ON THE VOLUME OF TUBES.*

By HERMANN WEYL.

1. The problem. In a lecture before the Mathematics Club at Princeton last year Professor Hotelling stated the following geometric problem \(^1\) as one of primary importance for certain statistical investigations:

Let there be given in the \(n\)-dimensional Euclidean space \(E_n\) or spherical space \(S_n\) a closed \(v\)-dimensional manifold \(C_v\). The solid spheres of given radius \(a\) around all the points of \(C_v\) cover a certain part \(C_v(a)\) of the embedding space \(E_n\) or \(S_n\), the volume \(V(a)\) of which is to be determined. We call \(C_v(a)\) an \((n,v)\)-tube (of radius \(a\) around \(C_v\)).

For small values of \(a\) one will have in the first approximation

\[
V(a) = \Omega ma^m \cdot k_v,
\]

where \(\Omega ma^m\) is the volume of the solid \(m\)-dimensional sphere

\[
\sigma_m(a) : \quad t_1^2 + \cdots + t_m^2 \leq a^2
\]

\((m = n - v)\), and \(k_v\) the area of the "surface" \(C_v\). Professor Hotelling showed that this formula is exact in \(E_n\) and a similar formula prevails in \(S_n\), for \(v = 1\). I shall here treat the problem for higher dimensionalities \(v\). The result in \(E_n\) is a formula consisting of \(1 + [\frac{1}{2}v]\) terms, of the following type (§ 3):

\[
V(a) = \Omega m \cdot \sum_{e} \frac{a^{m+e}}{(m+2)(m+4)\cdots(m+e)} k_e
\]

\((e\ even, \ 0 \leq e \leq v)\),

where \(k_e\) is a certain integral invariant of the surface \(C_v\) determined by the intrinsic metric nature of \(C_v\) only, and thus independent of its embedding in \(E_n\). I shall express these invariants (§ 4) in terms of the Riemannian tensor of \(C_v\). An analogous result is obtained for \(S_n\).

2. The fundamental formulas for the volume of tubes. If an \(n\)-dimensional manifold \(M_n\) consisting of points \(u\) and locally referred to

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\(^*\) Received October 14, 1938.

\(^1\) See his paper "Tubes and spheres in \(n\)-spaces, and a class of statistical problems" which precedes this article in this Journal, pp. 440-460.
parameters \( u^1, \cdots, u^n \) is mapped upon the Euclidean space \( E_n \) with the coördinates \( (x_1, \cdots, x_n) = r \),

\[
(3) \quad r = r(u) = r(u^1, \cdots, u^n),
\]

then the volume \( V \) of the image of \( M_n \) in \( E_n \) may be computed by means of the formula

\[
(4) \quad V = \int [r_1 \cdots r_n] du^1 \cdots du^n,
\]

where \([r_1 \cdots r_n]\) designates the determinant of the \( n \) columns \( r_i \), each consisting of the components of the vector

\[
r_i = \partial r/\partial u^i.
\]

This formula takes account of the ± orientation and multiplicity with which the mapping \( u \rightarrow r \) covers the several parts of \( E_n \). The covering will be locally a one-to-one mapping without folds and ramifications wherever \([r_1 \cdots r_n] > 0\). But even if this condition is satisfied everywhere, multiple covering might occur. This question is essentially one of topological rather than differential geometric nature. It is with this reservation in mind that in the following we apply formula (4).

When dealing with the spherical space \( S_n \) we employ homogeneous coördinates \( (x_0, x_1, \cdots, x_n) = r \), the set \( \rho x_i \) meaning the same point as \( x_i \), whatever the factor \( \rho \neq 0 \). Sometimes we use the normalization

\[
r^2 = x_0^2 + x_1^2 + \cdots + x_n^2 = 1.
\]

\( S_n \) then appears as the unit sphere in the Euclidean \( E_{n+1} \). (4) must be replaced by the formula

\[
(5) \quad V = \int \frac{[rr_1 \cdots r_n]}{(r^2)^{(n+1)/2}} du^1 \cdots du^n
\]

as one easily verifies by observing the following facts: (1) the integrand is orthogonally invariant; (2) it is not affected by the gauge factor \( \rho = \rho(u) \) because

\[
(\rho r)_i = \rho \cdot r_i + \frac{\partial \rho}{\partial u^i} \cdot r;
\]

(3) at the point \( r = (1, 0, \cdots, 0) \) the integrand reduces to the "Euclidean" value

\[
\begin{vmatrix}
\frac{\partial x_1}{\partial u^1}, & \cdots, & \frac{\partial x_n}{\partial u^1} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_1}{\partial u^n}, & \cdots, & \frac{\partial x_n}{\partial u^n}
\end{vmatrix}
\]
After these preliminary remarks I now turn to our problem in $E_n$. Let a piece of the $v$-dimensional manifold $C_v$ be given in the Gaussian representation

$$ r = r(u^1 \cdots u^v). $$

At each point we can determine $m = n - v$ normal vectors $\mathbf{n} = \mathbf{n}(1), \cdots, \mathbf{n}(m)$ satisfying the equations

$$ r_a \cdot \mathbf{n} = 0 \quad (a = 1, \cdots, v) $$

which are mutually normalized by

$$ \mathbf{n}(p) \cdot \mathbf{n}(q) = \delta_{pq} \quad (p, q = 1, \cdots, m). $$

$r_a$ is the derivative $\partial r / \partial u^a$. In using the radius vector (6) and these normals, the part $C_v(a)$ of the space covered by the spheres of radius $a$ around the points of $C_v$ allows the representation

$$ \mathbf{\xi} = r + t_1 \mathbf{n}(1) + \cdots + t_m \mathbf{n}(m), \quad (t_1^2 + \cdots + t_m^2 \leq a^2), $$

in terms of the parameters $u^1, \cdots, u^v, t_1, \cdots, t_m$. Hence its volume $V(a)$ is the integral

$$ \int \left[ \mathbf{\xi}_1, \cdots, \mathbf{\xi}_v, \mathbf{n}(1), \cdots, \mathbf{n}(m) \right] dt_1 \cdots dt_m du^1 \cdots du^v. $$

Following Gauss we describe the surface $C_v$ embedded in $E_n$ by its metric ground form

$$ (dr)^2 = \sum_{\alpha, \beta} g_{\alpha \beta} du^\alpha du^\beta, \quad (g_{\alpha \beta} = r_\alpha \cdot r_\beta) $$

together with the linear pencil of the second fundamental forms

$$ - \sum_{p=1}^m \left\{ t_p \sum_{\alpha, \beta} G_{\alpha \beta}(p) du^\alpha du^\beta \right\}, $$

which is the scalar product of

$$ d^2 r = \sum_{\alpha, \beta} r_{\alpha \beta} du^\alpha du^\beta $$

with an arbitrary normal $\mathbf{n} = t_1 \mathbf{n}(1) + \cdots + t_m \mathbf{n}(m)$.

$$ G_{\alpha \beta}(p) = G_{\beta \alpha}(p) = - r_\alpha \cdot \mathbf{n}(p) = r_\alpha \cdot \mathbf{n}_\beta(p). $$

A Greek subscript $\alpha$ attached to the vectors $r$, $\mathbf{n}$ and $\mathbf{\xi}$ always denotes differentiation by $u^\alpha$. 

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Each vector at the point \( u \) of \( C_v \) is a linear combination of the basic vectors \( r_\alpha, n(p) \). On applying this remark to \( n_\alpha(p) \) we set

\[
n_\alpha(p) = \sum_\beta G^{\alpha \beta}(p) \cdot r_\beta + \cdots,
\]

where \( \cdots \) indicates a linear combination of the normal vectors \( n(p) \). By scalar multiplication with \( r_\beta \) one finds

\[
G^{\alpha \beta}(p) = \sum_\lambda g_{\alpha \lambda} G^{\beta \lambda}(p).
\]

From (7) one infers that

\[
r_\alpha = \sum_\beta \{ \delta_\alpha^\beta + \sum_{p=1}^m t_p G^{\alpha \beta}(p) \} r_\beta + \cdots.
\]

Therefore the integrand in (8)

\[
= \det \{ \delta_\alpha^\beta + \sum_p t_p G^{\alpha \beta}(p) \} \cdot [r_1 \cdots r_\nu n(1) \cdots n(m)].
\]

Because of the general identity

\[
[a_1 \cdots a_n]^2 = \det (a_i a_k),
\]

\[
[r_1 \cdots r_\nu n(1) \cdots n(m)]^2 = |g_{\alpha \beta}|,
\]

and considering that

\[
ds = |g_{\alpha \beta}|^{\frac{1}{2}} du^1 \cdots du^\nu
\]

is the area element of \( C_v \), one arrives at the fundamental formula

\[
V(a) = \int_{C_v} \left\{ \int \cdots \int |\delta_\alpha^\beta + \sum_p t_p G^{\alpha \beta}(p)| \, dt_1 \cdots dt_m \right\} ds
\]

in the Euclidean case. The integrand is independent of the choice of the parameters \( u^\alpha \) on \( C_v \).

In the spherical case, let the manifold \( C_v \) be given by the parametric representation (6) with the normalization \( r^2 = 1 \). Therefore \( r \cdot r_\alpha = 0 \). The mutually orthogonal normal vectors \( n = n(1), \cdots, n(m) \) satisfy the equations

\[
r \cdot n = 0, \quad r_\alpha \cdot n = 0.
\]

From both equations there follows

\[
r \cdot n_\alpha = 0.
\]
The part \( C_v(\alpha) \) of the space \( S_n \) covered by the \( m = (n - r) \)-dimensional solid spheres of spherical radius \( \alpha \) is represented by

\[
\varepsilon = r + t_1 n(1) + \cdots + t_m n(m),
\]

where the argument \( u \) in \( r, n(1), \cdots, n(m) \) ranges over the whole \( C_v \), while the parameters \( t_1, \cdots, t_m \) are bound by

\[
t_1^2 + \cdots + t_m^2 \leq a^2, \quad (a = \tan \alpha).
\]

According to equation (5) the volume \( V(\alpha) \) of \( C_v(\alpha) \) is given by the integral of

\[
\left( \frac{\prod_{1}^{r} n(1) \cdots n(m)}{(r^2)^{(n+1)/2}} \right) du \cdots du' dt_1 \cdots dt_m
\]

extended with respect to \( u', \cdots, u^r \) over the whole of \( C_v \), with respect to \( t_1, \cdots, t_m \) over the sphere \( \sigma_m(a) \). Application of the same procedure as before results in the formula

\[
V(\alpha) = \int_{C_v} \left\{ \int_{(t_1^2 + \cdots + t_m^2 \leq a^2)} \delta_{\alpha}^p + \sum_p t_p G_{\alpha}^p(p) \right\} dt_1 \cdots dt_m \div (1 + t_1^2 + \cdots + t_m^2)^{(n+1)/2} ds.
\]

3. Evaluation. For any function \( \phi(t) = \phi(t_1, \cdots, t_m) \), let \( \langle \phi(t) \rangle_t \) designate its mean value over the sphere

\[
t_1^2 + \cdots + t_m^2 = 1.
\]

The mean value \( \langle t_1^{e_1} \cdots t_m^{e_m} \rangle_t \) of a monomial is obviously zero unless all exponents \( e_p \) are even. In the latter case one has the well-known formula

\[
\langle t_1^{e_1} \cdots t_m^{e_m} \rangle_t = \frac{e_1 \cdots e_m}{m(m + 2) \cdots (m + e - 2)},
\]

where

\[
0 = 1, \quad e = 1 \cdot 3 \cdots (e - 1) \quad \text{[for } e = 2, 4, \cdots \text{].}
\]

[ (12) is most easily proved by multiplying the monomial by

\[
e^{-t_1^2} \cdots e^{-t_m^2} = e^{-(t_1^2 + \cdots + t_m^2)}
\]

and then integrating over

\[ -\infty < t_p < \infty \quad (p = 1, \cdots, m). \]
One thus obtains
\[ \int t_1^{e_1} \cdots t_m^{e_m} dw_t \cdot \int_0^\infty e^{-r^2} e^{e_1 \cdots e_m} d\nu = \prod_p \left( \int_{-\infty}^{+\infty} e^{-r^2} e^{e_1 \cdots e_m} d\nu \right), \]
with \( \int \cdots \cdot dw_t \) indicating the "solid angle" integration over the sphere (11), and hence
\[ \frac{1}{2} \int t_1^{e_1} \cdots t_m^{e_m} dw_t = \frac{\Gamma \left( \frac{1 + e_1}{2} \right) \cdots \Gamma \left( \frac{1 + e_m}{2} \right)}{\Gamma \left( \frac{m + e}{2} \right)}. \]

In particular, for the surface \( \omega_m = \int dw_t \) of the sphere,

\[ (13) \quad \frac{1}{2} \omega_m = \left[ \Gamma \left( \frac{1}{2} \right) \right]^m / \Gamma \left( \frac{m}{2} \right). \]

Division results in the desired equation
\[ \langle t_1^{e_1} \cdots t_m^{e_m} \rangle_t = \frac{\Gamma \left( \frac{1 + e_1}{2} \right) \cdots \Gamma \left( \frac{1 + e_m}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \cdots \Gamma \left( \frac{1}{2} \right)} \times \frac{\Gamma \left( \frac{m + e}{2} \right)}{\Gamma \left( \frac{m}{2} \right)} \]

\[ = \frac{e_1 \cdots e_m}{m(\frac{m}{2} + 1) \cdots (m + e - \frac{3}{2})}. \]

The volume of the solid sphere \( \sigma_m(a) \) amounts to
\[ \omega_m \int_0^\infty r^{m-1} dr = \frac{\omega_m}{m} \cdot a^m; \]
hence \( \Omega_m = \omega_m/m \). Specialization of (13) for \( m = 2 \) yields \( [\Gamma(\frac{1}{2})]^2 = \pi \).

The numbers \( \omega_m, \Omega_m \) are best defined by the recursive formulas readily derived from (13):
\[ \omega_{m+2} = \frac{2\pi}{m} \cdot \omega_m \quad (m \geq 1); \quad (\omega_1 = 2, \quad \omega_2 = 2\pi). \]
\[ \Omega_{m+2} = \frac{2\pi}{m + 2} \cdot \Omega_m \quad (m \geq 0); \quad (\Omega_0 = 1, \quad \Omega_1 = 2). \]

We expand the determinant
\[ \psi(t_1 \cdots t_m) = |\delta^\alpha + \sum_p t_p G^{\alpha}(p)| = \psi_0 + \psi_1 + \cdots + \psi_\nu \]
according to degrees in the variables \( t_1, \cdots, t_m \):
\[ \psi_0(t_1 \cdots t_m) = \sum \phi_{e_1} \cdots e_m t_1^{e_1} \cdots t_m^{e_m} \quad (e_1 + \cdots + e_m = e) \]
is homogeneous of degree $e$. $\psi_0 = 1$. This decomposition is conveniently described by introducing an artificial parameter $\lambda$:

$$| \delta_\alpha^\beta + \lambda \sum_p t_p G_\alpha^\beta(p) | = 1 + \lambda \psi_1 + \lambda^2 \psi_2 + \cdots.$$ 

We set

$$\langle \psi_e(t_1 \cdots t_m) \rangle_1 = \frac{H_e}{m(m + 2) \cdots (m + e - 2)}.$$ 

By its definition, $H_e$ is a point invariant of $C_v$. $H_e$ is zero for odd $e$, while for even $e$ one derives from (12) the explicit expression

$$H_e = \sum_j e_1 \cdots e_m \cdot e_1 \cdots e_m, \quad (e_p \text{ even, } e_1 + \cdots + e_m = e).$$

The integral over the solid sphere $\sigma_m(a)$,

$$\int_{\sigma_m(a)} \cdot \cdots \int_{\sigma_m(a)} \psi_e(t_1 \cdots t_m) dt_1 \cdots dt_m$$

then will turn out to be

$$\frac{\omega_m H_e}{m(m + 2) \cdots (m + e - 2)} \int_0^a r^{e+m-1} dr = \omega_m H_e \cdot \frac{a^{m+e}}{m(m + 2) \cdots (m + e)}.$$ 

Thus we find in the Euclidean case

$$V(a) = \sum_e \omega_m \frac{k_e}{m(m + 2) \cdots (m + 4) \cdots (m + e)}, \quad (e \text{ even, } 0 \leq e \leq v),$$

with the coefficients

$$k_e = \int_{C_v} H_e ds.$$ 

In the spherical case one gets

$$\int_{\sigma_m(a)} \cdot \cdots \int_{\sigma_m(a)} \psi_e(t) \frac{dt_1 \cdots dt_m}{(1 + t_1^2 + \cdots + t_m^2)^{(n+1)/2}}$$

$$= \frac{\omega_m H_e}{m(m + 2) \cdots (m + e - 2)} \int_0^a r^{e+m-1} dr \cdot \frac{a^{m+e}}{(1 + r^2)^{(n+1)/2}}.$$ 

On putting $r = \tan \rho$ the integral at the right side becomes

$$\int_0^a (\sin \rho)^{m+e-1} (\cos \rho)^{v-e} d\rho,$$

and instead of (14) one obtains

$$V(a) = \omega_m \cdot \sum_e k_e J_e(a), \quad (e \text{ even, } 0 \leq e \leq v),$$

(16)
where

\[(17) \quad m(m + 2) \cdots (m + e - 2) J_e(x) = \int_0^\alpha (\sin \rho)^{m+e-1}(\cos \rho)^{v-e} \, d\rho.\]

One may notice the recurrent equation

\[\frac{(\sin x)^{e+m}(\cos x)^{v-e-1}}{m(m+2)\cdots(m+e)} = J_e(x) - (v - e - 1) J_{e+2}(x).\]

**Theorem.** The volumes of \((n, v)\)-tubes in Euclidean and in spherical space are given by the formulas (14), (16) respectively, \(J_e(x)\) being defined by (17). \(k_e, (15)\), are certain integral invariants of \(C_v\), in particular \(k_0\) is its surface.

4. **Intrinsic nature of the invariants \(k_e\).** So far we have hardly done more than what could have been accomplished by any student in a course of calculus. However, some less obvious argument is needed for ascertaining that more explicit form of the point invariant \(H_e\) which enables one to replace the curvature \(G_{\alpha\beta}(p)\) by the Riemannian tensor \(R_{\mu\nu\alpha\beta}\) of \(C_v\). I repeat the definition of this tensor in terms of the metric ground tensor:

\[\sum_\lambda g_{\kappa\lambda} \Gamma_{\alpha\beta}^\lambda = \frac{1}{2} \left( \frac{\partial g_{\alpha\kappa}}{\partial u^\beta} + \frac{\partial g_{\beta\kappa}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\kappa} \right)\]

[definition of the affine connection \(\Gamma_{\alpha\beta}^\kappa\),]

\[R_{\kappa\lambda\alpha\beta} = \left( \frac{\partial \Gamma_{\kappa\lambda}^\beta}{\partial u^\alpha} - \frac{\partial \Gamma_{\kappa\alpha}^\beta}{\partial u^\lambda} \right) + \sum_\rho (\Gamma_{\kappa\rho}^\alpha \Gamma_{\rho\lambda}^\beta - \Gamma_{\rho\beta}^\alpha \Gamma_{\rho\lambda}^\kappa).\]

After raising the index \(\lambda\) according to

\[R_{\kappa\lambda\alpha\beta} = \sum_\mu g_{\kappa\mu} R_{\alpha\beta}^\mu,\]

\(R_{\alpha\beta}^\mu\) is not only skew-symmetric in \(\alpha\beta\), but also in \(\kappa\lambda\). As a part of the integrability conditions expressing the *Euclidean* nature of the embedding space \(E_n\), one has the relations \(^2\)

\[(18) \quad R_{\alpha\beta}^\lambda = \sum_{p=1}^{m} \{ g_{\alpha\nu}(p) G_{\beta\lambda}(p) - G_{\beta\nu}(p) G_{\alpha\lambda}(p) \} .\]

In the *spherical* case we look upon \(C_v\) as a surface in \(E_{n+1}\). To the set of \(m\) normals \(n(p), (p = 1, \cdots, m)\) one has simply to add \(n(0) = r\). Since

\[ u_\alpha(0) = r_\alpha \text{ or } G_{\alpha\beta}(0) = \delta_{\alpha\beta}, \]

(18) changes into the equation

\[ R^{\kappa\lambda}_{\alpha\beta} = (\delta_{\alpha\kappa}\delta_{\beta\lambda} - \delta_{\beta\kappa}\delta_{\alpha\lambda}) = \sum_{p=1}^{m} \{ G_{\alpha}\kappa(p) G_{\beta}\lambda(p) - G_{\beta}\kappa(p) G_{\alpha}\lambda(p) \}. \]

[It is a pity that the inadequate name "curvature," which ought to be reserved for \( G_{\alpha\beta}(p) \), has been attached to the Riemann tensor. In the paper just quoted I proposed the more descriptive term "vector vortex." The left side of (19), and also of (18), is the excess of the vortex of \( C_\nu \) over that of the embedding space. In this form the relation would hold with an arbitrary embedding Riemann space.]

We must try then to express the spherical average

\[ \langle \det (\delta_{\alpha\beta} + \lambda \sum_{p} t_{p} G_{\alpha\beta}(p)) \rangle, \]

in terms of the quantities

\[ H \left( \begin{array}{c} \kappa \\ \lambda \\ \alpha \\ \beta \end{array} \right) = \sum_{p} G_{\alpha}\kappa(p) G_{\beta}\lambda(p) - \sum_{p} G_{\alpha}\lambda(p) G_{\beta}\kappa(p). \]

In this investigation the

\[ G_{\alpha\beta} = (G_{\alpha\beta}(1), \cdots, G_{\alpha\beta}(m)), \]

just as

\[ t = (t_1, \cdots, t_m), \]

may be looked upon as arbitrary vectors in an \( m \)-dimensional Euclidean space \( E_m \). Using for a moment the abbreviation

\[ z_{\alpha\beta} = (t \cdot G_{\alpha\beta}) = \sum_{p} t_{p} G_{\alpha\beta}(p), \]

one has

\[ \psi_{e} = \sum_{a_1 < \cdots < a_e} \begin{vmatrix} z_{a_1} & \cdots & z_{a_1}^{a_e} \\ \vdots & \ddots & \vdots \\ z_{a_e} & \cdots & z_{a_e}^{a_e} \end{vmatrix}. \]

Hence we try to determine

\[ \langle \det (t \cdot G_{\alpha\beta}) \rangle, \]

where \( G_{\alpha\beta}, (\alpha, \beta = 1, \cdots, e) \) are any \( e^2 \) given vectors in \( E_m \).
Lemmas.

\[(21) \quad \langle \det (t \cdot G_e^\beta) \rangle_{(\alpha, \beta=1, \ldots, e)} = \frac{1}{m(m+2) \cdots (m+e-2)} \sum_{[\alpha, \beta]} \delta^{(\beta)}_{(\alpha)} H^{(\beta_1 \beta_2)}_{(\alpha_1 \alpha_2)} \cdots H^{(\beta_{e-1} \beta_e)}_{(\alpha_{e-1} \alpha_e)}.\]

\[\alpha_1 \cdots \alpha_e, \beta_1 \cdots \beta_e\] are the numbers 1, \ldots, e in any two arrangements, \(\delta^{(\beta)}_{(\alpha)} = \pm 1\) according as the permutation carrying the \(\alpha\)- into the \(\beta\)-arrangement is even or odd. The sum extends over all couplings of pairs

\[(22) \quad \begin{array}{c|c|c} (\alpha_1 \alpha_2) & (\alpha_3 \alpha_4) & \cdots \\ \hline (\beta_1 \beta_2) & (\beta_3 \beta_4) & \cdots \end{array}\]

By a "pair" \((\alpha, \alpha')\) we mean here two distinct numbers \(\alpha_1, \alpha_2,\) irrespective of their order. Indeed the term \(T^{(\beta)}_{(\alpha)}\) under the sum \(\sum_{[\alpha, \beta]}\) on the right side of \((21)\) does not change under reversal of an \(\alpha\)-pair, \((\alpha_1, \alpha_2) \rightarrow (\alpha_2, \alpha_1)\), or of a \(\beta\)-pair. Nor does it change under permutation of its \(e/2\) factors \(H\); therefore only the coupling of the \(\alpha\)-pairs with the \(\beta\)-pairs, but not the order of the \(e/2\) blocks of the scheme \((22)\) matters. Of the \(2^e \cdot \left(\frac{e}{2}\right)!\) equal terms arising from \(T^{(\beta)}_{(\alpha)}\) by inverting any of the \(e\) pairs of indices and by permuting the \(e/2\) factors \(H\), only one is retained in the sum.

Taking the lemma for granted, we find at once

\[(23) \quad H_e = \sum_{[\alpha, \beta]} \delta^{(\beta)}_{(\alpha)} H^{(\beta_1 \beta_2)}_{(\alpha_1 \alpha_2)} \cdots H^{(\beta_{e-1} \beta_e)}_{(\alpha_{e-1} \alpha_e)},\]

where the sum now extends to all couplings of pairs \((22)\) from the larger range \(1, 2, \ldots, v\) for which the \(\beta\)-sequence consists of the same \(e\) distinct figures as the \(\alpha\)-sequence. The invariant nature of the sum to the right is evidenced when we first write it as

\[\frac{1}{2^e (e/2)!} \sum_{\alpha_1, \ldots, \alpha_e} \sum_{\gamma_1, \ldots, \gamma_e} \pm H^{(\alpha_1, \alpha_2)}_{(\alpha_1, \alpha_2)} H^{(\alpha_3, \alpha_4)}_{(\alpha_3, \alpha_4)} \cdots, \quad (e/2 \text{ factors}),\]

the inner sum alternatingly running over the permutations \(1', \ldots, e'\) of \(1, \ldots, e\). The limitation of distinctness imposed upon \(\alpha_1, \ldots, \alpha_e\) can be canceled, as the inner sum vanishes if two of the \(\alpha\)’s coincide. Hence

\[(24) \quad H_e = \frac{1}{2^e (e/2)!} \sum_{1', \ldots, e'} \left\{ \pm \sum_{\alpha_1, \ldots, \alpha_e} H^{(\alpha_1, \alpha_2)}_{(\alpha_1, \alpha_2)} H^{(\alpha_3, \alpha_4)}_{(\alpha_3, \alpha_4)} \cdots \right\}.\]
The inner sum in which each $\alpha$ runs independently from 1 to $\nu$ is a scalar. We have thus arrived at the decisive

**Theorem.** The scalar $H_\nu$ on $C_\nu$ is determined by the formulas (23), (24) where $H \left( \frac{\lambda^\mu}{\alpha^p} \right)$ is the Riemann tensor or vortex $R_\alpha^\mu$ in the Euclidean case, and the vortex excess (19) in the spherical case.

These metric scalars $H_\nu$ deserve attention on their own merits: they are probably the simplest and most fundamental scalars built up by the Riemann tensor.

As a very special case of our theorem we find that the one term formulas

\[ V(a) = \Omega_m a^m \cdot k_0, \quad V(\alpha) = \omega_m J_0(\alpha) \cdot k_0 \]

prevail if $C_\nu$ is applicable on $E_\nu$ or $S_\nu$ respectively. $k_0$ denotes the surface of $C_\nu$. Professor Hotelling's result concerning the tubes around a curve, $\nu = 1$, is fully contained in this special case.

The lemma is proved by an invariant-theoretic argument as follows. We consider the $e^2$ vectors $G_\alpha^\beta$ as independent variables.

\[ \Phi = \langle \det (t \cdot G_\alpha^\beta) \rangle, \]

is an orthogonal invariant of these variables and therefore, according to the theory of orthogonal vector invariants, expressible as a polynomial in terms of the scalar products $(G_\alpha^\lambda \cdot G_\beta^\mu)$. Observing that $\Phi$ is linear and homogeneous in the components of the vectors of each row and each column of the scheme

\[ \begin{vmatrix} G_1^1, \cdots, G_1^e \\ \cdots \cdots \cdots \\ G_e^1, \cdots, G_e^e \end{vmatrix} \]

we realize that it must be a linear combination of terms

\[ (G_\alpha_{\alpha_1} \cdot G_\alpha_{\alpha_2}) \cdots (G_{\alpha_{e-1}} \cdot G_{\alpha_e}^e), \]

where the $\alpha$ and $\beta$ are any two arrangements of $1, \cdots, e$. Moreover $\Phi$ is skew-symmetric with respect to the columns. Hence, by summing alternatingly over the $e!$ permutations of the superscripts $\beta$ we find that $\Phi$ is a linear combination of the following functions

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\[
\sum_{(\beta)} \delta^{(\beta)}_{(\alpha)} (G_{a_1}^{\beta_1} \cdot G_{a_2}^{\beta_2}) (G_{a_3}^{\beta_3} \cdot G_{a_4}^{\beta_4}) \cdots = \sum_{(\beta)} \delta^{(\beta)}_{(\alpha)} H^{(\beta_1\beta_2)}_{(\alpha_1\alpha_2)} H^{(\beta_3\beta_4)}_{(\alpha_3\alpha_4)} \cdots.
\]

The first sum runs over all \(e!\) permutations \(\beta_1 \cdot \cdot \cdot \beta_e\) of \(1, \cdot \cdot \cdot, e\), the second over all their \(e!/2^{e/2}\) arrangements in "pairs"

\((\beta_1\beta_2), (\beta_3\beta_4), \cdots\).

By applying the same argument to the subscripts \(\alpha\) one concludes that \(\Phi\) is a constant multiple \(c\) of \(H_e\), (23).

The constant \(c\) is determined by the specialization

\[G_a^\beta = (\delta_a^\beta, 0, \cdots, 0)\]

for which

\[\Phi = \langle k_1^e \rangle = \frac{e!}{m(m+2)\cdots(m+e-2)}\]

and

\[H^{(\lambda_\mu)}_{(a\beta)} = \delta_a^\lambda \delta_\beta^\mu - \delta_a^\mu \delta_\beta^\lambda, \quad H_e = e!/2^{e/2}(\frac{1}{2}e)! = e!\).

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