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TUBES AND SPHERES IN n-SPACES, AND A CLASS OF STATISTICAL PROBLEMS.* [1]

By Harold Hotelling.

1. The geometrical and the statistical problems. With reference to a curve C with continuously turning tangent in a metrical space of any number of dimensions, we define a *tube* as the locus of points at a fixed distance θ, called the *radius*, from C, the distance being measured in each case along a geodesic perpendicular to C. A *sphere* or *geodesic sphere* is of course the locus of points at a fixed geodesic distance from a given point. Lengths and areas of geodesic circles on a surface have been investigated by Bertrand and Diguet,[2] who obtained the first two non-vanishing terms in the expansion in powers of the radius. We shall generalize this result for spheres in n dimensions in § 5, and for tubes in § 6. The first term in such an expansion is independent of the curvature properties of the space and of the curve, and may therefore be found from the case of euclidean space. We shall see that alternate terms in the series vanish. The problem is then to express the others in terms of known invariants; this will be done for the first non-vanishing terms following the euclidean ones. We shall also find exact and simple expressions for the volumes enclosed by tubes in euclidean and spherical spaces. In both these cases the volume enclosed is exactly the product of the length of the curve by the $(n-1)$-dimensional area of a cross-section. The qualification must however be made that overlapping regions must be counted with their appropriate multiplicities. A necessary condition for non-overlapping will be obtained. We shall confine our consideration to spaces of positive definite distance elements.

A special type of normal coördinates associated with the arbitrarily given curve is introduced in § 6. These may prove useful in a variety of geometrical and physical problems.

Tubes on a hypersphere play a part in theoretical statistics. For example, if a set of observations

$$x_1, x_2, \cdots, x_n$$
$$y_1, y_2, \cdots, y_n$$

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[1] Presented to the American Mathematical Society, September 6, 1938.

[2] *Journal de Mathématiques* (Liouville), Ser. 1, vol. 13 (1848), pp. 80-86. L. P. Eisenhart, Differential Geometry (1909), p. 209.

is used to determine the parameters b and p in the regression equation

$$Y = bf(x, p) \tag{1.1}$$

in such a way that

$$\sum_{\alpha=1}^{n} \{bf(x_\alpha, p) - y\}^2$$

is a minimum, then the correlation R between the fitted values and the observed values y (calculated without elimination of the mean) is the cosine of the angle made by two lines through the origin of cartesian coördinates in euclidean space of n-dimensions, one line through the point of coördinates y_α, the other through the point of coördinates Y_α $(\alpha = 1, 2, \cdots, n)$. On the hypothesis that the y_α have no real relation to the x_α, but are normally and independently distributed about zero with a common variance, we may regard the y-line as drawn to a random point on a unit hypersphere, on which there is a uniform distribution of such points. The other line is drawn to a point on the curve whose equations in terms of the parameter p may be taken as

$$Y_\alpha = f(x_\alpha, p), \qquad (\alpha = 1, \cdots, n).$$

or the functions on the right may be multiplied by any constant. The method of least squares is such that R is made a maximum; consequently the Y-line is drawn through a point of the unit hypersphere lying on the curve C into which the foregoing curve is projected from the center, in such a way that the geodesic distance between the intersections of the two lines with the hypersphere is a minimum. The probability that R exceeds any assigned or observed value is the ratio to the whole $(n-1)$-dimensional "area" of the hypersphere of that portion of it contained within a tube about C. If C is a great circle the solution of this problem is known; this is the case if $f(x, p)$ is a linear function of p. C is also a great circle if, when x is replaced by some function of a new variable ξ and p by some function of a variable π, $f(x, p)$ reduces to a linear function of π. In other cases C will be a curve other than a great circle. The determination whether an observed correlation is significant then requires the evaluation of the volume of a tube about C.

Similar considerations apply to other situations in which a parameter enters in a non-linear fashion. For example we may fit a regression equation of the form

$$Y = a + bf(x, p), \tag{1.2}$$

determining a, b and p so as to make $\Sigma(Y - y)^2$ a minimum. The reduction of its theory to that of (1.1) is effected in the following manner, which may easily be extended to other cases in which additional parameters enter into the regression equation linearly. Put

$$\bar{f}(p) = \Sigma f(x_a, p)/n, \qquad \bar{y} = \Sigma y_a/n$$
$$f'(x_a, p) = f(x_a, p) - \bar{f}(p), \qquad y'_a = y_a - \bar{y}.$$

The expression to be made a minimum,

$$\Sigma (Y_a - y_a)^2 = \Sigma \{a + bf(x_a, p) - y_a\}^2,$$

when expressed in terms of the quantities just introduced, reduces easily, since

$$\Sigma f'(x_a, p) = 0 = \Sigma y_a'$$

to

$$\Sigma \{bf'(x_a, p) - y_a'\}^2 + n\{a + b\bar{f}(p) - \bar{y}\}^2.$$

The minimizing of the first sum is of the same nature as in the case of the regression equation (1.1); when b and p are determined in this way, a is determined immediately so as to make the other term vanish. The distribution of the correlation between y and Y, eliminating the means in this case, is now determined by the volume of a tube on a unit hypersphere of $(n-2)$ dimensions in the flat space of $(n-1)$-dimensions whose equation is $\Sigma Y_a = 0$. The axis of the tube is the curve C whose equations are

$$\mu Y_a = f'(x_a, p) \qquad (a = 1, \cdots, n),$$

where μ is determined by the condition that $\Sigma Y a^2 = 1$.

The numerical process of fitting regression equations non-linear in parameters is considerably more laborious than in the linear case. It should be noticed that in all such problems, while transformations of parameters and also transformations of independent variates are permissible, it is not permissible to make a transformation of the dependent variate y without changing the hypotheses underlying the application of the method of least squares to the particular case. Thus, the common practice of taking logarithms of both sides of such a regression equation as $Y = be^{px}$ in order to reduce it to linear form leads to inexact results unless the errors in log y, rather than in y itself, can be regarded as normally distributed with a common mean and variance.

As generalizations of (1.1) and (1.2) we may consider regression equations involving two or more parameters in an essentially non-linear fashion. Outstanding among these are the harmonic of undetermined period,

$$Y = a + b \cos(kx + \epsilon), \tag{1.3}$$

or more generally, a sum of such harmonics, and the logistic used to describe the growth of populations and of individual organisms,

$$Y = \frac{b}{1 + me^{-ax}}. \tag{1.4}$$

It has not always been realized that periodogram analysis, at least in Schuster's original sense of fitting a harmonic of the form (1.3), is essentially a problem in least squares, and that the problem of significance is a special case of the general one of least squares. The only published exact test of significance is due to R. A. Fisher [3] and is predicated on the assumption that only those periods are to be considered that are submultiples of the whole range of observations available. Empirical scientists in search of periodicities in sunspots, light variation of stars, rainfall, and business fluctuations have however not confined themselves to such a limited set of trial periods. The procedure is rather to try a very large number of periods, perhaps greater than the number of observations, and select the one showing greatest intensity. This is virtually equivalent to solving by trial the normal equations corresponding to (1.3). The maximum intensity obtainable, divided by the mean square residual, will be a function of the correlation R between the observed values y and the values Y computed from the regression equation (1.3). The probability distribution of R in the absence of genuine periodicity, on the assumption of normally and independently distributed observations with a common mean and variance, may be found approximately for high values of R by the geometrical method. Indeed, by applying to (1.3) the same considerations by which the theory of (1.2) was reduced to that of (1.1), we arrive at the equations

$$\mu Y_a = f'(x_a, k, \epsilon), \qquad (a = 1, \cdots, n) \tag{1.5}$$

satisfying the conditions $\Sigma Y_a = 0$ and $\Sigma Y_a{}^2 = 1$. The right-hand member is simply the difference between $\cos(kx_a + \epsilon)$ and the mean of this expression for the various values of x_a (in applications, the times) corresponding to the observations. We may regard (1.5) as the equations of a two-dimensional surface with parameters k and ϵ, lying in the $(n-2)$-dimensional hypersphere whose equations are

$$\Sigma Y_a = 0, \quad \Sigma Y_a{}^2 = 1.$$

The probability of any particular value of R being exceeded is proportional to the volume of the hypersphere within a geodesic distance θ of this surface, where $R = \cos\theta$. If we confine the range of periods, that is, of values of k, so that the corresponding portion of the surface does not have too great curvatures, and if θ is not too great, it is evident that this probability will be exactly or approximately proportional to the area of the portion of the surface explored.

[3] "Tests of significance in harmonic analysis," *Proceedings of the Royal Society*, London, vol. 125 A (1929), pp. 54-59.

The method suggested by Fisher is equivalent to using a finite number, approximately $(n-3)/2$, of great circles on the hypersphere, at constant mutual geodesic distances from each other of a quarter of a great circle. Of these circles, the one nearest the sample point corresponds to the period of maximum intensity. The probability appropriate to a test of significance by this method is the ratio to the whole $(n-2)$-dimensional volume of the hypersphere of the sum of the volumes of all the tubes about the selected circles, of radii equal to the minimum distance from the sample point. The aggregate volume of all these tubes will evidently be less than the volume of the region within geodesic distance θ of the surface (1.5), which passes through the axes of the tubes. This merely means that the method allowing selection of any period in a continuous range gives a greater probability of a particular value of R being exceeded than does Fisher's method of confining attention to certain predetermined periods, as was to be expected. Also, if the critical probabilities are made equal for the two tests, some intensities significant by Fisher's method will not be significant by the method of continuous variability of period; while periods eliminated from consideration by Fisher's method will sometimes appear significant when they are admitted to consideration.

The logistic (1.4) may be dealt with similarly by finding the area of a surface of two dimensions in a hypersphere. But in this case the assumption of equal variances of the deviations for different values of y becomes questionable, and a transformation leading to a different form of the problem will usually be suggested by the application to be made. The logistic (1.4) satisfies the differential equation

$$\frac{d \log Y}{dx} = a - b'Y, \tag{1.6}$$

where $b' = a/b$. The assumptions ordinarily underlying the use of the logistic as a growth curve are more in keeping with the assumption of independence and uniform variance for the deviations between the two members of this differential equation than for the deviations between the members of (1.4). The parameters of (1.6) enter in a linear fashion, so that in its fitting classical methods are more appropriate than the relatively complex ones associated with the direct fitting of the integrated logistic equation (1.4), provided suitable estimates of the growth rate on the left of (1.6) are available. One method of dealing with this situation has been given by the author in an earlier paper.[4] An analogous method based on a difference equation

[4] "Differential equations subject to error, and population estimates," *Journal of the American Statistical Association*, vol. 22 (1927), pp. 283-314.

instead of a differential equation, had been given earlier by G. U. Yule;[5] it appears to be the better of the two from a practical standpoint when, as Yule assumes, the time intervals between observations are strictly uniform.

But in this paper we shall not deal further with problems involving more than a single non-linear parameter, nor shall we discuss the integrals whose evaluation is necessary for practical work with the examples indicated above. The subsequent sections are offered purely as contributions to geometry, except that the results of Section 3 are essential to the tests of significance just described. As is usual in differential geometry, we shall assume the functions involved to have in the neighborhoods concerned continuous finite derivatives of all orders essential to the argument. Latin indices will be used to indicate the values $1, 2, \cdots, n$, whereas Greek indices will take only the values $2, \cdots, n$ throughout the paper, except in § 3. Repetition of a Greek index within a term will denote summation from 2 to n; of a Latin index, summation from 1 to n.

2. Tubes in euclidean space. In terms of cartesian coördinates $x_1, x_2, \cdots, x_n$ let the curve C be defined by the equations

$$x_i = f_i(v_1), \tag{2.1}$$

where v_1 is the distance along the curve from some fixed point. We shall use primes to denote differentiation with respect to v_1. Denote by λ_{i1} the unit vector tangent to C, by λ_{i2} the unit first curvature vector of C, and by $\lambda_{i3} \cdots, \lambda_{in}$ a set of unit vectors orthogonal to each other and to λ_{i1} and λ_{i2}, so chosen that the determinant $|\lambda_{ij}| = +1$. Then $\lambda_{i1} = f_i'(v_1)$; also λ_{i1} equals its cofactor in the determinant.

Introducing curvilinear coördinates $v_1, v_2, \cdots, v_n$ by means of the relations

$$x_i = f_i(v_1) + v_a \lambda_{ia}(v_1), \tag{2.2}$$

where the last term represents a sum from 2 to n with respect to a, we have

$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(v_1, \cdots, v_n)} = \begin{vmatrix} \lambda_{11} + \lambda_{1a}' v_a & \lambda_{12} \cdots \lambda_{1n} \\ \cdots & \cdots \\ \lambda_{n1} + \lambda_{na}' v_a & \lambda_{n2} \cdots \lambda_{nn} \end{vmatrix}.$$

Expanding with reference to the first column we obtain, since the cofactor of the j-th element in this column equals λ_{i1},

$$J = 1 + \lambda_{i1} \lambda_{ia}' v_a. \tag{2.3}$$

[5] *Journal of the Royal Statistical Society*, vol. 88 (1925), pp. 1-58.

Upon differentiating the orthogonality condition $\lambda_{i1}\lambda_{i\alpha} = 0$ we obtain:

$$\lambda_{i1}\lambda_{i\alpha}' + \lambda_{i1}'\lambda_{i\alpha} = 0. \tag{2.4}$$

The elementary relation between the principal normal, radius of first curvature ρ_1, and rate of change of the direction of the tangent may be written

$$\lambda_{i1}' = \frac{\lambda_{i2}}{\rho_1}. \tag{2.5}$$

Substituting this in (2.4), making use of the orthogonality of the vectors, and substituting the result in (2.3) gives:[6]

$$J = 1 - \frac{v_2}{\rho_1}. \tag{2.6}$$

It is clear that $v_2, \cdots, v_n$ are distances from the curve C in directions perpendicular to the tangent and to each other. A tubular hypersurface of radius θ therefore has the equation

$$v_2^2 + v_3^2 + \cdots + v_n^2 = \theta^2.$$

This may also be regarded as the equation of a hypersphere in space of $n - 1$ dimensions. Upon integrating (2.6) with respect to $v_2, \cdots, v_n$ over the interior of this sphere, since the mean value of v_α is zero, we obtain merely the volume enclosed by the sphere, namely

$$\frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \theta^{n-1}. \tag{2.7}$$

Since this does not involve v_1, *the tubular volume corresponding to an arc is the product of* (2.7) *by the length of the arc.*

This result is exact, but takes no account of overlapping of the tube with itself. Overlapping may be of portions of the tube corresponding to non-consecutive arcs, or it may be a local phenomenon resulting from the curvature of the axial curve being excessive in relation to the radius of the tube. The first kind is not within the domain of differential geometry, and apparently nothing can be said about it without some further specializiation of the curve.

The second kind of overlapping, or kinking, will occur if and only if $J = 0$ at some point within the tube. Since $|v_2| < \theta$ within the tube, it is

[6] This method of evaluating J, which is simpler than my original reduction and does not require continuity of derivatives of the vectors of orders higher than the first, was pointed out by Dean L. P. Eisenhart, to whom I am indebted for reading this paper and making several suggestions.

evident from (2.6) that J vanishes within the tube if and only if $\theta > \rho_1$. Thus the condition for non-overlapping of the local sort is that the radius of the tube shall not exceed the radius of first curvature of the axial curve, regardless of curvatures of higher order.

3. Tubes on a hypersphere. In terms of cartesian coördinates $x_1, x_2 \cdots x_n$ in euclidean space of n dimensions we may write the equation of a unit hypersphere

$$\Sigma x^2 = 1. \tag{3.1}$$

In this section it will be convenient to use this type of notation, denoting summation from 1 to n by the sign Σ and frequently omitting subscripts. Let the curve C on this sphere be defined in terms of the arc length s by the n equations $x_i = x_i(s)$; let differentiation with respect to s be indicated by primes; and let $\alpha_i = x_i'$. Then

$$\Sigma\alpha^2 = 1, \tag{3.2}$$

and, by differentiation of (3.1),

$$\Sigma x\alpha = 0. \tag{3.3}$$

Differentiating the last equation and using (3.2) we have:

$$\Sigma x\alpha' = -1. \tag{3.4}$$

The radius ρ of first curvature relative to the euclidean space is given by

$$1/\rho^2 = \Sigma\alpha'^2. \tag{3.5}$$

If the n new quantities $\xi_1, \cdots, \xi_n$ are subject to the three equations

$$\Sigma x\xi = 0, \qquad \Sigma\alpha\xi = 0, \qquad \Sigma\xi^2 = 1, \tag{3.6}$$

it is evident that they have $n - 3$ degrees of freedom for each value of s. Hence we may write them as functions,

$$\xi_i = \xi_i(s, \phi_1, \cdots \phi_{n-3}), \tag{3.7}$$

of forms to be specified later. Restricting the x_i to be cartesian coördinates of a point on the curve C, and therefore functions only of s, we shall use $y_1, \cdots, y_n$ as cartesian coördinates of a general point on the hypersphere, whose equation $\Sigma y^2 = 1$ is satisfied identically by the expressions

$$y_i = x_i \cos\theta + \xi_i \sin\theta, \tag{3.8}$$

because of (3.1) and (3.6). As curvilinear coördinates on the hypersphere we shall use s, θ, and the variables $\phi_1, \cdots \phi_{n-3}$ appearing in (3.7). Taking them in this order and using primes to denote *partial* differentiation with respect to s, we have as the matrix of coefficients of the linear element,

$$
(3.9) \qquad \left\| \begin{matrix} \Sigma y'^2 & \Sigma y' \dfrac{\partial y}{\partial \theta} & \cdots & \\ \Sigma y' \dfrac{\partial y}{\partial \theta} & \Sigma \left(\dfrac{\partial y}{\partial \theta}\right)^2 & \cdots & \\ \sin\theta\, \Sigma y' \dfrac{\partial \xi}{\partial \phi_1} & \sin\theta\, \Sigma \dfrac{\partial y}{\partial \theta} \dfrac{\partial \xi}{\partial \phi_1} & \cdots & \sin^2\theta\, \Sigma \dfrac{\partial \xi}{\partial \phi_1} \dfrac{\partial \xi}{\partial \phi_{n-3}} \\ \cdot & \cdot & \cdot & \cdot \\ \sin\theta\, \Sigma y' \dfrac{\partial \xi}{\partial \phi_{n-3}} & \sin\theta\, \Sigma \dfrac{\partial y}{\partial \theta} \dfrac{\partial \xi}{\partial \phi_{n-3}} & \cdots & \sin^2\theta\, \Sigma \left(\dfrac{\partial \xi}{\partial \phi_{n-3}}\right)^2 \end{matrix} \right\| .
$$

From (3. 6) we have

$$
(3.10) \qquad \Sigma x\xi' = 0, \quad \Sigma \xi\xi' = 0, \quad \Sigma x \frac{\partial \xi}{\partial \phi_\gamma} = 0, \quad \Sigma \alpha \frac{\partial \xi}{\partial \phi_\gamma} = 0, \quad \Sigma \xi \frac{\partial \xi}{\partial \phi_\gamma} = 0 \quad (\gamma = 1, \cdots, n-3).
$$

From these and the preceding identities it is easy to see that the elements in the second row or second column of (3. 9) are all zero, excepting the element in the intersection of the second row and column, which equals unity. This shows, by a well known theorem (*RG*,[7] p. 58) that θ measures the distance from C along geodesics of the hypersphere perpendicular to C.

Denoting by E the element in the upper left-hand corner of the matrix, we have from (3. 8) and (3. 2),

$$
(3.11) \qquad E = \cos^2\theta + 2\cos\theta\sin\theta\, \Sigma \alpha\xi' + \sin^2\theta\, \Sigma \xi'^2.
$$

The other elements in the first column are given by

$$
\sin\theta\, \Sigma(\alpha\cos\theta + \xi'\sin\theta) \frac{\partial \xi}{\partial \phi_\gamma} = \sin^2\theta\, \Sigma \xi' \frac{\partial \xi}{\partial \phi_\gamma}
$$

by (3. 10). The element of $(n-1)$-dimensional volume on the hypersphere is the product of $ds\, d\theta d\phi_1, \cdots, d\phi_{n-3}$ by the square root of the determinant of (3. 9). It therefore equals

$$
(3.12) \qquad \sin^{n-3}\theta \sqrt{G}\, ds\, d\theta\, d\phi_1 \cdots d\phi_{n-3},
$$

where

$$
(3.13) \qquad G = \left| \begin{matrix} E & \sin\theta\, \Sigma \xi' \dfrac{\partial \xi}{\partial \phi_1} & \cdots & \sin\theta\, \Sigma \xi' \dfrac{\partial \xi}{\partial \phi_{n-3}} \\ \sin\theta\, \Sigma \xi' \dfrac{\partial \xi}{\partial \phi_1} & \Sigma \left(\dfrac{\partial \xi}{\partial \phi_1}\right)^2 & \cdots & \Sigma \dfrac{\partial \xi}{\partial \phi_1} \dfrac{\partial \xi}{\partial \phi_{n-3}} \\ \cdot & \cdot & \cdot & \cdot \\ \sin\theta\, \Sigma \xi' \dfrac{\partial \xi}{\partial \phi_{n-3}} & \Sigma \dfrac{\partial \xi}{\partial \phi_1} \dfrac{\partial \xi}{\partial \phi_{n-3}} & \cdots & \Sigma \left(\dfrac{\partial \xi}{\partial \phi_{n-3}}\right)^2 \end{matrix} \right| .
$$

This determinant will be evaluated with the help of a special orthogonal ennuple of vectors of the Schmidt type.[7] Denoting the i-th component of the j-th vector by λ_{ij} $(i, j, = 1, \cdots, n)$ we put $\lambda_{i1} = x_i$, and then define $\lambda_{i2}, \cdots, \lambda_{in}$ as linear functions of x_i and its successive derivatives with respect to s, such that each vector in the sequence involves a derivative of order higher by unity than the preceding vector, and such that the whole set is orthogonal and normal. Thus $\lambda_{i2} = \alpha_i$; and

$$\Sigma \lambda_{ik} \lambda_{im} = \delta_m^k \tag{3.14}$$

where δ_m^k is the Kronecker delta, equal to unity if $k = m$, and otherwise zero. The formulae RG (32.16) analogous to those of Frenet and Serret give in this case

$$\lambda_{ij}' = \frac{\lambda_{i,\,j+1}}{\rho_j} - \frac{\lambda_{i,\,j-1}}{\rho_{j-1}} \qquad (i, j = 1, \cdots, n), \tag{3.15}$$

where the convention is made that $1/\rho_0 = 1/\rho_n = 0$ and $\rho_1, \cdots, \rho_{n-1}$ are certain functions of s. (These λ's and ρ's are different from those of other sections of this paper.) Putting $j = 1$ in (3.15) and recalling that $\lambda_{i1}' = x_i' = \alpha_i$, and that λ_{i2} also equals α_i, shows that $\rho_1 = 1$.

The quantities z_j defined by the orthogonal transformation

$$z_j = \Sigma \lambda_{ij} \xi_i \tag{3.16}$$

must, according to (3.6), satisfy the conditions

$$z_1 = z_2 = 0, \quad \Sigma z^2 = 1. \tag{3.17}$$

Multiplying (3.16) by λ_{mj} and summing with respect to j, using (3.14) and (3.17), and then changing m to i, gives

$$\xi_i = \Sigma \lambda_{ij} z_j = \lambda_{i3} z_3 + \cdots + \lambda_{in} z_n. \tag{3.18}$$

We may regard $z_3, \cdots, z_n$ as cartesian coördinates of a point independent of s on the sphere (3.17), whose intrinsic dimensionality is $n - 3$. We shall regard $\phi_1, \cdots, \phi_{n-3}$ as spherical polar coördinates on this sphere, thus specializing the arbitrary functions (3.7), which now take the form (3.18), where the λ's involve only s and the z's involve only the ϕ's. Putting

$$u_{j\gamma} = \frac{\partial z_j}{\partial \phi_\gamma} \tag{3.19}$$

we therefore obtain by differentiating (3.18),

$$\frac{\partial \xi}{\partial \phi_\gamma} = \Sigma \lambda_{ij} u_{j\gamma}, \tag{3.20}$$

[7] L. P. Eisenhart, Riemannian Geometry, Princeton, 1926, Sec. 32. We shall refer later to this treatise as RG.

and also, with the help of (3.15),

$$\xi_i' = \Sigma \lambda_{ij}' z_j = \Sigma \left(\frac{\lambda_{i,\, j+1}}{\rho_j} - \frac{\lambda_{i,\, j-1}}{\rho_{j-1}} \right) z_j. \tag{3.21}$$

Separating this into two summations, putting $j + 1 = k$ in the first and $j - 1 = k$ in the second, introducing the notation

$$\Delta_k = \frac{z_{k-1}}{\rho_{k-1}} - \frac{z_{k+1}}{\rho_k}, \tag{3.22}$$

and making use of the relations $z_1 = z_2 = 1/\rho_0 = 1/\rho_n = 0$, we find:

$$\xi_i' = \Sigma \lambda_{ik} \Delta_k. \tag{3.23}$$

From (3.22) it follows that

$$\Delta_1 = 0, \tag{3.24}$$

and from (3.23), (3.14), (3.24) and (3.20),

$$\Sigma \xi'^2 = \Sigma \Delta^2 = \Delta_2^2 + \Delta_3^2 + \cdots + \Delta_n^2, \tag{3.25}$$

$$\Sigma \alpha \xi' = \Sigma \lambda_{i2} \xi'_i = \Delta_2, \tag{3.26}$$

$$\Sigma \xi' \frac{\partial \xi}{\partial \phi_\gamma} = \Sigma u_{i\gamma} \Delta_i, \tag{3.27}$$

$$\Sigma \frac{\partial \xi}{\partial \phi_\gamma} \frac{\partial \xi}{\partial \phi_\delta} = \Sigma u_{i\gamma} u_{i\delta}. \tag{3.28}$$

From (3.11), (3.26) and (3.25),

$$E = (\cos\theta + \Delta_2 \sin\theta)^2 + (\Delta_3^2 + \cdots + \Delta_n^2) \sin^2\theta. \tag{3.29}$$

Let the determinant in (3.13) be represented as the sum of two determinants, identical in all but the first column, in such a way that one of these determinants has as its first element $(\cos\theta + \Delta_2 \sin\theta)^2$, and otherwise has zeros in the first column. We may thus write, with the help of (3.25) to (3.29) inclusive

$$G = F + H^2 \sin^2\theta, \tag{3.30}$$

where

$$F = (\cos\theta + \Delta_2 \sin\theta)^2 \begin{vmatrix} \Sigma u_{i1}^2 \cdots & \\ \Sigma u_{i1} u_{i2} \cdots & \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ & \cdots \Sigma u_{i,\, n-3}^2 \end{vmatrix}, \tag{3.31}$$

and

$$H = \begin{vmatrix} \Delta_3 & \cdots & \Delta_n \\ u_{31} & \cdots & u_{n1} \\ \cdot & \cdot & \cdot \\ u_{3,\, n-3} & \cdots & u_{n,\, n-3} \end{vmatrix}.$$

Differentiating the last of (3.6) gives $\Sigma \xi \xi' = 0$. If in this we substitute (3.18) and (3.23), and use (3.14), we obtain

$$\Sigma z_k \Delta_k = 0.$$

Differentiating $\Sigma z^2 = 1$ we have, from (3. 19),

$$\Sigma z_k u_{k\gamma} = 0.$$

These equations establish a homogeneous linear relation among the columns of H. Hence $H = 0$, so that $G = F$ by (3. 30). Denoting by G' the determinant in the right-hand member of (3. 31), and noting from (3. 22) that $\Delta_2 = - z_3/\rho_2$ we thus obtain the element of volume (3. 12) in the form

$$(3.\ 32) \qquad \sin^{n-3}\theta\,(\cos\theta - z_3 \sin\theta/\rho_2)\ \sqrt{G'}\ ds\, d\theta\, d\phi_1 \cdots d\phi_{n-3}.$$

The integral of $\sqrt{G'}\, d\phi_1 \cdots d\phi_{n-3}$ over the sphere is simply the $(n-3)$-dimensional volume of this unit sphere, namely

$$\frac{2\pi^{(n-2)/2}}{\Gamma\left(\dfrac{n-2}{2}\right)}.$$

In integrating (3. 32), the integral resulting from the second term in the parenthesis vanishes because z_3 is measured perpendicularly from a diametral plane of the sphere, and so has a mean value zero. Integrating also with respect to θ and s, we have the simple result:

The volume enclosed by a tube of geodesic radius θ on a hypersphere having intrinsically $n-1$ dimensions is the product of the length of the axial curve by

$$\frac{\pi^{(n-2)/2} \sin^{n-2}\theta}{\Gamma(n/2)}.$$

Local self-overlapping will exist if (3. 32) vanishes within the tube. This will occur if and only if $\tan\theta > \rho_2$, where θ is the geodesic radius. To evaluate ρ_2 we first put $j = 2$ in (3. 15) and deduce, since $\lambda_{i1} = x_i$, $\lambda_{i2} = \alpha_i$ and $\rho_1 = 1$, that

$$\lambda_{i3} = \rho_2\,(x_i + \alpha'_i).$$

Squaring, summing with respect to i, and using (3. 14), (3. 1), (3. 4) and (3. 5), we find

$$\rho_2 = \frac{\rho}{\sqrt{1-\rho^2}}.$$

The condition $\tan\theta \leq \rho_2$ for absence of local self-overlapping is therefore equivalent to

$$\sin\theta \leq \rho.$$

This condition is also expressed by the statement that the geodesic radius of the tube must not exceed the maximum radius of geodesic curvature of C if there is to be no local overlapping.

As an application, we observe that in fitting the regression equation

$$Y = be^{px}$$

we obtain a curve which, for $p = \pm\infty$, has ends. At these ends, the radius of curvature becomes zero. Consequently, if the foregoing proposition regarding the volume of a tube is to be applied to evaluate the goodness of fit, it is necessary either to confine attention to values of $|p|$ less than some upper limit, or to make a special study of the volume in neighborhoods of the ends of the curve. In the former case the volumes of hemispherical caps over the ends should be added to that enclosed by the tube in determining the relevant probability.

4. An orthogonality property.[8] The following theorem concerns geodesic spheres in an arbitrary Riemannian space; it reduces to one of Gauss when the space is of two dimensions, and may be proved in a somewhat similar manner:[9]

The geodesic sphere defined as the locus of points at a fixed geodesic distance from a point O is perpendicular to the geodesics through O.

The differential equations of the geodesics in terms of the arc length s are (*RG*, (17.8)):

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{4.1}$$

Taking the geodesics through O as the coördinate lines along which $x^1 = s$ is the distance from O and the other coördinates $x^2, \cdots, x^n$ are constant, we have

$$\frac{dx^j}{ds} = \delta_1{}^j.$$

Substituting this in (4.1) gives

$$\left\{ \begin{matrix} i \\ 1\,1 \end{matrix} \right\} = 0. \qquad (i = 1, 2, \cdots, n).$$

Hence $[1\,1, \alpha] = 0$. Since the choice of coördinates implies that $g_{11} = 1$ identically, it then follows that

$$\frac{\partial g_{1\alpha}}{\partial x^1} = 0. \tag{4.2}$$

Consider also another system of coördinates $y^1, \cdots, y^n$ such that $y^i = \xi^i x^1$, where the ξ^i are any functions of $x^2, \cdots, x^n$ have finite derivatives in a neigh-

[8] In this and the following sections the notation is throughout that of *RG*. Latin indices will in all cases vary from 1 to n, Greek indices from 2 to n.

[9] L. P. Eisenhart, Differential Geometry, Boston, 1909, p. 207.

borhood of O. Denoting by g'_{jk} the components of the distance tensor in this coordinate system, we have

$$g_{1a} = g_j{}'_k \frac{\partial y^j}{\partial x^1} \frac{\partial y^k}{\partial x^a} = g'_{jk} \xi^j \frac{\partial \xi^k}{\partial x^a} x^1.$$

Thus at O, where $x^1 = 0$, $g_{ia} = 0$. Since (4.2) shows that g_{ia} is independent of x^1, it follows that $g_{ia} = 0$ everywhere. This proves the theorem. Another proof of this theorem, based on the transversality condition of the calculus of variations could also be given.

5. Spheres in a general curved space. Let $x^1, \cdots, x^n$ be normal coördinates with origin at the center 0 of a geodesic sphere of radius θ. The element of volume is

$$\sqrt{g}\, dx^1 \cdots dx^n, \tag{5.1}$$

where g is the determinant of the distance tensor g_{ij}. For a point on the sphere let $\xi^1, \cdots, \xi^n$ be defined by the equations

$$x^i = \xi^i \theta, \tag{5.2}$$

which are also the equations of the geodesics through 0 if θ is regarded as a parameter and the ξ^i as constants. The ξ^i may be regarded as cartesian coördinates of a point on a unit sphere in euclidean space of n dimensions. Let $M_n\{\phi\}$ denote the mean value over this $(n-1)$-dimensional sphere of a function ϕ. Obviously $M_n\{1\} = 1$, and $M_n\{\xi^i\} = 0$. From considerations of symmetry it is further obvious that the mean value of the product of any powers of the ξ^i vanishes unless each of the ξ^i enters into the product with an even exponent. By integration with respect to spherical polar coördinates, or in various other ways, it is easy to establish that

$$M_n\{(\xi^i)^2\} = \frac{1}{n}, \quad M_n\{(\xi^i)^4\} = \frac{3}{n(n+2)}, \tag{5.3}$$

and at the same time, that the $(n-1)$-dimensional volume itself is

$$A_{n-1} = \frac{2\pi^{(n/2)}}{\Gamma\left(\frac{n}{2}\right)}. \tag{5.4}$$

At the origin of normal coördinates, $g_{ij} = \delta^i{}_j$ and consequently $g = 1$. Upon expanding $\sqrt{g}$ in a series of powers of the x^i and substituting from (5.2) we have therefore

$$\sqrt{g} = 1 + \left[\frac{\partial \sqrt{g}}{\partial x^i}\right]_0 \xi^i \theta + \left[\frac{\partial^2 \sqrt{g}}{\partial x^i\, \partial x^j}\right]_0 \xi^i \xi^j \frac{\theta^2}{2} + \cdots. \tag{5.5}$$

The $(n-1)$-dimensional volume of the sphere is found by integrating this expression over the unit euclidean sphere and, multiplying by θ^{n-1}. In this process all terms of odd order in θ vanish, since they are multiplied by odd numbers of the ξ^i. We thus obtain

$$(5.6)\quad A_{n-1}\,\theta^{n-1}\left\{1+\frac{\theta^2}{2n}\Sigma\left[\frac{\partial^2\sqrt{g}}{(\partial x^i)^2}\right]_0 + \frac{\theta^4}{24n}\frac{3}{(n+2)}\left(\Sigma\left[\frac{\partial^4\sqrt{g}}{(\partial x^i)^4}\right]_0 + 2\sum_{i<j}\left[\frac{\partial^4\sqrt{g}}{(\partial x^i)^2(\partial x^j)^2}\right]_0\right)+\cdots\right\}.$$

In terms of normal coördinates we have from *RG* (18.8) and (18.9)

$$(5.7)\quad \left\{\begin{matrix} m \\ i\;j\end{matrix}\right\}_0 = 0,$$

$$(5.8)\quad \left(\frac{\partial\left\{\begin{matrix} m \\ i\;j\end{matrix}\right\}}{\partial x^k}+\frac{\partial\left\{\begin{matrix} m \\ k\;i\end{matrix}\right\}}{\partial x^j}+\frac{\partial\left\{\begin{matrix} m \\ j\;k\end{matrix}\right\}}{\partial x^i}\right)_0 = 0,$$

and further identities which we shall not use here, since we shall limit our consideration of the series (5.6) to evaluating the second term in invariant form. The Ricci tensor, *RG* (8.14), is

$$(5.9)\quad R_{ij}=\frac{\partial^2\log\sqrt{g}}{\partial x^i\,\partial x^j}-\frac{\partial\left\{\begin{matrix} k \\ i\;j\end{matrix}\right\}}{\partial x^k}+\left\{\begin{matrix} m \\ i\;k\end{matrix}\right\}\left\{\begin{matrix} k \\ m\;j\end{matrix}\right\}-\left\{\begin{matrix} m \\ i\;j\end{matrix}\right\}\frac{\partial\log\sqrt{g}}{\partial x^n}.$$

From *RG* (7.9), namely,

$$(5.10)\quad \frac{\partial\log\sqrt{g}}{\partial x^i}=\left\{\begin{matrix} k \\ k\;i\end{matrix}\right\},$$

we have

$$(5.11)\quad \frac{\partial^2\log\sqrt{g}}{\partial x^i\,\partial x^j}=\frac{\partial\left\{\begin{matrix} k \\ k\;i\end{matrix}\right\}}{\partial x^j}=\frac{\partial\left\{\begin{matrix} k \\ k\;j\end{matrix}\right\}}{\partial x^i}.$$

Putting $m=k$ in (5.8), summing for k, and using (5.11), gives

$$(5.12)\quad \left(\frac{\partial\left\{\begin{matrix} k \\ i\;j\end{matrix}\right\}}{\partial x^k}\right)_0=-2\left[\frac{\partial^2\log\sqrt{g}}{\partial x^i\,\partial x^j}\right]_0.$$

From (5.9), (5.12) and (5.7),

$$(5.13)\quad [R_{ij}]_0=3\left[\frac{\partial^2\log\sqrt{g}}{\partial x^i\,\partial x^j}\right]_0=3\left[\frac{\partial^2\sqrt{g}}{\partial x^i\,\partial x^j}\right]_0,$$

since $(\sqrt{g})_0=1$ and, by (5.10) and (5.7), $(\partial\sqrt{g}/\partial x^i)_0=0$. The scalar curvature of the space is defined as

$$(5.14)\quad R=g^{ij}R_{ij}.$$

Since $(g^{ij})_0 = \delta_j^i$, the value taken by R at O is $\Sigma(R_{ii})_0$. Combining this result with (5.13) and (5.6), we have for the $(n-1)$-dimensional volume,

$$(5.15) \qquad A_{n-1}\theta^{n-1}\left\{1 + \frac{R_0\theta^2}{6n} + \cdots\right\},$$

where R_0 is the scalar curvature at the center. Integrating with respect to θ we have as the n-dimensional volume enclosed by the sphere,

$$(5.16) \qquad \frac{\pi^{(n/2)}}{\Gamma\left(\frac{n+2}{2}\right)}\,\theta^n\left\{1 + \frac{R_0\theta^2}{6(n+2)} + \cdots\right\}.$$

The case $n = 2$ of these results is due to Bertrand and Diguet. Their results are obtained by putting $n = 2$, $R_0 = -2K$, where K is the Gaussian curvature of the surface, in (5.15) and (5.16).

6. Tubes in a general curved space. Referring once more to *RG* § 32, we make use of the special Schmidt orthogonal ennuple for which $\lambda^i_{1/} = dx^i/ds$ is the direction of the tangent to the curve C. Hence from the generalized Frenet formulae *RG* (32.16) for spaces of positive definite distance element,

$$(6.1) \qquad \lambda^i_{p/,j}\,\lambda^j_{1/} = \frac{\lambda^i_{p+1/}}{\rho_p} - \frac{\lambda^i_{p-1/}}{\rho_{p-1}},$$

where $\rho_1, \cdots, \rho_{n-1}$ are the successive curvatures of C, and $1/\rho_0 = 1/\rho_n = 0$. These equations hold in every coördinate system.

We shall denote by x^1 the arc distance along C from some fixed point, and define x^1 at other points of the space by the condition that it shall be constant on every geodesic perpendicular to C. We restrict attention to a region such that no two geodesics normal to C at points of the region meet again within the region, and every point of the region lies on such a geodesic. A point P of this region then lies on a unique geodesic normal to the curve C; let Q be the point at which this geodesic meets C, and let s be the geodesic distance QP. Let ξ^α be the cosine of the angle at Q between the direction QP and the vector $\lambda^i_{\alpha/}$ of the orthogonal set described above. We define [10] the α-th coördinate of P as $x^\alpha = \xi^\alpha s$. The equations of the geodesics QP in terms of the arc s as parameter are therefore

$$(6.2) \qquad x^1 = \text{constant}, \qquad x^\alpha = \xi^\alpha s.$$

Since the vectors of the Schmidt ennuple are mutually orthogonal and are tangent at points of C to the coördinate lines (i.e., the curves along each of

[10] It is important to bear in mind that in this section Greek indices take only the values 2, . . ., n, while Latin indices vary from 1 to n.

which only one coördinate varies), we have at points of C:

$$g_{ij} = \delta_j{}^i = g^{ij}, \qquad g = 1. \tag{6.3}$$

If we substitute the equations (6.2) in the differential equations (4.1) of geodesics, we obtain:

$$\left\{ \begin{matrix} i \\ \alpha\,\beta \end{matrix} \right\} \xi^\alpha \xi^\beta = 0.$$

These equations are valid throughout the region, though both the Christoffel symbols and the ξ^α depend on the point P of evaluation. But at a point of C the Christoffel symbols take on definite values, because of the continuity assumed at the end of § 1, while the last equations hold when any numbers whatever are substituted for the ξ^α. Since a quadratic form can vanish for all sets of values of the variables only if all the coefficients vanish, we must have at every point of C,

$$\left\{ \begin{matrix} i \\ \alpha\,\beta \end{matrix} \right\} = 0. \tag{6.4}$$

If we differentiate (4.1) with respect to s and substitute (6.2) we obtain similarly the equations, valid at all points of C,

$$\frac{\partial \left\{ \begin{matrix} i \\ \alpha\,\beta \end{matrix} \right\}}{\partial x^\gamma} + \frac{\partial \left\{ \begin{matrix} i \\ \gamma\,\alpha \end{matrix} \right\}}{\partial x^\beta} + \frac{\partial \left\{ \begin{matrix} i \\ \beta\,\gamma \end{matrix} \right\}}{\partial x^\alpha} = 0. \tag{6.5}$$

In what follows it will be understood that the expressions considered are evaluated on C. Since (6.4) holds at all points of C, irrespectively of the value of x^1, the derivative of the left member with respect to x^1 vanishes on the curve. In particular,

$$\frac{\partial \left\{ \begin{matrix} 1 \\ \alpha\,\beta \end{matrix} \right\}}{\partial x^1} = 0. \tag{6.6}$$

From the definition of the coördinates above it follows that the direction of the coördinate line along which only x^p varies must at points of C coincide with that of the unit vector $\lambda^i{}_{p/}$. Hence at such a point this coördinate line must satisfy

$$\frac{dx^i}{ds} = \lambda^i{}_{p/}.$$

But such a line must also by its very nature satisfy

$$\frac{dx^i}{ds} = \delta_p{}^i.$$

Therefore $\lambda^i{}_{p/} = \delta^i{}_p$ on C. Elsewhere $\lambda^i{}_{p/}$ has not been defined, but for convenience we define

$$\lambda^i{}_{p/} = \delta^i{}_p \tag{6.7}$$

at all other points of the space. The covariant derivative of this contravariant vector is by definition

$$\lambda^i{}_{p/,j} = \frac{\partial \lambda^i{}_{p/}}{\partial x^j} + \lambda_{p/}{}^k \left\{ \begin{matrix} i \\ k\, j \end{matrix} \right\}.$$

With (6.7) this gives

$$\lambda^i{}_{p/,j} = \left\{ \begin{matrix} i \\ p\, j \end{matrix} \right\}. \tag{6.8}$$

Substituting (6.7) and (6.8) in (6.1) we have, at points of C,

$$\left\{ \begin{matrix} i \\ p\, 1 \end{matrix} \right\} = \frac{\delta^i{}_{p+1}}{\rho_p} - \frac{\delta^i{}_{p-1}}{\rho_{p-1}}. \tag{6.9}$$

In particular,

$$\left\{ \begin{matrix} 1 \\ \alpha\, 1 \end{matrix} \right\} = - \frac{\delta^2{}_\alpha}{\rho_1}. \tag{6.10}$$

The components of the Ricci tensor (5.9) with subscripts $2, \cdots, n$ simplify on account of (6.4) and (6.6) to the form

$$R_{\alpha\beta} = \frac{\partial^2 \log \sqrt{g}}{\partial x^\alpha\, \partial x^\beta} - \frac{\partial \left\{ \begin{matrix} \gamma \\ \alpha\, \beta \end{matrix} \right\}}{\partial x^\gamma} + \left\{ \begin{matrix} 1 \\ \alpha\, 1 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 1\, \beta \end{matrix} \right\}. \tag{6.11}$$

The scalar curvature (5.14) may with the help of (6.3), (6.11) and (6.10) be expressed in the form

$$\begin{aligned} R &= R_{11} + g^{\alpha\beta} R_{\alpha\beta} \\ &= R_{11} + g^{\alpha\beta} \left(\frac{\partial^2 \log \sqrt{g}}{\partial x^\alpha\, \partial x^\beta} - \frac{\partial \left\{ \begin{matrix} \gamma \\ \alpha\, \beta \end{matrix} \right\}}{\partial x^\gamma} \right) + \frac{1}{\rho_1{}^2}. \end{aligned} \tag{6.12}$$

The *mean curvature* of the space at a point with respect to the direction $\lambda^i{}_{1/}$ of the curve is defined as

$$R' = R_{ij} \lambda^i{}_{1/} \lambda^j{}_{1/}.$$

In view of (6.7) this gives

$$R' = R_{11}. \tag{6.13}$$

The mean curvature with respect to any direction has the following geometrical meaning (*RG*, p. 113). With each of $n - 1$ directions orthogonal to the given direction and to each other, the given direction determines a pencil of geodesics forming a surface. The sum of the Gaussian curvatures of these $n-1$ surfaces is the negative of the mean curvature of the space for the given direction. From this we derive a geometrical interpretation of the scalar curvature. Since in normal coördinates

$$R = R_{11} + R_{22} + \cdots + R_{nn},$$

and since each term on the right is, like (6.13), the mean curvature with respect to a particular one of an orthogonal ennuple of directions, it follows that $-R$ is twice the sum of the Gaussian curvatures of all the $n(n-1)/2$ geodesic surfaces determined by these directions. If further we denote by S the scalar curvature of the hypersurface $x^1 =$ constant, we have from this interpretation that $-S$ is twice the sum of the Gaussian curvatures of those geodesic surfaces determined by the ennuple which lie in the hypersurface. From this it follows that

$$R = S + 2R'. \tag{6.14}$$

With reference to the hypersurface $x^1 =$ constant the components $g_{\alpha\beta}$ of the distance tensor have the same values as for the n-space. The same is therefore true of those Christoffel symbols whose indices have the values $2, \cdots, n$, and of the derivatives with respect to $x^2, \cdots, x^n$ of these symbols. We shall denote by h the $(n-1)$-rowed determinant of the $g_{\alpha\beta}$, and by $S_{\alpha\beta}$ the Ricci tensor of the hypersurface where it is pierced by C. Similarly to (5.11) we have:

$$\frac{\partial^2 \log \sqrt{h}}{\partial x^\alpha\, \partial x^\beta} = \frac{\partial \left\{ \begin{matrix} \gamma \\ \gamma\,\alpha \end{matrix} \right\}}{\partial x^\beta} = \frac{\partial \left\{ \begin{matrix} \gamma \\ \gamma\,\beta \end{matrix} \right\}}{\partial x^\alpha}. \tag{6.15}$$

If in (6.5) we replace i by γ and then sum with respect to γ from 2 to n we obtain with the help of (6.15)

$$\frac{\partial \left\{ \begin{matrix} \gamma \\ \alpha\,\beta \end{matrix} \right\}}{\partial x^\gamma} = -2\, \frac{\partial^2 \log \sqrt{h}}{\partial x^\alpha\, \partial x^\beta}. \tag{6.16}$$

The Ricci tensor $S_{\alpha\beta}$ is obtained from the right-hand member of (5.9) by replacing i, j, k, m respectively by $\alpha, \beta, \gamma, \delta$ and using h in places of g. With (6.4) and (6.16) this gives

$$S_{\alpha\beta} = -\frac{3}{2} \frac{\partial \left\{ \begin{matrix} \gamma \\ \alpha\,\beta \end{matrix} \right\}}{\partial x^\gamma}. \tag{6.17}$$

Since $S = g^{\alpha\beta} S_{\alpha\beta}$ we have from (6.17) and (6.14), (6.18) $$g^{\alpha\beta}\frac{\partial \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}}{\partial x^\gamma} = -\frac{2}{3}R + \frac{4}{3}R'.$$

Substituting this and (6.13) in (6.12) gives, after rearrangement,

(6.19) $$g^{\alpha\beta}\frac{\partial^2 \log\sqrt{g}}{\partial x^\alpha\,\partial x^\beta} = \frac{R+R'}{3} - \frac{1}{\rho_1{}^2}.$$

From (5.10), (6.10), (6.4) and (6.3) it follows that

$$\frac{\partial\sqrt{g}}{\partial x^\alpha} = -\frac{\delta_\alpha{}^2}{\rho_1}.$$

Hence (6.19) gives

(6.20) $$g^{\alpha\beta}\frac{\partial^2\sqrt{g}}{\partial x^\alpha\,\partial x^\beta} = \frac{R+R'}{3}.$$

For a fixed value of x^1 we may expand $\sqrt{g}$ in a series of powers of $x^2, \cdots, x^n$, replace x^α by $\xi^\alpha\theta$ to obtain a series resembling (5.5) but with Latin indices replaced by Greek, and then integrate over the $(n-2)$-dimensional volume of the sphere $\Sigma(\xi^\alpha)^2 = 1$. This gives for the volume element of a tube $\sqrt{G}dx^1\,d\theta$, where

$$\begin{aligned}\sqrt{G} &= A_{n-2}\,\theta^{n-2}M_{n-1}\{\sqrt{g}\}\\ &= A_{n-2}\,\theta^{n-2}\left[(\sqrt{g})_C + \theta\left(\frac{\partial\sqrt{g}}{\partial x^\alpha}\right)_C M_{n-1}\{\xi^\alpha\} + \frac{\theta^2}{2}\left(\frac{\partial^2\sqrt{g}}{\partial x^\alpha\,\partial x^\beta}\right)_C M_{n-1}\{\xi^\alpha\xi^\beta\} + \cdots\right].\end{aligned}$$

The symmetry considerations of § 5 together with (5.3), (5.4), (6.3) and (6.20) reduce this to

$$\sqrt{G} = \frac{2\pi^{(n-1)/2}\theta^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)}\left[1 + (R+R')\frac{\theta^2}{6(n-1)} + \cdots\right].$$

Integrating with respect to θ gives as the volume element of a tube of radius θ,

$$\frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\theta^{n-1}\left[1 + \frac{(R+R')\,\theta^2}{6\,(n+1)} + \cdots\right]dx^1.$$

It might have been thought on the basis of geometric visualization that this result could have been obtained from (5.16) by replacing n by $n-1$ and R_0 by S. But R_0 must be replaced, not by S, but by $R+R' = S+3R'$.

By an extension of the foregoing procedures it seems likely that a fairly straightforward calculation would give the terms of these series, and of the corresponding series (5.15) and (5.16), to any required degree. What is required is to express the symmetrical sums of higher derivativs of $\sqrt{g}$ in terms of invariants by formulae analogous to (6.20). Invariants available for the purpose are the higher covariant derivatives of the right-hand member of (6.20), and the contracted covariant derivatives of the Ricci tensor. It is a question of some interest whether the volume element of the tube also involves the various radii of curvature ρ_k of C. It does not involve them in either of the two cases we have examined fully, those of euclidean and of spherical space. If they do enter in other cases, these will doubtless call for the use of (6.9), a formula whose use could have been avoided in obtaining only the terms found above.

The conditions for non-overlapping in terms of the radius of first curvature found for euclidean and spherical spaces do not seem capable of generalization to arbitrary spaces. The condition applicable instead is that the radius of the tube shall be so small that no two geodesics through the curve and perpendicular to it shall meet again within the tube.

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