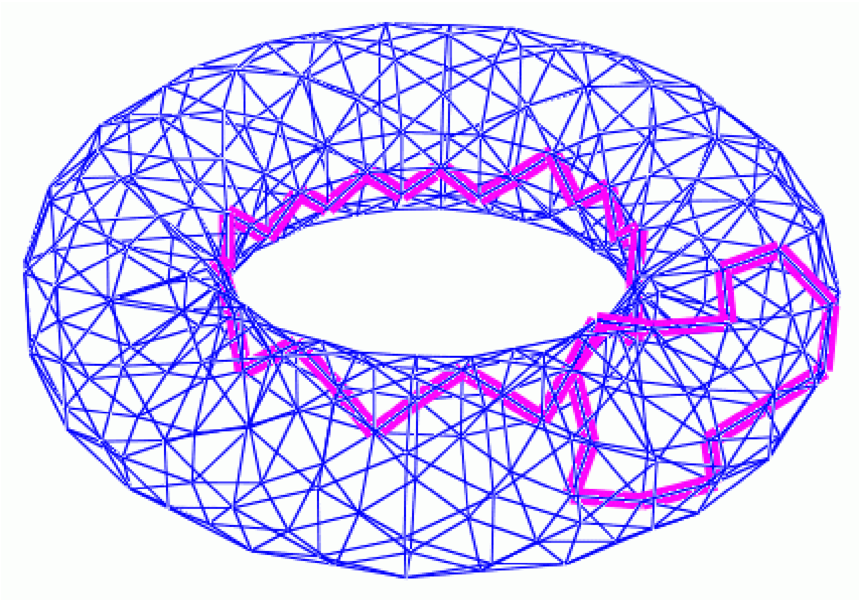
TOPOLOGY OF POINT CLOUD DATA



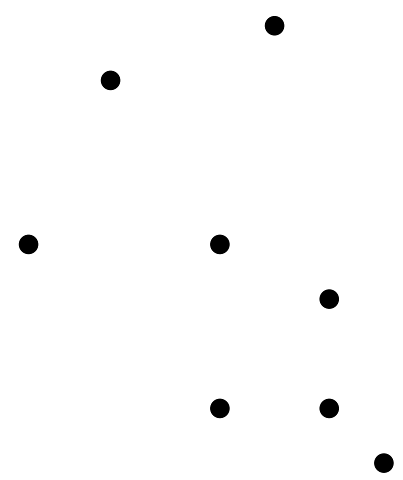
CS 468 – Lecture 8

3/3/4

Overview

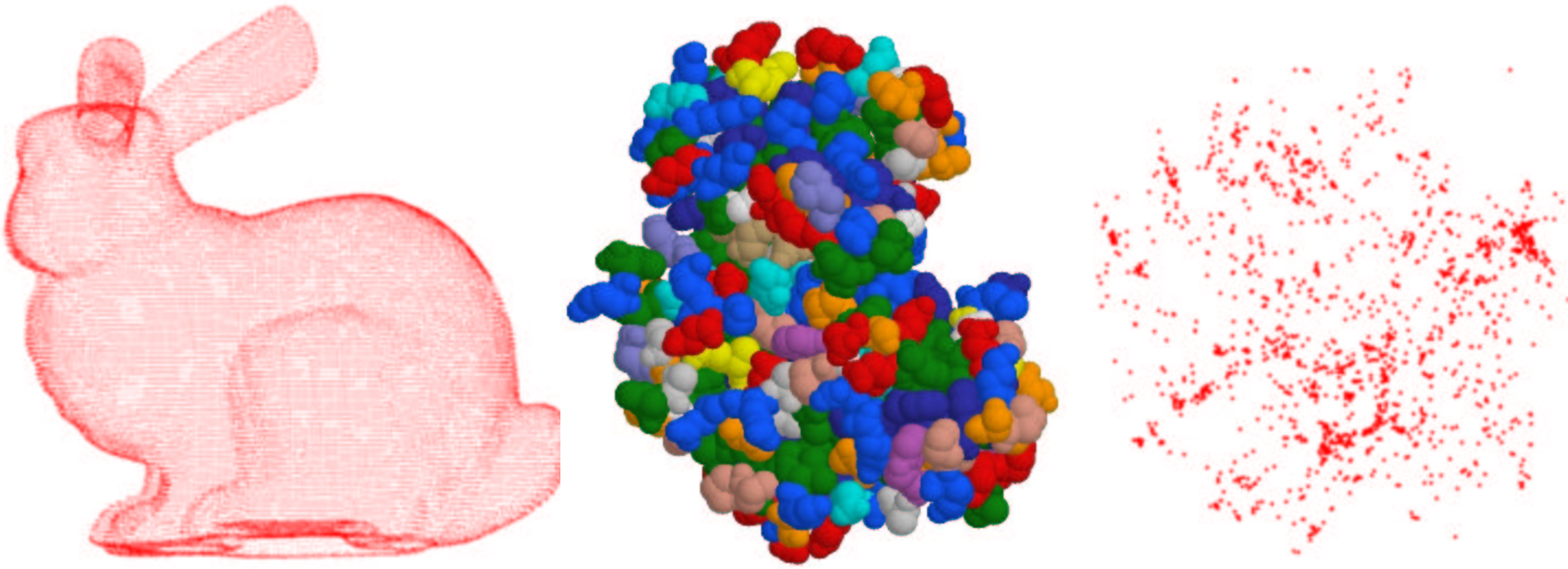
- Points
- Complexes
  - Cěch
  - Rips
  - Alpha
- Filtrations
- Persistence

POINTS



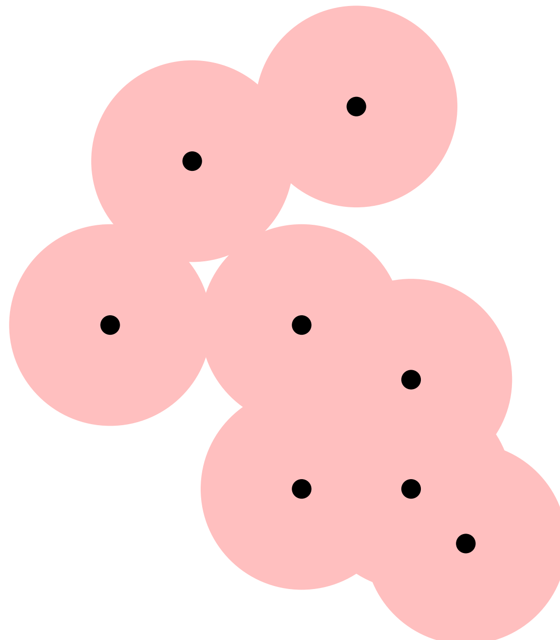
- m samples $M = \{m_1, m_2, \ldots, m_m\}$ from a manifold $\mathbb{M}$
- Samples are embedded, but intrinsic topology is lost
- Error: acquisition device noise and approximation

Point Cloud Data



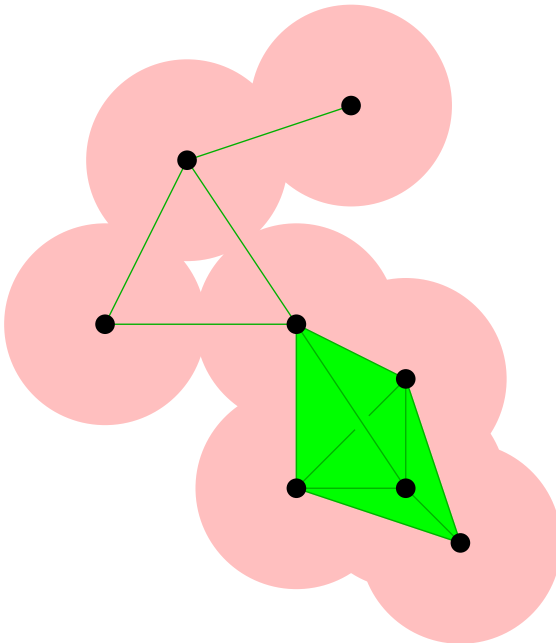
(a) Surface (b) Molecule (c) Universe

ϵ-BALLS



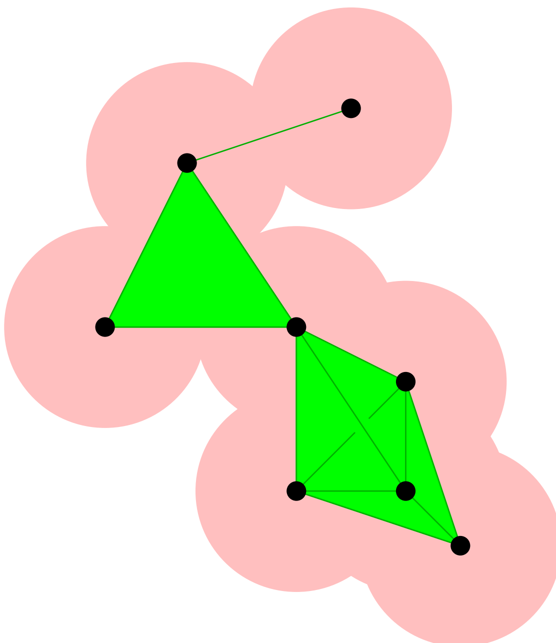
- ϵ-ball: $B_\epsilon(x) = \{y \mid d(x,y) < \epsilon\}$.
- Open sets and topology
- Manifold is $\tilde{\mathbb{M}} = \bigcup_{m_i \in M} B_\epsilon(m_i)$

Čech Complex



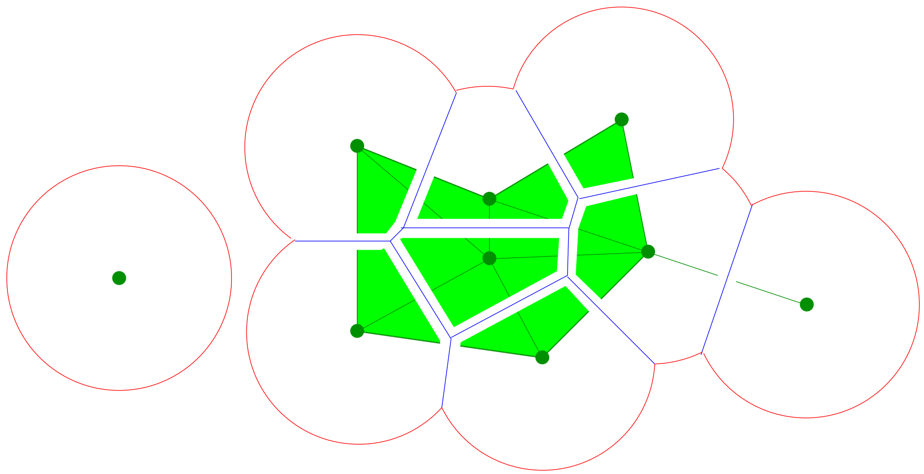
- $C_\epsilon(M) = \left\{ \text{conv}\, T \mid T \subseteq M, \bigcap_{m_i \in T} B_\epsilon(m_i) \neq \emptyset \right\}.$
- $\sum_{k=0}^{m} \binom{m}{k} = 2^{m+1} - 1$
- $C_\epsilon(M) \simeq \tilde{\mathbb{M}}$

RIPS COMPLEX



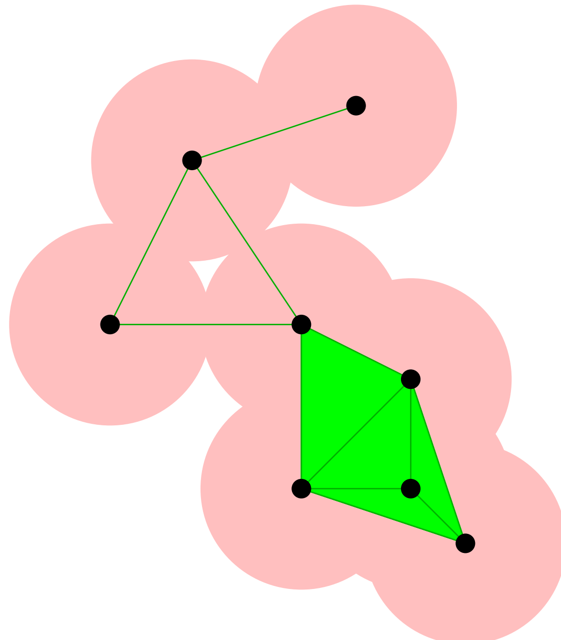
- $R_\epsilon(M) = \{\text{conv}\, T \mid T \subseteq M,\, d(m_i, m_j) < \epsilon, m_i, m_j \in T\}$.
- Still $O\left(\binom{m}{k}\right)$ for the kth skeleton
- Need $(k+1)$st skeleton for computing H_k

ALPHA COMPLEX



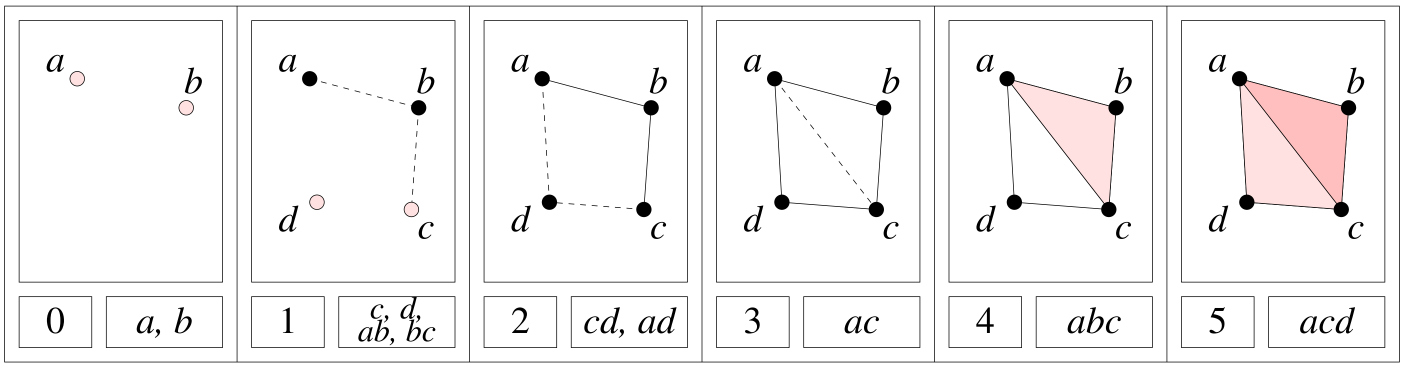
- $V(m_i) = \{x \in \mathbb{R}^3 \mid d(x, m_i) \leq d(x, m_j)\, \forall m_j \in M\}$
- $\hat{V}(m_i) = B_\epsilon(m_i) \cap V(m_i)$
- $A_\epsilon = \left\{ \mathrm{conv}\, T \mid T \subseteq M, \bigcap_{m_i \in T} \hat{V}(m_i) \neq \emptyset \right\}$
- $A_\epsilon(M) \simeq \tilde{\mathbb{M}}$, $A_\epsilon \subseteq D$, the Delaunay complex
- $O(n \log n + n^{\lceil d/2 \rceil})$

ALPHA COMPLEX



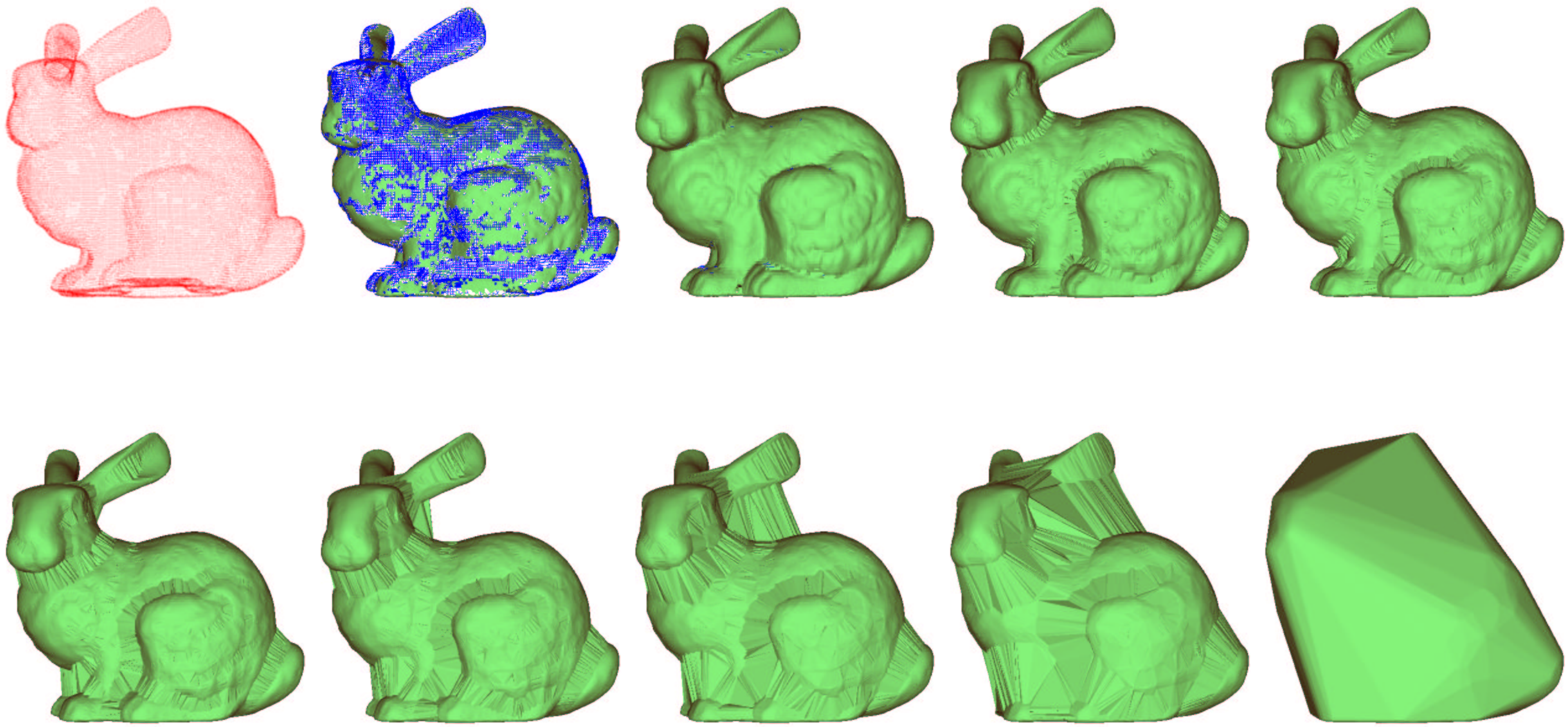
- Extendible to points with weights
- van der Waals model of molecules

FILTRATIONS



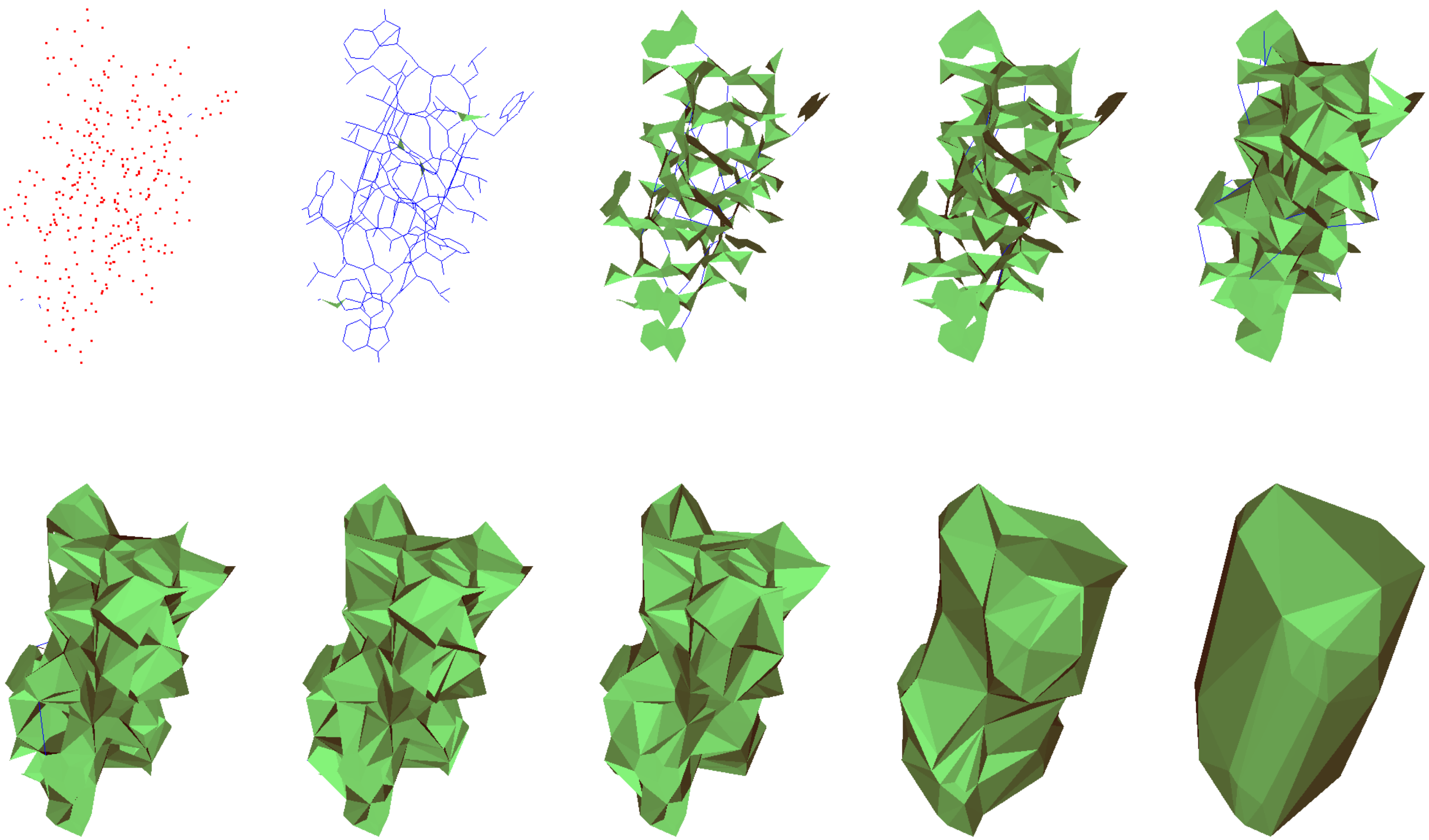
- Complexes C_ϵ, R_ϵ, A_ϵ, compute homology!
- Which ϵ? Vary and get a filtration!
- A filtration of a complex K is $\emptyset = K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m = K$.

BUNNY



- 34,834 points, 1,026,111 complexes

GRAMICIDIN A



- 312 atoms, 8,591 complexes

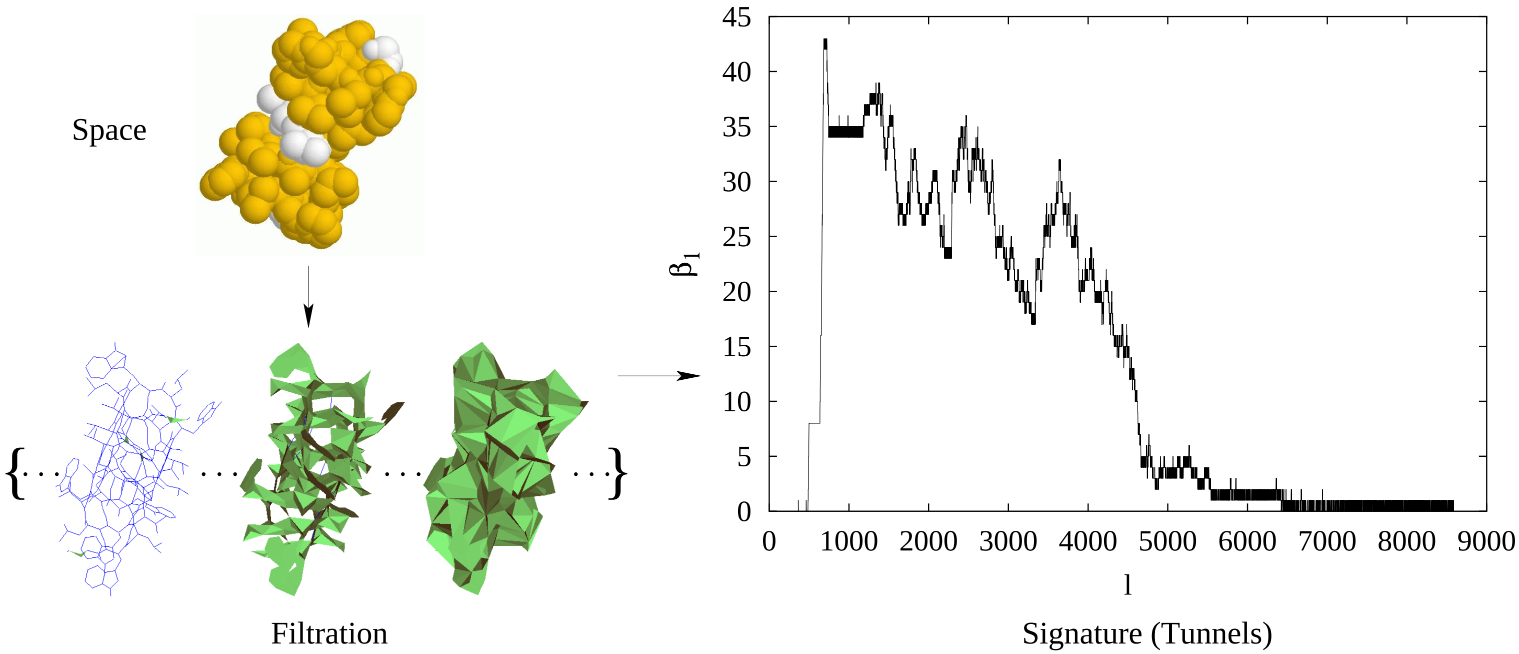
Approach

- Input: point cloud
- Procedure:
  - Put ϵ-balls around points
  - Compute complex K_ϵ
  - Compute homology of complex
- Varying ϵ gives us a filtration
- Incremental algorithm gives homology of filtration (demo)

Homology of a Filtration

- K^l is a filtration
- $\mathsf{Z}_k^l = \mathsf{Z}_k(K^l)$ and $\mathsf{B}_k^l = \mathsf{B}_k(K^l)$ are the kth cycle and boundary group of K^l, respectively.
- The kth homology group of K^l is $\mathsf{H}_k^l = \mathsf{Z}_k^l / \mathsf{B}_k^l$.
- The kth Betti number β_k^l of K^l is the rank of H_k^l.

PROBLEM



- Features
- Noise: spawned by noise, representation, etc.

Persistence

- K^l be a filtration.
- The p-persistent kth homology group of K^l is

$$\mathsf{H}_k^{l,p} \quad = \quad \mathsf{Z}_k^l / (\mathsf{B}_k^{l+p} \cap \mathsf{Z}_k^l),$$

- The p-persistent kth Betti number $\beta_k^{l,p}$ of K^l is the rank of $\mathsf{H}_k^{l,p}$.
- Well-defined
- $\eta_k^{l,p} : \mathsf{H}_k^l \to \mathsf{H}_k^{l+p}$,
- $\operatorname{im} \eta_k^{l,p} \simeq \mathsf{H}_k^{l,p}$.
- This lecture: $\mathbb{Z}_2$ homology

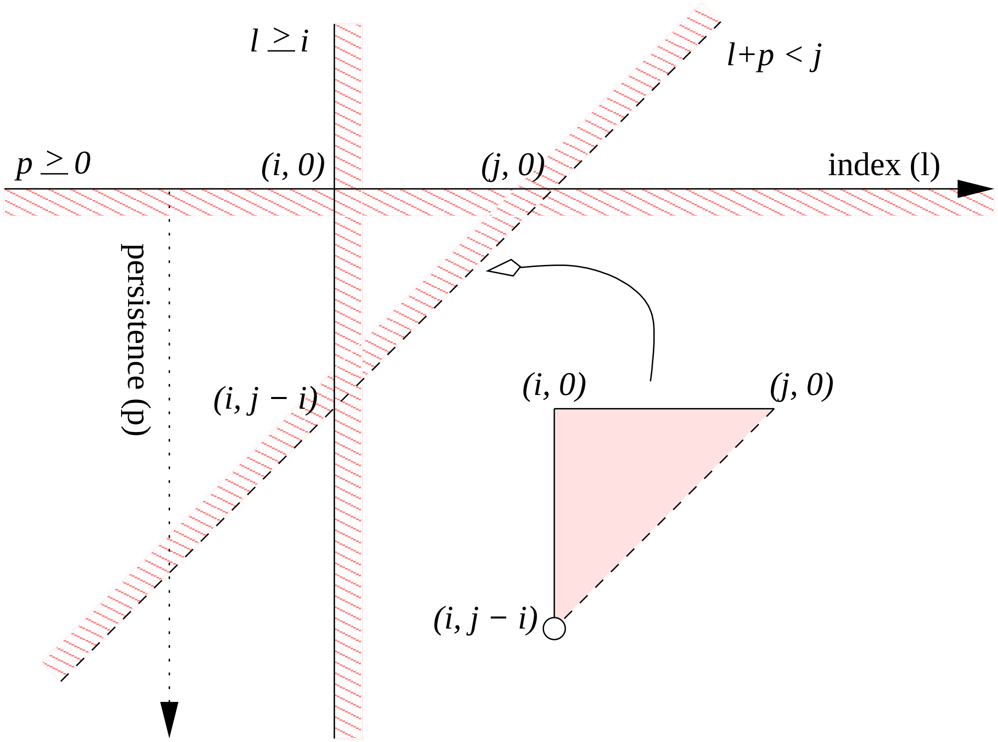
LIFETIMES

- Let z be a non-bounding k-cycle, created when σ enters complex at time i
- That is, $\beta_{k^{++}}$ at time i
- z creates a class of homologous cycles $[z]$
- $[z]$ is merged with the boundary class at time j when τ enters ($\beta_{k^{--}}$)
- τ destroys z and the cycle class $[z]$.
- The persistence of z, and its homology class $[z]$, is $j - i - 1$.
- σ is the creator (positive) and τ is the destroyer (negative) of $[z]$.
- If a cycle class does not have a destroyer, its persistence is ∞.

LIFETIME REGIONS

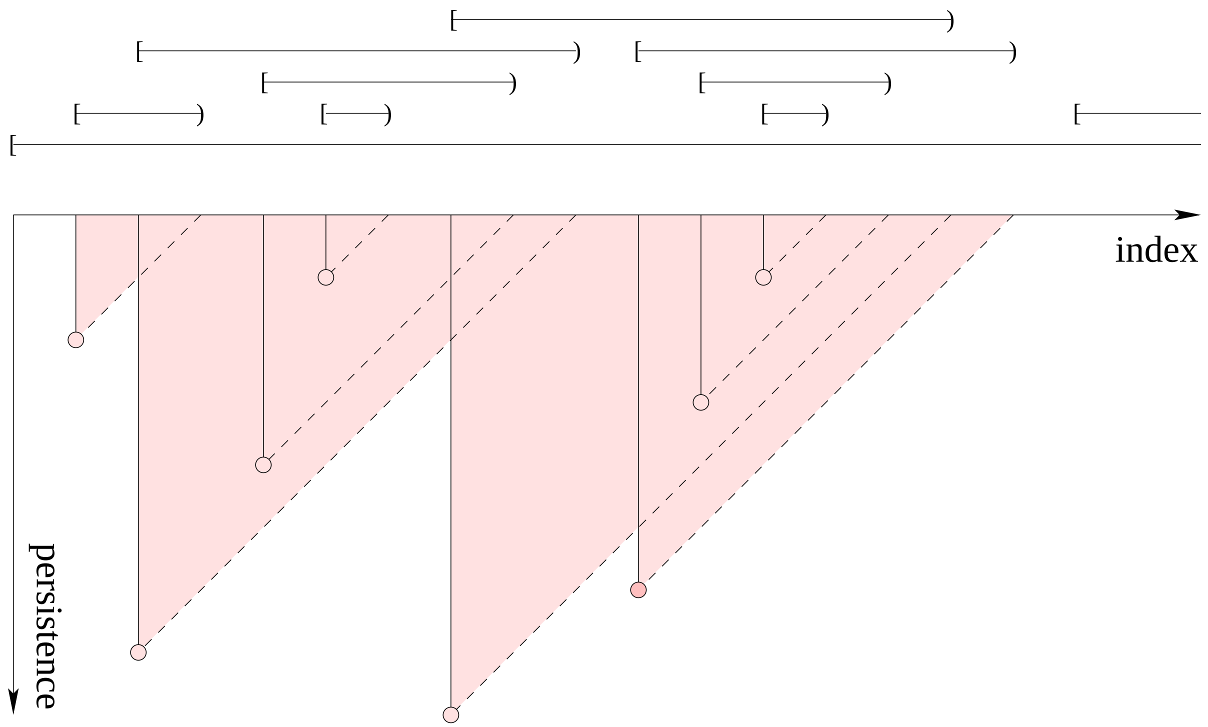
- $\mathsf{H}_k^{l,p} = \mathsf{Z}_k^l \,/\, (\mathsf{B}_k^{l+p} \cap \mathsf{Z}_k^l)$
- Basis element $z + \mathsf{B}_k$ lives during $[i, j)$
- $z \notin \mathsf{B}_k^l$ for $l \leq j$
- Therefore, $z \notin \mathsf{B}_k^{l+p}$ for $l + p < j$.
- $p \geq 0$
- $l \geq i$

TRIANGLE

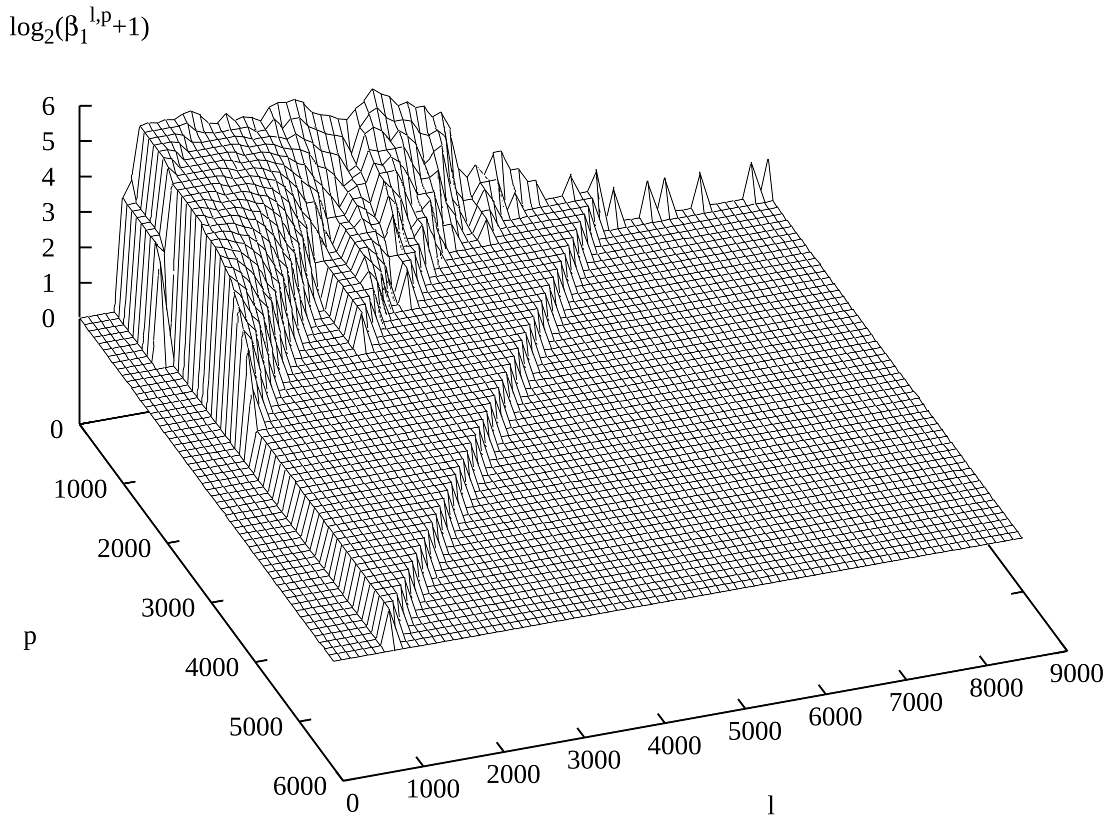


- $p \geq 0$
- $l \geq i$
- $l < j$

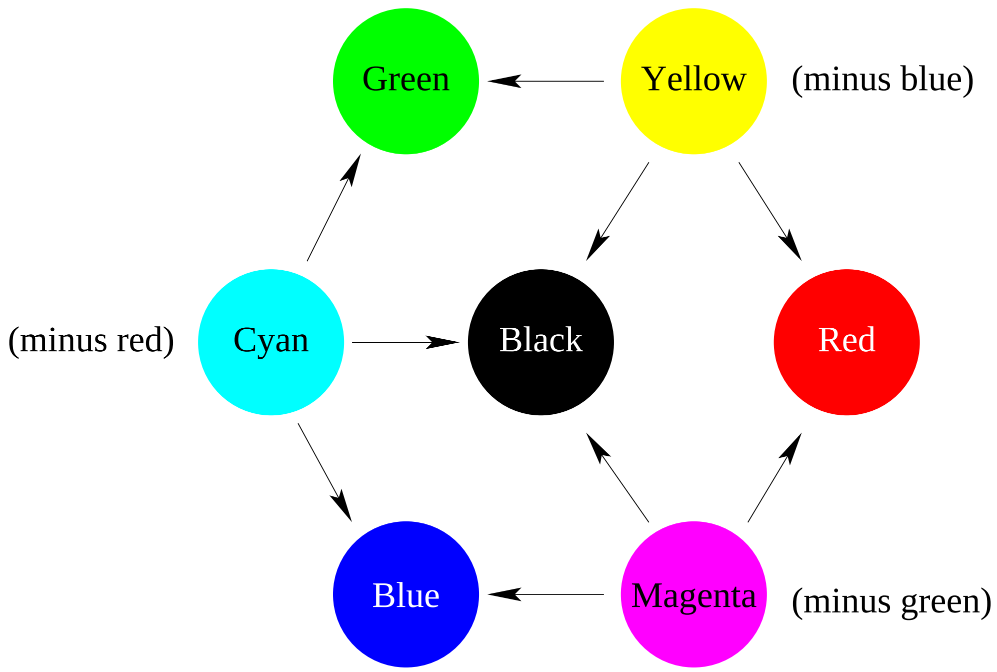
TRIANGLES



GRAPH OF $\log(\beta_1^{l,p} + 1)$

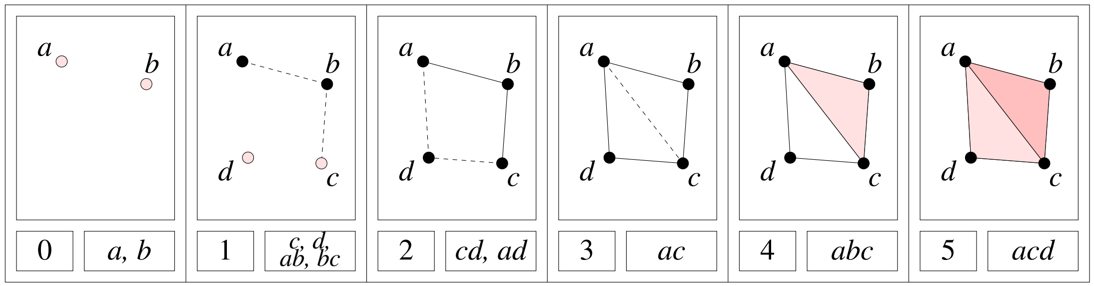


CMY COLOR SPACE



TOPOLOGY MAPS

Graded $\mathbb{F}[t]$-module



- Degrees of homogeneous elements:

a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	0	1	1	1	1	2	2	3	4	5

$$M_1 = \left[\begin{array}{c|ccccc} \partial_1 & ab & bc & cd & ad & ac \\ \hline d & 0 & 0 & t & t & 0 \\ c & 0 & 1 & t & 0 & t^2 \\ b & t & t & 0 & 0 & 0 \\ a & t & 0 & 0 & t^2 & t^3 \end{array}\right]$$

- $\deg \hat{e}_i + \deg M_k(i,j) = \deg e_j$

Column Echelon Form

$$\tilde{M}_1 = \left[\begin{array}{c|ccccc} & cd & bc & ab & z_1 & z_2 \\ \hline d & \boxed{\text{t}} & 0 & 0 & 0 & 0 \\ c & t & \boxed{1} & 0 & 0 & 0 \\ b & 0 & t & \boxed{\text{t}} & 0 & 0 \\ a & 0 & 0 & t & 0 & 0 \end{array}\right]$$

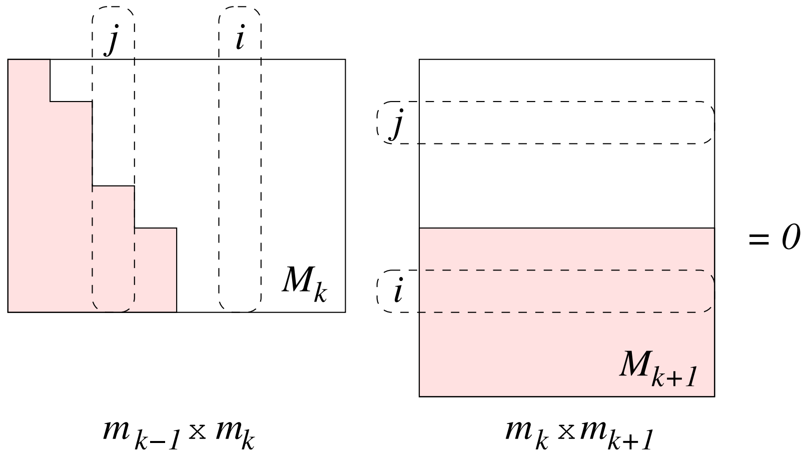
- Only column operations of type (1, 3)
- $z_1 = ad - cd - t \cdot bc - t \cdot ab$
- $z_2 = ac - t^2 \cdot bc - t^2 \cdot ab$
- $\{z_1, z_2\}$ form homogeneous basis for Z_1
- $\text{rank}\, M_k = \text{rank}\, \mathsf{B}_{k-1}$ is number of pivots

ECHELON FORM LEMMA

$$\tilde{M}_1 = \left[\begin{array}{c|ccccc} & cd & bc & ab & z_1 & z_2 \\ \hline d & \boxed{\text{t}} & 0 & 0 & 0 & 0 \\ c & t & \boxed{1} & 0 & 0 & 0 \\ b & 0 & t & \boxed{\text{t}} & 0 & 0 \\ a & 0 & 0 & t & 0 & 0 \end{array}\right]$$

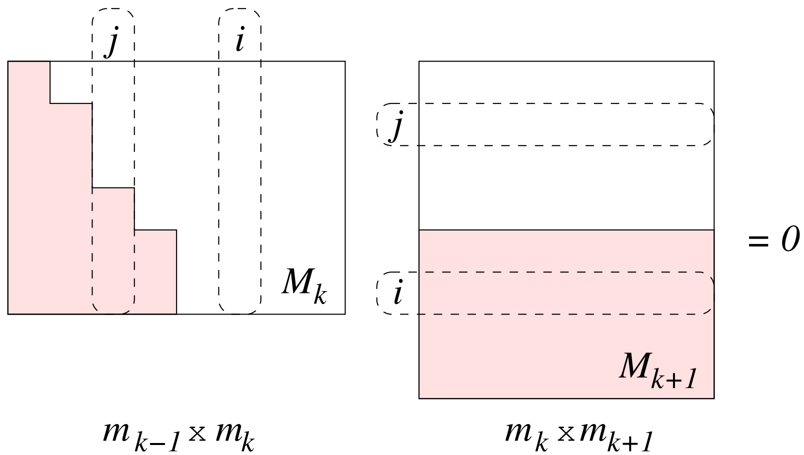
- (Lemma) The pivots in column-echelon form are the same as the diagonal elements in normal form. Moreover, the degree of the basis elements on pivot rows is the same in both forms.
- If only interested in degree of basis elements, read them off the column echelon form.

Basis Change Lemma



- Represent ∂_{k+1} in terms of the basis computed for Z_k
- $\partial_k \partial_{k+1} = \emptyset$, $M_k M_{k+1} = 0$
- (Lemma) To represent ∂_{k+1} relative to the standard basis for C_{k+1} and the basis computed for Z_k, simply delete rows in M_{k+1} that correspond to pivot columns in $\tilde{M}_k$.

PROOF



- Replace column i by (column i) $+ q$(column j) to eliminate element in pivot row j
- $\equiv$ replacing column basis element e_i by $e_i + qe_j$ in M_k
- $\equiv$ replacing row j with (row j) $- q$(row i) in M_{k+1}
- But row j is eventually zero and row i is not changed. QED

ALGORITHM

- No need for row operations
- Free columns correspond to positive simplices
- Pivot columns correspond to negative simplices
- No need for matrix representation
- Sparse matrix computation of Betti numbers based on persistence

DISCUSSION

- We can compute:
  - Cycles (components, cycles, voids)
  - Bounding manifolds
- (Demo)
- Points can be anything
  - samples from high dimensional manifolds: configuration spaces for robots (PRM), time-variant data, etc.
  - samples of tangent complex for data-set $\mathbb{M} \times \mathbb{S}^2$
- Need fast d-dim complex builder