TOPOLOGY OF POINT CLOUD DATA

CS 468 – Lecture 8
3/3/4
OVERVIEW

- Points
- Complexes
  - Čech
  - Rips
  - Alpha
- Filtrations
- Persistence
• $m$ samples $M = \{m_1, m_2, \ldots, m_m\}$ from a manifold $\mathbb{M}$
• Samples are embedded, but intrinsic topology is lost
• Error: acquisition device noise and approximation
POIN T CLOUD DATA

(a) Surface
(b) Molecule
(c) Universe
$\epsilon$-BALLS

- $\epsilon$-ball: $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$.
- Open sets and topology
- Manifold is $\tilde{M} = \bigcup_{m_i \in M} B_\epsilon(m_i)$
• $C_\epsilon(M) = \{\text{conv } T \mid T \subseteq M, \bigcap_{m_i \in T} B_\epsilon(m_i) \neq \emptyset\}$.

• $\sum_{k=0}^{m} \binom{m}{k} = 2^{m+1} - 1$

• $C_\epsilon(M) \simeq \tilde{M}$
• \( R_\epsilon(M) = \{\text{conv } T \mid T \subseteq M, \; d(m_i, m_j) < \epsilon, m_i, m_j \in T\} \).
• Still \( O\left(\binom{m}{k}\right) \) for the \( k \)th skeleton
• Need \((k + 1)\)st skeleton for computing \( H_k \)
• $V(m_i) = \{ x \in \mathbb{R}^3 \mid d(x, m_i) \leq d(x, m_j) \forall m_j \in M \}$
• $\hat{V}(m_i) = B_\epsilon(m_i) \cap V(m_i)$
• $A_\epsilon = \left\{ \text{conv } T \mid T \subseteq M, \bigcap_{m_i \in T} \hat{V}(m_i) \neq \emptyset \right\}$
• $A_\epsilon(M) \simeq \tilde{M}$, $A_\epsilon \subseteq D$, the Delaunay complex
• $O(n \log n + n^{\lceil d/2 \rceil})$
Alpha Complex

- Extendible to points with weights
- van der Waals model of molecules
Filtrations

- Complexes $C_\epsilon, R_\epsilon, A_\epsilon$, compute homology!

- Which $\epsilon$? Vary and get a filtration!

- A filtration of a complex $K$ is $\emptyset = K^0 \subset K^1 \subset \ldots \subset K^m = K$. 
BUNNY

- 34,834 points, 1,026,111 complexes
• 312 atoms, 8,591 complexes
Approach

• Input: point cloud

• Procedure:
  – Put $\epsilon$-balls around points
  – Compute complex $K_\epsilon$
  – Compute homology of complex

• Varying $\epsilon$ gives us a filtration

• Incremental algorithm gives homology of filtration (demo)
**Homology of a Filtration**

- $K^l$ is a filtration.
- $Z_k^l = Z_k(K^l)$ and $B_k^l = B_k(K^l)$ are the $k$th cycle and boundary group of $K^l$, respectively.
- The $k$th homology group of $K^l$ is $H_k^l = Z_k^l / B_k^l$.
- The $k$th Betti number $\beta_k^l$ of $K^l$ is the rank of $H_k^l$. 
Features

Noise: spawned by noise, representation, etc.
**PERSISTENCE**

- $K^l$ be a filtration.
- The $p$-persistent $k$th homology group of $K^l$ is
  \[ H_{k}^{l,p} = \mathbb{Z}_k^l / (B_{k}^{l+p} \cap \mathbb{Z}_k^l), \]
- The $p$-persistent $k$th Betti number $\beta_{k}^{l,p}$ of $K^l$ is the rank of $H_{k}^{l,p}$.
- Well-defined
- $\eta_{k}^{l,p} : H_{k}^{l} \to H_{k}^{l+p}$,
- $\text{im} \eta_{k}^{l,p} \cong H_{k}^{l,p}$.
- This lecture: $\mathbb{Z}_2$ homology
Lifetimes

- Let $z$ be a non-bounding $k$-cycle, created when $\sigma$ enters complex at time $i$
- That is, $\beta_{k^+}$ at time $i$
- $z$ creates a class of homologous cycles $[z]$
- $[z]$ is merged with the boundary class at time $j$ when $\tau$ enters ($\beta_{k^-}$)
- $\tau$ destroys $z$ and the cycle class $[z]$.
- The persistence of $z$, and its homology class $[z]$, is $j - i - 1$.
- $\sigma$ is the creator (positive) and $\tau$ is the destroyer (negative) of $[z]$.
- If a cycle class does not have a destroyer, its persistence is $\infty$. 
LIFETIME REGIONS

- \( H_{l,p}^{k} = \frac{Z_{l}^{k}}{(B_{l+p}^{k} \cap Z_{l}^{k})} \)
- Basis element \( z + B_{k} \) lives during \([i, j)\)
- \( z \not\in B_{l}^{k} \) for \( l \leq j \)
- Therefore, \( z \not\in B_{l+p}^{k} \) for \( l + p < j \).
- \( p \geq 0 \)
- \( l \geq i \)
\( p \geq 0 \)

\( l \geq i \)

\( l < j \)
GRAPH OF $\log(\beta_{l,p}^1 + 1)$
CMY Color Space

- Green
- Yellow (minus blue)
- Cyan (minus red)
- Black
- Red
- Blue
- Magenta (minus green)
Topology Maps
**Graded $\mathbb{F}[t]$-module**

- Degrees of **homogeneous** elements:

<table>
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<th>b</th>
<th>c</th>
<th>d</th>
<th>ab</th>
<th>bc</th>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

  \[ M_1 = \begin{bmatrix} \partial_1 & ab & bc & cd & ad & ac \\ d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{bmatrix} \]

- \( \deg \hat{e}_i + \deg M_k(i, j) = \deg e_j \)
## Column Echelon Form

\[ \tilde{M}_1 = \begin{bmatrix}
  \begin{array}{cccccc}
  cd & bc & ab & z_1 & z_2 \\
  d & t & 0 & 0 & 0 & 0 \\
  c & t & 1 & 0 & 0 & 0 \\
  b & 0 & t & 0 & 0 & 0 \\
  a & 0 & 0 & t & 0 & 0 
  \end{array}
\end{bmatrix} \]

- Only column operations of type (1, 3)
- \( z_1 = ad - cd - t \cdot bc - t \cdot ab \)
- \( z_2 = ac - t^2 \cdot bc - t^2 \cdot ab \)
- \( \{ z_1, z_2 \} \) form homogeneous basis for \( \mathbb{Z}_1 \)
- \( \text{rank } M_k = \text{rank } B_{k-1} \) is number of pivots
Echelon Form Lemma

- (Lemma) The pivots in column-echelon form are the same as the diagonal elements in normal form. Moreover, the degree of the basis elements on pivot rows is the same in both forms.
- If only interested in degree of basis elements, read them off the column echelon form.
**Basis Change Lemma**

- Represent $\partial_{k+1}$ in terms of the basis computed for $\mathbb{Z}_k$

- $\partial_k \partial_{k+1} = \emptyset, M_k M_{k+1} = 0$

- (Lemma) To represent $\partial_{k+1}$ relative to the standard basis for $\mathbb{C}_{k+1}$ and the basis computed for $\mathbb{Z}_k$, simply delete rows in $M_{k+1}$ that correspond to pivot columns in $\tilde{M}_k$. 
PROOF

- Replace column $i$ by (column $i$) + $q$(column $j$) to eliminate element in pivot row $j$
- $\equiv$ replacing column basis element $e_i$ by $e_i + qe_j$ in $M_k$
- $\equiv$ replacing row $j$ with (row $j$) − $q$(row $i$) in $M_{k+1}$
- But row $j$ is eventually zero and row $i$ is not changed. QED
ALGORITHM

- No need for row operations
- Free columns correspond to positive simplices
- Pivot columns correspond to negative simplices
- No need for matrix representation
- Sparse matrix computation of Betti numbers based on persistence
DISCUSSION

• We can compute:
  – Cycles (components, cycles, voids)
  – Bounding manifolds

• (Demo)

• Points can be anything
  – samples from high dimensional manifolds: configuration spaces for robots (PRM), time-variant data, etc.
  – samples of tangent complex for data-set \( \mathbb{M} \times \mathbb{S}^2 \)

• Need fast \( d \)-dim complex builder