BRAID GROUP COHOMOLOGIES AND ALGORITHM COMPLEXITY

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The topological complexity of an algorithm is defined as the number of its branchings (conditional transfer operators) (cf. [8]). In [8] Smale demonstrated a lower estimate of the minimal topological complexity of algorithms approximating all roots of polynomials of degree n,

$$\tau(n) \geqslant (\log_2 n)^{2/3}. \tag{1}$$

In this article we provide a two-sided estimate,

$$n-1 \geqslant \tau(n) \geqslant n - \min D_p(n), \tag{2}$$

where the minimum is taken over all primes p and $D_p(n)$ is the sum of digits in the p-ary decomposition of the number n. In particular, for $n = p^k$, $\tau(n) = n - 1$. Moreover, for $n = p^k$ the minimal topological complexity of algorithms computing only one root is also equal to $n - 1$.

Smale's method is of very general character: it provides an estimate of the topological complexity of any nonlinear ill-posed problem by means of a topological characteristic of the corresponding fibration, namely, its genus introduced and studied by Shvarts in [7] (and rediscovered in [8]). In the case of the problem on polynomial roots, an estimate of the genus [of which the inequality (1) is a corollary] was proved in [8] by means of Fuks' results on Artin's braid group cohomologies [4]. The right-hand part estimate in (2) is based on a more detailed study of these cohomologies.

The calculation of braid group cohomologies was initiated in [1, 3] in connection with the problem on representation of algebraic functions by compositions of functions in fewer variables. The study of genus of a covering associated with an algebraic function creates new obstacles for such a representation, see Sec. 6 of this article.

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1. Algorithms, Braids, Shvarts Genus

1.1. Problems. Let n be a natural number and ε a positive number. We consider algorithms solving one of the following two problems.

1.1.1. Problem 1. For each polynomial

$$x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n, \tag{3}$$

given by the set of its coefficients $a = (a_1, \ldots, a_n)$, satisfying the condition $|a_i| \leqslant 1 \, \forall i$, compute all of its roots with an error no greater than ε.

1.1.2. Problem 2. This differs from Problem 1 only in that one has to compute only one root of the polynomial.

We denote the domain of admissible entry data $\{a \in \mathbf{C}^n \mid \forall i \ |a_i| \leqslant 1\}$ by B^n.

1.2. Algorithms. We use the definition of an algorithm in [8]. We recall this definition as it applies to Problem 1.

An algorithm is a finite oriented tree (a cycle-free graph) with knots of the following four types.

1) A unique entry knot at which 2n real numbers Re a_i, Im a_i are entered (such that $(\mathrm{Re}\ a_i)^2 + (\mathrm{Im}\ a_i)^2 \leqslant 1$).

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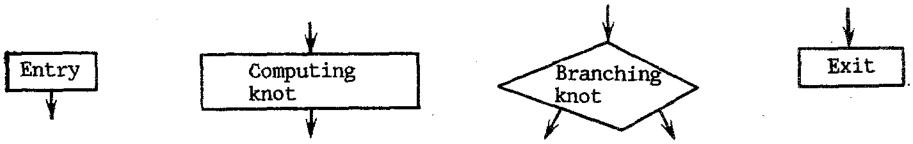

Fig. 1

2) Computing knots: at each of them are computed fixed real-valued rational functions in entry values $\mathrm{Re}\, a$, $\mathbf{Im}\, a$ and in the values of such functions computed at other computing knots located higher up in the algorithm.

3) Branching knots: in these knots the value of one of the rational functions in $\mathrm{Re}\, a$, $\mathrm{Im}\, a$ computed earlier is compared with zero and, depending on the result, the control is transferred along one of the two edges coming out of this knot.

4) Exit knots: at each of them some 2n rational functions in a computed earlier in the algorithm are declared, in an act of will, to be the real and imaginary parts of the roots of polynomial (3) and are denoted by $\mathrm{Re}\, z_i\, (a)$, and $\mathrm{Im}\, z_i\, (a)$, and this completes the implementation of the program.

(The condition that the graph has no cycles implies that the number of exits is 1 more than the number of branchings.)

Thus, each set of entry values $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ determines the path of action of the algorithm and, therefore (if in the course of this path the division by zero never occurs), also one of the exits as well as an ordered set of complex numbers $z_1\,(a), \ldots, z_n\,(a)$. An algorithm is called an ε-algorithm if for each set of empty data $a \in B^n$ the division by zero never occurs in the course of the algorithm and one can order the roots ξ_i of the corresponding polynomial (3) in such a way that $|\, z_i\,(a) - \xi_i\,(a)| \leqslant \varepsilon$ for all $i = 1,\ldots,n$.

1.3. The topological complexity of an algorithm is the number of branching knots in it. The topological complexity of a problem is the minimal topological complexity of algorithms solving it. We denote the topological complexity of Problem 1 by $\tau(\varepsilon, n)$. Obviously, the number $\tau(\varepsilon, n)$ does not decrease as ε is decreasing. We define $\tau(n)$ as $\lim\limits_{\varepsilon\to 0} \tau\,(\varepsilon, n)$.

1.4. Braids. Consider the subset Σ of B^n consisting of polynomials having multiple roots. Over the set $B^n - \Sigma$ one can define an n!-leaved covering $f_n\colon M^n \to B^n - \Sigma$: its fiber over a point a consists of all ordered sets of roots of the polynomial a. The exact (with $\varepsilon = 0$) solution of Problem 1 with entry data a includes a choice of one of the n! possible orderings of these roots.

Proposition (cf. [1, 2]). The spaces $B^n - \Sigma$ and M^n are spaces of the type $K(\pi, 1)$, where π is the braid group Br(n) and the Artin painted braid group I(n) of n threads, respectively. Both spaces have homotopy type of $(n - 1)$-dimensional CW-complexes. The cohomologies of these spaces with trivial coefficients were described in [2, 4, 9, 10].

1.5. Shvarts Genus. Let X and Y be normal Hausdorff spaces (for instance, manifolds or subsets of the Euclidean space) and let $f\colon X \to Y$ be a continuous map such that $Y = f(X)$. The genus of the map f is the smallest number j such that Y can be covered with j open sets over each of which there is a section of the map f. For instance the genus of each multileaved covering of the circle is equal to 2. The genus of a map f is denoted by $g(f)$.

1.5.1. THEOREM [8]. For each n there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the topological complexity of Problem 1 and the genus of the covering f_n are related by the inequality $\tau\,(\varepsilon, n) \geqslant g\,(f_n) - 1$.

Indeed, the covering and the system of sections involved in the definition of the genus are, in fact, determined by the algorithm. To each one of its exits corresponds a semialgebraic set $W_i \subset B^n$ consisting of entry values such that their input would lead to that specific exit. Let Σ_ε be the set of polynomials having a pair of roots at the distance $|\,\xi_i - \xi_j\,| \leqslant 2\varepsilon$. If ε is small, then the covering $f_n\colon M^n \to B^n - \Sigma$ is equivalent to its restriction to $B^n - \Sigma_\varepsilon$; in particular, it has the same genus. The algorithm determines over each set $W_i \cap (B^n - \Sigma_\varepsilon)$ an "ε-section" of the covering f_n, i.e., a map into M^n differing from a section by the distance ε. But, by the definition of Σ_ε, an ε-section can be deformed into an actual section. Finally, this section is extended to a section over some neighborhood of the set $W_i \cap (B^n - \Sigma_\varepsilon)$, and we obtain the required covering of $B^n - \Sigma_\varepsilon$.

The proof of the estimate (1) in [8] consists of this theorem along with an estimate of the genus of the covering f_n by means of homological characteristics of the space $B^n - \Sigma$ found in [4].

1.5.2. In conclusion, note that $g(f_n) \leqslant n$; generally, according to [7], the genus of any fibration over a k-dimensional basis does not exceed k + 1.

2. A Lower Estimate for Algorithms Computing All Roots

2.1. Fundamental Theorem. $g(f_n) > n - \min_p D_p(n)$, where $D_p(n)$ is the same as in formula (2).

COROLLARIES. 1. If n is a power of a prime, then $g(f_n) \geqslant n$.

2. For all n, $g(f_n) > n - \log_2 n$.

3. By virtue of Theorem 1.5.1, we can replace in Theorem 2.1 as well as in parts 1 and 2 of the present corollary $g(f_n)$ by $\tau(n) + 1$.

2.2. The Homological Genus of a Principal Covering. Suppose a fibration $f: X \to Y$ is a principal G-fibration, where G is a discrete group. Let $c: Y \to K(G, 1)$ be a classifying map of the covering f, i.e., this covering is isomorphic to that induced by the map c from the universal fibration over K(G, 1). Let A be an arbitratry G-module.

Definition. The homological A-genus of a principal covering $f: X \to Y$ is the smallest number i such that the map $c^*: H^j(K(G, 1), A) \to H^j(Y, c^*A)$ is trivial for all $j \geqslant i$.

The homological A-genus is denoted by $h_A(f)$.

THEOREM [7]. For each principal G-covering $f: X \to Y$ and for any G-module A, $g(f) \geqslant h_A(f)$.

2.3. In our case Y is the space $B^n - \Sigma \sim K(\mathrm{Br}(n), 1)$, G is the group S(n) of permutations of n elements, and the map $c = c(f_n): B^n - \Sigma \to K(S(n), 1)$ corresponds to the obvious homomorphism $\mathrm{Br}(n) \to S(n)$.

We will denote by the same symbol $\pm\mathbf{Z}$ the following three objects: a) the only nontrivial representation $S(n) \to \mathrm{Aut}(\mathbf{Z})$; b) the system of groups on the space K(S(n), 1) locally isomorphic to **Z** and turning over under translations over the paths corresponding to odd permutations; c) the local system on the space $B^n - \Sigma$ induced by the previous one under the map $c(f_n)$.

2.3.1. THEOREM. If n is a power of a prime, then $h_{\pm\mathbf{Z}}(f_n) = n$.

More generally, suppose that for some prime p, $n = n_1 + \ldots + n_t$, where $n_i = p^{k_i}$. Let $a \in B^n - \partial B^n$ be a polynomial having t distinct roots whose multiplicities are $n_1, \ldots, n_t$. Let $U \subset B^n$ be any neighborhood of the point a and let f_U be the restriction of the covering f_n to the domain $U - \Sigma$.

2.3.2. THEOREM. $h_{\pm\mathbf{Z}}(f_U) \geqslant n + 1 - t$.

This immediately implies Theorem 2.1. The proof of Theorems 2.3.1 and 2.3.2 occupies the rest of Sec. 2. Theorem 2.3.1 follows from the two statements below.

2.3.3. THEOREM. The group $H^{n-1}(\mathrm{Br}(n), \pm\mathbf{Z})$ is trivial if n is not a power of a prime and is isomorphic to $\mathbf{Z}_p$ if $n = p^k$.

2.3.4. THEOREM. The homomorphism $c^*: H^*(S(n), \pm\mathbf{Z}) \to H^*(B^n - \Sigma, \pm\mathbf{Z})$ is an epimorphism.

The last theorem has the following generalization. Let $\varkappa(n, m)$ denote the configuration space consisting of all unordered sets of n points in $\mathbf{R}^m$. The space $B^n - \Sigma$ is homotopically equivalent to $\varkappa(n, 2)$, and the space K(S(n), 1) can be realized as the limit of the spaces

$$\ldots \to \varkappa(n, m) \to \varkappa(n, m+1) \to \ldots, \tag{4}$$

in which all arrows are defined by the obvious embeddings $\mathbf{R}^m \to \mathbf{R}^{m+1} \to \ldots$. The map $\varkappa(n, 2) \to K(S(n), 1)$ determined by the sequence of maps (4) coincides with the map $c: B^n - \Sigma \to K(S(n), 1)$ corresponding to the homomorphism $\mathrm{Br}(n) \to S(n)$.

2.3.5. THEOREM. All homomorphisms $H^*(S(n), \pm\mathbf{Z}) \to H^*(\varkappa(n, m), \pm\mathbf{Z})$, $H^*(S(n), \mathbf{Z}_2) \to H^*(\varkappa(n, m), \mathbf{Z}_2)$ determined by the sequences of maps (4) are epimorphisms for all n and m. In particular, all intermediate maps $H^*(\varkappa(n, m+1), \pm\mathbf{Z}) \to H^*(\varkappa(n, m), \pm\mathbf{Z})$, $H^*(\varkappa(n, m+1), \mathbf{Z}_2) \to H^*(\varkappa(n, m), \mathbf{Z}_2)$ are also epimorphic.

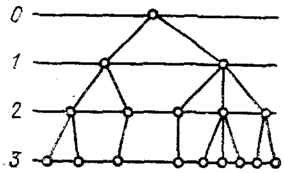

Fig. 2

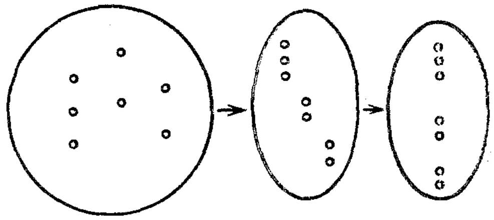

Fig. 3

2.4. The proof of Theorem 2.3.5 is based on the following cellular partition of the one-point compactifications of the configuration spaces (for m = 2 these partitions coincide with those constructed in [4]).

2.4.1. We fix a linear function ℓ on the real plane and the corresponding orientation of lines ℓ = const. We construct m + 1 parallel lines $L_i = \{\ell = i\}$, $i = 0, 1, \ldots, m$. Let $x_1, \ldots, x_m$ be coordinates in $\mathbf{R}^m$, let e_i be the corresponding orthonormal vectors, let π_j be the projection of the space $\mathbf{R}^m$ onto the j-dimensional plane $\{e_1, \ldots, e_j\}$ along $\{e_{j+1}, \ldots, e_n\}$. Let $\lambda \in \varkappa(m, n)$ be a family of n points $\lambda_1, \ldots, \lambda_n$ in $\mathbf{R}^m$. We put on each line L_j several points which are in a one-to-one correspondence with the set $\pi_j(\lambda)$, and the order of these points on the line L_j corresponds to the lexicographic order of the points of the set $\pi_j(\lambda)$ defined by the coordinates $x_1, \ldots, x_j$. For each $j < m$ we join a pair of selected points on the lines L_j and L_{j+1} with a segment if there exists a point in the set λ which is mapped by π_j and π_{j+1} to the points corresponding to these ones. In particular, the only point on L_0 is joined to all poitns on L_1. Each graph obtained in this manner and viewed up to homeomorphisms of the plane preserving the function ℓ as well as the orientation of the lines L_i is called a standard (n, m)-tree. Like in Sec. 3.2 in [4], we can see that partition of the configuration space $\varkappa(n, m)$ into sets of points corresponding to a single tree provides a cellular partition of the one-point compactification of $\varkappa(n, m)$.

Let $\Xi(n, m)$ be the cochain complex whose generators are the pairs (a transversally oriented cell of this partition, a basic section of the system $\pm\mathbf{Z}$ over it), a replacement of a transversal orientation or a basic section corresponds to multiplication of the generator by -1, the degree of such a generator is equal to the codimension of the cell, and the incidence coefficients are defined by the mutual behavior of the transversal orientations of incident cells of the adjacent dimensions, so $H^*(\Xi(n, m)) = H^*(\varkappa(n, m), \pm\mathbf{Z})$.

The complex $\Xi(n, m)$ has an obvious filtration: a generator lies in a subgroup F_j if the intersection of the corresponding tree with the line L_j consists of n points.

The embedding $\varkappa(n, m) \to \varkappa(n, m+1)$ defines a homomorphism $\Xi(n, m+1) \to \Xi(n, m)$ whose kernel is spanned by all cells not lying in F_m; each cell in F_m corresponding to the (n, m + 1)-tree T is thereby mapped to the cell corresponding to the intersection of T with the strip $\ell^{-1}([0, m])$ and equipped with the induced transversal orientation. This map is compatible with the cohomology map. Theorem 2.3.5 now follows from this geometrical statement whose verification is straightforward.

2.4.2. LEMMA. For all n, m the subgroup $F_m \subset \Xi(n, m+1)$ is a subcomplex.

2.5. The Proof of Theorem 2.3.3. In the case of m = 2 the cellular partition of 2.4.1 coincides with the one constructed in [4]: the trees of height 2 are in a one-to-one correspondence with ordered partitions of the number n, and the codimension of the cell corresponding to a partition $n = n_1 + \ldots + n_t$ is equal to $n - t$. The cell is denoted by $(n_1, \ldots, n_t)$.

Note also that for m = 2 (and, generally, for an even m) the space $\varkappa(n, m)$ is orientable and, therefore, one can consider the usual orientation of cells instead of their transversal orientation.

2.5.1. THEOREM. With a suitable choice of basic sections of the system $\pm\mathbf{Z}$ over cells of the complex $\Xi(n, 2)$ and orientations of these cells, the incidence coefficient $[(n_1, \ldots, n_t), (n_1', \ldots, n_{t-1}')]$ is determined by the following conditions:

if there exists k such that

$$n_j' = n_j \text{ for } j < k, \; n_k' = n_k + n_{k+1}, \; n_j' = n_{j+1} \text{ for } j > k, \tag{5}$$

then the incidence coefficient is equal to $(-1)^{k-1} C_{n_k'}^{n_k}$;

if there is no such k, then this coefficient is equal to 0 (moreover, these cells are not geometrically incident).

Theorem 2.3.3 immediately follows from this theorem applied to the case of $t = 2$. To prove Theorem 2.5.1, we will make the sections of the local system $\pm\mathbf{Z}$ over different cells compatible. For each cell $(n_1,\ldots,n_t)$ we take a path in $\varkappa(n, 2)$, which lies entirely in this cell, except for its end lying in a cell of maximal codimension (n), in such a wayy that if a point z_i in the set $\{z_1,\ldots,z_n\}$ corresponding to the beginning of the path lies to the right of z_j, then the points $\tilde{z}_i$ and $\tilde{z}_j$ obtained from them at the end of the path satisfy the condition $\operatorname{Im} z_i < \operatorname{Im} z_j$ (see Fig. 3). Fix some basic section of the system $\pm\mathbf{Z}$ over a cell (n) and extend it to the cell $(n_1,\ldots,n_t)$ using translations along such paths; obviously, the result does not depend on the choice of a path. Now, we define orientation of cells. Enumerate the points $z_1,\ldots,z_n$ of a set lying in this cell in such a way that for each $i < n$ either $\operatorname{Re} z_i < \operatorname{Re} z_{i+1}$ or $\operatorname{Re} z_i = \operatorname{Re} z_{i+1}$, $\operatorname{Im} z_i > \operatorname{Im} z_{i+1}$. The orientation of the cell $(n_1,\ldots,n_t)$ is defined by means of the differential form d (Re of the first group of n_1 points) $\wedge$ d (Re of the group of n_2 points) $\wedge\ldots\wedge$ d (Re of the group of n_t points) $\wedge \operatorname{Im} z_1 \wedge\ldots\wedge \operatorname{Im} z_n$. This family of orientation and compatibility of sections provides the coefficient promised in Theorem 2.5.1. Note that if the conditions (5) are satisfied, then the cell $(n_1,\ldots,n_{t-1})$ occurs in the boundary of the cell $(n_1,\ldots,n_t)$ exactly $C_{n_k'}^{n_k}$ times. All these occurrence provide the same contribution in the incidence coefficient: for each loop $(S^1, *) \to (\overline{(n_1,\ldots,n_t)}, (n_1',\ldots,n_{t-1}'))$ the respective orientations of these cells at its beginning and its end are distinct exactly when the local system $\pm\mathbf{Z}$ is turned over this loop.

2.5.2. The group $H^{n-1}(\mathrm{Br}(n), \pm\mathbf{Z})$ is generated by a single cell (n); by Theorem 2.5.1, it is not trivial only if all numbers $C_n^{n_1}$, $n_1(n - n_1) > 0$ do not generate the group $\mathbf{Z}$. This is true if and only if n is a power of a prime, and Theorem 2.3.1 is proved. The generator of the group $H^{n-1}(\mathrm{Br}(n), \pm\mathbf{Z})$ also has the following description.

Following [4], we consider the map $K(S(n), 1) \to BO(n - 1)$ corresponding to the standard homomorphism $S(n) \to O(n - 1)$ ($S(n)$ acts on $\mathbf{R}^n$ by permutations of the basis vectors and, therefore, it also acts on the hyperplane $\mathbf{R}^{n-1} = \{x \mid x_1 + \ldots + x_n = 0\}$). On the space $BO(n - 1)$ there is an orienting sheaf Or – a system of groups locally isomorphic to $\mathbf{Z}$ and controlling the orientation of the universal vector fibration. It is easily seen that the system $\pm\mathbf{Z}$ is induced on $K(S(n), 1)$, and therefore also on $K(\mathrm{Br}(n), 1)$, by this sheaf.

<u>THEOREM.</u> Under the homomorphism $H^{n-1}(BO(n-1), \mathrm{Or}) \to H^{n-1}(\mathrm{Br}(n), \pm\mathbf{Z})$ the Euler class of the universal fibration over $BO(n - 1)$ is mapped to the class of the cell (n).

<u>2.6. Proof of Theorem 2.3.2.</u> We may assume that the domain U is a small ball centered at a. Then the space $U - \Sigma$ is homotopically equivalent to the product

$$(B^{n_1} - \Sigma_1) \times \ldots \times (B^{n_t} - \Sigma_t), \tag{6}$$

where B^{n_i} is the polycircle in the space of polynomials of degree n_i with leading coefficient 1, Σ_i are sets of polynomials with multiple roots. The covering f_U is decomposed into a disjoined union of $C_n^{n_1,\ldots,n_t}$ copies of $n_1! \cdot \ldots \cdot n_t!$-leaved coverings each of which is isomorphic to the product of $n_i!$-leaved coverings induced by the coverings $f_{n_i}: M^{n_i} \to B^{n_i} - \Sigma_i$ under the projection of the product (6) onto its factor $B^{n_i} - \Sigma_i$. Therefore, in its restriction to $U - \Sigma$, the covering group $S(n)$ is reduced to its subgroup $S(n_1) \times \ldots \times S(n_t)$, and the map $U - \Sigma \to K(S(n), 1)$ classifying this covering can be written as a composition $U - \Sigma \to K(S(n_1), 1) \times \ldots \times K(S(n_t), 1) \to K(S(n), 1)$. The right arrow of this composition maps the local system $\pm\mathbf{Z}$ on $K(S(n), 1)$ into the tensor product of the systems $\pm\mathbf{Z}$ on $K(S(n_i), 1)$. Now, Theorem 2.3.2 follows from Theorem 2.3.1 and the Künneth formula.

3. A Lower Estimate of the Topological Complexity for Algorithms Computing One Root

Over the base $B^n - \Sigma$ of our $n!$-leaved covering f_n there is another n-leaved covering $\varphi_n: N^n \to B^n - \Sigma$, consisting of all pairs (a point $a \in B^n - \Sigma$, one of the roots of the polynomial a); the covering f_n can be viewed as the fibration of transformation groups associated with it. The following statement is proved just like Theorem 1.5.1.

<u>3.1. THEOREM.</u> There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ the topological complexity $\tau_1(\varepsilon, n)$ of Problem 1.1.2 satisfies the inequality $\tau_1(\varepsilon, n) \geqslant g(\varphi_n) - 1$.

Below, we will point out a topological obstacle to the inequality $g(\varphi_n) < n$; for $n = p^k$ this is a nontrivial obstacle, and we obtain the following theorem.

3.2. THEOREM. If n is a power of a prime, then $g(\varphi_n) = n$, and the topological complexity of Problem 1.1.2 is no less than $n - 1$.

3.3. Consider the fibration $\Theta : \nabla^n \to B^n - \Sigma$, which is obtained from the covering φ_n by fiberwise joining of $n - 1$ copies of its fibers [in particular, a fiber of the fibration Θ is homotopically equivalent to the wedge of $(n - 1)^{n-1}$ copies of the $(n - 2)$-dimensional sphere].

THEOREM (cf. [7]). $g(\varphi_n) < n$ if and only if the fibration Θ has a section.

The only obstacle $\alpha(\Theta)$ to the existence of such a section lies in the group $H^{n-1}(B^b - \Sigma, \tilde{\Theta})$, where $\tilde{\Theta}$ is the system of groups associated with Θ whose fiber is equal to the $(n - 2)$-nd homotopy group of the fiber of the fibration Θ. By the definition of the join and Hurewitz' theorem, $\tilde{\Theta}$ is the $(n - 1)$-st tensor power of the local system $\mathscr{H}$, whose fiber is the group of zero-dimensional homologies of the fiber of the covering φ_n reduced modulo a point. We have to compute the value of the obstacle $\alpha(\Theta)$ on the only cell of codimension $n - 1$. We order the leaves ξ_i of the covering φ_n over this cell in the decreasing order of the imaginary parts of the corresponding roots. A basis of the system $\mathscr{H}$ over this cell is given by the elements $\xi_1 - \xi_2, \ldots, \xi_{n-1} - \xi_n$.

3.4. THEOREM. The value of the obstacle $\alpha(\Theta) \in H^{n-1}(B^n, \otimes^{n-1}\mathscr{H})$ on the cell (n) is homologous to $(\xi_{n-1} - \xi_n) \otimes \ldots \otimes (\xi_1 - \xi_2)$.

For the proof, we will construct a section of the fibration Θ over the complement to the cell (n). We realize the join of $(n - 1)$ copies $A_1, \ldots, A_{n-1}$ of an n-point set as an $(n - 2)$-dimensional polyhedron whose vertices are all $n(n - 1)$ points $A_1 \cup \ldots \cup A_{n-1}$, and whose simplices are spanned by all sets of these points in such a way that none of these simplices has two vertices lying in the same set A_i (in our case all A_i are fibers of the covering φ_n; the $(n - 1)$-dimensional simplex spanned by points $\xi_{j_1} \in A_1, \ldots, \xi_{j_{n-1}} \in A_{n-1}$ is naturally denoted by $\xi_{j_1} \otimes \ldots \otimes \xi_{j_{n-1}}$.)

First we construct a section over the union of cells of the form $(n_1, \ldots, 1)$, i.e., over the set of families $\{z_1, \ldots, z_n\} \subset C^1$, such that $\operatorname{Re} z_n > \operatorname{Re} z_i$ for each $i < n$. Namely, for each such point $\{z_1, \ldots, z_n\}$ the image of this section belongs to the vertex of A_1 corresponding to z_n. This section is discontinuous near all cells of the form $(n_1, \ldots, \ell)$, $\ell > 1$. We will modify it as to remove these discontinuities near the union of cells of the form $(n_1, \ldots, 2)$. A point of this union corresponds to a family $\{z_1, \ldots, z_n\}$ such that $\operatorname{Re} z_n = \operatorname{Re} z_{n-1} > \operatorname{Re} z_i$ for all $i < n - 1$. For such a point we consider the vertex of the set A_2 corresponding either to z_n or to z_{n-1}, depending which point has a greater imaginary part. The part of the one-dimensional skeleton of the set $\Theta^{-1}(\{z_1, \ldots, z_n\})$ consisting of segments joining this point of A_2 to all points of A_1 is obviously contractible, so we can deform the section chosen earlier in a small neighborhood of the cells $(n_1, \ldots, 2)$ in such a way that it would become continuous there and its image would lie in this part of the 1-skeleton. We continue to act in a similar manner: at the r-th step there is a section over the union of cells of the form $(n_1, \ldots, i)$, $i \le r$, such that the image of this section lies in the union of simplices whose vertices lie only in $A_1, \ldots, A_r$. Near each point $\{z_1, \ldots, z_n\}$ of each cell of the form $(n_1, \ldots, r + 1)$ we remove the discontinuity of this section using a deformation whose image lies in the cone joining this union $\{A_1, \ldots, A_r\}$ to the vertex of the set A_{r+1} corresponding to the point z_i having the maximal imaginary part among the $r + 1$ points having the maximal real part. Finally, we obtain a section over the complement to the cell (n). Consider now a small $(n - 1)$-dimensional disk transversal to the cell (n). We identify the fibers of the fibration Θ over it, then the constructed section over its boundary defines an $(n - 2)$-dimensional spheroid in the fiber of this fibration. Let us compute it. Suppose our disk consists of families $\{z_1, \ldots, z_n\}$, $z_i \in C^1$, such that $\operatorname{Im} z_j = -j\varepsilon$ and the families of real parts run over a simplex in R^n spanned by the n points $u_i = \{x_i = \varepsilon, x_1 = \ldots x_i \ldots = x_n = -\varepsilon/(n - 1)\}$. We denote by Δ_i the face of this simplex spanned by the vertices $u_1, \ldots, \hat{u}_i, \ldots, u_n$. By the induction hypothesis, before the last step in the construction of the section over the complement to the cell (n) the obstacle to the section over Δ_i lying in $\{A_1, \ldots, A_{n-2}\}$ is equal to $(\xi_{n-1} - \xi_n) \otimes \ldots \otimes (\xi_{i+1} - \xi_{i+2}) \otimes (\xi_{i-1} - \xi_{i+1}) \otimes \ldots \otimes (\xi_1 - \xi_2)$. At the last step almost all of these obstacles are closed with cones connecting $\{A_1, \ldots, A_{n-2}\}$ with the first point in A_{n-1} and only one of them, that corresponding to Δ_1, is joined to the second point. Orienting these simplices Δ_i as elements of the boundary of our $(n - 1)$-dimensional simplex, we conclude that the sum of these cones in $\{A_1, \ldots, A_{n-1}\} = \Theta^{-1}(\cdot)$ is equal to

$$\left[\sum_{i=2}^{n}(-1)^i (\xi_{n-1} - \xi_n) \otimes \ldots \otimes (\xi_{i+1} - \xi_{i+2}) \otimes (\xi_{i-1} - \xi_{i+1}) \otimes \ldots \otimes (\xi_1 - \xi_2)\right] \otimes \xi_1 -$$

$$-(\xi_{n-1}-\xi_n)\otimes\ldots\otimes(\xi_2-\xi_3)\otimes\xi_2\equiv(\xi_{n-1}-\xi_n)\otimes\ldots\otimes(\xi_2-\xi_1).$$

3.5. THEOREM. If n is a power of a prime p, then the obstacle $\alpha(\Theta)$ is not trivial.

Indeed, under the obvious transformation of coefficients $\otimes^{n-1}\mathcal{H}\to\Lambda^{n-1}\mathcal{H}\cong\pm\mathbf{Z}$ the element α is mapped to the generator of this group, and all elements homologous to zero are mapped to multiples of p by Theorem 2.5.1.

4. An Algorithm of Complexity n − 1

THEOREM. For each n and any $\varepsilon > 0$ there exists an ε-algorithm for Problem 1.1.1 whose topological complexity is equal to $n - 1$.

Proof. We introduce a metric in the polycircle B^n: the distance between polynomials a_1, and is equal to the minimum of the number $\max_i|\xi_i(a_1)-\xi_i(a_2)|$ over all orderings of the roots $\xi_i(a_1)$, $\xi_i(a_2)$ of these polynomials. We partition B^n into sets $S_1,\ldots,S_n$: S_t consists of polynomials in which the real parts of the roots assume exactly t distinct values. Obviously, S_t is a real semialgebraic subset of B^n of codimension $n - t$ and its connected components are in a one-to-one correspondence with ordered partitions of the number n into a sum of t natural numbers. By the Weierstrass approximation theorem, for each $t = 1,\ldots,n$, there exists a polynomial $\chi_t: B^n\to\mathbf{R}$, such that the domain $V_t=\{\chi_t\leqslant 0\}$ lies in a $2^{-2t+1}\cdot\varepsilon$-neighborhood of the set S_t and contains its $2^{-2t}\cdot\varepsilon$-neighborhood (in particular, $V_n = B^n$). Note that each point a of the set $V_t - V_{t-1}$ lies in a $2^{-2t+1}\cdot\varepsilon$-neighborhood of exactly one component of the set S_t. Suppoe that this component corresponds to a partition $n = n_1 + \ldots + n_t$. We break the roots $\xi_1,\ldots,\xi_n$ of the polynomial a into piles of cardinalities $n_1,\ldots,n_t$ according to the ordering of their real parts; within each pile these real parts differ no more than by $2^{-2t+2}\cdot\varepsilon$. We assign to the point a the following 2n numbers: $\widetilde{\mathrm{Re}}_1(a)=\ldots=\widetilde{\mathrm{Re}}_{n_1}(a)=$ (the arithmetic mean of the real parts of the roots in the first pile), $\widetilde{\mathrm{Re}}_{n_1+1}=\ldots=\widetilde{\mathrm{Re}}_{n_1+n_2}=$ (the arithmetic mean of the real parts of the roots in the second pile), etc.; $\widetilde{\mathrm{Im}}_1(a)$, $\widetilde{\mathrm{Im}}_2(a)$, ..., $\widetilde{\mathrm{Im}}_{n_1}(a)$ are, respectively, the greatest, the second greatest, ..., the smallest of the numbers $\mathrm{Im}(\xi_j)$, $\tilde{t}=1,\ldots,n_1$; $\widetilde{\mathrm{Im}}_{n_1+1}(a)$, ..., $\widetilde{\mathrm{Im}}_{n_1+n_2}(a)$ are the similarly ordered imaginary parts of the roots in the second pile, etc. Acting in a similar manner for every point of the set $V_t - V_{t-1}$, we obtain 2n continuous functions on entire B^n and, using the Weierstrass theorem again, approximate them, with accuracy of $\varepsilon/2$, by the polynomials $\mathrm{Re}_1^t,\ldots,\mathrm{Re}_n^t,\mathrm{Im}_1^t,\ldots,\mathrm{Im}_n^t$.

The required algorithm is described as follows:

Entry. Compute $\chi_1(a)$. Is $\chi_1(a)\leqslant 0$? If so, then, compute the values of the polynomials $\mathrm{Re}_1^1,\mathrm{Re}_2^1,\ldots,\mathrm{Im}_n^1$ at the point a and output them. If not, then compute $\chi_2(a)$ and ask whether $\chi_2(a)\leqslant 0$. If so, then compute the polynomials $\mathrm{Re}_1^2,\ldots,\mathrm{Im}_n^2$ and output them. If not, then work with χ_3, etc.

5. Braid Group Cohomologies with Coefficients in Coxeter's Representation

Coxeter's representation of the group S(n) [and, therefore, also of Br(n)] is the representation $S(n)\to\mathrm{Aut}(\mathbf{Z}^n)$, which acts by permutations of the basis vectors. The restriction of this representation to the sublattice $\{c\mid c_1+\ldots+c_n=0\}\subset\mathbf{Z}^n$ is called the reduced Coxeter representation. These representations are denoted by X_n and $\tilde{X}_n$, respectively.

THEOREM. The group $H^i(\mathrm{Br}(n), X_n)$ is trivial for $i > n - 1$, isomorphic to $\mathbf{Z}$ for $i = 0$, $n - 1$, to $\mathbf{Z}^2$ for $i = 1$, and to $\mathbf{Z}^2\oplus$ torsion for $i = 2,\ldots,n - 2$. The group $H^i(\mathrm{Br}(n),\tilde{X}_n)$ is trivial for $i = 0$ and $i > n - 1$, isomorphic to $\mathbf{Z}$ for $i = 1$, $n - 1$, and to $\mathbf{Z}^2\oplus$ torsion for $i = 2, 3,\ldots,n - 2$.

The proof of the theorem will be done by induction on n. We denote by $\mathrm{Br}_1(n)$ the subgroup of index n in Br(n) consisting of braids in which the end of the n-th thread is located again at the n-th place. The representation X_n of the group Br(n) is induced by the trivial $\mathbf{Z}$-representation of $\mathrm{Br}_1(n)$. Therefore, $H^i(\mathrm{Br}(n),X_n)\cong H^i(\mathrm{Br}_1(n),\mathbf{Z})$. If $n = 2$, then $\mathrm{Br}_1(n)$ is the painted braid group of two threads which is isomorphic to $\mathbf{Z}$. Let now $n\geqslant 3$. Consider the Sérre–Hochschild spectral sequence for the homomorphism l: $\mathrm{Br}_1(n)\to\mathrm{Br}(n-1)$, given by erasing the n-th thread. The kernel of l is a free group with $n - 1$ generators, therefore, $H^i(\mathrm{Ker}\,l,\mathbf{Z})$ is trivial for $i\geqslant 2$, isomorphic to $\mathbf{Z}$ for $i = 0$, and to $\mathbf{Z}^{n-1}$ for $i = 1$. The action of the group $\mathrm{Br}(n - 1)$ on $H^0(\mathrm{Ker}\,l,\mathbf{Z})$ is trivial and on $H^1(\mathrm{Ker}\,l,\mathbf{Z})$ it is isomorphic to Coxeter's representation X_{n-1}. Therefore, acording to [1], the term $E_2^{p,0}$ is $\mathbf{Z}$ for $p = 0, 1,$

trivial for p = 2 or p > n − 1, and finite for p > 2, and $E_2^{p,1} \cong H^p(\mathrm{Br}(n-1), X_{n-1})$. The statement of the theorem on the cohomologies with coefficients in X_n is now obtained by induction on n. The statement on $H^i(\mathrm{Br}(n), \tilde{X}_n)$ is obtained from the first statement and the results of [1] by means of the exact sequence of coefficients $0 \to \tilde{X}_n \to X_n \to \mathbf{Z} \to 0$.

6. On Compositions of Algebraic Functions

The study of the cohomologies of the group Br(n) began in [1] in connection with the problem of representing algebraic functions by compositions of functions in fewer variables. The following theorem was proved in [3]. Let $D_2(n)$ be the number of ones in the binary expression of the number n.

6.1. THEOREM. A universal n-valued algebraic function (3) in variables $a_1, \ldots, a_n$ cannot be written as a complete composition of algebraic functions in k variables if $k < n - D_2(n)$.

(For the definition of complete composition, see [3].)

6.2. The proof of this theorem is based on the following two results. Let φ_n be the n-leaved covering over $B^n - \Sigma$ considered in Sec. 3.

6.2.1. LEMMA [3]. If a universal algebraic function (3) can be written as a complete composition of algebraic functions in k variables, then there exist a k-dimensional Stein variety K, an n-leaved covering $\mathfrak{A} \to K$ and a map $\varkappa : (B^n - \Sigma) \to K$ such that the covering φ_n is isomorphic to the map induced from the covering $\mathfrak{A}$.

Consider the n-dimensional vector fibration $E \to B^n - \Sigma$ whose fiber over a point a consists of all real-valued functions on the set of roots of the polynomial a. This fibration is isomorphic to the fibration induced by the universal vector fibration over BO(n) under the obvious homomorphism $\mathrm{Br}(n) \to S(n) \to O(n)$.

6.2.2. THEOREM [4]. 1. The map $H^*(BO(n), \mathbf{Z}_2) \to H^*(\mathrm{Br}(n), \mathbf{Z}_2)$ corresponding to this homomorphism is an epimorphism, i.e., the ring $H^*(\mathrm{Br}(n), \mathbf{Z}_2)$ is generated by the Stiefel–Whitney classes of the fibration E.

2. The group $H^{n-D_2(n)}(\mathrm{Br}(n), \mathbf{Z}_2)$ is not trivial.

It remains to note that under the hypotheses of Lemma 6.2.1 the fibration E is induced by some fibration over K. But, with $k < n - D_2(n)$, this contradicts Theorem 6.2.2 as well as the fact that the group $H^i(K)$ is trivial for i > k because of the functoriality of the Stiefel–Whitney classes.

The results of Sec. 2 permit to improve the estimate of Theorem 6.1.

6.3. THEOREM. In the statement of Theorem 6.1 one may replace $n - D_2(n)$ by any $n - D_p(n)$, where p is a prime.

This theorem is of purely methodological interest because in [6] an even stronger statement was proved: in Theorem 6.1 one can always replace $n - D_2(n)$ by $n - 1$. However, Lin's proof is algebraic whereas our proof of Theorem 6.3 is again topological and is based on the same Lemma 6.3.1. Namely, it follows from Theorem 2.1, Remark 1.5.2, and the fact that the genus of the induced fibration is no greater than the genus of the original one.

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EFFECTIVE SUFFICIENT CONDITIONS FOR THE SOLVABILITY OF THE INVERSE PROBLEM OF MONODROMY THEORY FOR SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Since the time of the classical papers of Fuchs [1] and Birkhoff [2, 3], the problem of constructing a system of ordinary differential equations with rational coefficients

$$\frac{d\Psi}{d\lambda}=A(\lambda)\Psi,\quad \lambda\in\mathbf{C},$$
$$A(\lambda)=\sum_{\nu=1}^{n}\sum_{k=0}^{r_\nu}\frac{A_{\nu,k}}{(\lambda-a_\nu)^{k+1}}+\sum_{k=1}^{r_\infty}A_{\infty,k}\lambda^{k-1} \tag{1}$$

from a given monodromy group (the Riemann–Hilbert problem) has been discussed in the literature. Here $A_{\nu,k}$ are m × m-matrices which are constant with respect to λ, $(m>1)$, $a_1, \ldots, a_n$, $a_\infty = \infty$ are n + 1 distinct fixed points on the Riemann sphere $\mathbf{C}P^1$. The monodromy group (cf., e.g., [4, 5] for precise definition) defines a transformation of a fundamental matrix solution Ψ upon passage about each singular point a_ν

$$(\lambda-a_\nu)\to(\lambda-a_\nu)e^{2\pi i}\Rightarrow\Psi\to\Psi M_\nu.$$

The matrices M_ν are called monodromy matrices.

In its most famous version the inverse problem formulated above is posed for systems of Fuchsian type (Hilbert's 21-st problem [6]). The system (1) is called a system of Fuchsian type if $A_{\infty,k} = 0$, $r_\nu = 0$ for all $k, \nu\geqslant 1$. For such systems the inverse problem of monodromy theory (IPMT) is solved for an arbitrary domain $\Omega\subset\mathbf{C}$, which is conformally equivalent to the disc [4, p. 135]. For systems of Fuchsian type on $\mathbf{C}P^1$ the IPMT is also solvable [7], but under specific supplementary restrictions on the monodromy group (cf. [4, 8] for details).

In the case of systems with irregular singular point $(r_\infty\geqslant 1)$ in the collection of monodromy data it is necessary to include, along with the monodromy matrices, the so-called Stokes matrices which are defined as follows. Let A_{∞,r_∞} have the distinct eigenvalues μ_α. We cover a neighborhood of infinity by a finite number of sectors Ω_k, $k = 1, 2,\ldots,2r_\infty + 1$ such that $\Omega_k\cap\Omega_{k+1}\neq\varnothing$ and each sector contains only one separating ray defined by the condition $\operatorname{Re}[(\mu_\alpha-\mu_\beta)\lambda^{r_\infty}]=0$, $\alpha, \beta = 1, 2, \ldots, m$. In each sector one can find a holomorphic nondegenerate solution Ψ_k of (1) with asymptotics [5]

$$\Psi_k=(I+\chi_1\lambda^{-1}+\chi^2\lambda^{-2}+O(\lambda^{-3}))\cdot\exp\Big(\sum_{j=1}^{r_\infty}D_j\lambda^j+D_0\ln\lambda\Big),\quad \lambda\to\infty, \tag{2}$$

where $D_j=\operatorname{diag}(d_{1j},\ldots,d_{mj})$, $j=0,\ldots,r_\infty$.

The Stokes matrices S_k relate solutions Ψ_k in the intersection of neighboring sectors: $\Psi_{k+1}(\lambda)=\Psi_k(\lambda)S_k$, $k=1, 2, \ldots, 2r_\infty$.

For systems with a unique singularity of irregular type at infinity a weaker result than that given above for Fuchsian systems is established. Namely, for $|\lambda| > \lambda_0$ there exists a system (1) with matrix A which has asymptotics $A(\lambda)=\sum_{k=1}^{r_\infty}A_{\infty,k}\lambda^{k-1}+O(\lambda^{-1})$, $\lambda\to\infty$, such that its Stokes factors and matrices D_j coincide with the given ones. The matrix A(λ) is non-

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