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COMPLEXITY OF BEZOUT'S THEOREM. I: GEOMETRIC ASPECTS

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CHAPTER I: THE MAIN RESULT AND THE STRUCTURE OF ITS PROOF

I-1. INTRODUCTION

This paper represents a step in the general program of establishing principles for solving nonlinear systems of equations efficiently.

Let $\mathscr{H}_{(d)}$ be the vector space of all homogeneous polynomial systems $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$ of degree $d = (d_1, \dots, d_n)$ (so that degree $f_i = d_i$).

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Consider the computational problem, given $f \in \mathscr{H}_{(d)}$, solve $f(\zeta) = 0$. What does this mean? A reasonable answer is: exhibit $x \in \mathbb{C}^{n+1}$ such that x is an approximate zero of f restricted to N_x, $f/N_x : N_x \to \mathbb{C}^n$ where $N_x = x + \{y \in \mathbb{C}^{n+1} / \langle y, x \rangle = 0\}$ and $\langle\ ,\ \rangle$ is the Hermitian inner product on $\mathbb{C}^{n+1}$. See $(*)$ of Theorem 1 of the next section for the precise definition of approximate zero. In particular, Newton's method for f/N_x, starting at x, converges quadratically to some $\zeta \in N_x$ with $f(\zeta) = 0$, and ε relative accuracy is obtained with $\log|\log \varepsilon|$ further steps.

N_x is the Hermitian orthogonal complement to the vector $x \in \mathbb{C}^{n+1}$ through x, and can be thought of as the tangent space to complex projective n-space. In order to apply Newton's method, f is restricted to an n-dimensional subspace. The choice of N_x is natural from the projective space point of view and optimizes some of our estimates. That x is an approximate zero of f/N_x is invariant under scaling; i.e., λx is an approximate zero of $f/N_{\lambda x}$, $\lambda \neq 0$, if x is an approximate zero of f/N_x. Thus we say x is an *approximate zero* of f in the *projective sense* with associated actual zero ζ.

A curve $F : [0, 1] \to \mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ satisfying $f_t(\zeta_t) = 0$, $F_t = (f_t, \zeta_t)$ is called a *homotopy-path.*

An important computational problem is to produce from the input (f_t) and an approximation z_0 of ζ_0, a sequence z_i, $i = 1, \dots, k$, which *fits* (ζ_t) in the following sense. Each z_i is an approximate zero of f_{t_i} in the projective sense with associated actual zero ζ_{t_i}, $0 = t_0 < \cdots < t_{i-1} < t_i < \cdots < t_k = 1$.

Projective Newton's method proposed by Shub [24] yields z_i from z_{i-1} by applying Newton's method to $f_{t_i}/N_{z_{i-1}}$. The problem we deal with here is how small can k be taken to obtain $z_1, \dots, z_k$ fitting (ζ_t).

The answer we demonstrate is that the controlling factor is the distance along $P(\mathscr{H}_{(d)})$, $\rho(F)$, in the corresponding projective spaces, of the curve F_t in $\mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ to the discriminant locus Σ' (an irreducible algebraic variety of ill-posed problems). The other factors are the length L of the curve (f_t), a modest contribution from the degree of f and a small constant.

More precisely:

Main Theorem. *Let $F_t = (f_t, \zeta_t)$ be a homotopy-path in $\mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$, and let z_0 satisfy*

$$\frac{\|z_0 - \zeta_0\|}{\|\zeta_0\|} \leq \frac{C_1 \rho(F)}{D^{3/2}}, \qquad C_1 = .035\dots.$$

Let

$$l > \frac{C_2 L D^{3/2}}{\rho(F)^2}, \qquad C_2 = 8.35\dots;$$

then l projective Newton steps are enough to produce $z_1, \dots, z_l$ fitting (ζ_t). Here $D = \max(d_i)$.

The number of variables n does not enter directly into the complexity l, but $\zeta(F) \leq \sqrt{n}$ so it is implicitly there.

The problem of finding the starting point can be dealt with by choosing a universal $f_{(d)} \in \mathscr{H}_{(d)}$, or from aspects of the particular problem. The invariant $\rho(F)$ needs to be studied from a geometric probability point of view. Part II is devoted to these problems.

For the problem of fitting *all* the solution curves of given f_t, we find similar results, using the distance in $\mathscr{H}_{(d)}$ to the discriminant locus Σ. This will lead to the complexity of algorithms for finding all the zeros of a given system $f \in \mathscr{H}_{(d)}$. Moreover, we provide a similar complexity analysis for the (nonhomogeneous) general polynomial system $f : \mathbb{C}^n \to \mathbb{C}^n$ using a more traditional (yet global) form of Newton's method.

One novel feature in our development is unitary invariance. For example, if U is a unitary transformation then x is an approximate zero of f in the projective sense iff $U(x)$ is an approximate zero of $f \circ U^{-1}$ in the projective sense. In general, one would not expect the particular coordinate representation of the polynomial system to be reflected in the basic invariants of the theory. The distances of solutions to each other seem basic. Our way of dealing with this is by using a fully unitarily invariant theory. This has an added feature of forcing a more elegant development of the mathematics.

Our proof of the main theorem puts into a very general setting theorems of Eckart-Young, Houth and Demmel on the condition number and the reciprocal of the distance to the algebraic variety of ill-posed problems.

The most important work on this problem previously is that of Renegar [20]. That paper was very helpful to us.

Very roughly, work on algorithms for Bezout's problem can be divided into two distinct schools. One is algebraic, represented for example by Brownawell [2], Grigoriev [9], Heintz [10], Canny [3], Renegar [21], and Ierardi [12], and a second more numerical analysis approach represented here. The algebraic algorithms tend to be less numerically stable (see, e.g., Morgan [18]).

The convergence and practice of path-following algorithms (or homotopy methods) may be seen for example in Allgower-Georg [1], Garcia-Zangwill [7], Hirsch-Smale [11], Keller [13], Li-Sauer-Yorke [16], Morgan [17], Wright [34], and Zulehner [35]. One variable complexity results on these algorithms is in Shub-Smale [25, 26] and Smale [27, 28].

The proof of the Main Result is quite long. The rest of Chapter I is devoted to giving the structure of this proof by displaying some intermediate theorems. In fact, in these theorems there is an effort to isolate some main concepts.

We would like to thank Matt Grayson for his calculations of some of our constants.

I-2. Complexity of path following in Banach spaces

Here we state some general results on complexity which are valid in a wide setting, yet form the framework of the main proofs on the complexity of Bezout's Problem. These ideas revolve about an invariant $\alpha(f, x)$ proposed in Smale [29, 30]. Subsequent work of Royden [23], Wang-Han [32] sharpened and broadened these first results, and that work is incorporated into the present treatment, Theorems 1 and 2 below. Moreover, robustness results of the

α-theory, only suggested in Smale [30] and in Renegar-Shub [22], are formulated in Theorems 3 and 4. Besides applications to polynomial systems, these theorems may be used to analyze complexity of linear programming algorithms.

It turns out profitable to give a complete demonstration of both the old and new results of this α-theory together. Thus the theorems stated in this section are proved in Chapter II. For some motivation and broader perspective one can see Smale [30] as well as the other references.

Throughout this section and Chapter II, $\mathbb{E}$ and $\mathbb{F}$ denote Banach spaces. In the main applications they are both $\mathbb{C}^m$, m-dimensional complex Cartesian space (or subspaces) with a norm defined by the standard Hermitian inner product.

We consider analytic maps $f : D_r(x_0) \to \mathbb{F}$ where $x_0 \in \mathbb{E}$ and $D_r(x_0) = \{x \in \mathbb{E} | \, \|x - x_0\| \le r\}$. For $x \in D_r(x_0)$ let

$$Df(x) : \mathbb{E} \to \mathbb{F}$$

denote the derivative of f at x (see Lang [15] for our way of doing calculus). If $Df(x)$ is not an isomorphism all the following α, β, γ are ∞ (or not defined). Otherwise define for $f : D_r(x_0) \to \mathbb{F}$ and $x \in D_r(x_0)$

$$\begin{aligned} \beta(f, x) &= \|Df(x)^{-1} f(x)\|, \\ \gamma(f, x) &= \sup_{k>1} \left\| \frac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}, \\ \alpha(f, x) &= \beta(f, x)\gamma(f, x). \end{aligned}$$

Newton's method (when defined) constructs a sequence of points $x_1, x_2, \ldots$ in $D_r(x_0)$ by the formula

$$x_n = x_{n-1} - Df(x_{n-1})^{-1} f(x_{n-1}), \qquad n = 1, \ldots .$$

We also write

$$N_f(x) = x - Df(x)^{-1} f(x) \quad \text{and} \quad x_n = N_f(x_{n-1}).$$

Thus $\beta(f, x_n) = \|x_{n+1} - x_n\|$.

We also frequently use these quantities:

$$\tau(\alpha) = \frac{(1+\alpha) - \sqrt{(1+\alpha)^2 - 8\alpha}}{4}, \quad \text{for } 0 \le \alpha \le 3 - 2\sqrt{2} \sim .1715$$

and

$$\alpha_0 = \frac{1}{4}(13 - 3\sqrt{17}) \sim .157671.$$

Theorem 1. *Let* $f : D_r(x_0) \to \mathbb{F}$ *be analytic,* $\beta = \beta(f, x_0)$, $\gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ *and suppose* $r \ge \frac{\tau(\alpha)}{\gamma}$. *Then if* $\alpha \le \alpha_0$, *the Newton iterates* $x_1, x_2, \ldots$ *are defined, converge to* $\zeta \in D_r(x_0)$ *with* $f(\zeta) = 0$ *and for all* $n \ge 1$

$$(*) \qquad \|x_{n+1} - x_n\| \le \left(\frac{1}{2}\right)^{2^n - 1} \|x_1 - x_0\|.$$

Moreover $\|\zeta - x_0\| \le \frac{\tau(\alpha)}{\gamma}$, *and* $\|\zeta - x_1\| \le \frac{\tau(\alpha)-\alpha}{\gamma}$.

A point $x_0 \in \mathbb{E}$ is called an *approximate zero* (of f) if $(*)$ is satisfied. In that case ζ is called the *associated* (true) *zero*. The following is an easy consequence of Theorem 1.

Corollary. *Let* $f : \mathbb{E} \to \mathbb{F}$ *be analytic,* $x_0 \in \mathbb{E}$ *satisfy* $\alpha = \alpha(f, x_0) \le \alpha_0$ *and have associated zero* ζ. *Then the nth Newton iterate* z_n *of* z_0 *is within* ε *of* ζ *provided* $n \ge (\log|\log \frac{\tau(\alpha)}{\varepsilon\gamma}|) + 1$.

Remark. The $\frac{1}{2}$ in Theorem 1 may be replaced by any λ, $0 < \lambda < 1$, with α_0 redefined. See §II-1.

Consider as an example the following family of real-valued functions (which have a universal quality as we will see):

$$h_{\beta,\gamma}(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \qquad \beta, \gamma > 0.$$

Let $\alpha = \beta\gamma$ satisfy $(\alpha + 1)^2 - 8\alpha > 0$ or equivalently $0 < \alpha < 3 - 2\sqrt{2}$. Then $h_{\beta,\gamma}$ has two distinct real positive roots at

$$\frac{\tau(\alpha)}{\gamma} = \frac{(\alpha + 1) \pm \sqrt{(\alpha + 1)^2 - 8\alpha}}{4\gamma}.$$

Moreover $d^2 h_{\beta,\gamma}(t)/dt^2 > 0$ as long as $0 < t < \frac{1}{\gamma}$. Thus Newton's method starting at 0 converges to the smaller root since by convexity the Newton sequence is monotone. Let $t_n = N_{h_{\beta,\gamma}}(t_{n-1})$ where $t_0 = 0$.

Theorem 2 (Domination Theorem). *Let* $f : D_r(x_0) \to \mathbb{F}$ *be analytic,* $\beta = \beta(f, x_0)$, $\gamma = \gamma(f, x_0)$, $\alpha = \beta\gamma$ *and suppose* $r \ge \frac{\tau(\alpha)}{\gamma}$ *and* $\alpha \le \alpha_0$. *These values of* β, γ *define* $h_{\beta,\gamma}$, *and the sequence* t_n. *Then*

$$\|x_n - x_{n-1}\| \le t_n - t_{n-1}, \qquad n = 1, 2, \dots,$$

where x_n *is the Newton sequence of* f *starting at* x_0.

The Domination Theorem yields the last sentence of Theorem 1 as follows.

$$\|x_0 - \zeta\| \le \sum_{n=0}^{\infty} \|x_{n+1} - x_n\| \le \sum_{n=0}^{\infty} t_{n+1} - t_n = \frac{\tau(\alpha)}{\gamma},$$

$$\|x_1 - \zeta\| \le \sum_{n=1}^{\infty} \|x_{n+1} - x_n\| \le \sum_{n=1}^{\infty} t_{n+1} - t_n = \frac{\tau(\alpha)}{\gamma} - \beta. \quad \square$$

We next deal with the question, "How does α vary with the initial condition?"

Let

$$\psi(u) = 2u^2 - 4u + 1, \qquad 0 \le u \le 1 - \frac{\sqrt{2}}{2},$$

so that $0 \le \psi(u) \le 1$.

Proposition 1. *Let* $f : D_r(x_0) \to \mathbb{F}$ *be analytic. For* $x \in D_r(x_0)$, *we have*

$$\alpha(f, x) \le \frac{\alpha(f, x_0)(1-u)+u}{\psi(u)^2}$$

where $u = \gamma(f, x_0)\|x_0 - x\|$ *and* $u < 1 - \frac{\sqrt{2}}{2}$.

Proposition 1 plays a role in the proof of the following result which permits repeated applications of Newton's method.

Theorem 3. *There are universal constants*

$$\bar{\alpha} \text{ about } .08019667, \qquad \bar{u} \text{ about } .02207$$

with this property. Let $f : D_r(\zeta) \to \mathbb{F}$ *be analytic with* $\gamma = \gamma(f, \zeta) \le \bar{\gamma}$ *(some constant),* $\beta = \beta(f, \zeta) \le \frac{\bar{\alpha}}{\bar{\gamma}}$, *and* $r \ge \frac{\tau(\alpha)}{\bar{\gamma}}$. *Suppose* $x \in D_r(\zeta)$ *satisfies* $\|x - \zeta\| \le \frac{\bar{u}}{\bar{\gamma}}$ *and* ζ_1 *is the associated zero of* ζ. *Then* $\|x_1 - \zeta_1\| \le \frac{\bar{u}}{\bar{\gamma}}$, *where* $x_1 = N_f(x)$.

Theorem 3 can be readily seen to have global implications. A *homotopy* $f_t : \mathbb{E} \to \mathbb{F}$ is a family of analytic maps $0 \le t \le 1$ with the induced map $[0, 1] \times \mathbb{E} \to \mathbb{F}$ continuous. An *associated path* is a continuous map $[0, 1] \to \mathbb{E}$, $t \to \zeta_t$, satisfying for each $t \in [0, 1]$, (a) $f_t(\zeta_t) = 0$ and (b) the derivative $Df(\zeta_t) : \mathbb{E} \to \mathbb{F}$ is an isomorphism. Sometimes we call $\{f_t, \zeta_t\}$ a *homotopy-path.*

The central algorithm in this paper (as used in Smale [28]) is designed to follow a path associated to a homotopy and works this way. To a subdivision $T = \{t_0 = 0, t_1, \dots, t_k = 1\}$, $t_i < t_{i+1}$, $|T| = k$, x_0 with $\|x_0 - \zeta_0\| < \delta$, define inductively by Newton's method

$$(*) \qquad x_i = N_{f_{t_i}}(x_{i-1}), \qquad i = 1, \dots, k.$$

If $|t_i - t_{i-1}|$ and $\|x_0 - \zeta_0\|$ are small enough, then $\|x_i - \zeta_{t_i}\|$ is small for all $i = 1, \dots, k$. More precisely we will say that *Newton's method follows the homotopy-path* $\{f_t, \zeta_t\}$, relative to T and δ provided x_i of $(*)$ is well defined, $\alpha(f_{t_i}, x_i) < \alpha_0$, and ζ_{t_i} is the associated actual zero to the approximate zero x_i of f_{t_i}, $i = 1, \dots, |T|$.

Note that in this case the number of Newton steps to reach an ε-approximation of the zero ζ_1 of f_1 is given by

$$|T| + \log\log \frac{\tau(\alpha)}{\varepsilon\gamma}$$

where $\alpha = \alpha(f_1, x_1)$, $\gamma = \gamma(f_1, x_1)$. The central complexity measure is $|T|$.

The most important example of a homotopy is a linear homotopy: $f_t = tf_1 + (1-t)f_0$, $f_0, f_1 : \mathbb{E} \to \mathbb{F}$. Often in this case a zero (or all the zeros) of f_0 is known.

Theorem 4. *Let* $F = \{f_t, \zeta_t\}$ *be a homotopy with an associated path as above. Let* $\Delta = \frac{1}{k}$, k *an integer, and* $\bar{\gamma} > 0$ *be such that* $\beta(f_{t'}, \zeta_t) \le \frac{\bar{\alpha}}{\bar{\gamma}}$ *and* $\gamma(f_{t'}, \zeta_t) \le \bar{\gamma}$ *if* $|t' - t| \le \Delta$.

Let $\|z_0 - \zeta_0\| \le \frac{\bar{u}}{\bar{\gamma}}$. *Then if* $T = \{0, \Delta, 2\Delta, \ldots, k\Delta = 1\}$, *Newton's method follows* F. *In fact*

$$\|z_i - \zeta_{t_i}\| \le \frac{\bar{u}}{\bar{\gamma}}, \qquad i = 1, \ldots, k.$$

The proof of Theorem 4 from Theorem 3 is practically apparent. One uses only the appropriate continuity of the associated zero which comes from Theorem 1.

Theorem 4 indicates the importance of estimating $\beta(f_{t'}, \zeta_t)$, $\gamma(f_{t'}, \zeta_t)$ and much of the rest of this paper is devoted to just that.

I-3. Complexity for polynomial systems in terms of the condition number, μ

We start this section with some general background on polynomial systems. It turns out that the affine (usual) and projective developments shed light on each other, and in fact we treat, in part, the affine problem in a homogeneous context. In both cases a representation of the unitary group plays an important role in our study.

Subsequently, the condition number $\mu = \mu(f, x)$ for $f : \mathbb{C}^n \to \mathbb{C}^n$, $x \in \mathbb{C}^n$, is introduced. This is a modified version of the simple $\|Df(x)^{-1}\|$. Our condition number must assume a more technical definition for several reasons, mainly related to natural scalings.

The algorithms to follow a path of a homotopy are modeled on that of Theorem 4 (previous section) and we are able to estimate the appropriate α invariants in terms of the degree and the condition number of the homotopy. The passage from Theorem 3 to Theorem 4 in the previous section gives the underlying idea of how we then obtain complexity results. As usual, this section is on the overall structure with full proofs in Chapter III.

We turn to describing spaces of polynomial systems together with a unitarily invariant metric. This metric while natural and used in the theory of group representations (Stein-Weiss [31]) is not traditional in the numerical analysis literature of equations. It has been suggested by Kostlan [14] and seems to be well suited to purposes of complexity, and corresponding estimates appear to be more elegant. Unitary invariance plays a central role in our approach to complexity.

We use $\mathscr{P}_{(d)}$ to denote the linear space of all polynomial systems $f : \mathbb{C}^n \to \mathbb{C}^n$, $f = (f_1, \ldots, f_n)$, each f_i a polynomial of n-variables of degree $\le d_i$, and $d = (d_1, \ldots, d_n)$, $d_i \ge 0$.

Let $\mathscr{H}_{(d)}$ be the homogeneous counterpart. That is, $f \in \mathscr{H}_{(d)}$ is a map $\mathbb{C}^{n+1} \to \mathbb{C}^n$ of the form $f = (f_1, \ldots, f_n)$ where each f_i is a homogeneous polynomial of degree exactly d_i. We suppose $0 \in \mathscr{H}_{(d)}$ so that $\mathscr{H}_{(d)}$ is a linear space.

Note that there is a natural linear isomorphism $\Phi : \mathscr{P}_{(d)} \to \mathscr{H}_{(d)}$ given by homogenization as follows. Let $f = (f_1, \dots, f_n) \in \mathscr{P}_{(d)}$, so that

$$f_i(z_1, \dots, z_n) = \sum_{\substack{|\alpha| \le d_i \\ \alpha = (\alpha_1, \dots, \alpha_n)}} a_\alpha z^\alpha$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $|\alpha| = \sum \alpha_i$. Then $\Phi(f) = (\Phi_1(f_1), \dots, \Phi_n(f_n))$ with $\Phi_i(f_i)(z_0, \dots, z_n) = \sum_\alpha a_\alpha z^\alpha z_0^{d_i - |\alpha|}$. The inverse of Φ is given by setting $z_0 = 1$.

For homogeneous polynomials $g, f : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d, let

$$\langle f, g \rangle_d = \sum_{|\alpha| = d} a_\alpha \bar{b}_\alpha \left(\frac{\alpha_1! \cdots \alpha_n!}{d!} \right)$$

where $f(z) = \sum a_\alpha z^\alpha$, $g(z) = \sum b_\alpha z^\alpha$. This induces a Hermitian inner product on $\mathscr{H}_{(d)}$. Simply write, for $f, g \in \mathscr{H}_{(d)}$,

$$\langle f, g \rangle = \sum_i \langle f_i, g_i \rangle_{d_i}.$$

Proposition 1 (Kostlan [14]). *Let the unitary group act on $\mathbb{C}^{n+1}$ in the canonical way and on $\mathscr{H}_{(d)}$ by the induced representation. Then $\langle\ \rangle$ on $\mathscr{H}_{(d)}$ is invariant.*

In other words $\langle fu^{-1}, gu^{-1} \rangle = \langle f, g \rangle$ for all $f, g \in \mathscr{H}_{(d)}$, and $u : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ unitary. Of course $\|fu^{-1}\| = \|f\|$.

By the isomorphism $\Phi : \mathscr{P}_{(d)} \to \mathscr{H}_{(d)}$ we obtain an induced Hermitian structure on $\mathscr{P}_{(d)}$. We will denote the corresponding norm on f by simply $\|f\|$ for each $\mathscr{P}_{(d)}$ and $\mathscr{H}_{(d)}$. We sometimes use the same symbol for $f \in \mathscr{P}_{(d)}$ and $\Phi(f) \in \mathscr{H}_{(d)}$.

If $\mathbb{E}$ is a linear space over $\mathbb{C}$, let $P(\mathbb{E})$ denote the corresponding projective space of lines through the origin in $\mathbb{E}$. So $P(\mathbb{E}) = (\mathbb{E} - 0)/\mathbb{C}^*$, $\mathbb{C}^* = \mathbb{C} - 0$. A Hermitian structure on $\mathbb{E}$ induces a Riemannian structure on $P(\mathbb{E})$ (of constant curvature). Thus we have canonical metrics on $P(\mathscr{P}_{(d)})$, $P(\mathscr{H}_{(d)})$. Sometimes for $f \in \mathscr{H}_{(d)}$, the same symbol will denote the corresponding element of $P(\mathscr{H}_{(d)})$.

We define, for $x \in \mathbb{C}^{n+1}$,

$$\mathrm{Null}(x) = \{v \in \mathbb{C}^{n+1}, \langle v, x \rangle = 0\}$$

and an affine subspace

$$N_x = x + \mathrm{Null}(x) \subset \mathbb{C}^{n+1}.$$

Let $e_0 = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$.

We will use the notation $\Delta(y_i)$ to mean the diagonal matrix whose ith element is y_i.

We are ready to define the condition number $\mu(f, x)$ of $f \in \mathscr{P}_{(d)}$ at $x \in \mathbb{C}^n$. The idea is to take μ to be $\|Df(x)^{-1}\|$, e.g., as in Wilkinson [33], but it is important to take into account the special polynomial nature of f. We also want to make μ compatible with homogenization. Moreover, for sharper estimates on complexity it is convenient to have a further factor of $d_i^{\frac{1}{2}}$. Thus define

$$\mu(f, x) = \|f\| \, \|Df(x)^{-1}\Delta((d_i)^{\frac{1}{2}}\|x\|_1^{d_i-1})\|$$

or 1, whichever is larger, and where $\|x\|_1 = (\sum_1^n x_i^2 + 1)^{\frac{1}{2}}$.

If f is treated as an element of $\mathscr{H}_{(d)}$ this is equivalent to

$$\mu(f, x) = \|f\| \, \|Df|_{N_{e_0}}(x)^{-1}\Delta(d_i^{\frac{1}{2}}\|x\|^{d_i-1}) \text{ or } 1$$

where $x = (x_0, \ldots, x_n)$, $x_0 = 1$.

We will use the two versions interchangeably.

The projective version of the condition number is: For $f \in \mathscr{H}_{(d)}$ and $x \in \mathbb{C}^{n+1}$,

$$\mu_{\text{proj.}}(f, x) = \|f\| \, \|Df|_{N_x}(x)^{-1}\Delta(d_i^{\frac{1}{2}}\|x\|^{d_i-1})\| \text{ or } 1$$

whichever is larger.

For $f \in \mathscr{H}_{(d)}$, let

$$\eta(f, x) = \frac{\|\Delta(d_i^{-\frac{1}{2}}\|x\|^{-d_i})f(x)\|}{\|f\|}.$$

For $f \in \mathscr{P}_{(d)}$, $\eta(f, x)$ is the same except $\|x\|$ is replaced by $\|x\|_1$.

Let $f \in \mathscr{P}_{(d)}$, $x \in \mathbb{C}^n$. Appropriate versions of $\beta(f, x)$, $\gamma(f, x)$ are

$$\beta_0(f, x) = \frac{\beta(f, x)}{\|x\|_1},$$
$$\gamma_0(f, x) = \gamma(f, x)\|x\|_1.$$

Thus $\alpha(f, x) = \beta_0(f, x)\gamma_0(f, x)$.

The projective case goes as follows. For $f \in \mathscr{H}_{(d)}$, $x \in \mathbb{C}^{n+1}$,

$$\beta_0(f|_{N_x}, x) = \beta(f|_{N_x}, x)/\|x\|,$$
$$\gamma_0(f|_{N_x}, x) = \gamma(f|_{N_x}, x)\|x\|.$$

These definitions are invariant under scalings of f and x, so they make sense on $P(\mathscr{H}_d)$ and $P(\mathbb{C}^{n+1})$.

Proposition 2. *For* $f \in \mathscr{P}_{(d)}$, $x \in \mathbb{C}^n$,

$$\beta_0(f, x) \leq \mu(f, x)\eta(f, x).$$

For $f \in \mathscr{H}_{(d)}$, $x \in \mathbb{C}^{n+1}$,

$$\beta_0(f|_{N_x}, x) \leq \mu_{\text{proj.}}(f, x)\eta(f, x).$$

The proof of Proposition 2 is obtained by putting together the definitions with $\|Ab\| \leq \|A\| \, \|b\|$.

In Chapter III we will show

Proposition 3.

$$\gamma_0(f, x) \le \frac{\mu(f, x) D^{\frac{3}{2}}}{2}, \qquad f \in \mathscr{P}_d, \ x \in \mathbb{C}^n,$$

$$\gamma_0(f|_{N_x}, x) \le \frac{\mu_{\text{proj.}}(f, x) D^{\frac{3}{2}}}{2}, \qquad f \in \mathscr{H}_d, \ x \in \mathbb{C}^{n+1}.$$

For $f, g \in \mathscr{H}_{(d)}$, let $d_P(f, g) = \min_{\lambda \in \mathbb{C}} \frac{\|f - \lambda g\|}{\|f\|}$. $d_P(f, g)$ is independent of scaling of both f and g and defines a distance function on $P(\mathscr{H}_{(d)})$. To see that this function is in fact a metric we compare it to the standard metric on $P(\mathscr{H}_{(d)})$. There is up to scaling a unique unitarily invariant Hermitian metric on $P(\mathscr{H}_{(d)})$. One way to get the existence of one is simply to restrict the Hermitian structure on $\mathscr{H}_{(d)}$ to $\text{Null}(f)$ at each f. This Hermitian structure defines a unique Riemannian structure and a distance $d_R(f, g)$ for $f, g \in P(\mathscr{H}_{(d)})$.

Proposition 4. $d_P(f, g) = \sin d_R(f, g)$ *for* $f, g \in P(\mathscr{H}_{(d)})$.

Proof.

$$d_P(f, g) = \min_{\lambda \in \mathbb{C}} \frac{\|f - \lambda g\|}{\|f\|} = \frac{\|f - \frac{\langle f, g\rangle g}{\langle g, g\rangle}\|}{\|f\|} = \left(1 - \frac{|\langle f, g\rangle|^2}{\|f\|^2 \|g\|^2}\right)^{\frac{1}{2}}$$

by expanding the norm in the numerator as a Hermitian product. Now

$$\left(1 - \frac{|\langle f, g\rangle|^2}{\|f\|^2 \|g\|^2}\right)^{1/2} = \sin \arccos \frac{|\langle f, g\rangle|}{\|f\| \, \|g\|}$$

so we have only to see that $d_R(f, g) = \arccos \frac{|\langle f, g\rangle|}{\|f\| \, \|g\|}$. To see this last, use unitary invariance and the uniqueness of unitary structure on $\mathbb{C}^N$ up to isomorphism. We may assume that d_R and $\arccos \frac{|\langle f, g\rangle|}{\|f\| \, \|g\|}$ are defined on $\mathbb{C}P(1)$ corresponding to $\mathbb{C}^2$ spanned by f and g. Moreover in affine coordinates we may assume $f = (1, 0)$, $g = (1, x_0)$ where the metric (see Mumford [19]) is

$$ds^2 = \frac{dx\, d\bar{x}}{1 + |x|^2} - \frac{(x\, d\bar{x})(\bar{x}\, dx)}{(1 + |x|^2)^2} = \frac{dx\, d\bar{x}}{(1 + |x|^2)^2}.$$

Now integrate on the path $(1, tx_0)$ for $0 \le t \le 1$.

$$\int_0^1 \frac{|x_0|}{1 + t^2 |x_0|^2} dt = \arctan |x_0| = \arccos\left(\frac{1}{(1 + |x_0|^2)^{1/2}}\right)$$

which verifies the formula in this case. Note that to see that d_P is a metric it is enough to note that $\sin(A + B) \le \sin A + \sin B$ for $0 \le A, B \le \frac{\pi}{2}$.

The diameter of $P(\mathscr{H}_{(d)})$ equals one with d_P (and equals $\frac{\pi}{2}$ for d_R). Note $d_P \le d_R$.

Proposition 5. *Let* $f, g \in \mathscr{H}_{(d)}$, $\zeta \in \mathbb{C}^{n+1}$. *Then*

$$\text{(a)} \quad \mu(g, \zeta) \le \frac{\mu(f, \zeta)(1 + d_P(f, g))}{1 - D^{1/2} d_P(f, g)\mu(f, \zeta)} \quad \textit{for } \zeta_0 = 1,$$

$$\text{(b)} \quad \mu_{\text{proj.}}(g, \zeta) \le \frac{\mu_{\text{proj.}}(f, \zeta)(1 + d_P(f, g))}{1 - D^{1/2} d_P(f, g)\mu_{\text{proj.}}(f, \zeta)} \quad \textit{for } \zeta \ne 0$$

as long as the denominators are positive.

Remark. $D^{1/2}$ in (b) may be omitted by a different proof.

For a while now we restrict ourselves to the affine case. So in §I-2, $\mathbb{E}$ and $\mathbb{F}$ both become $\mathbb{C}^n$ and a homotopy $f_t : \mathbb{C}^n \to \mathbb{C}^n$ is a (continuous) curve in $\mathscr{P}_d$. Then ζ_t is a curve in $\mathbb{C}^n$ with $f_t(\zeta_t) = 0$ and $Df_t(\zeta_t) : \mathbb{C}^n \to \mathbb{C}^n$ nonsingular.

Theorem 1. *Suppose* $C_0 \in [0, 1]$ *and* $\{f_t, \zeta_t\}$ *is a homotopy-path in* $\mathscr{P}_{(d)} \times \mathbb{C}^n$, $t, t' \in [0, 1]$, $\mu = \mu(f_t, \zeta_t)$ *with*

$$d_P(f_{t'}, f_t) \le 2C_0 \left(\frac{1 - C_0}{1 + C_0}\right)^2 \frac{1}{D^{3/2}\mu^2}.$$

Let

$$\bar{\gamma} = \frac{D^{3/2}}{2}\mu\left(\frac{1 + C_0}{1 - C_0}\right).$$

Then

$$\beta(f_{t'}, \zeta_t)\bar{\gamma} \le C_0,$$

and

$$\gamma(f_{t'}, \zeta_t) \le \gamma_0(f_{t'}, \zeta_t) \le \bar{\gamma}.$$

For example one could take C_0 as α_0 or $\bar{\alpha}$ from the previous section. If $C_0 = \bar{\alpha}$, then

$$2C_0 \left(\frac{1 - C_0}{1 + C_0}\right)^2 = \alpha^*$$

is about .11629.

We give the short proof here (assuming the previous propositions). Choose in Propositions 2 and 3, $f = f_{t'}$, and $x = \zeta_t$. Then

$$\beta(f_{t'}, \zeta_t) \le \mu(f_{t'}, \zeta_t)\eta(f_{t'}, \zeta_t).$$

Lemma 1 (easy).

$$\eta(f_{t'}, \zeta_t) \le d_P(f_{t'}, f_t) = \Delta_P.$$

By Proposition 5

$$\mu(f_{t'}, \zeta_t) \le \frac{\mu(1 + \Delta_P)}{(1 - D^{1/2}\Delta_P\mu)}$$

so we obtain using the lemma,

$$\beta(f_{t'}, \zeta_t) \le \mu\Delta_P\left(\frac{1 + \Delta_P}{1 - D^{1/2}\Delta_P\mu}\right).$$

Lemma 2.

$$\frac{1+\Delta_P}{1-D^{1/2}\Delta_P\mu} \leq \frac{1+C_0}{1-C_0}.$$

Proof. Using the hypothesis on Δ_P and the fact that $D \geq 2$, $\mu \geq 1$, one sees that $\mu D^{1/2}\Delta_P \leq C_0$. Then $\Delta_P \leq C_0$ and the lemma is proved.

The estimate on β and γ of Theorem 1 now follows by making the appropriate substitutions.

We now will give an estimate of the number of steps of the algorithm of §I-2 described right after Theorem 3. For a homotopy-path $F = \{f_t, \zeta_t\}$ in $\mathscr{P}_{(d)} \times \mathbb{C}^n$, define $L = L(F)$ to be the length in the metric d_P of the curve f_t, $0 \leq t \leq 1$. Define the condition number

$$\mu = \mu(F) = \max_{0 \leq t \leq 1} \mu(f_t, \zeta_t).$$

Note that μ takes different meanings in different contexts.

Theorem 2. *Let $F = \{f_t, \zeta_t\}$ be a homotopy-path in $\mathscr{P}_{(d)} \times \mathbb{C}^n$. Let*

$$k \geq \frac{LD^{3/2}}{\alpha^*}\mu^2.$$

Then k Newton steps are sufficient to follow the path ζ_t, $[0 \leq t \leq 1]$, in the sense of §I-2.

Again we give the short proof which follows from Theorem 1, but first note:

Remark. For the main case of a linear homotopy $f_t = tf + (1-t)f_0$, recall that L is less than or equal to the diameter of projective space, which is 1.

Proof. One may choose t_i so that

$$d_P(f_{t_i}, f_{t_{i-1}}) \leq \frac{L}{k}, \qquad i = 1, \dots, k.$$

Then since $\frac{L}{k} \leq \alpha^*/D^{3/2}\mu$, the hypotheses of Theorem 1 are satisfied where $C_0 = \bar{\alpha}$. The conclusions of Theorem 1 put us into the situation of Theorem 3 of §I-2 and this finishes the proof of Theorem 2.

The next theorem is an important step in analyzing the projective version of Newton's method.

Theorem 3. *There exist numbers $\alpha_{\text{proj.}} \sim .07364$, $u_{\text{proj.}} \sim .0203\ldots$ with the following. Suppose $f \in \mathscr{H}_{(d)}$, $\bar{\gamma} \geq D^{1/2}$ and $x \in N_\zeta$ satisfies*

(a) $\eta(f, \zeta)\mu_{\text{proj.}}(f, \zeta) \leq \alpha_{\text{proj.}}/\bar{\gamma}$,
(b) $\|x - \zeta\|/\|\zeta\| \leq u_{\text{proj.}}/\bar{\gamma}$,
(c) $\gamma_0(f, \zeta) \leq \bar{\gamma}$.

Then

$$\frac{\|x' - \zeta'\|}{\|\zeta'\|} \leq \frac{u_{\text{proj.}}}{\bar{\gamma}}$$

where $x' = N_{f|N_x}(x)$, $x' \in N_{\zeta'}$ *and* $\lambda\zeta'$ *is the associated zero of* x *for* $f|_{N_x}$, *some* $\lambda \in \mathbb{C}$.

Remark 1. If we take $\bar{\gamma} = D^{3/2}\mu/2$, $\mu = \mu_{\text{proj.}}(f, \zeta)$ then (c) is automatically satisfied by Proposition 3.

Remark 2. That $f \in \mathscr{H}_{(d)}$ is not crucial to the proof. f_i homogeneous complex analytic of degree d_i with large enough radius of convergence around ξ suffices with $\bar{\gamma} \geq \max(1, D^{1/2})$, $D = \max_{i=1,\dots,n} |d_i|$. Note also that the expression $\eta\mu$ of (a) does not involve $\|f\|$ and is defined for all such f.

A homotopy $f_t : \mathbb{C}^{n+1} \to \mathbb{C}^n$ in the homogeneous case is a curve in the space $\mathscr{H}_{(d)}$, $0 \leq t \leq 1$. An associated path ζ_t is a curve in $\mathbb{C}^{n+1}$ satisfying $f_t(\zeta_t) = 0$.

Let

$$\mu = \mu_{\text{proj.}}(F) = \max_t \mu_{\text{proj.}}(f_t, \zeta_t), \qquad F = \{f_t, \zeta_t\},$$

be the condition number of the homotopy-path F.

We will now prove the main theorem, §1, with $\rho(F)$ replaced by $\frac{1}{\mu(F)}$. In the next section, the main theorem asserts these quantities are equal and so that result will finish the proof of the main theorem.

Let C_1 be the first positive root of

$$C\left(1 + \frac{C}{D^{3/2}\mu^2}\right)^2 = 2\alpha_{\text{proj.}}\left(1 - \frac{C}{D\mu}\right)^2$$

for $D = 2$, $\mu = 1$ and let $C_2 = \frac{1}{C_1}$. Then $C_2 = 8.35\ldots$.

Let

$$\Delta = \frac{1}{C_2 D^{3/2}\mu^2} \quad \text{and} \quad \bar{\gamma} = \frac{\mu(1+\Delta)D^{3/2}}{2(1 - D^{1/2}\Delta\mu)}.$$

Choose t_i, $i = 0, \dots, k = [\frac{L}{\Delta}]$, such that for $s \in [t_i, t_{i+1}]$, $d_P(f_s, f_{t_i}) \leq \Delta$. Here $[x]$ is the smallest integer $\geq x$. Let $\mu' = \sup_{s \in [t_i, t_{i+1}]} \mu(f_s, \zeta_{t_i})$. Then

$$\mu' \leq \frac{\mu(1+\Delta)}{1 - D^{1/2}\Delta\mu}$$

by Proposition 5 and for $s \in [t_i, t_{i+1}]$,

$$\gamma_0(f_s, \zeta_{t_i}) \leq \frac{\mu(1+\Delta)D^{3/2}}{2(1 - D^{1/2}\Delta\mu)} = \bar{\gamma}$$

by Proposition 3. This gives condition (c) of Theorem 3. We now check conditions (a) and (b). For (a), note that $\eta(f_s, \zeta_{t_i}) \leq \Delta$ for $s \in [t_i, t_{i+1}]$ by Lemma 1. Also

$$\begin{aligned}\Delta\bar{\gamma}\mu' &\leq \frac{1}{C_2 D^{3/2}\mu^2}\,\frac{\mu(1+\Delta)D^{3/2}}{2(1 - D^{1/2}\Delta\mu)}\,\frac{\mu(1+\Delta)}{(1 - D^{1/2}\Delta\mu)} \\ &= \frac{C_1\left(1 + C_1/D^{3/2}\mu^2\right)^2}{2\left(1 - C_1/D\mu\right)^2} \leq \alpha_{\text{proj.}}\end{aligned}$$

by the definition of C_1. Thus $\eta \le \alpha_{\text{proj.}}/\gamma\mu'$.

For (b), by hypothesis,

$$\frac{\|z_0 - \zeta_0\|}{\|\zeta_0\|} \le C_1 \frac{\rho(F)}{D^{3/2}} \le \frac{u_{\text{proj.}}}{\bar{\gamma}}.$$

Apply Theorem 3 inductively to obtain

$$\frac{\|x_i - \zeta_{t_i}\|}{\|\zeta_{t_i}\|} \le \frac{u_{\text{proj.}}}{\bar{\gamma}}$$

for all i.

Finally from Proposition 2 of §III-2

$$\alpha_x \le \frac{\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}})^2 u_{\text{proj.}}}{\psi(u_{\text{proj.}})^2} \le .024,$$

the last using a pocket calculator. Here α_x stands for α of $f_{t_{i+1}}$ at x_{t_i}. Thus x_{t_i} is an approximate zero of $f_{t_{i+1}}$ and so certainly $x_{t_{i+1}}$ is. Theorem 1 of §I-2 applies to yield log log estimates, i.e.

$$\frac{\|N^j_{f_{t_{i+1}}|N_{x_{t_i}}}(x_{t_{i+1}}) - \zeta_{t_{i+1}}\|}{\|\zeta_{t_{i+1}}\|} \le \frac{1}{2^{2^j}}$$

where $\zeta_{t_{i+1}}$ here means the associated root of x_{t_i} in $N_{x_{t_i}}$. □

I-4. Complexity in terms of the distance to the discriminant variety Σ

The goal of this section is to replace the condition number in the estimates of the previous section by the distance to the discriminant variety. To that end, we extend an idea going back to Eckart and Young [6] and developed especially by Demmel [4].

Consider the product space $\mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ with quotient $\mathscr{H}_{(d)} \times P_n$ where $P_n = P(\mathbb{C}^{n+1})$ is n-dimensional complex projective space. Let

$$\widehat{V} = \{(f, z) \in \mathscr{H}_{(d)} \times P_n | f(z) = 0\}.$$

We may consider the projection of $\widehat{V}$ onto the second factor $\widehat{V} \to P_n$ as a vector space bundle with fiber over z given by $\widehat{V}_z = \{f \in \mathscr{H}_{(d)} \mid f(z) = 0\}$. The associated bundle $\pi_2 : V \to P_n$ with fiber $P(\widehat{V}_z)$ is a smooth algebraic hypersurface $V \subset P(\mathscr{H}_{(d)}) \times P_n$ (see Shub [24]).

Let Σ' be the algebraic hypersurface in V given as the set of $(f, \xi) \in V$ such that $Df(\xi) : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is singular (i.e., of rank less than n).

While we have considered V as a bundle $\pi_2 : V \to P_n$ we may also consider the projection $\pi : V \to P(\mathscr{H}_{(d)})$ on the first factor.

Remark. Σ' may alternately be described as the set of critical points of π. That is, $(f, x) \in V$ is in Σ' precisely when $D\pi(f, x) : T_{(f,x)}(V) \to T_f(P(\mathscr{H}_{(d)}))$ is singular.

The image under π, $\pi(\Sigma') = \Sigma \subset P(\mathscr{H}_{(d)})$, is the algebraic hypersurface of polynomial systems with a degenerate zero (i.e. $f \in \Sigma$ if and only if there is some $\zeta \in P_n$ with $f(\zeta) = 0$ and $Df(\zeta)$ singular).

This variety $\Sigma \subset P(\mathscr{H}_{(d)})$ is called the discriminant variety. It is familiar in the affine one variable case as the set of all polynomials with nonvanishing leading coefficient having a multiple root. The map $\pi : V \to P(\mathscr{H}_{(d)})$ is an n-dimensional generalization (homogenized) of the well-known map taking roots of a polynomial onto the coefficients by the symmetric functions (in one variable; sometimes the "Vieta map").

The variety Σ has played an important role in recent complexity analysis of polynomial zero finding since it consists of "ill-posed problems". For one variable Newton method see especially Smale [27, 28], Shub-Smale [25, 26]. For the many variable case see Renegar [20]. On the algebraic side, a similar situation prevails; see Canny [3], Heintz [10], Ierardi [12], and Renegar [21]. In both cases, however, there is a subvariety of the discriminant locus of a more seriously ill-posed system which contains an infinite number of zeros.

An underlying theme in much of this literature is the idea that the condition number is bounded by the reciprocal of the distance to Σ. This theme also comes from numerical analysis (see for example Demmel [4, 5]) even more explicitly. We sharpen and develop that idea here with Theorem 1 below.

Our account continues with one version of a result seen in undergraduate numerical analysis texts. Let $\|A\|_F$ be the Frobenius norm of a matrix $A \in \mathscr{M}(n)$, the set of all $n \times n$ matrices. Thus

$$\|A\|_F = (\Sigma |a_{ij}|^2)^{1/2}.$$

Let $S \subset \mathscr{M}(n)$ be the subset of singular matrices and let $d_F(A, S)$ be the distance from A to S in the Frobenius norm.

Proposition 1 (Eckart and Young [6]).

$$\|A^{-1}\| = \frac{1}{d_F(A, S)}.$$

The proof is in Golub and Van Loan [8].

Here $\|A^{-1}\|$ refers (as always here) to the usual operator norm induced from the Hermitian structure on $\mathbb{C}^n$.

Next we define a function ρ on V which represents the distance to the discriminant variety. For $(f, x) \in V$, take $\rho(f, x)$ as the distance in the fiber V_x of $\pi : V \to P_n$ over x of (f, x) to $\Sigma' \cap V_x$. Recall that this fiber is the projectified subspace $\{f \in \mathscr{H}_{(d)} \mid f(x) = 0\}$ of $\mathscr{H}_{(d)}$, and the distance is computed in the d_P metric. Thus ρ is ultimately defined by our unitarily invariant norm on $\mathscr{H}_{(d)}$.

Theorem 1. *Let* $f \in \mathscr{H}_{(d)}$, $x \in \mathbb{C}^{n+1}$, $f(x) = 0$. *Then* $\mu_{\text{proj.}}(f, x) = \frac{1}{\rho(f, x)}$.

On one hand Proposition 1 is used to prove Theorem 1; on the other hand it is a special case of Theorem 1. In the case of one variable polynomials, there is the work of Hough and Demmel [4] giving upper and lower bounds for the condition number of f at x in terms of a version of our $\rho(f, x)^{-1}$.

In the passage from Proposition 1 to Theorem 1 we use heavily unitary invariance. Unitary invariance has already played a role in the proof of Proposition 3 of I-3 and continues to do so throughout many of our proofs.

In more detail the unitary group $U(n+1)$ acts on $\mathscr{H}_{(d)} \times \mathbb{C}^{n+1}$ by sending (f, z) to (fu^{-1}, uz) for $u \in U(n+1)$. This action induces actions on $\mathscr{H}_{(d)} \times P_n$ leaving $\widehat{V}$ invariant and on $P(\mathscr{H}_{(d)}) \times P_n$ leaving V invariant as well as $\Sigma' \subset V$ invariant.

As a corollary to Theorem 1 we have immediately that, in the main complexity results of §I-3, we may replace $\mu_{\text{proj.}}(f, x)$ by $\frac{1}{\rho(f,x)}$.

Next we give a result which corresponds to Theorem 1 with the x eliminated. Quite simply for $f \in \mathscr{H}_{(d)}$, let

$$\mu_{\text{proj.}}(f) = \max_{\substack{x \\ f(x)=0}} \mu_{\text{proj.}}(f, x),$$

$$\rho(f) = \min_{\substack{x \\ f(x)=0}} \rho(f, x).$$

Then $\mu_{\text{proj.}}(f)$ may be thought of as the condition number of f.

Corollary. *Let $f \in \mathscr{H}_{(d)}$. Then*

$$\mu_{\text{proj.}}(f) = \frac{1}{\rho(f)}.$$

Remark. It is easily seen that $\rho(f) \geq d_P(f, \Sigma)$ so that $\mu(f) \leq 1/d_P(f, \Sigma)$.

We now proceed to an analysis of the condition number in the affine case. The situation here is more complicated.

Define $\Sigma_0' = V \cap \Sigma_\infty$ where

$$\Sigma_\infty = \{(f, z) \in P(\mathscr{H}_{(d)}) \times P_n \mid z_0 = 0\}.$$

Thus $(f, z) \in \Sigma_0'$ means that f has z as a zero at ∞. Let $\Sigma_0 = \pi(\Sigma_0')$, $\pi : V \to P(\mathscr{H}_{(d)})$. We may consider Σ_0 as contained in $\mathscr{P}_{(d)}$ via $\Phi^{-1} : \mathscr{H}_{(d)} \to \mathscr{P}_{(d)}$ (abusing notation). This way Σ_0 consists of all polynomial systems $f : \mathbb{C}^n \to \mathbb{C}^n$ with the property that the highest order homogeneous parts of f_i have a common nontrivial zero. It was observed in Hirsch-Smale [11] that if $f \notin \Sigma_0$, then f is proper. From the construction Σ_0' and hence Σ_0 are varieties and in fact irreducible (compare Shub [24]) hypersurfaces in V, $P(\mathscr{H}_d)$ respectively.

The following proposition gives a bound on the zeros of a polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$.

Proposition 2. *Let $f \in \mathscr{P}_{(d)}$, $x \in \mathbb{C}^n$ and $f(x) = 0$. Then*

$$\|x\|_1 = \left(1 + \sum_1^n |x_i|^2\right)^{1/2} \leq \frac{\|\Delta(d_i^{1/2})f\|}{d(f, \Sigma_0)} \leq \frac{D^{1/2}\|f\|}{d(f, \Sigma_0)}.$$

Let $\rho_0(f) = d(f, \Sigma_0)/\|f\|$. Thus $(1 + \sum_1^n |x_i|^2)^{1/2} \le D^{1/2}/\rho_0(f)$ if $f(x) = 0$.

The following gives us the affine condition numbers in terms of the projective ones.

Theorem 2. *Let* $f \in \mathscr{P}_{(d)}$, $\xi \in \mathbb{C}^n$. *Then*

$$\mu(f, \xi) \le \|\xi\|_1 \mu_{\text{proj.}}(f, (1, \xi)).$$

Corollary. *Let* $f \in \mathscr{P}_{(d)}$, $x \in \mathbb{C}^n$ *with* $f(x) = 0$. *Then*

$$\mu(f, x) \le \frac{D^{1/2}}{\rho_0(f)\rho(f, x)},$$

$$\mu(f) \le \frac{D^{1/2}}{\rho_0(f)\rho(f)}.$$

The corollary uses both Theorems 1, 2 and the proposition. This yields another version of Theorems 1 and 2 of the previous section.

We summarize this section by reviewing what must be proved in Chapter IV. These results are Proposition 1, Theorem 1, Proposition 2 and Theorem 2.

CHAPTER II: THE ABSTRACT THEORY

II-1. Point estimates

We prove the results stated in §I-2. The proof of the first part of Theorem 1 of §I-2 is given. We proceed directly with a general result which is rather technical sounding. It is used in proving all of the theorems of §I-2.

Use the basic notation of §I-2 and besides let

$$\psi(c, u) = 1 - 2(c+1)u + (c+1)u^2.$$

Proposition 1. *Let* $f : D_r(z) \to \mathbb{F}$ *be an analytic map and* $z' \in D_r(z)$. *Let* $\beta = \beta(f, z)$, $\beta' = \beta(f, z')$ *and* $c, \delta > 0$ *satisfy*

$$\frac{\|Df(z)^{-1}D^k f(z)\|}{k!} \le c\delta^{k-1}, \qquad k = 2, 3, \ldots.$$

If $u = \|z - z'\|\delta$ *and* $\psi(c, u) > 0$, *i.e.,* $u < \sqrt{c^2 + c}/(c+1)$, *then*

$$\beta' \le \frac{(1-u)^2}{\psi(c, u)}\left(\beta + \left(\frac{cu}{1-u} + 1\right)\|z' - z\|\right).$$

Moreover, if $c' = \frac{c}{\psi(c, u)}$, $\delta' = \frac{\delta}{1-u}$, *then*

$$\frac{\|Df(z')^{-1}D^k f(z')\|}{k!} \le c'(\delta')^{k-1}, \qquad k = 2, 3, \ldots.$$

Finally, if $\kappa = \beta\delta$, $\kappa' = \beta'\delta'$, *then*

$$\kappa' \le \frac{(1-u)}{\psi(c, u)}\left(\kappa + \left(\frac{cu}{1-u} + 1\right)u\right).$$

Note that the κ' estimate is a consequence of the β' estimate and the definitions of δ', κ, κ'.

We write down the special case of Proposition 1 for $c = 1$ and $\delta = \gamma(f, z)$. Let $\psi(u) = \psi(1, u)$.

Proposition 2. *Let $f : D_r(z) \to \mathbb{F}$ be analytic, $z' \in D_r(z)$ with $\psi(u) > 0$ where $u = \|z' - z\|\gamma(f, z)$. Then*

$$\beta(f, z') \le \frac{(1-u)}{\psi(u)}((1-u)\beta(f, z) + \|z' - z\|),$$

$$\gamma(f, z') \le \frac{\gamma(f, z)}{\psi(u)(1-u)},$$

$$\alpha(f, z') \le \frac{(1-u)\alpha(f, z) + u}{\psi(u)^2}.$$

This proves Proposition 1 of §I-2.

We now prove Proposition 1.

Lemma 1. *Let $A, B : \mathbb{E} \to \mathbb{F}$ be bounded linear maps with A invertible such that $\|A^{-1}B - I\| < c < 1$. Then B is invertible and $\|B^{-1}A\| < \frac{1}{1-c}$.*

Proof. Using the series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $\|x\| < 1$, $A^{-1}B = I - (I - A^{-1}B)$ is invertible. So B is invertible and $\|B^{-1}A\| = \|(A^{-1}B)^{-1}\| \le \frac{1}{1-c}$.

The following very easy lemma is left to the reader to prove.

Lemma 2.

$$\frac{\psi(c, u)}{(1-u)^2} = 1 - c\left(\left(\frac{1}{1-u}\right)^2 - 1\right).$$

If $\psi(c, u) > 0$, then $c((\frac{1}{1-u})^2 - 1) < 1$.

Lemma 3. *With notations and hypotheses of Proposition* 1

(1) *$Df(z')$ is invertible,*

(2) $\|Df(z')^{-1}Df(z)\| \le (1-u)^2/\psi(c, u)$,

(3) $\|Df^{-1}(z')D^k f(z')/k!\| \le c'(\delta')^{k-1}$.

Proof. The Taylor series of $Df(z')$ about z is

$$Df(z') = Df(z) + \sum_{k=2} \frac{D^k f(z)(z' - z)^{k-1}}{(k-1)!}.$$

So

$$\begin{aligned}\|Df(z)^{-1}Df(z') - I\| &\le \sum_{k=2} \frac{k\|Df(z)^{-1}D^k f(z)\|}{k!}\|z' - z\|^{k-1} \\ &\le c\sum_{k=2} k\delta^{k-1}\|z' - z\|^{k-1} \\ &\le c\left(\left(\frac{1}{1-u}\right)^2 - 1\right).\end{aligned}$$

This bound is less than 1 by Lemma 2 and thus Lemma 1 applies to yield

$$\|Df(z')^{-1}Df(z)\| \le \frac{1}{1 - c((\frac{1}{1-u})^2 - 1)}.$$

By Lemma 2 we obtain Lemma 3 parts (1) and (2).

For part (3) of Lemma 3 we use part (2) as follows.

$$\begin{aligned}\left\|\frac{Df(z')^{-1}D^k f(z')}{k!}\right\| &\le \|Df(z')^{-1}Df(z)\| \left\|\sum_{l=0} \frac{Df(z)^{-1}}{k!}\frac{D^{k+l}f(z)}{l!}(z'-z)^l\right\| \\ &\le \frac{(1-u)^2}{\psi(c,u)}\sum_{l=0}\frac{(k+l)!}{k!l!}c\delta^{k+l-1}\|z'-z\|^l \\ &\le \frac{c(1-u)^2}{\psi(c,u)}\delta^{k-1}\sum_{l=0}\frac{(k+l)!}{k!l!}u^l \\ &\le \frac{c(1-u)^2}{\psi(c,u)}\delta^{k-1}\left(\frac{1}{1-u}\right)^{k+1} \le \frac{c}{\psi(c,u)}\left(\frac{\delta}{1-u}\right)^{k-1} \\ &\le c'(\delta')^{k-1}\end{aligned}$$

proving Lemma 3.

By Lemma 3 it is sufficient for the proof of Proposition 1 to estimate β', or the first part of the following lemma.

Lemma 4.

(a) $\|Df(z)^{-1}f(z')\| \le \beta + (\frac{cu}{1-u} + 1)\|z' - z\|$.

(b) *Let* $z' = z - Df(z)^{-1}f(z)$. *Then*

$$\|Df^{-1}(z)f(z')\| \le \frac{cu}{1-u}\|z' - z\|.$$

Proof.

$$\begin{aligned}\|Df(z)^{-1}f(z')\| &\le \left\|Df(z)^{-1}\left(f(z) + \sum_{k=1}^{\infty}\frac{D^k f(z)}{k!}(z'-z)^k\right)\right\| \\ &\le \|Df(z)^{-1}f(z) + z' - z\| + \sum_{k=2}^{\infty}\frac{\|Df(z)^{-1}D^k f(z)\|}{k!}\|z'-z\|^k.\end{aligned}$$

But

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{\|Df(z)^{-1}D^k f(z)\|}{k!}\|z'-z\|^k &\le \left(\sum_{k=2}^{\infty} c\delta^{k-1}\|z'-z\|^{k-1}\right)\|z'-z\| \\ &\le \frac{cu}{1-u}\|z'-z\|.\end{aligned}$$

To prove (b) note that in this case $\|Df(z)^{-1}f(z) + z' - z\| = 0$. For (a) we have

$$\|Df(z)^{-1}f(z')\| \le \beta + \|z'-z\| + \frac{cu}{1-u}\|z'-z\|.$$

This proves Lemma 4. Proposition 1 follows,

$$\begin{aligned}\beta' = \|Df(z')^{-1}f(z')\| &\le \|Df(z')^{-1}Df(z)\|\,\|Df(z)^{-1}f(z')\| \\ &\le \frac{(1-u)^2}{\psi(c,u)}\left(\beta + \|z'-z\| + \frac{cu}{1-u}\|z'-z\|\right)\end{aligned}$$

by Lemma 3 and the above.

Proposition 3. *Under the conditions of Proposition* 1, *let* $z' = z - Df(z)^{-1}f(z)$. *Then*

$$\beta' \le \frac{\beta c}{\psi(c,\kappa)}\kappa(1-\kappa).$$

Moreover take $\delta = \gamma$ *and* $c = 1$, *to obtain*

$$\begin{aligned}\gamma' &\le \frac{\gamma}{\psi(\alpha)(1-\alpha)}, \\ \beta' &\le \frac{\beta\alpha(1-\alpha)}{\psi(\alpha)}, \\ \alpha' &\le \frac{\alpha^2}{\psi(\alpha)^2}\end{aligned}$$

as in Smale [29].

Note that for

$$z' = z - Df(z)^{-1}f(z)$$

we have $u = \kappa$ and

$$\begin{aligned}\beta(z') &= \|Df(z')^{-1}f(z')\| \\ &\le \|Df(z')^{-1}Df(z)\|\,\|Df(z)^{-1}f(z')\| \\ &\le \frac{(1-u)^2}{\psi(c,u)}\cdot\frac{cu}{(1-u)}\|z'-z\| \\ &\le \frac{c\kappa(1-\kappa)}{\psi(c,\kappa)}\beta\end{aligned}$$

using Proposition 1 and Lemma 4(b). Now recall that for $\delta = \gamma$ we may take $c = 1$ and $\kappa = u = \alpha$. Thus the inequality for γ' follows from Proposition 2, the inequality for β' from the above both by substitution and the inequality for α' by multiplying the inequalities for γ' and β'. This proves Proposition 3.

We next make a change of variables from c to σ as follows:

$$\begin{aligned}\sigma &= \frac{c\kappa}{(1-\kappa)^2}, \quad 0 \le \sigma \le \frac{1}{4}, \\ \sigma' &= \frac{c'\kappa'}{(1-\kappa')^2}.\end{aligned}$$

We are continuing the use of notation from Proposition 1.

Proposition 4. *Under the hypotheses of Proposition* 3,

(a) $\kappa' \leq \frac{\sigma\kappa}{1-2\sigma+\sigma\kappa}$,

(b) $\beta' \leq \frac{\beta(1-\kappa)\sigma}{1-2\sigma+\sigma\kappa}$ $(\leq \frac{\sigma}{1-2\sigma}\beta)$,

(c) $\sigma' \leq (\frac{\sigma}{1-2\sigma})^2$.

Proof. Observe that

$$\frac{\psi(c, \kappa)}{(1-\kappa)^2} = 1 - 2\sigma + \sigma\kappa.$$

Then (b) follows from Proposition 3 and an easy substitution, and (a) follows from (b) since $\kappa' = \beta'\delta'$ and $\delta' \leq \frac{1}{1-\kappa}$ by Proposition 1 recalling that $u = \kappa$. It remains to confirm (c).

Let $\psi = \psi(c, \kappa)$. Then

$$\sigma' = \frac{c'\kappa'}{(1-\kappa')^2} \leq \frac{c}{\psi}\frac{c\kappa^2}{\psi}\frac{1}{(1-\kappa')^2} = \sigma^2\left(\frac{(1-\kappa)^2}{\psi(1-\kappa')}\right)^2.$$

But

$$\frac{(1-\kappa)^2}{\psi(1-\kappa')} \leq \frac{(1-\kappa)^2}{\psi - c\kappa^2} = \frac{(1-\kappa)^2}{(1-2(c+1)\kappa+\kappa^2)} = \frac{1}{1-2c\kappa/(1-\kappa)^2} = \frac{1}{1-2\sigma}$$

proving Proposition 4 from Proposition 3.

We now suppose the hypotheses of Theorem 1 of §I-2.

Proposition 5. *For* $0 \leq \sigma \leq \frac{1}{4}$, *let* $0 \leq \lambda \leq 1$ *satisfy* $\sigma = \lambda/(1+\lambda)^2$. *Then*

$$\beta_k \leq \lambda^{2^k - 1}\beta_0, \qquad k = 0, 1, 2, \ldots .$$

For the proof of Proposition 5, we use the following lemma.

Lemma 5. *Let* $\beta_k = \beta(f, z_k)$ *and* σ_j *be the* j*th iterate under* $\sigma \to (\frac{\sigma}{1-2\sigma})^2$ *of* $\sigma_0 = \sigma$. *Then*

(a) $\beta_k \leq \prod_{j=0}^{k-1}(\frac{\sigma_j}{1-2\sigma_j})\beta$,

(b) $(\frac{\sigma_j}{1-2\sigma_j}) \leq \lambda^{2^j}$.

Since $\prod_{j=0}^{k-1}\lambda^{2^j} = \lambda^{2^k-1}$, Proposition 5 is a consequence of the lemma. Moreover, (a) is a consequence of Proposition 4(b). Note that

$$\sigma' \leq \left(\frac{\lambda/(1+\lambda)^2}{1-2\lambda/(1+\lambda)^2}\right)^2 = \frac{\lambda^2}{(1+\lambda^2)^2}.$$

It follows that

$$\sigma_k \leq \frac{\lambda^{2^k}}{(1+\lambda^{2^k})^2}.$$

But since

$$(1-2\sigma_k)(1+\lambda^{2^k})^2 \geq 1$$

we have

$$\frac{\sigma_k}{1-2\sigma_k} \leq \lambda^{2^k}$$

proving (b) of the lemma.

Take $c = 1$. Then $\kappa = \alpha$ and

$$\sigma = \frac{\alpha}{(1-\alpha)^2} = \frac{\lambda}{(1+\lambda)^2}.$$

Choose $\lambda = \frac{1}{2}$, so $\alpha = \alpha_0 = \frac{1}{4}(13 - 3\sqrt{17})$ and Proposition 5 implies the first part of Theorem 1 of §I-2. Recall that the rest of Theorem 1 follows from Theorem 2. The next section is devoted to a proof of that theorem.

II-2. The domination theorem

For the proof of the domination theorem (Theorem 2 of §I-2), we will use some lemmas. The first concerns the monotonicity of the functions defining the inequalities in Proposition 4 of the previous section. Let

$$K(\kappa, \sigma) = \frac{\sigma\kappa}{1-2\sigma+\sigma\kappa},$$

$$B(\beta, \kappa, \sigma) = \frac{\beta(1-\kappa)\sigma}{1-2\sigma+\sigma\kappa},$$

$$S(\sigma) = \left(\frac{\sigma}{1-2\sigma}\right)^2.$$

Lemma 1. *Suppose* $0 \leq \kappa_1 \leq \kappa_2 \leq 1$, $0 \leq \sigma_1 \leq \sigma_2 \leq \frac{1}{4}$ *and* $0 \leq \beta_1 \leq \beta_2$. *Then*

(a) $K(\sigma_1, \kappa_1) \leq K(\sigma_2, \kappa_2)$,
(b) $B(\beta_1, \kappa_1, \sigma_1) \leq B(\beta_2, \kappa_1, \sigma_1)$,
(c) $S(\sigma_1) \leq S(\sigma_2)$,
(d) $S(\sigma_1) \leq \frac{1}{4}$, $0 \leq K(\sigma_1, \kappa_1)$ *and* $0 \leq B(\beta_1, \kappa_1, \sigma_1)$.

Proof. For (a), (b), (c), compute the derivatives of K, B, S with respect to the appropriate variables and note that they are nonnegative. The proof of (d) is straightforward.

Next introduce the functions

$$h_{\beta,\kappa,\sigma}(t) = \beta - t + \frac{(1-\kappa)^2}{\kappa}\sigma\frac{\frac{\kappa}{\beta}t^2}{1-\frac{\kappa}{\beta}t}.$$

We have purposefully not simplified the expression for ease of manipulation. By direct calculus we prove

Lemma 2.

(a) $Dh_{\beta,\kappa,\sigma}(t) = -1 + \frac{(1-\kappa)^2}{\kappa}\sigma(-1 + 1/(1-\frac{\kappa}{\beta}t)^2)$,
(b)

$$\frac{D^{(i)}h_{\beta,\kappa,\sigma}(t)}{i!} = \frac{(1-\kappa)^2}{\kappa}\sigma\frac{(\frac{\kappa}{\beta})^{i-1}}{(1-\frac{\kappa}{\beta}t)^{i+1}}, \quad \textit{for } i \geq 2,$$

(c) $h_{\beta,\kappa,\sigma}(\beta)=\sigma(1-\kappa)\beta$,
(d) $Dh_{\beta,\kappa,\sigma}(\beta)=-(1-2\sigma+\kappa\sigma)$.

Let

$$T(\beta,\kappa,\sigma)=(B(\beta,\kappa,\sigma),K(\sigma,\kappa),S(\sigma))$$

and $B^j(\beta,\kappa,\sigma)$ be the β component of the jth iterate T^j of T.

Lemma 3.

$$\frac{h_{\beta,\kappa,\sigma}(\beta+s)}{-h'_{\beta,\kappa,\sigma}(\beta)}=\beta'-s+\frac{(1-\kappa')^2}{\kappa'}\sigma'\frac{\frac{\kappa'}{\beta'}s^2}{(1-\frac{\kappa'}{\beta'}s)}$$

where $(\beta',\kappa',\sigma')=T(\beta,\kappa,\sigma)$.

Proof. We prove this algebraic identity as follows: the Taylor series for $h_{\beta,\kappa,\sigma}(s+\beta)$ is:

$$\sigma(1-\kappa)\beta+h'_{\beta,\kappa,\sigma}(\beta)s+\frac{(1-\kappa)^2}{\kappa}\sigma\sum_{i\geq 2}\frac{(\frac{\kappa}{\beta})^{i-1}s^i}{(1-\kappa)^{i+1}}$$

using the previous lemma.

$$\begin{aligned}\frac{h_{\beta,\kappa,\sigma}(s+\beta)}{-h'_{\beta,\kappa,\sigma}(\beta)}-\frac{\sigma(1-\kappa)\beta}{1-2\sigma+\kappa\sigma}+s&=\frac{\sigma s}{\kappa(1-2\sigma+\kappa\sigma)}\sum_{i\geq 2}\left(\frac{\kappa s}{(1-\kappa)\beta}\right)^{i-1}\\&=\frac{\sigma}{\kappa(1-2\sigma+\kappa\sigma)}\frac{(\frac{\kappa}{\beta(1-\kappa)})s^2}{(1-\frac{\kappa}{\beta(1-\kappa)}s)}.\end{aligned}$$

Comparing terms finishes the proof.

Newton's method has the following basic property.

Proposition 1. *Let* L *be a linear automorphism of* $\mathbb{F}$, $A:\mathbb{F}\to\mathbb{E}$ *an affine isomorphism,* $U\subset\mathbb{E}$, *and* $f:U\to\mathbb{E}$. *Then*

$$N_{L\cdot f\cdot A}=A^{-1}N_fA.$$

Let $\mathrm{Tr}(b)$ denote the translation by b.

Lemma 4.

$$\mathrm{Tr}\left(-\sum_{j=0}^{n-1}B^j(\beta,\kappa,\sigma)\right)N_{h_{\beta,\kappa,\sigma}}\mathrm{Tr}\left(\sum_{0}^{n-1}B^j(\beta,\kappa,\sigma)\right)=N_{h_{T^n(\beta,\kappa,\sigma)}}$$

for $n\geq 1$.

Proof. For $n=1$

$$\mathrm{Tr}(-\beta)N_{h_{\beta,\kappa,\sigma}}\mathrm{Tr}(\beta)=N_g=N_{h_{T(\beta,\kappa,\sigma)}},$$

$$g=\frac{h_{\beta,\kappa,\sigma}\cdot\mathrm{Tr}(\beta)}{-h'_{\beta,\kappa,\sigma}(\beta)}.$$

The first equality follows from the last proposition and the second equality from Lemma 3. Now induction finishes the proof.

Recall that in the setting of the domination theorem we have

$$h_{\beta,\gamma}(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t} = h_{\beta,\alpha,\sigma_\alpha}(t) \quad \text{where } \sigma_\alpha = \frac{\alpha}{(1-\alpha)^2}$$

and t_n is the nth iterate of 0 by Newton's method so $t_n = N^n_{h_{\beta,\alpha,\sigma_\alpha}}(0)$.

Lemma 5. *For* $n \geq 1$

$$t_n - t_{n-1} = B^{n-1}(\beta, \alpha, \sigma_\alpha).$$

Proof. For $n = 1$, this is obvious. Since $t_0 = 0$, induction gives us at stage $n-1$ that

$$(*) \qquad t_{n-1} = \sum_{j=0}^{n-2} B^j(\beta, \alpha, \sigma_\alpha).$$

It follows from Lemma 4 that

$$N_{h_{T^{n-1}(\beta,\alpha,\sigma_\alpha)}}(0) = N_{h_{(\beta,\alpha,\sigma_\alpha)}}\left(\sum_{j=0}^{n-2} B^j(\beta, \alpha, \sigma_\alpha)\right) - \sum_{j=0}^{n-2} B^j(\beta, \alpha, \sigma_\alpha);$$

substituting $(*)$ gives

$$N_{h_{T^{n-1}(\beta,\alpha,\sigma_\alpha)}}(0) = N_{h_{(\beta,\alpha,\sigma_\alpha)}}(t_{n-1}) - t_{n-1} = t_n - t_{n-1}$$

where the last equality is the definition of t_n. Now $B^{n-1}(\beta, \alpha, \sigma_\alpha)$ is by definition the β component of $T^{n-1}(\beta, \alpha, \sigma_\alpha)$ so

$$B^{n-1}(\beta, \alpha, \sigma_\alpha) = N_{h_{T^{n-1}(\beta,\alpha,\sigma_\alpha)}}(0) = t_n - t_{n-1}.$$

Proof of the Domination Theorem (Theorem 2, §I-2). By Proposition 4, §II-1, Lemma 1 and induction it follows that

$$\beta(f, x_{n-1}) \leq B^{n-1}(\beta, \alpha, \sigma_\alpha) \quad \text{for } n \geq 1.$$

Now $\|x_n - x_{n-1}\| = \beta(f, x_{n-1})$ by definition and $B^{n-1}(\beta, \alpha, \sigma_\alpha) = t_n - t_{n-1}$ by Lemma 5; thus $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$.

II-3. Robustness

Here we give the proof of Theorem 3 of §I-2. Toward the proof of Theorem 3 consider the function $a(t) = \tau(t) - t$ with $\tau(t) = (1 + t - \sqrt{(1+t)^2 - 8t})/4$ as usual.

Lemma 1. *The map a is a differentiable homeomorphism, $a : [0, t_0] \to [0, u_0]$ where $t_0 = 3 - \sqrt{8}$, $u_0 = \frac{3\sqrt{2}-4}{2}$, $a'(t) > 0$, $t \in (0, t_0)$.*

Proof. It is easy to check that $a(0) = 0$, $a(t_0) = u_0$,

$$a'(t) = \frac{1}{4}\left(\frac{3-t}{\sqrt{1-6t+t^2}} - 3\right) > 0, \qquad 0 < t < t_0,$$

$a'(0) = 0$, $a'(t_0) = \infty$. Thus $a^{-1} : [0, u_0] \to [0, t_0]$ is well defined and differentiable on the interior.

Lemma 2. *Suppose $b > 0$. The function $\frac{a(bs)}{s} = \frac{\tau(bs)-bs}{s}$ is monotone increasing for $0 < s \le \frac{3-\sqrt{8}}{b}$.*

It is sufficient to show that this function has a positive derivative. We leave this as an exercise for the reader.

Define

$$s(t, u) = \frac{t(1-u)+u}{\psi(u)^2}.$$

We remind the reader that

$$\psi(u) = 2u^2 - 4u + 1.$$

Lemma 3. *The functions $\frac{(1-u)^2}{\psi(u)}$ and $\frac{(1-u)}{\psi(u)}$ are monotone increasing for $0 \le u < 1 - \frac{\sqrt{2}}{2}$.*

Once again check the positivity of the derivative.

Lemma 4. *$s(t, u)$ is monotone increasing in t and u for $0 < t \le 1$ and $0 < u < 1 - \frac{\sqrt{2}}{2}$.*

The numerator increases and the denominator decreases to zero in this range.

Let

$$\alpha(u) = \frac{\psi(u)^2 a^{-1}\left(\frac{u}{\psi(u)(1-u)}\right) - u}{1-u}$$

for $0 \le u \le u_1$ where u_1 is defined by $u_1/\psi(u_1)(1-u_1) = u_0$.

Note for $0 \le u \le u_1$, $a(s(\alpha(u), u)) = \frac{u}{\psi(u)(1-u)}$. The following is straightforward.

Lemma 5. *$\alpha(0) = 0$, $\alpha(u_1) < 0$ and $\alpha'(u) > 0$ for small $u > 0$.*

Definition. Let $\bar{u}$ be the first positive zero of $\alpha'(u)$ and $\bar{\alpha} = \alpha(\bar{u})$,

$$\bar{u} = 0.02207\ldots, \qquad \bar{\alpha} = 0.08019667\ldots .$$

Let $G(\gamma, u) = \frac{\gamma}{\psi(u)(1-u)}$ for $0 \le \gamma$ and $0 \le u < 1 - \frac{\sqrt{2}}{2}$. Let $\bar{s} = s(\bar{\alpha}, \bar{u})$ and $\bar{G} = G(\bar{\gamma}, \bar{u})$.

Lemma 6. *As in Theorem 3 (§I-2) let $\alpha = \beta\gamma$, $\beta = \beta(f, \zeta)$, $\gamma = \gamma(f, \zeta)$. Let also $\alpha_x = \beta_x\gamma_x$, $\beta_x = \beta(f, x)$, $\gamma_x = \gamma(f, x)$. Suppose $\alpha \le \bar{\alpha}$, $\gamma \le \bar{\gamma}$, $u \le \bar{u}$ where $u = \bar{\gamma}\|x - \zeta\|$ and $\beta \le \frac{\bar{\alpha}}{\bar{\gamma}}$. Then*

$$\frac{a(\alpha_x)}{\gamma_x} \le \frac{a(\bar{s})}{\bar{G}}.$$

Proof.

$$\frac{a(\alpha_x)}{\gamma_x} \le \frac{a(\beta_x G(\gamma, u))}{G(\gamma, u)} \le \frac{a(\beta_x \bar{G})}{\bar{G}}$$

by Proposition 2 of §II-1, Lemma 2 applied twice and the monotonicity of G.

Now

$$\beta_x G(\bar{\gamma}, \bar{u}) \le \frac{(1-u)}{\psi(u)}((1-u)\beta + \|x-\zeta\|)\frac{\bar{\gamma}}{\psi(\bar{u})(1-\bar{u})}$$

by Proposition 2 of §II-1 again. This is

$$\le \left(\frac{(1-u)^2}{\psi(u)}\bar{\alpha} + \frac{(1-u)u}{\psi(u)}\right)\frac{1}{\psi(\bar{u})(1-\bar{u})}$$

by the hypotheses, and is

$$\le \left(\frac{(1-\bar{u})^2}{\psi(\bar{u})}\bar{\alpha} + \frac{(1-\bar{u})\bar{u}}{\psi(\bar{u})}\right)\frac{1}{\psi(\bar{u})(1-\bar{u})} = s(\bar{\alpha}, \bar{u}) = \bar{s}$$

by Lemma 3. By Lemma 2 $a(\beta_x \bar{G}) \le a(\bar{s})$ and thus $a(\alpha_x)/\gamma_x \le a(\bar{s})/\bar{G}$.

Proof of Theorem 3 (§I-2). By Theorem 1 (§I-2)

$$\|x_1 - \xi_1\| \le \frac{a(\alpha_x)}{\gamma_x} \le \frac{a(\bar{s})}{\bar{G}} \quad \text{by Lemma 6}$$

$$= \frac{\bar{u}}{\bar{\gamma}} \quad \text{by the definition of } \bar{s} \text{ and } \bar{G}.$$

CHAPTER III: REDUCTION TO THE ANALYSIS OF THE CONDITION NUMBER

Here we give the proofs of the statements in §I-3. Some references back to that section are inevitable.

III-1. The higher derivative estimate

Our main goal of this section is to prove the estimate on γ of Proposition 3 of §I-3 and Proposition 5 of that section. We start with

Proposition 1. *Let* $f : \mathbb{C}^{n+1} \to \mathbb{C}$ *be a homogeneous polynomial of degree* d. *Then*

$$|f(x)| \le \|f\| \, \|x\|^d \quad \textit{for all } x \in \mathbb{C}^{n+1}.$$

Proof. Let $x \in \mathbb{C}^{n+1}$ and $y = (\|x\|, 0, \dots, 0)$. Take a unitary automorphism $U : \mathbb{C}^{n+1} \hookleftarrow$ satisfying $U^{-1}y = x$. Then

$$\frac{|f(x)|}{\|x\|^d} = \frac{|fU^{-1}(Ux)|}{\|x\|^d} = \frac{|g(y)|}{\|y\|^d}$$

where $g = fU^{-1} = \sum_\alpha b_\alpha x^\alpha$ and $\|g\| = \|f\|$ by Proposition 1 of §I-3. We have

$$\frac{|g(y)|}{\|y\|^d} = \frac{|b_{d,0,\dots,0}| \|x\|^d}{\|y\|^d} = |b_{d,0,\dots,0}| \le \|g\| = \|f\|. \quad \square$$

Proposition 2. *If* $f \in \mathscr{H}_{(d)}$, *then* $\|\Delta(\|x\|^{-d_i})f(x)\| \le \|f\|$.

Proof. From the previous proposition we know that

$$\|x\|^{-d_i}|f_i(x)| \le \|f_i\|, \qquad i = 1, \dots, n.$$

Just square both sides and sum over i.

Proposition 3. *Let* f *be a homogeneous polynomial of degree* d. *Then*

$$\|D^k f(x)(w_1, \dots, w_k)\| \le d(d-1)\cdots(d-k+1)\|f\|\,\|x\|^{d-k}\|w_1\|\cdot\|w_k\|$$

for all $x, w_i \in \mathbb{C}^{n+1}$.

The proof uses two lemmas.

Lemma 1. *Let* $U : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *be a unitary automorphism,* $f : \mathbb{C}^{n+1} \to \mathbb{C}$ *a homogeneous polynomial and* $x, w \in \mathbb{C}^{n+1}$. *Then*

$$D(f \circ U^{-1})(U(x))(Uw) = Df(x)(w).$$

The lemma follows from the chain rule,

$$D(f \circ U^{-1})U(x) = Df(x)U^{-1}.$$

Apply this to Uw.

Next let $\mathscr{F}_d$ be the space of homogeneous polynomials $\mathbb{C}^{n+1} \to \mathbb{C}$ and $f \in \mathscr{F}_d$, $w \in \mathbb{C}^{n+1}$. Then $Df(x)(w)$ is a polynomial of degree $d-1$ in x and can thus be considered as an element say $Df(w)$ of $\mathscr{F}_{d-1}$.

Lemma 2. $\|Df(w)\|_{\mathscr{F}_{d-1}} \le d\|f\|_{\mathscr{F}_d}\|w\|$.

Here the subscripts on the norms are temporary. It is sufficient to prove Lemma 2 for $\|w\| = 1$ by scaling and then for $w = e_0 = (1, 0, \dots, 0)$ by choosing unitary U with $Uw = e_0$ and using Lemma 1 together with the unitary invariance of the norm.

Then since

$$Df(x)(w) = \sum_{\substack{\alpha \\ \alpha_0 \ne 0}} \alpha_0 a_\alpha x_0^{\alpha_0 - 1} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

we have

$$\begin{aligned}\|Df(w)\|^2_{\mathscr{F}_{d-1}} &= \sum_{\substack{\alpha \\ \alpha_0 \ne 0}} |\alpha_0|^2 |a_\alpha|^2 \frac{(\alpha_0 - 1)!\alpha_1!\cdots\alpha_n!}{(d-1)!} \\ &= d \sum_{\substack{\alpha \\ \alpha_0 \ne 0}} |\alpha_0||a_\alpha|^2 \frac{\alpha_0!\cdots\alpha_n!}{d!} \\ &\le d^2 \sum_{\alpha} |a_\alpha|^2 \frac{\alpha_0!\cdots\alpha_n!}{d!} = d^2\|f\|^2_{\mathscr{F}_d}.\end{aligned}$$

This proves Lemma 2.

Note considering $D^2 f(w_1, w_2)$ as a polynomial in x we have from Lemma 2 applied twice

$$\|D^2 f(w_1, w_2)\|_{\mathcal{F}_{d-2}} \leq d(d-1)\|f\|_{\mathcal{F}_d}\|w_1\|\,\|w_2\|$$

and similarly by induction:

$$\|D^k f(w_1, \dots, w_k)\|_{\mathcal{F}_{d-k}} \leq d(d-1)\cdots(d-k+1)\|f\|_{\mathcal{F}_d}\|w_1\|\cdots\|w_k\|.$$

Now apply Proposition 1 to obtain the assertion of Proposition 3.

Lemma 3. *Let $d \geq k \geq 2$ be positive integers. Then*

$$\max_{k>1}\left(\frac{d(d-1)\cdots(d-k+1)}{d^{1/2}k!}\right)^{1/(k-1)}$$

is at $k = 2$.

Proof. Observe

$$\left(\prod_{i=1}^{k-1}\frac{(d-i)}{i+1}\right)^{1/k-1} > \frac{d-k}{k+1}$$

for $2 < k \leq d$ since each of the $k-1$ terms in the product is bigger than $\frac{d-k}{k+1}$.
Now for $2 \leq k \leq d-1$

$$\left(\frac{\left(\frac{d(d-1)\cdots(d-k+1)}{d^{1/2}k!}\right)^{1/(k-1)}}{\left(\frac{d(d-1)\cdots(d-k)}{d^{1/2}(k+1)!}\right)^{1/k}}\right)^k = \frac{d^{1/2(k-1)}\left(\prod_{i=1}^{k-1}\left(\frac{d-i}{i+1}\right)\right)^{1/(k-1)}}{\frac{d-k}{k+1}} > 1.$$

Thus

$$\left(\frac{d(d-1)\cdots(d-k+1)}{d^{1/2}k!}\right)^{1/k-1}$$

is a decreasing function of k.

Lemma 4. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d. Then*

$$\left(\frac{\|D^k f(x)(w_1, \dots, w_k)\|}{d^{1/2}\|x\|^{d-k}k!\|f\|\,\|w_1\|\cdots\|w_k\|}\right)^{1/(k-1)} \leq \frac{d^{1/2}(d-1)}{2}$$

for every $k > 1$.

This follows from Proposition 3 and Lemma 3. Recall from the introduction that $D = \max(d_i)$.

Theorem 1. *Let $f \in \mathcal{H}_{(d)}$ and $x \in \mathbb{C}^{n+1}$. Then*

$$\left(\frac{\|\Delta(\|x\|^{d_i-k}d_i^{-1/2})^{-1}D^k f(x)\|}{k!\|f\|}\right)^{1/(k-1)} \leq \frac{D^{1/2}(D-1)}{2} \leq \frac{D^{3/2}}{2}.$$

Proof of Theorem 1. By the definition of $\|\ \|$,

$$\left(\frac{\|\Delta(\|x\|^{d_i-k}d_i^{1/2})^{-1}D^k f(x)\|}{k!\|f\|}\right)^{1/(k-1)} = \left(\sum\left(\frac{\|D^k f_i(x)\|}{\|x\|^{d_i-k}k!\|f\|d_i^{1/2}}\right)^2\right)^{1/2(k-1)}$$

and by Lemma 4

$$\le \left(\sum \left(\left(\frac{d_i^{1/2}(d_i-1)}{2}\right)^{k-1} \frac{\|f_i\|}{\|f\|}\right)^2\right)^{1/2(k-1)}$$
$$\le \frac{D^{1/2}(D-1)}{2}. \quad \square$$

We next prove Proposition 3 of §I-3.

For $f : \mathbb{C}^n \to \mathbb{C}^n$, $x \in \mathbb{C}^n$,

$$\begin{aligned}\gamma_0(f, x) &= \gamma(f, x)\|x\|_1 \\ &= \max_{k>1} \left\| \frac{Df(x)^{-1} D^k f(x)}{k!} \right\|^{1/(k-1)} \|x\|_1 \\ &= \max_{k>1} \left\| \frac{Df(x)^{-1} \Delta(d_i^{1/2}) \Delta(\|x\|_1^{d_i-1}) \Delta(d_i^{-1/2}) \Delta(\|x\|_1^{-(d_i-k)}) D^k(f)}{k!} \right\|^{1/(k-1)} \\ &\le \max_{k>1} \mu(f, x)^{1/(k-1)} \frac{D^{3/2}}{2}\end{aligned}$$

by Theorem 1. The last is less than $\mu(f, x)D^{3/2}/2$ since $\mu(f, x) \ge 1$. This proves the first part. The proof of the second part is essentially the same.

Lemma 5. *Let $A, B : \mathbb{E} \to \mathbb{F}$ be bounded linear maps of Banach spaces where B is invertible, and $\|A - B\| \|B^{-1}\| < 1$. Then A is invertible and $\|A^{-1}\| \le \|B^{-1}\|/(1 - \|B^{-1}\| \|A - B\|)$.*

Proof. $\|I - AB^{-1}\| \le \|A - B\| \|B^{-1}\|$ so, by Lemma 1, §II-1,

$$\|BA^{-1}\| \le \frac{1}{1 - \|A - B\| \|B^{-1}\|} \quad \text{and} \quad \|A^{-1}\| \le \|B^{-1}\| \|BA^{-1}\|.$$

We next prove Proposition 5a of §I-3. For $\lambda \ne 0$, $\lambda \in \mathbb{C}$,

$$\mu(g, \zeta) = \mu(\lambda g, \zeta) = \|(\Delta(d_i^{-1/2}\|\zeta\|^{-(d_i-1)}) D(\lambda g)|_{N_{e_0}}(\zeta))^{-1}\| \|\lambda g\|$$
$$\le \frac{\|(\Delta(d_i^{-1/2}\|\zeta\|^{-(d_i-1)}) Df|_{N_{e_0}}(\zeta))^{-1}\| \|\lambda g\|}{1 - \|\Delta(d_i^{-1/2}\|\zeta\|^{-(d_i-1)}) D(f - \lambda g)|_{N_{e_0}}(\zeta)\| \|(\Delta(d_i^{-1/2}\|\zeta\|^{-(d_i-1)}) Df|_{N_{e_0}}(\zeta))^{-1}\|}$$

by Lemma 5 as long as the denominator is positive. Thus

$$\mu(g, \zeta) \le \frac{\mu(f, \zeta)\frac{\|\lambda g\|}{\|f\|}}{1 - \|\Delta(d_i^{1/2})(f - \lambda g)\| \frac{\mu(f, \zeta)}{\|f\|}}$$

which follows from Proposition 3, and

$$u(g, \zeta) \le \frac{\mu(f, \zeta)(1 + \frac{\|f - \lambda g\|}{\|f\|})}{1 - D^{1/2} \frac{\|f - \lambda g\|}{\|f\|} \mu(f, \zeta)}.$$

We apply the last inequality to that λ for which $d_P(f, g) = \frac{\|f - \lambda g\|}{\|f\|}$ which by hypothesis makes the denominator positive.

The proof of 5(b) is the same replacing μ by $\mu_{\text{proj.}}$ and N_{e_0} by Null_ζ.

III-2. ANALYSIS OF PROJECTIVE NEWTON METHOD

We give the proof of Theorem 3 of §I-3. Part of this proof is very similar to that of Theorem 3 of §I-2 in §II-3, where we use the same notation.

For $0 \le \alpha \le \alpha_0$, $0 \le u < 1 - \frac{\sqrt{2}}{2}$, let

$$\hat{\alpha}(u, \alpha) = (\bar{\kappa}(u, \alpha))^2 \left(\frac{(1-u)\alpha + u}{\psi(u)^2}\right) = \bar{\kappa}(u, \alpha)^2 S(\alpha, u)$$

where

$$\bar{\kappa}(u, \alpha) = \frac{(1+u^2)^{1/2}}{1 - u((2u-u^2)/(1-u)^2 + 2\alpha)(1-u)^2/\psi(u)}.$$

For each u let $\alpha(u)$ be the maximum of α such that

$$(*) \qquad a(\hat{\alpha})(1 + \sqrt{2}\tau(\hat{\alpha})) \le \frac{\bar{\kappa}(u, \alpha)u}{\psi(u)(1-u)}.$$

Recall that $a(t) = \tau(t) - t$ is defined in §II-3.

Let $\alpha_{\text{proj.}}$ be the maximum of $\alpha(u)$ (over u) and $u_{\text{proj.}}$ the least u such that $\alpha(u) = \alpha_{\text{proj.}}$.

Lemma 1. *Both $\alpha_{\text{proj.}}$ and $u_{\text{proj.}}$ are defined uniquely and are positive. Moreover approximately*

$$\alpha_{\text{proj.}} = .07364\ldots, \qquad u_{\text{proj.}} = .0203\ldots .$$

The proof is very close to the proof of Theorem 3 of §I-2 in §II-3. One must check the monotonicity and boundary conditions of $\bar{\kappa}$. We leave the details to the reader.

Proposition 1. *Let $f \in \mathcal{H}_{(d)}$, $x, \zeta \in \mathbb{C}^{n+1}$ with $x \in N_\zeta$ where $N_\zeta = \zeta + \text{Null}_\zeta$, $N_x = x + \text{Null}_x$, $\text{Null}_\zeta = \{v \in \mathbb{C}^{n+1} \mid \langle v, \zeta\rangle = 0\}$ etc. Let $r_0 = \frac{\|x-\zeta\|}{\|\zeta\|}$, $u = r_0\gamma_0(f|_{N_\zeta}, \zeta)$. Then $\|Df(x)|_{N_x}^{-1} Df(x)|_{N_\zeta}\| \le \kappa$ where*

$$\kappa = \frac{(1+r_0^2)^{1/2}}{1 - r_0((2-u)u/(1-u)^2 + D\mu\eta)(1-u)^2/\psi(u)}$$

where $\mu = \mu_{\text{proj.}}(f, \zeta)$, $\eta = \eta(f, \zeta)$ and as long as the denominator remains positive.

Proof. For the proof we use a series of lemmas.

Lemma 2. *Let $L : \mathbb{C}^{n+1} \to \mathbb{C}^n$ have rank n. Suppose $\mathbb{C}^{n+1} = V^n \oplus V^1$ as unitary direct sum where V^n has dimension n and V^1 dimension 1. With respect to this splitting write $L(x+y) = Ax + By$, $A = L/V^n$, $B = L/V^1$. Let W^n be an n-dimensional subspace of $\mathbb{C}^{n+1}$ which is given as the graph of a linear map $\sigma : V^n \to V^1$,*

$$W^n = \{(x, \sigma(x)) \mid x \in V^n\}.$$

If A is invertible and $\|A^{-1}B\sigma\| < 1$ then $L \mid W^n$ is also invertible and

$$\|(L \mid W^n)^{-1}A\| \leq \frac{(1+\|\sigma\|^2)^{1/2}}{1-\|A^{-1}B\sigma\|}.$$

Proof. We wish to solve the equation

$$(L \mid W^n)^{-1}A(v) = x + \sigma(x)$$

where $x \in V^n$ or

$$(L \mid W^n)(x+\sigma(x)) = A(v)$$

or yet

$$Ax + B\sigma(x) = A(v).$$

Inverting A we have

$$x + A^{-1}B\sigma x = v.$$

This last equation can be solved for x by $(1-t)^{-1} = 1 + t + t^2 + \cdots$, $x = (I + A^{-1}B\sigma)^{-1}(v)$ and $\|x\| \leq \frac{1}{1-\|A^{-1}B\sigma\|}\|v\|$. Finally since $\|x + \sigma(x)\| \leq (1+\|\sigma\|^2)^{1/2}\|x\|$, multiplying gives

$$\|x+\sigma(x)\| \leq \frac{(1+\|\sigma\|^2)^{1/2}}{1-\|A^{-1}B\sigma\|}\|v\|.$$

Lemma 3. *Let $x \in \mathrm{Null}(\zeta)+\zeta$. Then Null_x is the graph of $\sigma : \mathrm{Null}(\zeta) \to \mathbb{C}(\frac{\zeta}{\|\zeta\|})$ where $\sigma(w) = -\frac{\langle w, x-\zeta\rangle}{\|\zeta\|}$. (Here $\mathbb{C}(\frac{\zeta}{\|\zeta\|})$ means the subspace generated by $\frac{\zeta}{\|\zeta\|}$.)*

Proof. Given $w \in \mathrm{Null}_\zeta$ we want to find $\sigma(w)$ such that $\langle w+\sigma(w)\frac{\zeta}{\|\zeta\|}, x\rangle = 0$. Solving for $\sigma(w)$

$$\left\langle \sigma(w)\frac{\zeta}{\|\zeta\|}, x\right\rangle = -\langle w, x\rangle$$

but now

$$\left\langle \sigma(w)\frac{\zeta}{\|\zeta\|}, x\right\rangle = \frac{\sigma(w)}{\|\zeta\|}(\langle \zeta, x-\zeta\rangle + \langle \zeta, \zeta\rangle) = \sigma(w)\|\zeta\|$$

and

$$\langle w, x\rangle = \langle w, x-\zeta\rangle + \langle w, \zeta\rangle = \langle w, x-\zeta\rangle$$

since $x - \zeta \in \mathrm{Null}_\zeta$ and $w \in \mathrm{Null}_\zeta$.

Lemma 4.

$$\frac{\|(Df|_{N_x}(x))^{-1}Df(x)x\|}{\|x\|} \leq \mu_{\mathrm{proj.}}(f, x)\eta(f, x)D.$$

Proof. $Df(x)x = \Delta(d_i)f(x)$ is Euler's identity and

$$\begin{aligned}&\frac{\|(Df|_{N_x}(x))^{-1}\Delta(d_i)f(x)\|}{\|x\|}\\ &\quad\leq \|(Df|_{N_x}(x))^{-1}\Delta(d_i^{1/2}\|x\|^{d_i-1})\|\,\|\Delta(d_i)\|\,\|\Delta(d_i^{-1/2}\|x\|^{-d_i})f(x)\|\\ &\quad\leq \mu_{\mathrm{proj.}}(f, x)\eta(f, x)D.\end{aligned}$$

Lemma 5. *Let* $x \in N_\zeta$. *Then*

$$\|(Df|_{N_\zeta}(x))^{-1}Df(x)\zeta\| \le \|\zeta\|\frac{(1-u)^2}{\psi(u)}\left(\mu_{\text{proj.}}(f, \zeta)\eta(f, \zeta)D + \frac{(2-u)u}{(1-u)^2}\right).$$

Proof.

$$\|(Df|_{N_\zeta}(x))^{-1}Df(x)\zeta\| \le \|(Df|_{N_\zeta}(x))^{-1}Df|_{N_\zeta}(\zeta)\| \, \|(Df|_{N_\zeta}(\zeta))^{-1}Df(x)\zeta\|.$$

The first term of the product $\le \frac{(1-u)^2}{\psi(u)}$ by Lemma 3, §II-1. The second term satisfies

$$\|D(f|_{N_\zeta}(\zeta))^{-1}Df(x)\zeta\| = \left\|\sum_{k=0}^{\infty} D(f|_{N_\zeta}(\zeta))^{-1}\frac{D^{k+1}f(\zeta)}{k!}(x-\zeta)^k\zeta\right\|$$

$$\le \|(Df|_{N_\zeta}(\zeta))^{-1}Df(\zeta)\zeta\| + \|\zeta\|\sum_{k=1}^{\infty}(k+1)(\gamma(\zeta)\|x-\zeta\|)^k$$

which by Lemma 4 and summation of the series is less than or equal to

$$\|\zeta\|\left(\mu_{\text{proj.}}(f, \zeta)\eta(f, \zeta)D + \frac{1}{(1-u)^2} - 1\right).$$

Now multiply the two estimates together.

Proof of Proposition 1. We apply Lemma 2 with

$$\sigma(w) = -\frac{\langle w, x-\zeta\rangle}{\|\zeta\|},$$
$$B = Df(x)\frac{\zeta}{\|\zeta\|},$$
$$A = Df|_{N_\zeta}(x).$$

Thus $\|\sigma\| \le r_0$,

$$\|A^{-1}B\sigma\| \le \|A^{-1}B\| \, \|\sigma\| \le \frac{\|(Df|_{N_\zeta}(x))^{-1}Df(x)\zeta\| r_0}{\|\zeta\|}$$

$$\le \frac{(1-u)^2}{\psi(u)}\left(D\mu_{\text{proj.}}(f, \zeta)\eta(f, \zeta) + \frac{(2-u)u}{(1-u)^2}\right)r_0 < 1$$

where the last inequality is a hypothesis of Proposition 1. □

Proposition 2. *As in Proposition* 1, *let* $\beta_\zeta = \beta_0(f|_{N_\zeta}, \zeta)$, $\beta_x = \beta_0(f|_{N_x}, x)$ *etc. Then*

$$\beta_x \le \kappa\frac{(1-u)((1-u)\beta_\zeta + r_0)}{\psi(u)}\frac{\|\zeta\|}{\|x\|},$$
$$\gamma_x \le \frac{\kappa\gamma_\zeta}{\psi(u)(1-u)}\frac{\|x\|}{\|\zeta\|},$$
$$\alpha_x \le \frac{\kappa^2((1-u)\alpha_\zeta + u)}{\psi(u)^2}.$$

This proposition is a consequence of Proposition 2 of §II-1, and the previous proposition.

Proposition 3. *Let $x' \in N_{\zeta'}$ where $\lambda\zeta'$ is the zero of $f|_{N_x}$ associated to x by Newton's method on $f|_{N_x}$ and where x' is one Newton iterate of x. (We suppose $\alpha(f|_{N_x}, x) = \alpha_x \le \alpha_0$.) Then*

$$r_0' = \frac{\|x' - \zeta'\|}{\|\zeta'\|} \le \left(\frac{\tau(\alpha_x) - \alpha_x}{\gamma_0(x)}\right)\left(1 + \sqrt{2}\frac{\tau(\alpha_x)}{\gamma_0(x)}\right).$$

First we prove a lemma.

Lemma 6. *Let $\xi', x' \in N_x$. Let $\pi(x')$ be the radial projection of x' into the space $N_{\xi'}$. Then*

(a)

$$\pi(x') = \frac{\|\xi'\|^2}{\|\xi'\|^2 - \langle \xi' - x', \xi' - x\rangle} x' = \frac{\|\xi'\|^2}{\langle x', \xi'\rangle} x'.$$

(b) *If $\|x - \xi'\| \ge \|x' - \xi'\|$ then*

$$\|\pi(x') - \xi'\| \le \|x' - \xi'\|\left(1 + \sqrt{2}\frac{\|\xi' - x\|}{\|x\|}\right).$$

Proof. (a) Since $\langle \xi' - x', x\rangle = 0$,

$$\|\xi'\|^2 - \langle \xi' - x', \xi' - x\rangle = \langle x', \xi'\rangle$$

so to prove (a) it suffices to prove $\langle \|\xi'\|^2 x'/\langle x', \xi'\rangle - \xi', \xi'\rangle = 0$ which is immediate.

b)

$$\begin{aligned}
\|\pi(x') - \xi'\| &= \left\| x' - \xi' + \left(\frac{\|\xi'\|^2}{\|\xi'\|^2 - \langle \xi' - x', \xi' - x\rangle} - 1\right) x'\right\| \\
&\le \|x' - \xi'\| + \left\| \frac{\langle \xi' - x', \xi' - x\rangle}{\|\xi'\|^2 - \langle \xi' - x', \xi' - x\rangle} x' \right\| \\
&\le \|x' - \xi'\| + \left\| \frac{\langle \xi' - x', \xi' - x\rangle}{\langle x', \xi'\rangle} x' \right\| \\
&\le \|x' - \xi'\| + \frac{\|x' - \xi'\| \, \|\xi' - x\| \, \|x'\|}{|\langle x', \xi'\rangle|} \\
&= \|x' - \xi'\|\left(1 + \frac{\|\xi' - x\| \, \|x'\|}{|\langle x', \xi'\rangle|}\right).
\end{aligned}$$

Now we note that

$$2|\langle x', \xi'\rangle| \ge |\langle x', \xi'\rangle + \langle \xi', x'\rangle| = |\, \|x'\|^2 + \|\xi'\|^2 - \|x' - \xi'\|^2|$$

which follows from expanding $\langle x' - \xi', x' - \xi' \rangle$. Substituting we have by Pythogoras

$$\begin{aligned}\left(\frac{\|\xi' - x\|\,\|x'\|}{|\langle x', \xi'\rangle|}\right)^2 &\le \frac{4\|\xi' - x\|^2\|x'\|^2}{|\,\|x'\|^2 + \|\xi'\|^2 - \|x' - \xi'\|^2|^2} \\ &= \frac{4\|\xi' - x\|^2(\|x\|^2 + \|x - x'\|^2)}{(\|x\|^2 + \|x - x'\|^2 + \|x\|^2 + \|x - \xi'\|^2 - \|x' - \xi'\|^2)^2} \\ &\le \frac{4\|\xi' - x\|^2(\|x\|^2 + \|x - x'\|^2)}{(2\|x\|^2 + \|x - x'\|^2)^2} \\ &\le \frac{4\|\xi' - x\|^2}{(2\|x\|^2 + \|x - x'\|^2)} \le 2\left(\frac{\|\xi' - x\|}{\|x\|}\right)^2;\end{aligned}$$

substituting above yields

$$\|\pi(x') - \xi'\| \le \|x' - \xi'\|\left(1 + \sqrt{2}\frac{\|\xi' - x\|}{\|x\|}\right).$$

Proof of Proposition 3. Let $\pi(x')$ be the radial projection of x' on $N_{\lambda\zeta'}$. Then

$$\frac{\|x' - \zeta'\|}{\|\zeta'\|} = \frac{\|\pi(x') - \lambda\zeta'\|}{\|\lambda\zeta'\|},$$

and

$$\|x - \lambda\zeta'\| \le \frac{\tau(\alpha_x)}{\gamma_x},$$

$$\|x' - \lambda\zeta'\| \le \frac{\tau(\alpha_x) - \alpha_x}{\gamma_x}$$

by Theorem 1, §I-2. Therefore

$$\frac{\|\pi(x') - \lambda\zeta'\|}{\|\lambda\zeta'\|} \le \frac{\tau(\alpha_x) - \alpha_x}{\|\lambda\zeta'\|\gamma_x}\left(1 + \sqrt{2}\frac{\tau(\alpha_x)}{\gamma_x\|x\|}\right)$$

by Lemma 6 and since $\|\lambda\zeta'\| \ge \|x\|$, we are done.

After these preliminary results we go directly to the proof of Theorem 3 (§I-3). So suppose $\eta = \eta(f, \zeta) \le \alpha_{\text{proj.}}/\bar{\gamma}\mu$ as in the hypotheses of the theorem. Let

$$\hat{\alpha} = \hat{\alpha}(u_{\text{proj.}}, \alpha_{\text{proj.}}).$$

We will show

(a) $\alpha_\zeta \le \alpha_{\text{proj.}}$,
(b) $\kappa \le \bar{\kappa}(u_{\text{proj.}}, \alpha_{\text{proj.}}) = \bar{\kappa}$,
(c) $\alpha_x \le \hat{\alpha}$,
(d) $\frac{\|x' - \zeta'\|}{\|\zeta'\|}\bar{\gamma} \le a(\hat{\alpha})(1 + \sqrt{2}\tau(\hat{\alpha}))\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})/\bar{\kappa}$.

First note that (d) with the definition of $\alpha_{\text{proj.}}$, $u_{\text{proj.}}$ yields Theorem 3. Next (a) is a consequence of the bounds on η, γ_0 and Proposition 2 of §I-3. Here is the argument for (b). Observe

(i) $r_0\bar{\gamma} \leq u_{\text{proj.}}$ by hypothesis and $\bar{\gamma} > 1$ so $r_0 < u_{\text{proj.}}$,
(ii) $u = r_0\gamma_0 \leq r_0\bar{\gamma} \leq u_{\text{proj.}}$,
(iii)
$$r_0 D\mu\eta \leq \frac{u_{\text{proj.}}}{\bar{\gamma}} \cdot D\frac{\alpha_{\text{proj.}}}{\bar{\gamma}} = \frac{D}{\bar{\gamma}^2} u_{\text{proj.}}\alpha_{\text{proj.}} \leq u_{\text{proj.}}\alpha_{\text{proj.}}$$
by the hypotheses.

Finally note that $\bar{\kappa}(u, \alpha)$ is monotone in u and α, as long as the denominator doesn't vanish.

Part (c) is a consequence of Proposition 2 and (a), (b).

For (d) we have by Proposition 3 that

$$(*) \qquad \frac{\|x' - \zeta'\|}{\|\zeta'\|} \leq \frac{a(\alpha_x)}{\gamma_{0x}}\left(1 + \sqrt{2}\frac{\tau(\alpha_x)}{\gamma_{0x}}\right).$$

By Proposition 2

$$\gamma_{0x} \leq \frac{\kappa\gamma_{0\zeta}}{\psi(u)(1-u)}\frac{\|x\|}{\|\zeta\|}$$

so by (ii) and the monotonicity of κ, $\frac{1}{\psi(u)(1-u)}$ and the hypothesis that $\gamma_{0\zeta} \leq \bar{\gamma}$ we have

$$\gamma_{0x} \leq \frac{\bar{\kappa}(\alpha_{\text{proj.}}, u_{\text{proj.}})\bar{\gamma}\frac{\|x\|}{\|\zeta\|}}{\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})}.$$

Let

$$H(\bar{\gamma}) = \frac{\bar{\kappa}(\alpha_{\text{proj.}}, u_{\text{proj.}})\frac{\|x\|}{\|\zeta\|}}{\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})}\bar{\gamma}.$$

Then by Lemma 2 of §II-3 and the assumption that $\alpha_x \leq \alpha_0$,

$$\frac{a(\alpha_x)}{\gamma_{0x}} \leq \frac{a(\beta_{0x}H(\bar{\gamma}))}{H(\bar{\gamma})}.$$

By Proposition 2

$$\beta_{0x}H(\bar{\gamma}) \leq \frac{\kappa(1-u)((1-u)\beta_\zeta + r_0)}{\psi(u)}\frac{\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}})\bar{\gamma}}{\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})}.$$

Then $\beta_{0\zeta} \leq \alpha_{\text{proj.}}/\bar{\gamma}$ by Proposition 2 of §I-3 and the hypothesis that $n\ \mu_{\text{proj}} \leq \alpha_{\text{proj}}/\bar{\gamma}$. Also by hypothesis $r_0\bar{\gamma} < u_{\text{proj.}}$. Thus by Lemma 3 of §II-3 and (b)

$$\beta_{0x}H(\bar{\gamma}) \leq \hat{\alpha}(u_{\text{proj.}}, \alpha_{\text{proj.}}) = \hat{\alpha}.$$

By Lemma 1 of §II-3

$$a(\beta_{0x}H(\bar{\gamma})) \leq a(\hat{\alpha}).$$

Thus

$$\frac{a(\alpha_x)}{\gamma_{0x}} \leq \frac{a(\hat{\alpha})}{H(\bar{\gamma})}.$$

As $\frac{\|x\|}{\|\zeta\|} \geq 1$,

$$H(\bar{\gamma}) \geq \frac{\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}})}{\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})}\bar{\gamma}$$

and we have

$$(**) \qquad \frac{a(\alpha_x)}{\gamma_{0x}} \leq \frac{a(\hat{\alpha})\psi(u_{\text{proj.}})(1 - u_{\text{proj.}})}{\bar{\gamma}\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}})}.$$

Now we consider the term $\tau(\alpha_x)/\gamma_{0x}$. By Lemma 2 of §II-3 $\tau(bs)/s$ is also monotone increasing in s and $\tau(t)$ is monotone in t by Lemma 1 of §II-3. Thus as above

$$\frac{\tau(\alpha_x)}{\gamma_{0x}} \leq \frac{\tau(\hat{\alpha})}{\frac{\bar{\gamma}\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}})}{\psi(u_{\text{proj.}})(1-u_{\text{proj.}})}}$$

since $\bar{\gamma} \geq D^{1/2} \geq 1$, $\kappa(\alpha_{\text{proj.}}, u_{\text{proj.}}) > 1$ and $\psi(u_{\text{proj.}})(1 - u_{\text{proj.}}) < 1$ the denominator is > 1. Hence $\tau(\alpha_x)/\gamma_{0x} \leq \tau(\hat{\alpha})$. And

$$(***) \qquad \left(1 + \sqrt{2}\frac{\tau(\alpha_x)}{\gamma_{0x}}\right) \leq (1 + \sqrt{2}\tau(\hat{\alpha})).$$

Multiplying $(**)$ and $(***)$ and substituting in $(*)$ finishes the proof of (d) and hence the theorem.

CHAPTER IV: CHARACTERIZING THE CONDITION NUMBER

IV-1. The projective case $\mu = 1/\rho$

In this section we prove Theorem 1 of §I-4. We begin with the same notation and a preliminary proposition.

Given two n-dimensional complex vector spaces V_1, V_2 with Hermitian structures and a linear map $A : V_1 \to V_2$ we define the Frobenius norm of A, $\|A\|_F$ as $\|M\|_F$ where M is a matrix representation of A with respect to any orthonormal bases of V_1 and V_2. By the following standard lemma, $\|A\|_F$ is well defined.*

Lemma 1. *Let A, V_1, V_2 be $n \times n$ matrices with V_1, V_2 unitary. Then $\|V_1 A V_2\|_F = \|A\|_F$.*

Let $0 \neq x \in \mathbb{C}^{n+1}$. Let $L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$ be the subspace of linear maps vanishing at x. Let $\mathscr{L}_x \subset \mathscr{H}_{(d)}$ be the subspace of maps $f = (f_1, \dots, f_n)$ of the form $f_i(z) = (\langle z, x\rangle^{d_i - 1}/\langle x, x\rangle^{d_i - 1})L_i$ and $L = (L_1, \dots, L_n) \in L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$. Let $D_x : \mathscr{H}_{(d)} \to L(\mathbb{C}^{n+1}, \mathbb{C}^n)$ be the derivative $f \to D_x f$. Recall $\widehat{V}_x = \{f \in \mathscr{H}_{(d)} \mid f(x) = 0\}$. For $f \in \widehat{V}_x$, $D_x f(x) = 0$ since f is constantly zero on the ray through x. Thus we may consider $D_x : \widehat{V}_x \to L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$. Let $G_x = \{f \in \widehat{V}_x \mid D_x f = 0\}$.

* $\|A\|_F$ is the same as the Hilbert-Schmidt norm $(\text{trace}(A^* A))^{1/2}$.

Proposition 1. (a) $\widehat{V}_x$ *in the Hermitian direct sum* $\mathscr{L}_x \oplus G_x$.

(b) *For* $h \in \mathscr{L}_x$, $\|h\| = \|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})Dh(x)|_{\text{Null}_x}\|_F$.

For the proof of this proposition we prove two lemmas.

Let $u : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a unitary transformation and $\hat{u} : \widehat{V}_x \to \widehat{V}_{ux}$ the induced isometry $\hat{u}(f) = f \circ u^{-1}$.

Lemma 2. *Let* $u : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *be unitary. Then*

(a) $\hat{u}(\mathscr{L}_x) = \mathscr{L}_{ux}$,

(b) $\hat{u}(G_x) = G_{ux}$.

Proof. (a) Let $f \in \mathscr{L}_x$ so

$$f = (f_1, \dots, f_n), \qquad f_i(z) = \frac{\langle z, x\rangle^{d_i-1}}{\langle x, x\rangle^{d_i-1}} L_i(z),$$

with $L = (L_1, \dots, L_n) \in L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$. But then $L \circ u^{-1} = (L_1 \circ u^{-1}, \dots, L_n \circ u^{-1}) \in L_{ux}(\mathbb{C}^{n+1}, \mathbb{C}^n)$ and $f \circ u^{-1} = (f_1 \circ u^{-1}, \dots, f_n \circ u^{-1})$ where

$$f_i \circ u^{-1}(z) = \frac{\langle u^{-1}z, x\rangle^{d_i-1}}{\langle x, x\rangle^{d_i-1}}, \qquad L_i \circ u^{-1}(z) = \frac{\langle z, ux\rangle^{d_i-1}}{\langle ux, ux\rangle^{d_i-1}} (L \circ u^{-1})_i z.$$

This shows $\hat{u}(\mathscr{L}_x) \subset \mathscr{L}_{ux}$, but as $\hat{u}^{-1} = \widehat{u^{-1}}$, $\hat{u}^{-1}(\mathscr{L}_{ux}) \subset \mathscr{L}_x$ and $\hat{u}(\mathscr{L}_x) = \mathscr{L}_{ux}$.

(b) By the chain rule, $D(f \circ u^{-1}(u(x))) = 0$ iff $Df(x) \circ u^{-1} = 0$ which holds iff $Df(x) = 0$.

Lemma 3. *Let* $L \in L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$ *and* $f(z) = (f_1(z), \dots, f_n(z))$ *where* $f_i(z) = (\langle z, x\rangle^{d_i-1}/\langle x, x\rangle^{d_i-1})L_i(z)$. *Then*

(a) $Df(x) = L$,

(b) $\|f\| = \|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})Df(x)|_{\text{Null}_x}\|_F$.

Proof. (a) $f(z) = \Delta(\langle z, x\rangle^{d_i-1}/\langle x, x\rangle^{d_i-1})L(z)$ so

$$\begin{aligned} Df(x)v &= \Delta\left(D\left(\frac{\langle z, x\rangle^{d_i-1}}{\langle x, x\rangle^{d_i-1}}\right)(x)v\right)L(x) + \Delta\left(\frac{\langle x, x\rangle^{d_i-1}}{\langle x, x\rangle^{d_i-1}}\right)L(v) \\ &= 0 + L(v). \end{aligned}$$

(b) Let u be the unitary transformation mapping x to $\|x\|e_0$.

By Lemma 2

$$f \circ u^{-1} = h = (h_1, \dots, h_n)$$

where

$$h_i(z) = \frac{z_o^{d_i-1}}{\|x\|^{d_i-1}} \sum_{j=1}^{n} a_{ij} z_j, \qquad z = (z_0, \dots, z_n)$$

and where

$$L_i \circ u^{-1} = \sum_{j=1}^{n} a_{ij} z_j.$$

Thus

$$\|f\| = \|f \circ u^{-1}\| = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|a_{ij}|^2}{(\|x\|^{d_i-1})^2 d}$$
$$= \|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)}) L \circ u^{-1}|_{\mathrm{Null}_{e_0}}\|_F$$
$$= \|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)}) L|_{\mathrm{Null}_x}\|_F$$

by Lemma 1.

Proof of Proposition 1. (a) $D_x : V_x \to L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$ is linear. The kernel is G_x by definition, and $D_x : \mathcal{L}_x \to L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$ is an isomorphism by Lemma 3(a). Thus V_x is the direct sum of $\mathcal{L}_x$ and G_x. We need only check they are orthogonal. As $\mathcal{L}_{\lambda x} = \mathcal{L}_x$ and $G_{\lambda x} = G_x$ for $\lambda \in \mathbb{C}$, $\lambda \neq 0$ it is sufficient to do this for $\|x\| = 1$ and by Lemma 2 for $x = e_0$. In this case $\mathcal{L}_{e_0} = (f_1, \ldots, f_n)$ and

$$f_i(z) = \sum_{j=1}^{n} a_{ij} z_j z_0^{d_i-1}.$$

If $g = (g_1, \ldots, g_n)$ and $g(e_0) = 0$ then $g_i(z) = \sum a_{ij} z_i z_0^{d_i-1} + \sum_\alpha a_\alpha z^\alpha$ where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ and $\alpha_0 \leq d_i - 2$. Then

$$D\left(\sum_\alpha a_\alpha z^\alpha\right)(e_0) = 0$$

where $\alpha_0 \leq d_i - 2$. Thus for $Dg(e_0) = 0$ all the $a_{ij} = 0$. This establishes the orthogonality.

1(b) is Lemma 3(b).

Proof of Theorem 1. First we prove that $\mu_{\mathrm{proj.}}(f, x) \geq \frac{1}{\rho(f,x)}$: Let $(g, x) \in \Sigma'$ be such that $\rho(f, x) = dp(f, g) = \frac{\|f-g\|}{\|f\|}$ and let $f - g = h$. First we claim that $h \in \mathcal{L}_x$. By Proposition 1

$$h = h_{\mathcal{L}_x} + h_{G_x}, \qquad h_{\mathcal{L}_x} \in \mathcal{L}_x, h_{G_x} \in G_x,$$

and $\|h\| \geq \|h_{\mathcal{L}_x}\|$ with equality iff $h = h_{\mathcal{L}_x}$. Since $D_x h_{G_x} = 0$, $D(g + h_{G_x})(x) = Dg(x)$ and $(g + h_{G_x}, x) \in \Sigma'$ but $dp(f, g + h_{G_x}) \leq \|h_{\mathcal{L}_x}\|/\|f\|$. Thus $\|h\| = \|h_{\mathcal{L}_x}\|$ and $h = h_{\mathcal{L}_x}$.

That $g \in \Sigma' \cap V_x$ means that $D_x(f - h)$ is singular or

$$\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})(D_x(f-h))$$

is singular. It follows that

$$d_F(\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)}) Df(x)|_{\mathrm{Null}_x}, S)$$
$$\leq \|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)}) Dh(x)|_{\mathrm{Null}_x}\|_F = \|h\|$$

by Proposition 1. By Proposition 1 of §I-4

$$\|(\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})Df(x)|_{\mathrm{Null}_x})^{-1}\| \geq \frac{1}{\|h\|}$$

and hence $\mu_{\mathrm{proj.}}(f, x) \geq \frac{\|f\|}{\|h\|} = \frac{1}{\rho(f, x)}$. Now we prove the opposite inequality

$$\mu_{\mathrm{proj.}}(f, x) \leq \frac{1}{\rho(f, x)}.$$

Suppose $\|(\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})Df(x)|_{\mathrm{Null}_x})^{-1}\| = \kappa$. So $\mu_{\mathrm{proj.}}(f, x) = \kappa\|f\|$. Then by Proposition 1 of §I-4 there is a linear map $B : \mathrm{Null}_x \to \mathbb{C}^n$ such that $\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})(Df(x)|_{\mathrm{Null}_x} - B) \in S$ and $\|\Delta(d_i^{-1/2}\|x\|^{-(d_i-1)})B\|_F = \frac{1}{\kappa}$. Extend B to $\widehat{B} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ by making $\widehat{B}(x) = 0$ so $\widehat{B} \in L_x(\mathbb{C}^{n+1}, \mathbb{C}^n)$. Let $\widehat{B}_i$ be the ith coordinate of B and $h = (h_1, \ldots, h_n)$ where

$$h_i = (\langle z, x\rangle^{d_i-1}/\langle x, x\rangle)\widehat{B}_i(z).$$

By Lemma 3(a) $Dh|_{\mathrm{Null}_x} = \widehat{B}|_{\mathrm{Null}_x} = B$ and $f - h \in \Sigma' \cap V_x$. By Proposition 1 $\|h\| = \frac{1}{\kappa}$ and $\rho(f, x) \leq \frac{\|h\|}{\|f\|} = \frac{1}{\kappa\|f\|} = \frac{1}{\mu_{\mathrm{proj.}}(f, x)}$.

IV-2. Bounds on zeros and the affine case

We first prove Proposition 2 of §I-4. It follows immediately from

Theorem 1. *Let $f \in \mathscr{H}_{(d)}$ and $x \neq 0 \in \mathbb{C}^{n+1}$ with $f(x) = 0$. Then*

$$d(f, \Sigma_0) \leq \frac{|x_0|}{\|x\|}\|\Delta(d_i^{1/2})f\|.$$

For the proof we first construct a perturbation $H \in \mathscr{H}_{(d)}$. Let $H_i(z) = \hat{f}_i(x)(\langle z, x\rangle^{d_i}/\langle x, x\rangle^{d_i})$ be the ith coordinate of H where $\hat{f}_i$ is the "highest order homogeneous part" of f_i. Precisely $\hat{f}_i(z) = f_i(z)|_{z_0=0}$ or yet $\hat{f}_i$ consists of the sum of monomials of f_i which do not contain z_0. Note that we have immediately that $H(x) = \hat{f}(x)$ so that $f - H \in \Sigma_0$.

The theorem is thus a consequence of

Lemma 1. *Under the hypotheses of Theorem* 1

$$\|H_i\| \leq \frac{|x_0|}{\|x\|}\|d_i^{1/2}f_i\|.$$

Proof of Lemma 1. Using unitary invariance of the norm it is easy to see that

$$\|H_i\| = \frac{|\hat{f}_i(x)|}{\|x\|^{d_i}} = \frac{1}{\|x\|}\left(\frac{|f_i(x) - \hat{f}_i(x)|}{\|x\|^{d_i-1}}\right) = \frac{|x_0|}{\|x\|}\frac{|g(x)|}{\|x\|^{d_i-1}}$$

where $z_0g(z) = f_i(z) - \hat{f}_i(z)$ and degree g is $d_i - 1$. Thus by Proposition 1 of §III-1, $\|H_i\| \leq (|x_0|/\|x\|)\|g\|$.

Thus for Lemma 1 and Theorem 1 it is sufficient to prove

Lemma 2. $\|g\|^2 \le d_i \|f_i\|^2$.

Note that all the terms of $f_i(x) = \sum_{|\alpha|=d_i} a_\alpha x^\alpha$ where $\alpha_0 = 0$ have been subtracted off by $\hat{f}_i$, and the coefficients of g are the remaining a_α with α_0 reduced by one, i.e.,

$$g(x) = \sum_{|\alpha|=d_i-1} a_{(\alpha_0+1,\ldots,\alpha_n)} x^\alpha.$$

Therefore

$$\|g\|^2 = \sum_{|\alpha|=d_i-1} |a_{\alpha_0+1,\ldots,\alpha_n}|^2 \frac{(\alpha_0)!\cdots\alpha_n!}{(d_i-1)!}$$

and

$$\|f_i\|^2 \ge \sum_{|\alpha|=d_i-1} |a_{\alpha_0+1,\ldots,\alpha_n}|^2 \frac{(\alpha_0+1)!\cdots\alpha_n!}{d_i!}$$

so $\|g\|^2 \le d_i \|f_i\|^2$. □

We proceed to the proof of Theorem 2 of §I-4.

Lemma 3. *Let $x, \xi \in \mathbb{C}^{n+1}$ such that $\langle x, \xi\rangle \ne 0$. Let $\pi_\xi : \mathbb{C}^{n+1} \to \mathrm{Null}_\xi$ and $\pi_x : \mathbb{C}^{n+1} \to \mathrm{Null}_x$ be the orthogonal projections. Then*

$$\frac{\|\pi_x \pi_\xi(x)\|}{\|\pi_\xi(x)\|} = \left|\left\langle \frac{\xi}{\|\xi\|}, \frac{x}{\|x\|}\right\rangle\right|$$

and $\pi_\xi(x)$ is orthogonal to $\mathrm{Null}_x \cap \mathrm{Null}_\xi$.

Proof. $\pi_\xi(x) = x - \frac{\langle x,\xi\rangle\xi}{\langle \xi,\xi\rangle}$. Now if $w \in \mathrm{Null}_x \cap \mathrm{Null}_\xi$ then $\langle x, w\rangle$ and $\langle \xi, w\rangle$ are both zero, so $\langle \pi_\xi(x), w\rangle = 0$ and $\pi_\xi(x)$ is orthogonal to $\mathrm{Null}_x \cap \mathrm{Null}_\xi$. Let $v = \pi_\xi(x)$.

$$\pi_x(v) = v - \frac{\langle v, x\rangle}{\langle x, x\rangle^2} x.$$

Note that $\langle v, x\rangle = \langle v, v\rangle$ since $v = \pi_\xi(x)$. Thus $\pi_x(v) = v - (\|v\|^2/\|x\|^2)x$.

$$\|v\|^2 = \|\pi_x v\|^2 + \frac{\|v\|^4}{\|x\|^4}\|x\|^2$$

and

$$\frac{\|\pi_x v\|^2}{\|v\|^2} = 1 - \frac{\|v\|^2}{\|x\|^2} = \frac{\|x - v\|^2}{\|x\|^2} \quad \text{by Pythagoras}$$

$$= \frac{|\langle x, \xi\rangle|^2}{\|\xi\|^2\|x\|^2} \quad \text{by the definition of } v.$$

Lemma 4. *Let $x, \xi \in \mathbb{C}^{n+1}$ such that $\langle x, \xi\rangle \ne 0$. Let $\pi_\xi : \mathbb{C}^{n+1} \to \mathrm{Null}_\xi$ be the orthogonal projection. Then $\|(\pi_\xi \mid \mathrm{Null}_x)^{-1}\| = \frac{\|\xi\|\,\|x\|}{|\langle x,\xi\rangle|}$.*

Proof. Let $v_1, \dots, v_{n-1}$ be an orthonormal basis of $\text{Null}_x \cap \text{Null}_\xi$. Then $v_1, \dots, v_n$, $\pi_x\xi/\|\pi_x\xi\|$ is an orthonormal basis of Null_x and $v_1, \dots, v_n$, $\pi_\xi\pi_x\xi$ is an orthonormal basis of Null_ξ. Let

$$v = \sum_{i=1}^n a_i v_i + a_{n+1}\frac{\pi_\xi\pi_x\xi}{\|\pi_\xi\pi_x\xi\|}.$$

Then

$$\|v\| = \left(\sum |a_i|^2\right)^{1/2}, \qquad \pi_\xi^{-1}(v) = \sum_{i=1}^n a_i v_i + a_{n+1}\frac{(\pi_x\xi)}{\|\pi_\xi\pi_x\xi\|}$$

and

$$\begin{aligned}\|\pi_\xi^{-1}(v)\| &= \left(\sum_{i=1}^n |a_i|^2 + \frac{|a_{n+1}|^2\|\pi_x\xi\|^2}{\|\pi_\xi\pi_x\xi\|^2}\right)^{1/2} \\ &= \left(\sum_{i=1}^n |a_i|^2 + |a_{n+1}|^2\frac{\|\xi\|^2\|x\|^2}{|\langle x, \xi\rangle|^2}\right)^{1/2}\end{aligned}$$

by the previous lemma. This is less than or equal to $\frac{\|\xi\|\,\|x\|}{|\langle x, \xi\rangle|}(\|v\|)$ with equality if all $a_i = 0$ for $i = 1, \dots, n$, $a_{n+1} \neq 0$. □

Proposition 1. *Let* $A : \mathbb{C}^{n+1} \to \mathbb{C}^n$ *be linear. Suppose* $A(\xi) = 0$ *and* $A \mid \text{Null}_\xi$ *is invertible. Let* $x \in \mathbb{C}^{n+1}$ *such that* $\langle x, \xi\rangle \neq 0$. *Then* $A \mid \text{Null}_x$ *is invertible and*

$$\|(A \mid \text{Null}_x)^{-1}\| \le \frac{\|\xi\|\,\|x\|}{|\langle x, \xi\rangle|}\|(A \mid \text{Null}_\xi)^{-1}\|.$$

Proof.

$$\begin{aligned}\|(A\,\text{Null}_x)^{-1}\| &= \|(A \mid \text{Null}_\xi) \circ \pi_\xi \mid \text{Null}_x)^{-1}\| \\ &\le \|(\pi_\xi \mid \text{Null}_x)^{-1}\|\|(A \mid \text{Null}_\xi)^{-1}\|\end{aligned}$$

and the previous lemma finishes the proof.

Theorem 2 of §I-4 now follows from Proposition 1. Let

$$A = \Delta(d_i^{-1/2})\Delta(\|\xi\|^{-(d_i-1)})Df(\xi)$$

and $x = e_0$. Then $\|x\| = 1$ and $\langle x, \xi\rangle = e_0$.

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