 we modify our fixed point map by replacing M with N, obtaining

$$\psi(\overline{p}) = \arg\max_{p\in S} \, [p \cdot z(\overline{p}) - \|p - \overline{p}\|^2].$$

Once again ψ is continuous and all its fixed points are Walras equilibria. Our perturbation $-||p-\bar{p}||^2$ still simplifies matters, even with dealing with excess demand *functions*. We apply similar maps in other special cases, e.g., with constant-returns-to-scale technologies. Much of the paper is devoted to these special cases, because many readers will find these cases to be all they are really interested in.[4] Our map ψ is quite different from the standard mpa (deriving from Nash's matrix game map) that is exposited in most textbooks, but ψ turns out to be closely related to the

[3]The function N has nevertheless often been used to prove the existence of equilibrium. In one such approach the prices p are called "better" than the prices $\overline{p}$ if $N(p,\overline{p}) > 0$. Walras equilibrium then exists if it can be shown that this partial ordering on prices has a maximal element. The problem is thus reduced to one of maximizing a (nontransitive) binary relation, for which see Nikaido [20], Fan [8], Sonnenschein [22], and Aliprantis and Brown [1]. For a lucid exposition of these ideas, see Border [3].

[4]An interesting feature of each successive Walras existence proof is that Brouwer's fixed point theorem must be augmented by Farkas' Lemma (when technology is given by a finite number of activities), the separating hyperplane theorem (when technology is given more generally by a cone), and the MinMax theorem (when technological possibilities are given by arbitrary convex sets).

maps used by Todd [23] and Kehoe [12] to compute equilibria of economies with fixed coefficient technologies.

An advantage to the map ψ based on N over the map φ based on M is that ψ requires less information on the part of the auctioneer to implement: just like with Walrasian tatonnement, the bigger the excess demand the greater the price increase. However, like Walrasian tatonnement, the algorithm $\overline{p}(t+1) = \psi(\overline{p}(t))$ does not necessarily converge. It is an open question whether the algorithm $\overline{p}(t+1) = \varphi(\overline{p}(t))$ converges more generally.

The only technical point in this paper occurs in showing that the function $M(p, \overline{p})$ is continuous, which is tantamount to showing that the "better than" correspondence $Z_+(\bar{p})$ is upper semi-continuous (USC) and lower semi-continuous (LSC). The standard Kakutani based argument requires proving that the excess demand correspondence $Z(\bar{p})$ is USC, which is accomplished by invoking the Maximum Principle. Kakutani's fixed point theorem does not require Z to be LSC, which is fortunate, because the Maximum Principle does not guarantee LSC and in general Z is not LSC. The impression the student is sometimes left holding is that LSC is less central than USC, but we should not forget that the Maximum Principle cannot be applied unless the budget correspondence of each agent is USC and LSC. Here we introduce new lemma called the Satisficing Principle, which could perhaps stand just behind the Maximum Principle as a useful tool in the theory of choice, because it guarantees LSC and USC.

The Satisficing Principle supposes that an agent maximizing a quasi-concave utility subject to a convex constraint is satisfied with a payoff $w(\alpha) < v(\alpha)$, where $v(\alpha)$ is the maximum achievable utility given the exogenous parameters α, and w is any continuous function. It asserts that the correspondence $W(\alpha)$ of all choices achieving payoff at least $w(\alpha)$ is lower semi-continuous (LSC) as well as upper semi-continuous (USC) in α. The Satisficing Principle complements the Maximum Principle, which guarantees that $v(\alpha)$ is continuous and that the set of choices achieving $v(\alpha)$ is USC but not necessarily LSC. One immediate application of the Satisficing Principle is that the Walrasian budget correspondence is LSC and USC when the endowment is strictly positive. More importantly, since the Walrasian indirect utility function is continuous, and by non-satiation, strictly less than the maximal utility achievable without a budget constraint, the Satisficing Principle guarantees the LSC and USC of $D^h_+(p)$, and hence of $Z_+(p)$.

The Satisficing Principle is stated and proved in Section 4, where it is also used to give a Brouwer FPT proof that quasi-concave games have Nash equilibria. In some sense the whole idea of this paper comes down to replacing optimization with satisficing improvement; first for the game players and the auctioneer, by subtracting $||\sigma_n - \overline{\sigma}_n||^2$ or $\|p - \overline{p}\|^2$, and second for the households, in substituting $Z_+(\overline{p})$ for $Z(\overline{p})$.

1 Games and Nash Equilibrium

1.1 Concave Games

Let a game G among N players be defined by compact and convex strategy spaces $\Sigma_1, ..., \Sigma_N$ in finite-dimensional Euclidean spaces, and by continuous payoff functions u_1, ..., u_N, where for each $n \in N$, $u_n : \Sigma \equiv \Sigma_1 \times \cdots \times \Sigma_N \to \mathbb{R}$. We call G a concave game if for any fixed $\overline{\sigma}_{-n} \equiv (\overline{\sigma}_1, ..., \overline{\sigma}_{n-1}, \overline{\sigma}_{n+1}, ..., \overline{\sigma}_N) \in \Sigma_{-n} \equiv \Sigma_1 \times \cdots \times \Sigma_{n-1} \times \Sigma_{n+1} \times \cdots \times \Sigma_N$, $u_n(\sigma_n, \overline{\sigma}_{-n})$ is concave in σ_n.

The two player matrix games are defined by $r \times s$ matrices A and B. Player α has strategy space $\Sigma_\alpha \equiv \{p \in \mathbb{R}^r_+ : \sum_{i=1}^r p_i = 1\}$ and player β has strategy space $\Sigma_\beta = \{q \in \mathbb{R}^s_+ : \sum_{j=1}^s q_j = 1\}$. The payoffs are defined by $u_\alpha(p, q) \equiv p'Aq$ and $u_\beta(p, q) \equiv p'Bq$. Since u_n is linear on Σ_n for $n = \alpha$ and β, these matrix games are indeed concave games.

Given a game $G = (\Sigma_1, ..., \Sigma_N; u_1, ..., u_N)$, a Nash equilibrium is a choice $\overline{\sigma} = (\overline{\sigma}_1, ..., \overline{\sigma}_N) \in \Sigma$ such that for all $n \in N$ and all $\sigma_n \in \Sigma_n$,

$$u_n(\overline{\sigma}) \geq u_n(\sigma_n, \overline{\sigma}_{-n}) \ .$$

THEOREM: *Every concave game has a Nash equilibrium.*

PROOF: Define the function

$$\varphi_n : \Sigma \to \Sigma_n \ \text{ by}$$

$$\varphi_n(\overline{\sigma}_1, ..., \overline{\sigma}_n, ..., \overline{\sigma}_N) = \underset{\sigma_n \in \Sigma_n}{\arg\max} \, [u_n(\sigma_n, \overline{\sigma}_{-n}) - \|\sigma_n - \overline{\sigma}_n\|^2].$$

Observe that the maximand is the sum of a continuous, concave function in σ_n, and a negative quadratic function in σ_n, and hence is continuous and strictly concave. Since Σ_n is compact and convex, φ_n is a well-defined function. Furthermore, the maximand is continuous in the parameter $\overline{\sigma} = (\overline{\sigma}_1, ..., \overline{\sigma}_n)$, hence φ_n is a continuous function.

Now define $\varphi : \Sigma \to \Sigma$ by $\varphi = (\varphi_1, ..., \varphi_N)$. Clearly φ is continuous, and so by Brouwer's theorem it has a fixed point $\varphi(\overline{\sigma}) = \overline{\sigma}$.

Suppose for some $\sigma_n \in \Sigma_n$, $u_n(\sigma_n, \overline{\sigma}_{-n}) - u_n(\overline{\sigma}) \equiv E > 0$. Then by concavity of u_n, for $0 < \varepsilon < 1$, $u_n(\varepsilon\sigma_n + (1-\varepsilon)\overline{\sigma}_n, \overline{\sigma}_{-n}) - u_n(\overline{\sigma}) \geq \varepsilon E > 0$ while $\|(\varepsilon\sigma_n + (1-\varepsilon)\overline{\sigma}_n) - \overline{\sigma}_n\|^2 = \varepsilon^2\|\sigma_n - \overline{\sigma}_n\|^2 < \varepsilon E$, if ε is small enough, contradicting the definition of φ_n. Hence $\overline{\sigma}$ is a Nash equilibrium. □

Nash [1950] suggested the correspondence $\psi_n : \Sigma \rightrightarrows \Sigma_n$ defined by $\psi_n(\overline{\sigma}) = \arg\max_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \overline{\sigma}_{-n})$. Since u_n is not necessarily strictly concave, $\psi_n(\overline{\sigma})$ may contain multiple elements.

The maximand above is simply a perturbation of the Nash maximand. It guarantees that a player will always make some improvement when there is an opportunity to improve, but he will not necessarily move all the way to his best response. Another difference is that the Nash correspondence ψ_n throws away some information, since

ψ_n actually is defined on Σ_{-n}. The map φ_n depends on all the coordinates, including Σ_n.

Nash [18] also showed that for matrix games, Brouwer's Fixed Point Theorem sufficed. He suggested using the excess return functions $z_\alpha(\bar{p}, \bar{q}) = A\bar{q} - (\bar{p}'A\bar{q})1$ and $z_\beta(\bar{p}, \bar{q}) = \bar{p}'B - (\bar{p}B\bar{q})1$, which specify the surplus each agent can get by playing each pure strategy instead of his designated mixed strategy. He then defined the map

$$f(\bar{p}, \bar{q}) = \left(\frac{\bar{p} + [A\bar{q} - (\bar{p}'A\bar{q})1]^+}{1 + [A\bar{q} - (\bar{p}'A\bar{q})1]^+ \cdot 1}, \frac{\bar{q} + [\bar{p}'B - (\bar{p}'B\bar{q})1]^+}{1 + [\bar{p}'B - (\bar{p}'B\bar{q})1]^+ \cdot 1} \right),$$

where for any vector y, $[y]^+$ is the vector with ith coordinate $\max(0, y_i)$, and 1 is the vector of all 1's, or just the scalar 1, depending on the context. A fixed point of the Nash map can be shown to be a Nash equilibrium by observing that $\bar{p}'[A\bar{q} - (\bar{p}'A\bar{q})1] = 0$. Indeed this same trick is copied in the now standard existence proof for Walrasian equilibrium, where it crops up as Walras law. The Nash map f expoits the special form of matrix games.

The map φ can be used for any concave game. In the special case of matrix games it reduces to

$$\varphi(\bar{p}, \bar{q}) = h(\bar{p}, \bar{q}) \equiv \left(\Pi_{\Sigma_\alpha}\left(\bar{p} + \tfrac{1}{2}A\bar{q}\right), \ \Pi_{\Sigma_\beta}\left(\bar{q} + \tfrac{1}{2}\bar{p}'B\right)\right),$$

where $\Pi_A(x)$ is the closest point in A to x. The map h has been used to prove the existence of Nash equilibrium in matrix games by Lemke–Howson [14], and to study the index of matrix game Nash equilibrium by Gul–Pearce–Stacchetti [11]. To see that φ reduces to h for matrix games, one needs to use the Kuhn–Tucker theorem. Indeed, one needs the Kuhn–Tucker theorem to verify that a fixed point of h is a Nash equilibrium.[5]

2 Walrasian Economies

2.1 The Walrasian Economy

Let us represent an economy by

$$E = \left\{ H, \, (X^h, \, e^h, \, u^h)_{h \in H}, \, F, \, (Y_f)_{f \in F}, \, (\theta^h_f)^{h \in H}_{f \in F} \right\},$$

where H is a finite set of households, $X^h \subset \mathbb{R}^L$ is the consumption set of household h, e^h is the endowment, and u^h is the utility function of agent $h \in H$, F is a finite set of firms, Y_f is the technology of firm $f \in F$, and $\theta^h_f \in \mathbb{R}_+$ is the ownership share of firm f by agent h, $\sum_{h \in H} \theta^h_f = 1$ for all $f \in F$. We assume in addition that $\forall h \in H$,

[5] By the Kuhn–Tucker theorem, $\varphi(\bar{p}, \bar{q}) - (\varphi_\alpha(\bar{p}, \bar{q}), -\varphi_\beta(\bar{p}, \bar{q}))$ satisfies $A\bar{q} - 2(\varphi_\alpha(\bar{p}, \bar{q}) - \bar{p}) - \lambda e + \Lambda = 0$, where $\Lambda \geq 0$ is a diagonal matrix with $\Lambda_{jj} > 0$ only if $\varphi_{\alpha j}(\bar{p}, \bar{q}) = 0$. By the Kuhn–Tucker theorem, the map $h(\bar{p}, \bar{q}) = (h_\alpha(\bar{p}, \bar{q}), h_\beta(\bar{p}, \bar{q}))$ satisfies $-2(h_\alpha(\bar{p}, \bar{q}) - \frac{1}{2}A\bar{q} - \bar{p}) + \mu e + \Omega = 0$, where $\Omega \geq 0$ is a diagonal matrix with $\Omega_{jj} > 0$ only if $h_{\alpha j}(\bar{p}, \bar{q}) = 0$. Our definition of φ avoids the need for the Kuhn–Tucker theorem.

(1) X^h is closed, convex, and bounded from below: $\exists \underline{d}^h$ such that $\underline{d}^h \leq x$ for all $x \in X^h$

(2) $e^h \in X^h$ and $\exists d^h \in X^h$ with $d^h \ll e^h$

(3a) $u^h : X^h \to \mathbb{R}$ is continuous

(3b) u^h is quasi-concave, i.e., $[u^h(x) > u^h(y)$ and $0 < \lambda < 1] \Rightarrow [u^h(\lambda x + (1-\lambda)y) > u^h(y)]$, for all $x, y \in X^h$

(3c) u^h is non-satiated, i.e., $\forall y \in X^h$, $\exists x \in X^h$ with $u^h(x) > u^h(y)$

and for all $f \in F$,

(4) Y_f is a closed convex subset of $\mathbb{R}^L$, and $0 \in Y_f$

and furthermore,

(5) $\exists p^* \in \mathbb{R}^L_{++}$ with $p^* \cdot Y_f \leq \pi_f < \infty$, $\forall f \in F$

(6) the map $\sigma : Y_1 \times \cdots \times Y_F \to \mathbb{R}^L$ defined by $\sigma(y_1, ..., y_F) = y_1 + \cdots + y_F$ is proper, i.e., $\sigma^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^L$ is compact.

2.2 Walras Equilibrium

A Walras equilibrium (WE) for the economy E is a tuple $(\overline{p}, (\overline{x}^h)_{h\in H}, (\overline{y}_f)_{f\in F}) \in \mathbb{R}^L_+ \times \mathsf{X}_{h\in H} X^h \times \mathsf{X}_{f\in F} Y_f$ satisfying

(1) $\sum_{h\in H} \overline{x}^h \leq \sum_{h\in H} e^h + \sum_{f\in F} \overline{y}_f$

(2) $\overline{y}_f \in \arg\max_{y_f\in Y_f} \overline{p} \cdot y_f$, $\forall f \in F$

(3) $\overline{x}^h \in B^h(\overline{p}) = \{x \in X^h : \overline{p} \cdot x \leq \overline{p}e^h + \sum_{f\in F} \theta^h_f \max_{y_f\in Y_f} \overline{p}y_f \equiv I^h(p)\}$, $\forall h \in H$

(4) $\overline{x}^h \in \arg\max_{x\in B^h(\overline{p})} u^h(x)$.

By non-satiation we know that at a WE each agent spends all his income, so the budget inequality in (3) reduces to equality, and we therefore conclude that in a WE,

$$\sum_{h\in H} \overline{x}^h_i < \sum_{h\in Y} e^h_i + \sum_{f\in F} \overline{y}_{fi} \Rightarrow \overline{p}_i = 0. \tag{1.1}$$

2.3 Easy Consequences of the Assumptions

An irreversibility assumption similar in spirit to (6) was proposed by Debreu. It has the consequence that $\widehat{Y} \equiv \sigma^{-1}(\{y \in \mathbb{R}^L : p^* \cdot y \leq \sum_{f\in F} \pi_f \text{ and } \overline{e} + y \geq \underline{d}\})$ is compact in $Y_1 \times \cdots \times Y_F$, where $\overline{e} \equiv \sum_{h\in H} e^h$ and $\underline{d} \equiv \sum_{h\in H} \underline{d}^h$. We may therefore find $\widehat{Y}_f \subset Y_f$ for all $f \in F$ that are compact, convex, contain 0, and such that $\widehat{Y} \subset \widehat{Y}_1 \times \cdots \times \widehat{Y}_F$.

Furthermore, let us define $\widehat{X}^h = \{x \in X^h : \forall i \text{ with } 1 \leq i \leq L,\ x_i \leq 1 + (p^* \cdot \overline{e} + \sum_{f\in F} \pi_f)/p_i^*\}$. Then by quasi-concavity of the utilities, restricting the consumption sets from X^h to $\widehat{X}^h$ and restricting the technologies from Y_f to $\widehat{Y}_f$ gives rise to an economy $\widehat{E}$ with exactly the same Walras equilibria as E. Thus without loss of generality, we may add assumption (7):

(7) X^h and Y^f are compact for all $h \in H$ and $f \in F$.

We list three more simple observations. Lemmas 1 and 2 rely on the definitions of USC and LSC, and on the Satisficing Principle, all of which are deferred to Section IV. Only Lemma 3 is directly used in the Walras existence proof.

LEMMA 1: *The budget correspondence $B^h(p)$ is USC, LSC, non-empty valued, convex-valued and compact-valued on $S = \{p \in \mathbb{R}^L_+ : \sum p_\ell = 1\}$.*

PROOF:

$$B^h(p) = \{x \in X^h : p \cdot x \leq I^h(p)\} = \{x \in X^h : -p \cdot x \geq -I^h(p)\} .$$

Since $e^h >> d^h$, $-I^h(p) \leq -p \cdot e^h < -p \cdot d^h \leq \arg\max_{x\in X^h} -p \cdot x$, so the lemma follows from the compactness of X^h, the continuity of $I^h(p)$, and the Satisficing Principle $\square$

Let $v^h(p) \equiv \max_{x\in B^h(p)} u^h(x)$ be the so-called indirect utility function of agent h. Since $B^h(p)$ is USC and LSC, non-empty valued and compact-valued, by the Maximum Principle, $v^h(p)$ must be continuous on S. Furthermore, let

$$D^h(p) \equiv \arg\max_{x\in B^h(p)} u^h(x)$$

be the demand correspondence of agent h. Again by the Maximum Principle, $D^h(p)$ is USC. Unfortunately, $D^h(p)$ may not be LSC, as is well known.

A central element of the existence proof given in Section B is the replacement of the demand correspondence $D^h(p)$, which may fail to be LSC, with the "demand or better" correspondence $D^h_+(p)$, which is always LSC. McKenzie [16] used a similar correspondence.

LEMMA 2: *$D^h_+(p) = \{x \in X^h : u^h(x) \geq v^h(p)\}$ is USC, LSC, non-empty-valued, convex-valued, and compact-valued for $p \in S$. Hence so is the better than excess demand $Z_+(p) = \sum_{h\in H} D^h_+(p) - \sum_{h\in H} e^h - \sum_{f\in F} Y_f$.*

PROOF: Apply the Satisficing Principle, noting that X^h and u^h are independent of p, and that $v^h(p)$ is continuous.

LEMMA 3: *The minimum expenditure function*

$$M(p,\overline{p}) = \min_{z\in Z_+(\overline{p})} p\cdot z$$

is continuous in $(p,\overline{p}) \in S\times S$*, and concave in* p *for any fixed* $\overline{p}\in S$*.*

PROOF: Lemma 2 and the Maximum Principle guarantee the continuity of $M(p,\overline{p})$. For any fixed $\overline{p}$, $M(p,\overline{p})$ is the minimum of a family of linear functions in p, hence it must be concave. □

In the next section we prove in passing that $M(p,\overline{p}) = \sum_{h\in H} M^h(p,\overline{p})$ where $M^h(p,\overline{p}) = \min\{p\cdot x : x\in D_+^h(\overline{p})\} - I^h(p)$.

2.4 Existence of Walras Equilibrium

We now construct an existence proof of Walras equilibrium for general quasi-concave preferences and convex production sets, that uses only the domain of prices S, and only Brouwer's fixed point theorem.

THEOREM: *Let* $E = (H, (u^h)_{h\in H}, F, (Y_f)_{f\in F}, (\theta_f^h)_{f\in F}^{h\in H})$ *be a Walras economy satisfying assumptions* (1)–(6)*. Then* E *has a Walras Equilibrium* $(\overline{p}, (\overline{x})_{h\in H}, (\overline{y}_f)_{f\in F})$*.*

Recalling that $Z_+(\overline{p}) = \sum_{h\in H} D_+^h(\overline{p}) - \sum_{h\in H} e^h - \sum_{f\in F} Y_f(\overline{p})$ is the at least as good as excess demand, and that $M(p,\overline{p}) = \min_{z\in Z_+(\overline{p})} p\cdot z$, define $\varphi : S\to S$ by

$$\begin{aligned}\varphi(\overline{p}) &= \arg\max_{p\in S} \left[M(p,\overline{p}) - \|p-\overline{p}\|^2\right] \\ &= \arg\max[\min_{z\in Z_+(\overline{p})} p\cdot z - \|p-\overline{p}\|^2].\end{aligned}$$

Since M is concave in p for any fixed $\overline{p}$, and $\|p-\overline{p}\|^2$ is quadratic, the maximand is strictly concave, so it has a unique maximum and $\varphi(\overline{p})$ is a function. Since M is continuous (equivalently, since $Z_+(\overline{p})$ is USC and LSC), φ is a continuous function. Therefore by Brouwer's fixed point theorem, φ has a fixed point $\overline{p}$.

At the fixed point $\overline{p}$,

$$\overline{p} \in \arg\max_{p\in S} M(p,\overline{p}) = \arg\max_{p\in S} \min_{z\in Z_+(\overline{p})} p\cdot z.$$

This is because of our familiar argument that the first term of the maximand is concave, and the second term has derivative zero around $p=\overline{p}$.

We now invoke the convexity of $Z_+(\overline{p})$ for the first time to derive from the Maxmin theorem the existence of $\overline{z}\in Z_+(\overline{p})$ such that

$$\overline{p}\cdot\overline{z} = \max_{p\in S} p\cdot\overline{z} = \min_{z\in Z_+(\overline{p})} \overline{p}\cdot z$$

Since $\overline{z} \in Z_+(\overline{p}) = \sum_{h\in H} D^h_+(\overline{p}) - \sum_{h\in H} e^h - \sum_{f\in F} Y_f$, we can find $\overline{x}^h \in D^h_+(p)$ and $\overline{y}_f \in Y_f$ such that $\overline{z} = \sum_{h\in H} \overline{x}^h - \sum_{h\in H} e^h - \sum_{f\in F} \overline{y}_f$. Furthermore, from the fact that $\overline{z} \in \arg\min_{z\in \widehat{Z}_+(\overline{p})} \overline{p} \cdot z$, we deduce that $\overline{y}_f \in \arg\max_{y_f\in Y_f} p \cdot y_f$. Furthermore, $p \cdot \overline{x}^h \leq I^h(p)$, otherwise replacing $\overline{x}^h$ with some element of $D^h(p)$ would improve on $\overline{z}$. Hence indeed $\overline{x}^h \in D^h(p)$.

It now follows that

$$\sum_{h\in H} \overline{p} \cdot \overline{x}^h \leq \sum_{h\in H} I^h(\overline{p}) = \sum_{h\in H} [\overline{p} \cdot e^h + \sum_{f\in F} \theta^h_f \overline{p} \cdot \overline{y}_f] = \sum_{h\in H} \overline{p} \cdot e^h + \sum_{f\in F} \overline{p} \cdot \overline{y}_f \ .$$

Hence $\overline{p} \cdot \overline{z} \leq 0$. But then from $\arg\max_{p\in S} p \cdot \overline{z} = \overline{p} \cdot \overline{z} \leq 0$, we deduce that $\overline{z} \leq 0$. $\square$

One notable aspect of the proof is that convexity (of $Z_+(p)$) was not needed until after we found a fixed point $\overline{p}$. In the usual proof, the excess demand $Z(p)$ is required to be convex in order to guarantee the existence of a fixed point. McKenzie [16] showed that one could always reduce convex technologies to CRS-technologies by adding F auxiliary commodities, representing the contributions of the owners to each firm. The fixed point map must then be carried out in a simplex of dimension $L + F - 1$. In the above proof the domain is the original $L - 1$ dimensional simplex.

3 Walras Equilibrium with Strictly Convex Preferences

In this section we specialize the general Walrasian economy given in Section II to cases where we can work with excess demand *functions.* For these cases it is already known that Brouwer's Theorem suffices to prove the existence of Walras equilibrium. But we show here that the perturbation $-||p - \overline{p}||^2$ can still simplify matters.

3.1 Pure Exchange and Strictly Convex Technologies

Let $S = \{p \in \mathbb{R}^L_+ \ : \ \sum_{i=1}^L p_i = 1\}$ be the usual price simplex.

Let z be called an excess demand function whenever $z : S \to \mathbb{R}^L$ is a continuous function satisfying Walras Law: $p \cdot z(p) = 0 \ \forall p \in S$.[6]

We define a Walras equilibrium for the excess demand function z as a price vector $\overline{p} \in S$ satisfying

$$z(\overline{p}) \leq 0 \ .$$

[6]Suppose that, in addition to assumptions (1)–(7) from Section 2, for all $h \in H$,

$$[u^h(x) \geq u^h(y)] \Rightarrow [u^h(\lambda x + (1-\lambda)y) > u^h(y)]$$

if $0 < \lambda < 1$, $x \neq y$ and $x, y \in X^h$, and for all $f \in F$

$$[x \neq y \in Y_f, \ 0 < \lambda < 1] \Rightarrow [\exists z \in Y_f \text{ with } z \gg \lambda x + (1-\lambda)y] \ .$$

Then $z(p) = \sum_{h\in H} D^h(p) - \sum_{h\in H} e^h - \sum_{f\in F} \arg\max_{y_f\in Y_f} p \cdot y_f$ is a continuous function satisfying Walras Law. In the special case $Y_f = \{0\} \ \forall f \in F$, we have a pure exchange economy.

Note that by Walras Law, $z_i(\overline{p}) = 0$ unless $\overline{p}_i = 0$, in which case we may have $z_i(\overline{p}) < 0$.

THEOREM: *Every excess demand function has a Walras equilibrium.*

PROOF: Define the map $\varphi : S \to$S by

$$\varphi(\overline{p}) \equiv \arg\max_{p \in S} [p \cdot z(\overline{p}) - \|p - \overline{p}\|^2].$$

Observe that the maximand is the sum of a linear function in p and a quadratic function in p, hence it is strictly concave and continuous in p. Since S is compact and convex, $\varphi(\overline{p})$ is a single point, and so φ is a function. By the maximum principle, φ is a continuous function (since the parameters $z(\overline{p})$ and $\overline{p}$ move continuously as $\overline{p}$ varies).

Hence by Brouwer's Fixed Point Theorem, φ has a fixed point $\overline{p}$. At this fixed point, we cannot have $p \cdot z(\overline{p}) > 0 = \overline{p} \cdot z(\overline{p})$ for any $p \in S$, because then the first term of the maximand would have a positive derivative at $\overline{p}$ in the direction $p - \overline{p}$, while the second term has derivative 0 at $\overline{p}$ in every direction, contradicting the optimality of $\overline{p}$.

Hence $p \in S \Rightarrow p \cdot z(\overline{p}) \leq 0$, which implies $z(\overline{p}) \leq 0$. □

Debreu's [7] proof of Walras equilibrium uses the correspondence $\psi(z) = \arg\max_{p \in S} p \cdot z$. As Debreu said, ψ is motivated by the principle that when there is excess demand in some commodity, $z_i > 0$, prices should go up, at least where excess demand is greatest. The only drawback to Debreu's construction is that $\psi(z)$ may be multi-valued, thus forcing the use of Kakutani's Fixed Point Theorem. The function $\varphi(p)$ is obtained by a slight perturbation of Debreu's construction.

The best known continuous function for proving Walras equilibrium is obtained by imitating the Nash [18] fixed point map for matrix games: $g_i(p) \equiv \{p_i + [z_i(p)]^+\}/\{1 + \sum_{j=1}^{L} [z_j(p)]^+\}$, where $[x]^+ = \max\{x, 0\}$, for $i = 1, ..., L$. A simple, but slightly awkward argument, using Walras law, shows that a fixed point of g is a Walras equilibrium.

The function $\varphi(p)$ is (surprisingly) identical to the map $h(p) = \Pi_S(p + \frac{1}{2}z(p))$, where $\Pi_S(x)$ is the closest point in S to x.[7] By deriving φ from the above maximization, one can see transparently that a fixed point is a Walrasian equilibrium. To show that a fixed point of h on the boundary of S is an equilibrium, the Kuhn–Tucker theorem must be invoked.

[7] By the Kuhn–Tucker theorem, $\varphi(\overline{p}) = \arg\max_{p \in S} [p \cdot z(\overline{p}) - \|p - \overline{p}\|^2]$ satisfies $(\varphi(\overline{p}) - \overline{p}) = \frac{1}{2}z(\overline{p}) - \lambda e + \Lambda$ where $\Lambda \geq 0$ is a diagonal matrix with $\Lambda_j > 0$ only if $\varphi_j(\overline{p}) = 0$. Similarly by the Kuhn–Tucker theorem $h(\overline{p}) = \arg\min_{p \in S} \|p - [\overline{p} + \frac{1}{2}z(\overline{p})]\|^2$ satisfies the same equation.

3.2 Production with Constant Returns-to-Scale Technologies

We now consider CRS production. A constant returns-to-scale (CRS) technology is a set $Y \subset \mathbb{R}^L$ such that Y is a closed, convex, cone ($y \in Y$ implies $ty \in Y$ for all $t \geq 0$; in particular, $0 \in Y$). Furthermore we suppose that Y allows for free disposal; $z \leq y$ and $y \in Y$ implies $z \in Y$. Finally, we suppose there is some $p^* \in S$ with $p^* \cdot Y \leq 0$, i.e., $p^* \cdot y \leq 0$ for all $y \in Y$.

A Walras equilibrium with production for an excess demand function, CRS-technology pair (z, Y) is a price $\overline{p} \in S$ such that $z(\overline{p}) \in Y$ and $\overline{p}Y \leq 0$. Note that by Walras Law the production plan $z(\overline{p})$ chosen makes zero profits, while alternatives either lose money or do no better.

The central example of a CRS-technology is an activity analysis production technology given by the matrix $B = [-I \;\; A]$ where I is the $L \times L$ identity matrix and A is an $L \times n$ vector of activities. Each column of the B matrix represents an "activity." Positive elements correspond to outputs, negative entries in B correspond to inputs. The first L columns of B represent pure disposal. The activity matrix B determines the CRS-technology

$$Y = \{Bx | x \in \mathbb{R}_+^{L+n}\} \; .$$

Clearly Y is a convex, closed cone allowing for free disposal. If for some vector $W \gg 0$, $\{x \in \mathbb{R}_+^{L+n} \, : \, Bx + W \geq 0\}$ is bounded, then there must be a $p^* \in S$ with $p^* \cdot Y \leq 0$.

TECHNOLOGY LEMMA: *If Y is a* CRS-*technology and for some vector $z \in \mathbb{R}^L$, $[p \in S$ and $pY \leq 0] \Rightarrow pz \leq 0$, then $z \in Y$.*

PROOF: Suppose $z \notin Y$. Since Y is closed and convex, by the separating hyperplane theorem we can strictly separate Y and z, that is find some $\overline{p} \in \mathbb{R}^L$ such that $\overline{p} \cdot Y < \overline{p} \cdot z$. But Y is a cone, so $\overline{p} \cdot Y$ bounded above implies $\overline{p} \cdot Y \leq 0$; also $0 \in Y$, so we have $\overline{p} \cdot Y \leq 0 < \overline{p} \cdot z$. By free disposal, $\overline{p} \cdot Y \leq 0$ implies $\overline{p} \geq 0$. Scaling $\overline{p}$, we get $p \in S$ and $pY \leq 0 < p \cdot z$, contradicting the hypothesis. □

THEOREM: *Every excess demand function,* CRS-*technology pair (z, Y) has a Walras equilibrium.*

PROOF: We seek $\overline{p} \in S_Y \equiv \{p \in S \, : \, p \cdot Y \leq 0\}$ with $z(\overline{p}) \in Y$. By the technology lemma, it suffices to find $\overline{p} \in S_Y$ such that $p \in S_Y \Rightarrow p \cdot z(\overline{p}) \leq 0 = \overline{p} \cdot z(\overline{p})$.

By hypothesis, S_Y is non-empty. Furthermore, $S_Y \equiv \bigcap_{y \in Y} \{p \in S \, : \, p \cdot y \leq 0\}$ is the intersection of closed and convex sets, and so is closed and convex.

Define $\varphi \, : \, S_Y \to S_Y$ by

$$\varphi(\overline{p}) \equiv \underset{p \in S_Y}{\arg\max} \, [p \cdot z(\overline{p}) - \|p - \overline{p}\|^2] \; .$$

As we argued earlier, φ is a continuous function. Since S_Y is compact and convex, Brouwer's Fixed Point Theorem guarantees φ has a fixed point $\overline{p}$.

Again as we argued earlier, at the fixed point $\overline{p}$, $p \in S_Y \Rightarrow p \cdot z(\overline{p}) \leq \overline{p} \cdot z(\overline{p}) = 0$. □

The idea that Brouwer's theorem alone can be used to prove the existence of Walras equilibrium with production is due to McKenzie [16] who also used the set S_Y. His mapping is much more elaborate than φ, but it allows for excess demand correspondences.

Todd [23] suggested the map $h(p) = \Pi_{S_Y}[p + z(p)]$. (A similar map is in Kehoe [12].) He showed by the Kuhn–Tucker theorem that a fixed point of h must be a Walras equilibrium, when Y is given by an activity analysis technology. The map φ is identical, its only advantage being a perhaps more transparent proof that a fixed point is a Walras equilibrium (and the incorporation of general CRS Y).

3.3 Monotonic Preferences and Boundary Behavior

In Sections 3.1 and 3.2 we assumed that the excess demand function z is continuous on all of S, including at $p \in S$ where some prices p_i may be zero. We now consider the possibility that preferences might be strictly monotonic, so that excess demand becomes infinite as p approaches the boundary, and z is not even defined on all of S. Let S^0 be the interior of S, and ∂S be its boundary. For every $\varepsilon > 0$, let $S^\varepsilon \equiv \{p \in S : p \geq \varepsilon e\}$ be the trimmed simplex, and ∂S^ε its boundary, where $1 = (1, ..., 1)$.

We say that (z, Y) is an excess demand function, CRS-technology pair with proper boundary behavior whenever $z : S^0 \to \mathbb{R}^L$ is a continuous function satisfying Walras Law for all $p \in S^0$, and such that $\exists \varepsilon > 0$ and $\exists p^* \in S^\varepsilon$, satisfying

$$p^* \cdot Y \leq 0 \ . \tag{1}$$

$$p \in \partial S^\varepsilon \Rightarrow p^* \cdot z(p) > 0 \ , \tag{2}$$

When preferences are strictly monotonic, $p \to \partial S \Rightarrow$ some $z_i(p) \to \infty$. Since excess demand is bounded from below by the aggregate endowment of goods, strict monotonicity implies that for any $p^* \gg 0$, $p^* \cdot z(p) > 0$ if p is close enough to the boundary. Thus proper boundary behavior is automatically satisfied by excess demand functions derived from strictly monotonic preferences, provided we can find some strictly positive prices p^* at which $p^* \cdot Y \leq 0$. This latter condition is trivially verified if for example there is some indispensable input like labor that is never produced.[8]

THEOREM: *Every monotonic excess demand function,* CRS-*technology pair with proper boundary behavior has a Walras equilibrium.*

PROOF: S^ε is compact and convex. Hence $S^\varepsilon_Y \equiv S^\varepsilon \cap S_Y$ is also compact and convex. Define $\varphi : S^\varepsilon_Y \to S^\varepsilon_Y$ by

$$\varphi(\overline{p}) \equiv \arg\max_{p \in S^\varepsilon_Y} \left[p \cdot z(\overline{p}) - \|p - \overline{p}\|^2\right] \ .$$

[8]For a refinement of this boundary condition, see Neuefeind [19].

As before, φ is a continuous function, hence it has a fixed point $\overline{p}$. Again by the familiar argument, $p \in S_Y^\varepsilon \Rightarrow p \cdot z(\overline{p}) \leq \overline{p} \cdot z(\overline{p}) = 0$.

If some $\overline{p}_i = \varepsilon$, then by proper boundary behavior, $p^* \cdot z(\overline{p}) > 0$, a contradiction, since $p^* \in S_Y^\varepsilon$. Hence $\overline{p} \gg \varepsilon e$. But then by concavity of the maximand, $p \in S_Y \Rightarrow p \cdot z(\overline{p}) \leq 0$. By the technology lemma, $z(\overline{p}) \in Y$, so $\overline{p}$ is a Walras equilibrium. □

4 The Satisficing Principle and Quasi-Concave Games

4.1 The Satisficing Principle

Recall that the famous Maximum Principle asserts that the best response correspondence is upper semi-continuous (USC). The USC property is the crucial hypothesis in Kakutani's fixed point theorem for correspondences. Kakutani's theorem is used instead of Brouwer precisely because the best response correspondence may not be lower semi-continuous (LSC). What I show below is that if we replace maximization with almost maximization (satisficing), then the satisficing correspondence is LSC and USC. For the purpose of proving existence of equilibrium, we shall see that nothing is lost by replacing best response with better than.

Let $\mathcal{A} \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^n$, and let $\psi : \mathcal{A} \rightrightarrows X$ be a correspondence associating with each $\alpha \in \mathcal{A}$ a subset $\psi(\alpha) \subset X$. We say that ψ is upper semi-continuous (USC) if

$$\left.\begin{array}{l} \alpha_n \to \alpha \\ x_n \to x \\ x_n \in \psi(\alpha_n) \end{array}\right\} \Rightarrow x \in \psi(\alpha)$$

for any $\{x_n, x\} \subset X$, $\{\alpha_n, \alpha\} \subset \mathcal{A}$. We say that ψ is lower semi-continuous (LSC) iff

$$\left.\begin{array}{l} \alpha_n \to \alpha \\ x \in \psi(\alpha) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \exists x_n \to x \\ x_n \in \psi(\alpha_n) \end{array}\right.$$

for any $\{\alpha_n, \alpha\} \subset \mathcal{A}$ and $x \in X$.

We say that ψ is USC or LSC at a point $\overline{\alpha} \in \mathcal{A}$ if the above conditions hold when $\alpha = \overline{\alpha}$. Clearly ψ is USC or LSC if it is USC or LSC at each point $\overline{\alpha} \in \mathcal{A}$.

SATISFICING PRINCIPLE: Let $u : X \times \mathcal{A} \to \mathbb{R}$ be a continuous function, where $X \times \mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^m$, and X is convex. Let u be quasi-concave in X, for any fixed $\alpha \in \mathcal{A}$. Let $\beta : \mathcal{A} \rightrightarrows X$ be a non-empty, USC and LSC, convex-valued correspondence. Let $v : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the maximum value function defined by $v(\alpha) \equiv \sup_{x \in \beta(\alpha)} u(x, \alpha)$. Finally, let $w : \mathcal{A} \to \mathbb{R}$ be continuous and satisfy $w(\alpha) < v(\alpha)$ for all $\alpha \in \mathcal{A}$. Then the correspondence $W : \mathcal{A} \rightrightarrows X$ defined by

$$W(\alpha) \equiv \{x \in \beta(\alpha) : u(x, \alpha) \geq w(\alpha)\}$$

is USC and LSC, and non-empty and convex-valued.

If in addition $\beta(\alpha) = \beta$ for all $\alpha \in \mathcal{A}$, and $u(x, \alpha) = u(x)$ for all $(x, \alpha) \in X \times \mathcal{A}$, then the same conclusion holds even with a weak inequality $w(\alpha) \leq v(\alpha) \equiv v$ for all $\alpha \in \mathcal{A}$.

PROOF: The non-emptiness and convex-valuedness of W are evident. USC follows as in the maximum principle, and does not depend on the convexity of X or the quasi-concavity of u, or on the strict inequality $w(\alpha) < v(\alpha)$. Simply note that if $\{x_n \in W(\alpha_n)$ for all n, and $\alpha_n \to \alpha$ and $x_n \to x\}$, then by USC of β, $x \in \beta(\alpha)$. By hypothesis, $u(x_n, \alpha_n) \geq w(\alpha_n)$. Passing to the limit, and recalling the continuity of u and w, $u(x, \alpha) \geq w(\alpha)$, so $x \in W(\alpha)$.

To prove LSC of W, let $\alpha_n \to \alpha$ and let $x \in W(\alpha)$. By hypothesis, there is some $\overline{x} \in \beta(\alpha)$ with $u(\overline{x}, \alpha) > w(\alpha)$. From the LSC of β, we can find $\overline{x}_n \in \beta(\alpha_n)$, $\overline{x}_n \to \overline{x}$. From the continuity of u and w, for large n, say, $n \geq N$, $u(\overline{x}_n\, \alpha_n) > w(\alpha_n)$.

From the LSC of β, we can also find $\widehat{x}_n \in \beta(\alpha_n)$ with $\widehat{x}_n \to x$. For each $n \geq N$, let $t(n)$ be the smallest $t \in [0, 1]$ such that $u((1-t)\widehat{x}_n + t\overline{x}_n, \alpha_n) \geq w(\alpha_n)$. Let $x_n \equiv (1-t(n))\widehat{x}_n + t(n)\overline{x}_n$, for $n \geq N$, and let x_n be any point in $W(\alpha_n)$ for $n < N$. By convexity of β, $x_n \in \beta(\alpha_n)$, and hence $x_n \in W(\alpha_n)$. I claim $t(n) \to 0$, and $x_n \to x$.

Suppose otherwise. Then there is a subsequence $t(n_k) \to \tilde{t} > 0$. Note that for $n \geq N$, if $x_n \neq \hat{x}_n$, then $u(x_n, \alpha_n) = w(\alpha_n)$. Let $\tilde{x} = (1-\tilde{t})x + \tilde{t}\overline{x}$. By the continuity of u and w, $u(\tilde{x}, \alpha) = w(\alpha)$. But from the quasi-concavity of u, $u(\tilde{x}, \alpha) > w(\alpha)$, since $u(x, \alpha) \geq w(\alpha)$ and $u(\overline{x}, \alpha) > w(\alpha)$ and $\tilde{t} > 0$, a contradiction. Thus LSC is proved.

Now, suppose neither β nor u depends on α. To verify LSC even when $w(\alpha) = v(\alpha)$, let $x \in \beta(\alpha) = \beta$, $u(x, \alpha) = u(x) = w(\alpha) = v(\alpha) = v$. Let $\alpha_n \to \alpha$. Then $x \in \beta(\alpha_n) = \beta$ for all n. Furthermore, $u(x, \alpha) = v \geq w(\alpha_n)$ for all n. Hence $x \in W(\alpha_n)$ for all n. □

4.2 Quasi-Concave Games

We can weaken the hypothesis that u_n is concave in σ_n to the hypothesis of quasi-concavity: $u_n(\sigma_n, \overline{\sigma}_{-n}) > u_n(\overline{\sigma}_n, \overline{\sigma}_{-n})$ implies $u_n(\lambda\sigma_n + (1-\lambda)\overline{\sigma}_n, \overline{\sigma}_{-n}) > u_n(\overline{\sigma}_n, \overline{\sigma}_{-n})$ for all $0 < \lambda < 1$. The result is called a quasi-concave game.

THEOREM: *Every quasi-concave game has a Nash equilibrium.*

PROOF: Let $v_n(\overline{\sigma}_{-n}) \equiv \max_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \overline{\sigma}_{-n})$ define a continuous function from Σ_{-n} to $\mathbb{R}$, called the "indirect utility function." Let $\delta_n(\overline{\sigma}) = v_n(\overline{\sigma}_{-n}) - u_n(\overline{\sigma})$, let $\delta(\overline{\sigma}) = \max_{n \in N} \delta_n(\overline{\sigma})$, let $w_n(\overline{\sigma}) = v_n(\overline{\sigma}_{-n}) - \frac{1}{2}\delta(\overline{\sigma})$, and let

$$W_n(\overline{\sigma}_n, \overline{\sigma}_{-n}) = \{\sigma_n \in \Sigma_n \,:\, u_n(\sigma_n, \overline{\sigma}_{-n}) \geq w_n(\overline{\sigma}_{-n})\} \;.$$

Suppose G has no Nash equilibrium. Then for each $\overline{\sigma} \in \Sigma$, $\delta(\overline{\sigma}) > 0$ and for each n, $w_n(\overline{\sigma}) < v_n(\overline{\sigma})$. Moreover, for some player n, $u_n(\overline{\sigma}) < w_n(\overline{\sigma})$, so $\overline{\sigma}_n \notin W_n(\overline{\sigma})$. By the Satisficing Principle, W_n is in non-empty, USC, LSC, and convex-valued. Define $\varphi_n \,:\, \Sigma_n \times \Sigma_{-n} \to \Sigma_n$ by

$$\varphi_n(\overline{\sigma}_n, \overline{\sigma}_{-n}) = \min_{\sigma_n \in W_n(\overline{\sigma}_n, \overline{\sigma}_{-n})} \|\sigma_n - \overline{\sigma}_n\|^2 \;.$$

Clearly φ_n is a function, since W_n is convex-valued. Furthermore, if W_n is USC and LSC, then by the Maximum Principle, φ_n is a continuous function. Let $\varphi =$

$(\varphi_1, ..., \varphi_N)$. If G has no Nash equilibrium, then φ is a continuous function with no fixed point, a contradiction. □

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