Geom Dedicata
DOI 10.1007/s10711-009-9402-1

ORIGINAL PAPER

Taming 3-manifolds using scalar curvature

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Received: 6 July 2009 / Accepted: 14 July 2009

Abstract In this paper we address the issue of uniformly positive scalar curvature on noncompact 3-manifolds. In particular we show that the Whitehead manifold lacks such a metric, and in fact that $\mathbb{R}^3$ is the only contractible noncompact 3-manifold with a metric of uniformly positive scalar curvature. We also describe contractible noncompact manifolds of higher dimension exhibiting this curvature phenomenon. Lastly we characterize all connected oriented 3-manifolds with finitely generated fundamental group allowing such a metric.

Keywords Positive scalar curvature · Noncompact manifolds · Whitehead manifold

Mathematics Subject Classification (2000) 53C21 · 19K56 · 57N10 · 57M40

1 Introduction

If M is an n-dimensional endowed with a Riemannian metric g, then its scalar curvature $\kappa : M \to \mathbb{R}$ satisfies the following property. At each point $p \in M$ there is an expansion

$$\mathrm{Vol}_M(B_\varepsilon(p)) = \mathrm{Vol}_{\mathbb{R}^n}(B_\varepsilon(0)) \left(1 - \frac{\kappa(p)}{6(n+2)}\,\varepsilon^2 + \cdots\right)$$

Research partially supported by NSF Grants DMS-0405867, DMS-0805913 and DMS-0600216.

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Published online: 06 August 2009

for all sufficiently small $\varepsilon > 0$. A complete Riemannian metric g on a manifold M is said to have uniformly positive scalar curvature if there is fixed constant $\kappa_0 > 0$ such that $\kappa(p) \geq \kappa_0 > 0$ for all $p \in M$. In the compact setting, obstructions to such metrics are largely achieved in one of two ways: (1) the minimal surface techniques in dimensions at most 7 by Schoen-Yau and more recently in all dimensions by Christ-Lohkamp; (2) the K-theoretic Dirac index method for spin manifolds by Atiyah-Singer and its generalizations by Hitchin, Gromov, Lawson and Rosenberg.

In the realm of noncompact manifolds it is now well recognized that the original approach by [9] and [26] proving that no compact manifold of nonpositive sectional curvature can be endowed with a metric of positive scalar curvature, is actually based on a restriction on the coarse quasi-isometry type of complete noncompact manifolds. Block and Weinberger [2] investigate the problem of complete metrics for noncompact symmetric spaces when no quasi-isometry conditions are imposed. They prove that, if G is a semisimple Lie group with maximal compact subgroup K and irreducible lattice Γ, then the double quotient $M \equiv \Gamma \backslash G/K$ can be endowed with a complete metric of uniformly positive scalar curvature if and only if Γ is an arithmetic group with $\mathrm{rank}_{\mathbb{Q}}\Gamma \geq 3$. This theorem includes previously derived results by Gromov-Lawson in rational rank 0 and 1, in view of the characterization of compactness in terms of rational rank by Borel and Harish-Chandra. In the case when the rational rank exceeds 2, Chang proves that any metric on M with uniform positive scalar curvature fails to be coarsely equivalent to the natural one [5].

For noncompact (spin) manifolds that are diffeomorphic to interiors of manifolds with boundary, there is a reasonable plan of attack: one can define a index of a Dirac operator that lies in a real K-theory of pairs which presumably obstructs the existence of a complete uniformly positive scalar curvature metric. We note that the condition for a space to be the interior of a manifold with boundary involves a mixture of fundamental group and homological conditions, together with the Siebenmann obstruction in $KO(\mathbb{Z}\pi_1^{\infty})$. Here π_1^{∞} is the inverse limit of the fundamental groups of complements of a sequence of compact sets that exhaust the manifold.

It is fairly straightforward to connect this notion to index theory when the fundamental group at infinity injects into $\pi_1(M)$. As a result, for groups which coarsely embed in Hilbert space, this obstruction must vanish. See [27] and [28] in addition to the argument in [2]. If the fundamental group at infinity does not inject into $\pi_1(M)$, the analytic setting for proving such a vanishing result is less apparent, and will be discussed in a sequel article. In this paper, we will discuss 3-dimensional noncompact manifolds, in which venue there is relevant fundamental group information at infinity completely different than that in the case for interiors of compact manifolds.

Our first main result asserts that the only non-compact contractible 3-manifold with uniformly positive scalar curvature is $\mathbb{R}^3$. Uncountably many such non-compact contractible 3-manifolds are known, the most famous being the Whitehead manifold W, which we discuss first in Sect. 2. This manifold has a trivial fundamental group at infinity but it not simply connected at infinity, and thereby demonstrates much richer structure than $\pi_1^{\infty}(W)$ is able to detect. Here we will give an ad hoc argument mixing 3-manifold topology with known facts about the Novikov conjecture to prove this result. We will also prove a general taming theorem, described below. Our later paper will unify this case and the tame situation for which the fundamental group at infinity does not inject into that of the entire manifold.

Our theorems are as follows:

Theorem 1 *If M^3 is a three-dimensional contractible manifold with a complete metric of uniformly positive scalar curvature, then M^3 must be homeomorphic to $\mathbb{R}^3$.*

Theorem 2 *For all $n \geq 3$ there is a contractible manifold M^n of dimension n with no complete metric of uniformly positive scalar curvature.*

Theorem 3 *Suppose that M is a connected oriented 3-manifold whose fundamental group is finitely generated, and M has a complete Riemannian metric with uniformly positive scalar curvature. Then we have the following.*

(1) *The space M is homeomorphic to a connected sum of space forms (quotients of the 3-sphere) and copies of $S^2 \times S^1$. In fact it can be compactified to a compact manifold $\widetilde{M}$ so that the set $P = \widetilde{M} - M$ of boundary points is a totally disconnected set.*
(2) *Further, if M is homotopy equivalent to a finite complex (or even has finite second Betti number), then P can be taken to be finite.*

Remark Compact 3-manifolds can be geometrized by the work of Perelman, and their positive scalar curvature properties are well known. For instance, Theorem 3 implies Theorem 1 in light of the Poincaré conjecture. The final section of the paper gives a version of Theorem 3 that strongly relates to Perelman's work, and allows for infinite homotopy type. It is also worth noting here that the complement N of a Cantor set in S^3 is a non-tame manifold with a complete uniformly positive scalar curvature metric, showing that the homotopy condition in Theorem 3 is necessary. In fact, the space N is the universal cover of a connected sum $L^3 \# L^3$ of Lens spaces, which has a uniformly positive scalar curvature metric by [8] and [26].

2 The Whitehead manifold

Definition A topological space V is said to be *simply connected at infinity* if, for all compact subsets C of V, there is a compact set D in X containing C so that the induced map $\pi_1(V - D) \to \pi_1(V - C)$ is trivial.

The following theorem of Stallings indicates that there is topologically a unique noncompact manifold that is both contractible and simply connected at infinity.

Theorem (Stallings) *Let $n \geq 4$ and consider a contractible manifold V. The following are equivalent.*

(1) *The manifold V is homeomorphic to $\mathbb{R}^n$.*
(2) *The manifold V is simply connected at infinity.*
(3) *There are compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$ such that $\bigcup_{i=1}^{\infty} K_i = V$ and $\pi_1(V - K_i)$ is trivial for all i.*
(4) *For every sequence $K_1 \subset K_2 \subset K_3 \subset \cdots$ of compact subsets with $\bigcup_{i=1}^{\infty} K_i = V$, the sequence $\pi_1(V - K_1) \leftarrow \pi_1(V - K_2) \leftarrow \cdots$ is protrivial; i.e. $\varprojlim \pi_1(V - K_i)$ is trivial.*

Stalling's theorem is however not true in dimension 3. The Whitehead manifold is an open contractible 3-manifold that satisfies (4) but is not homeomorphic to $\mathbb{R}^3$. It is constructed in the following manner: we start with a copy of the three-sphere S^3 and identify an unknotted torus T_1 inside of it. Take another torus T_2 inside T_1 such that T_2 forms a thickened Whitehead link with a tubular neighborhood of the meridian curve in T_1. The torus T_2 is null-homotopic in the complement of the meridian of T_1. Inductively embed T_n in T_{n-1} in the same manner and let $K = \bigcap_{n=1}^{\infty} T_n$. Define the Whitehead manifold to be $W = S^3 \backslash K$, which is noncompact without boundary. By the Hurewicz theorem and Whitehead's theorem, it follows that W is contractible. Although $\pi_1^{\infty}(W)$ is trivial, it is not simply connected at infinity, so the Whitehead manifold is not homeomorphic to $\mathbb{R}^3$. This space is a standard example of a nontame manifold.

Some properties of the Whitehead manifold W are given as follows ([1,3]):

(1) $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$;
(2) Every homeomorphism of W to itself is orientation-preserving;
(3) The one-point compactification S^3/K is not a manifold but $(S^3/K) \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.

We can imagine the Whitehead manifold as the following union:

$$W = T_1^c \cup (T_1 - T_2) \cup_{N_2} (T_2 - T_3) \cup_{N_3} (T_3 - T_4) \cup_{N_4} \cdots$$

where N_i is the boundary of T_i for each i and T_1^c is the complement of T_1 in S^3. For each i let $A_i = T_i - T_{i+1}$ and let ∂_i^0 be the outer boundary of A_i and ∂_i^1 the inner boundary. Then there are injections $\pi_1(\partial_i^0) \to \pi_1(A_i)$ and $\pi_1(\partial_i^1) \to \pi_1(A_i)$.

Let K_i be those compact sets such that $W = \bigcup_{i=1}^n K_i$ and $S^3 - \mathrm{Int}(K_i) = T_i$. Then there are maps $\pi_1(W - K_{i+1}) \to \pi_1(W - K_i)$ which gives rise to maps

$$\pi_1(A_i) *_{\pi_1(\partial_i^1)} \pi_1(A_{i+1}) *_{\pi_1(\partial_{i+1}^1)} \cdots \longrightarrow \pi_1(A_{i+1}) *_{\pi_1(\partial_{i+1}^1)} \pi_1(A_{i+2}) *_{\pi_1(\partial_{i+2}^1)} \cdots$$

The inverse limit of this diagram is very small, but the $\pi_1(S^3 - K_i)$ are infinitely generated.

3 Positive scalar curvature and the Whitehead manifold

To prove that $\mathbb{R}^3$ is the only contractible 3-manifold with a complete Riemannian metric of uniformly positive scalar curvature, we first eliminate the Whitehead manifold by an index-theoretic argument. In the next section, we show that the general topological results of this section imply that all contractible 3-manifolds have similar enough topological structure to the Whitehead manifold for the same proof to apply.

We will rely on the following three basic theorems of Papakyriakopolous:

Theorem 1 (Dehn Lemma) *Let M be a 3-manifold with boundary. If $\ell \in \partial M$ is an embedded loop which is trivial in $\pi_1(M)$, then there is an embedded disk $D^2 \subset M$ such that $\ell = D^2 \cap \partial M$.*

Theorem 2 (Loop Theorem) *Let M be a 3-manifold with boundary. Suppose that the map $i_*\colon \pi_1(\partial M) \to \pi_1(M)$ is not injective. Then there is an essential loop $\ell \in \partial M$, i.e. ℓ is not trivial in $\pi_1(\partial M)$ such that $i_*(\ell)$ is trivial in $\pi_1(M)$.*

Theorem 3 (Sphere Theorem) *If M is an oriented 3-manifold such that $\pi_2(M)$ is nontrivial, then there is an embedded essential 2-sphere S^2 in M.*

Corollary *If L is a link in S^3 and $S^3 - L$ is not aspherical, then $\pi_2(S^3 - L) \neq 0$ and there is an essential S^2 splitting the link.*

Proof Let $M = S^3 - L$ and let $\widetilde{M}$ be the universal cover. Then $H_1(\widetilde{M}) = 0$ and $H_n(\widetilde{M}) = 0$ for all $n \geq 3$. If $\pi_2(\widetilde{M}) = 0$, then $H_2(\widetilde{M}) = 0$ by the Hurewicz theorem, so all homotopy groups of $\widetilde{M}$ vanish. Therefore M is aspherical, a contradiction. The sphere theorem then gives an embedded essential 2-sphere in M. If it does not split the link, then it is nullhomotopic, a contradiction. □

Lemma *Suppose that $K \subset S^1 \times D^2$ is a knot. The following are equivalent.*

(1) *The knot K lies in some ball B^3 in $S^1 \times D^2$.*
(2) *The space $(S^1 \times D^2) - K$ is not aspherical.*
(3) *The fundamental group $\pi_1(\partial(S^1 \times D^2))$ does not inject into $\pi_1((S^1 \times D^2) - K)$.*
(4) *There is an embedded disk $D \subseteq S^1 \times D^2$ with $\partial D \subseteq \partial(S^1 \times D^2)$ and not null-homotopic in $\partial(S^1 \times D^2)$ such that $D \cap K = \emptyset$.*

Proof The equivalence of (1) and (4) is clear. If (4) holds, then let D^2 be chosen so that $D^2 \cap K = \emptyset$ and ∂D^2 is a nontrivial loop in $\pi_1(S^1 \times D^2)$. The loop can be contracted in $S^1 \times D^2 - K$, thus giving (3). Conversely, if (3) holds, then by the Loop Theorem, there is a loop $\ell \in \partial(S^1 \times D^2)$ that vanishes in $S^1 \times D^2 - K$. Choose a tubular neighborhood $N(K)$ of K such that ℓ vanishes in $M = S^1 \times D^2 - N(K)$. By the Dehn Lemma there is a disk D such that $D^2 \cap \partial(S^1 \times D^2 - N(K)) = \ell$, so $D \cap K = \emptyset$. Statements (1) and (2) are equivalent by the Sphere Theorem and homological arguments. □

Corollary *In our construction at the end of Sect. 2, each A_i is aspherical.*

Theorem 3.1 *Any complete Riemannian metric on the Whitehead manifold does not have uniformly positive scalar curvature in the complement of any compact subset.*

Proof Suppose that the Whitehead manifold W has a metric of uniformly positive scalar curvature and write

$$W = (S^1 \times D^2) \cup_{T^2} A_1 \cup_{T^2} A_2 \cup_{T^2} \cdots$$

as at the end of Sect. 2. Note that each A_i is the complement in a solid torus of a non-split link, and is aspherical by the above Lemma. We cut the manifold W along the first T^2 and glue the noncompact piece with its double to obtain a two-ended manifold N of the form $\cdots \overline{A_2} \cup \overline{A_1} \cup A_1 \cup A_2 \cdots$, where $\overline{A_i}$ is A_i with the opposite orientation. This N has uniformly positive scalar curvature away from the glueing between A_1 and $\overline{A_1}$. In particular, the manifold N has uniformly positive scalar curvature outside of a compact set. Consider $\Gamma = \pi_1(N) = \pi_1(W - B) *_{\pi_1(T^2)} \pi_1(W - B)$, where $B = S^1 \times D^2$ with boundary T^2. □

By the equivariant version[1] of Roe's partitioned index theorem proved in [2], we know that the index of the Dirac class $[D_{T^2}]$ vanishes in $K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^*-algebra of Γ. We will show that this result yields a contradiction.

Consider the commutative diagram given as follows:

$$\begin{tikzcd} K_0(T^2) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & K_0(B\Gamma) \arrow[r, "\phi"] & K_0(C_r^*\Gamma) \\ K_0(C_r^*\mathbb{Z}^2) \arrow[urr, "\psi"'] & & \end{tikzcd}$$

Note that $K_0(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. The Fredholm index lies in the first copy of $\mathbb{Z}$; the projection of the K-homology class of the Dirac operator onto the second copy of $\mathbb{Z}$ is nonzero.

[1] This theorem states that, if Z is a noncompact spin manifold with uniformly positive scalar curvature at infinity, and if V is a spin hypersurface that partitions Z into two pieces, then the index of the Dirac operator on V vanishes in $K_{n-1}(C_r^*(\pi_1(Z)))$, where n is the diemnsion of Z.

This second copy of $\mathbb{Z}$ injects into the K-homology of $B\Gamma$ (note that the manifold N is an Eilenberg-MacLane space) since the torus separates the noncompact manifold N into two noncompact pieces. In fact, the map $K_0(T^2) \to K_0(B\Gamma)$ is injective.

Therefore the image of $[D_{T^2}] \in K_0(T^2)$ under the map α is nonzero in $K_0(B\Gamma)$. The rational strong Novikov conjecture holds for all two-dimensional cohomology classes ([7], [13]). Via the Chern map, we can conclude that the image of $\alpha([D_{T^2}])$ under the map $\phi\colon K_0(B\Gamma) \to K_0(C_r^*\Gamma)$ is nonzero (note that $K_0(T^2)$ is torsion-free, so the image of the nonzero class $[D_{T^2}]$ under the map $K_0(T^2) \to K_0(B\Gamma)$ has infinite order, hence its image under the composition $\phi \circ \alpha$ is rationally nonzero, and therefore integrally nonzero). As a result the index $\mathrm{ind}(D_{T^2})$ in $K_0(C_r^*\Gamma)$ is nonzero, contradicting the above result.

Remark Instead of the 2-dimensional Novikov Conjecture, one can use Pimsner's theorem [21] to show that the Baum–Connes conjecture holds for the group Γ, using the fact that injections must commute with direct limits.

Theorem 3.2 *For all $n \in \mathbb{Z}_{\geq 2}$, there is a contractible manifold M^n having no complete Riemannian metric with uniformly positive scalar curvature outside a compact set.*

Proof We first note that $\mathbb{R}^2$ and the Whitehead manifold do not have metrics of uniformly positive scalar curvature. For tame manifolds, i.e. manifolds diffeomorphic to the interiors of compact manifolds, a sufficient condition for the absence of a metric with uniformly positive scalar curvature outside a compact set is that the fundamental class of the boundary be nonzero in $K_{n-1}(B\pi_1(\partial M))$ and that the strong Novikov conjecture hold for $\pi_1(\partial M)$. For $n = 4$, the Mazur manifolds [15] are such examples. Indeed, many of these Mazur manifolds have hyperbolic structures, and therefore in this case the desired result follows by the Gromov-Lawson theorem [9]. In dimensions n exceeding four, any n-dimensional homology sphere bounds a contractible manifold after perhaps changing the differentiable structure (see [10]). We merely need to build homology spheres which represent nontrivial cycles in their group homology (and which satisfy the Novikov conjecture). But this result is achieved in [17]. □

4 General contractible 3-manifolds and positive scalar curvature

Lemma 4.1 *Suppose that M^3 is a contractible noncompact 3-manifold and let $K \subseteq M^3$ be compact. Then there is a separating (hyper)surface Σ disjoint from K partitioning M into a compact piece K' containing K and a noncompact piece V such that $\pi_1(\Sigma) \to \pi(V)$ is injective.*

Proof First we know that a contractible n-manifold is one-ended if $n \geq 3$, and as a result we can choose our Σ to be connected. For every point x in Σ, consider the unit normal vector $n(x)$ pointing in the direction of the noncompact piece V. Since M is contractible, it is also orientable. The orientation on M together with the nonzero normal vector field $n(x)$ on Σ give us an orientation for Σ. □

If the map $\pi_1(\Sigma) \to \pi_1(V)$ is not injective, then there is some nontrivial loop ℓ in Σ that is nullhomotopic in V. By the Dehn Lemma there is an embedded disk D in V such that $D \cap \Sigma = \ell$. Execute surgery on Σ via this disk to produce a surface Σ' of lower genus. Iterate this process finitely many times to produce a separating hypersurface with the desired property.

Let Γ be a discrete group and X be a metric space with a proper and free isometric action of Γ. Let $C_0(X)$ be the algebra of all complex-valued continuous function on X which vanish at infinity.

Let H be a Hilbert space and let ϕ be a $*$-homomorphism from $C_0(X)$ to the C^*-algebra $B(H)$ of all bounded operators on H such that $\phi(f)$ is a noncompact operator in $B(H)$ for any nonzero function $f \in C_0(X)$. We further assume that H has a Γ-action compatible with ϕ in the sense that $\phi(\gamma f)h = (\gamma(\phi(f))\gamma^{-1})h$ for all $\gamma \in \Gamma$, $f \in C_0(X)$ and $h \in H$. Such a triple $(C_0(X), \Gamma, \phi)$ is called a *covariant system*.

The following concepts were introduced by Roe [22].

(1) Let T be a bounded linear operator acting on H. The *support* of T, denoted by $\mathrm{Supp}(T)$, is defined to be the complement (in $X \times X$) of the set of all points $(x, y) \in X \times X$ for which there exist $f \in C_0(X)$ and $g \in C_0(X)$ satisfying $\phi(f)T\phi(g) = 0$ and $f(x) \neq 0$ and $g(y) \neq 0$;

(2) The operator T is said to have *finite propagation* if there exists $r \geq 0$ such that $d(x, y) \leq r$for all $(x, y) \in \mathrm{Supp}(T)$;

(3) The operator T is said to have Γ*-bounded propagation* if there exist a finite subset $F \subseteq \Gamma$ and $r \geq 0$ such that $\min\{d(gx, y) : g \in F\} \leq r$ for all $(x, y) \in \mathrm{Supp}(T)$;

(4) The operator T is said to be *locally compact* if $\phi(f)T$ and $T\phi(f)$ are compact for all $f \in C_0(X)$.

We define $C^*_{\Gamma,b}(X)$ to be the operator norm-closure of all Γ-invariant locally compact operators with Γ-bounded propagation on H. It is not diffcult to verify that $C^*_{\Gamma,b}(X)$ is independent of the choice of H (up to $*$-isomorphism). Let $C^*_\Gamma(X)$ be the operator norm-closure of all Γ-invariant locally compact operators with finite propagation on H. Notice $C^*_\Gamma(X)$ is an equivariant version of the Roe algebra [22] and is a subalgebra of $C^*_{\Gamma,b}(X)$.

Theorem 4.2 *Let M be a noncompact n-manifold with fundamental group Γ. Suppose that M has a complete Riemannian metric with uniformly positive scalar curvature outside a compact set. Let D be the Dirac operator on M with Γ-lift $\widetilde{D}$ and let $C^*_{\Gamma,b}(\widetilde{M})$ be the operator norm-closure of all Γ-invariant locally compact operators with Γ-bounded propagation on the universal cover $\widetilde{M}$. Then ind$(\widetilde{D}) = 0$ in $K_*(C^*_{\Gamma,b}(\widetilde{M}))$.*

Proof The index map is a composite $K^\Gamma_*(\widetilde{M}) \to K_*(C^*_\Gamma(\widetilde{M})) \to K_*(C^*_{\Gamma,b}(\widetilde{M}))$ but factors through $K_*(\mathcal{K} \otimes C^*_r\Gamma)$ by the positivity assumption on scalar curvature. Here $\mathcal{K}$ is the algebra of compact operators on M. It suffices therefore to prove that the map

$$i_*\colon K_*(\mathcal{K} \otimes C^*_r\Gamma) \to K_*(C^*_{\Gamma,b}(\widetilde{M}))$$

is identically zero.

When $* = 0$, we consider a geodesic ray $\mathbb{R}_+ \to M$ embedded in M. We write

$$L^2(\mathbb{R}_+) = \bigoplus_{n=0}^{\infty} L^2[n, n+1] = L^2[0, 1] \otimes \ell^2(\mathbb{N}).$$

□

If p_1 is a rank 1 projection on $L^2[0, 1]$, then $p_1 \otimes p_0$ generates $K_0(\mathcal{K}) \cong \mathbb{Z}$, where $p_0\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is the projection onto the function δ_0. Let $S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the shift operator given by $(S\xi)(n) = \xi(n+1)$ for all $\xi \in \ell^2(\mathbb{N})$ and $n \in \mathbb{N}$. Note that S has adjoint S^* given by

$$(S^*\xi)(n) = \begin{cases} \xi(n-1) & \text{if} \quad n \geq 1, \\ 0 & \text{if} \quad n = 0 \end{cases}$$

for all $\xi \in \ell^2(\mathbb{N})$. Let q be a projection in the matrix algebra of $C_r^*\Gamma$ representing an element in $K_0(C_r^*\Gamma)$. Set $p = q \otimes p_1$. If $T = p \otimes S$, then $T^*T = p \otimes I$ and $TT^* = p \otimes (I - p_0)$. Since T^*T and TT^* are Murray-von Neumann equivalent, so $[p \otimes I] = [p \otimes (I - p_0)]$ in $K_0(C^*(\mathbb{R}_+) \otimes C_r^*\Gamma)$, where $C^*(\mathbb{R}_+)$ is the Roe algebra for $\mathbb{R}_+$. Hence $[p \otimes p_0]$ is zero in $K_0(C^*(\mathbb{R}_+) \otimes C_r^*\Gamma)$. With the map $K_0(C^*(\mathbb{R}_+) \otimes C_r^*\Gamma) \to K_0(C^*_{\Gamma,b}(\widetilde{M}))$ (induced by the Γ-equivariant embedding of the lift of the geodesic ray into $\widetilde{M}$), it follows that i_* is the zero map.

Now let $* = 1$. We proceed by an Eilenberg swindle argument. Again let us write

$$L^2(\mathbb{R}_+) = L^2[0, 1] \otimes \ell^2(\mathbb{N})$$

and let $u \in (\mathcal{K}(L^2[0, 1]) \otimes C_r^*\Gamma)^+$ be a unitary representing an element in $K_*(\mathcal{K} \otimes C_r^*\Gamma)$. Then $u \otimes I$ lies in $K_*(\mathcal{K} \otimes C_r^*\Gamma)$. Using the shift operator S as above, we define

$$W = \bigoplus_{n=1}^{\infty}(I \otimes S)^n(u \otimes I)(I \otimes S^*)^n : \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+) \to \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+).$$

Now define

$$W_1 = \bigoplus_{n=1}^{\infty}(I \otimes S)^{n+1}(u \otimes I)(I \otimes S^*)^{n+1}$$

and

$$W_2 = I \oplus \left(\bigoplus_{n=1}^{\infty}(I \otimes S)^{n+1}(u \otimes I)(I \otimes S^*)^{n+1}\right).$$

Then $[W] = [W_1] = [W_2]$ in $K_*(C^*(\mathbb{R}_+) \otimes C_r^*\Gamma)$. Therefore $[u \otimes I] = [I]$ in $K_*(C^*(\mathbb{R}_+) \otimes C_r^*\Gamma)$, and thus it is trivial in $K_*(C^*_{\Gamma,b}(\widetilde{M}))$.

Theorem 4.3 *Suppose that M is an oriented n-manifold with $\Gamma = \pi_1(M)$ and Σ is a compact separating codimension 1 hypersurface partitioning M into M_0 and M_1. Denote by Σ_Γ the Γ-lift of Σ. Assume that the strong Novikov conjecture holds for Γ and that the image of $[D_\Sigma]$ is nonzero under the map $f_* : K^\Gamma_{*-1}(\Sigma_\Gamma) \to K_*(B\Gamma)$. Then ind$(\widetilde{D})$ is nonzero in $K_*(C^*_{\Gamma,b}(\widetilde{M}))$.*

Proof For each $r > 0$, let N_r be the r-neighborhood of Σ in M_1, with metric inherited from $\widetilde{M}$. Let Σ_Γ be the Γ-lift of Σ. Define $C^*_{\Gamma,M_1}(\Sigma_\Gamma) \equiv \lim_{r\to\infty} C^*_{\Gamma,b}(\widetilde{N_r})$. Then $C^*_{\Gamma,M_1}(\Sigma_\Gamma)$ is an ideal in $C^*_{\Gamma,b}(\widetilde{M})$. It is not hard to show that the map $j_* : K_*(C^*_{\Gamma,b}(\Sigma_\Gamma)) \to K_*(C^*_{\Gamma,M_1}(\Sigma_\Gamma))$ is an isomorphism. Consider the commutative diagram given by

$$\begin{tikzcd}
K_*^\Gamma(\widetilde{M}) \arrow[r] \arrow[d] & K_*(C^*_{\Gamma,b}(\widetilde{M})) \arrow[d] \\
K_*^\Gamma(\widetilde{M}_1, \Sigma_\Gamma) \arrow[r] \arrow[d] & K_*(C^*_{\Gamma,b}(\widetilde{M}_1)/C^*_{\Gamma,M_1}(\Sigma_\Gamma)) \arrow[d] \\
K_{*-1}^\Gamma(\Sigma_\Gamma) \arrow[r] & K_{*-1}(C^*_{\Gamma,M_1}(\Sigma_\Gamma))
\end{tikzcd}$$

Now $K_{*-1}(C^*_{\Gamma,M_1}(\Sigma_\Gamma)) \cong K_{*-1}(C^*_{\Gamma,b}(\Sigma_\Gamma)) \cong K_{*-1}(C^*_r\Gamma)$ by the fact that Σ is compact. The Dirac class $[\widetilde{D}]$ in $K_*^\Gamma(\widetilde{M})$ has image $[\widetilde{D}_{\Sigma_\Gamma}]$ in $K_{*-1}^\Gamma(\Sigma_\Gamma)$ under the composition of the first horizontal map with the two vertical maps on the right hand side in the above diagram, whose index is nonzero in $K_{*-1}(C^*_r\Gamma)$ because the strong Novikov conjecture is assumed for Γ. Therefore the index $\mathrm{ind}(\widetilde{D})$ is nonzero in $K_*(C^*_{\Gamma,b}(\widetilde{M}))$, as required. □

Theorem 4.4 *If a noncompact contractible 3-manifold M has a complete Riemannian metric with uniformly positive scalar curvature outside a compact set, then it is homeomorphic to $\mathbb{R}^3$.*

Proof If M has a complete Riemannian metric with uniformly positive scalar curvature outside some compact set, let K be any compact set in M. By Lemma 4.1, there is an orientable hypersurface Σ that partitions M into a compact subset K' containing K and a noncompact piece V. Moreover Σ can be chosen so that $\pi_1(\Sigma)$ injects into $\pi_1(V)$. Let N be the doubling of V. If Σ is aspherical, then N is aspherical. Let $\pi_1(N) = \Gamma$. Notice that $H_2(\Sigma)$ injects into $H_2(N)$. This can be seen as follows: there is a noncompact 1-dimensional submanifold of N whose intersection number with Σ is 1. Hence $H_2(B\pi_1(\Sigma))$ injects into $H_2(B\Gamma)$. By [14], the strong Novikov conjecture holds for Γ. Therefore N and Γ satisfy the conditions of Theorem 4.3, contradicting the result of Theorem 4.2. It follows that Σ must be a sphere. By the Poincaré conjecture, the union $\Sigma \cup K'$ is a ball. Since K is arbitrary, it follows that M is a union of nested balls, so must be $\mathbb{R}^3$. □

5 General 3-manifolds and positve scalar curvature

Theorem 5.1 *Suppose that M is a connected oriented 3-manifold whose fundamental group is finitely generated, and M has a complete Riemannian metric with uniformly positive scalar curvature. Then we have the following.*

(1) *The space M is homeomorphic to a connected sum of space forms (quotients of the 3-sphere) and copies of $S^2 \times S^1$. In fact it can be compactified to a compact manifold $\widetilde{M}$ so that the set $P = \widetilde{M} - M$ of boundary points is a totally disconnected set.*
(2) *Further, if M is homotopy equivalent to a finite complex (or even has finite second Betti number), then P can be taken to be finite.*

Proof The proof is again based on the same ideas. We start with an exhaustion of M and then improve it so that (1) the boundary components separate different ends and (2) the fundamental groups of the boundary components inject into the fundamental groups of their ends.

If there is ever a surface of positive genus among the collection of boundary components, then we contradict uniformly positive scalar curvature (outside a compact set) by the earlier arguments in the proof of Theorem 4.4, i.e. by doubling the manifold and using the geometrization theorem and an index theory argument based on Theorem 4.2 and Theorem 4.3 (the geometrization theorem implies that the doubled manifold is aspherical in this case). Then at each stage we obtain finite unions of 2-spheres as the separating surfaces. For sufficiently large exhaustions, the (annular) three-manifolds bounding these 2-spheres must be simply connected; otherwise the fundamental group is infinitely generated by van Kampen's Theorem.

As they are simply connected, the Poincaré conjecture implies that these annular pieces are all multiply punctured spheres, and the assumption that the fundamental group is finitely generated implies that the geometry at infinity is asymptotically a tree. The set of boundary points of the compactification is precisely the space of end points of the tree. We therefore have a noncompact manifold M with uniformly positive scalar curvature that is homeomorphic to a connected sum $N\#A$, where N is a compact 3-manifold and A is a multiply punctured sphere. The prime decomposition theorem for 3-manifolds states that N is a finite connected sum of manifolds with finite fundamental group (i.e. space forms), copies of $S^1 \times S^2$ and Eilenberg-Maclane spaces $K(\pi, 1)$.

We claim that none of these summands can be Eilenberg-Maclane. If so, we can express M as $K\#L$ for some finitely presented group π and some noncompact manifold L, where K is a compact $K(\pi, 1)$ manifold. Since every noncompact manifold admits a proper (coarse) map onto a ray R, consider the corresponding proper map $f\colon K\#L \to K \vee R$, where $K \vee R$ means the one-point union of K and the ray R at the endpoint of R. Let X and Y be respectively the covering spaces of $K\#L$ and $K \vee R$ with proper and free actions of π satisfying $X/\pi = K\#L$ and $Y/\pi = K \vee R$. One can define a higher index of the Dirac operator D on $K\#L$ in $K_1(C^*_\pi(X))$ (denoted by $\mathrm{ind}(D)$), where $C^*_\pi(X)$ is the operator norm closure of all π-invariant and locally compact operators with finite propagation acting on the Hilbert space of all L^2-sections of the spinor bundle on X. The map f induces a proper π-equivariant map g from X to Y. This g induces a homomorphism g_* from $K_1(C^*_\pi(X))$ to $K_1(C^*_\pi(Y))$, where $C^*_\pi(Y)$ is the operator norm closure of all π-invariant and locally compact operators with finite propagation acting on the Hilbert space $L^2(Y)$ (the measure on the lift of the ray R is given by the Lebesgue measure). By a (controlled) Mayer-Vietoris sequence argument, one can show that

$$K_1(C^*_\pi(Y)) = K_1(C^*_r(\pi)) \oplus \mathbb{Z}.$$

It is not difficult to check the first component of $g_*(\mathrm{ind}(D))$ in the above decomposition corresponds to the higher index of the Dirac operator on K in $K_1(C^*_r(\pi))$, which is nonzero since the strong Novikov conjecture holds for π [14]. This implies that $\mathrm{ind}(D)$ is nonzero. This result contradicts the existence of a uniformly positive scalar curvature metric on M. We have therefore proven (1).

Under the assumptions of (2), the tree has only finitely many places of valence exceeding two; otherwise the second homology of the 3-manifold is infinitely generated.

Remark One can show that $X = K\#L$ lacks a uniformly positive scalar curvature metric by the following alternate observation. Since π is torsion-free, it contains a copy of $\mathbb{Z}$. Choose an embedded curve γ in X which is not homotopic to zero, and consider the covering space

$\widetilde{X} \to X$ corresponding to the infinite cyclic subgroup of π generated by $[\gamma]$. There is a lifting of γ to an embedded curve $\widetilde{\gamma} \subseteq \widetilde{X}$ which generates $\pi_1(\widetilde{X}) \cong \mathbb{Z}$.

By homological calculations, the class of the small normal circle is of infinite order in $H_1(\widetilde{X} - S^1, \mathbb{Z})$. In such a situation, the manifold $\widetilde{X}$ is said to *carry a small circle*. Gromov and Lawson [8] prove that manifolds that carry small circles cannot have a metric of uniformly positive scalar curvature.

Remark It is possible to have a complete Riemannian manifold with uniformly positive scalar curvature and an infinitely generated fundamental group and finitely generated first and second homology: consider simply an infinite connected sum of Poincaré dodecahedral spheres.

Question: Are there contractible 4-manifolds other than $\mathbb{R}^4$ which can be endowed with complete metrics of uniformly positive scalar curvature? Note that we know little about the curvature of any of the uncountably many differentiable structures on $\mathbb{R}^4$ itself.

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