Replacement of fixed sets for compact group actions: The 2ρ theorem

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To Tom Farrell and Lowell Jones, with admiration.

§1. Introduction and Main Results.

The replacement problem for stratified spaces asks whether any manifold (simple) homotopy equivalent to the bottom stratum of a stratified space X is the bottom stratum of a stratified space stratified (simple) homotopy equivalent to X.

This is impossible in general, as one sees by considering the where $X = M \times [0,1]$ and one works rel $M \times 0$.

On the other hand, the classic theorem of Browder, Casson, Haefliger, Sullivan and Wall (see [Wa1] Corollary 11.3.1) asserts that if X = (W, M) is a pair consisting of a manifold with a codimension c submanifold, and c>2, then replacement is always possible in the topological and PL locally flat settings[2]. In that work, the key technical core of the result is the stability theorem (and its topological analogue); that is, the map of classifying spaces,

$$G_c/PL_c \to G/PL$$

is, for $c > 2$, a homotopy equivalence. In this case, the replacement of the submanifold M can be achieved without altering the manifold W.

If we move on to the setting of group actions (even just on manifolds), and we consider the issue of replacing fixed sets, then the situation is more subtle and not directly related to stability properties of classifying spaces. For instance, when the group S^1 is acting semi-freely on a manifold X, and M is a codimension 4 component of fixed set of the action, then the (equivariant) replacement problem boils down to one for the submanifold M in X/S^1, and since the neighborhood of M has quotient a manifold, the theorem of Browder et al, yields replacement. In [CW1] this was analyzed and it was

[1] The authors are partially supported by DARPA, NSF grants DMS 0504721 and 0805913, and Hong Kong RGC General Research Fund, No.605005 and No.604408, respectively.

[2] For a discussion of the codimension two situation, see [CS2,3]. As the homotopy type of the top stratum can change in these results, and, for the non-locally flat results, even the number of strata can change, these results do not fit into the framework we are discussing here.

shown that replacing the fixed set can force a change in the structure of the global manifold acted upon.

Moreover, in [CW2] it is proven that replacement of a component of the fixed set of a semi-free circle action is possible generally iff the codimension of that component is 0 mod 4. When the codimension is 2 mod 4 then the restriction map $S^{S^1}(M\times D(V) \text{ rel } \partial) \to S(M)$ is, in fact, trivial, and the fixed set is thus *rigidly determined by the action on the complement.*

We use the standard notation that for an orthogonal or unitary representation, V, its unit disk is denoted by D(V), and its unit sphere by S(V). Also following standard usage, the structure set of a stratified space will be denoted by S(X). Indeed, this symbol will often denote the spectrum whose 0-th homotopy group is that set. (See [We1] for definitions and the spectrum structure.) The symbol $S^G(M)$ is the same as S(M/G) and has an isovariant interpretation. We expect that this overuse of the letter S will cause no more confusion in this paper than it does throughout the literature.

Note: Throughout this paper we will make the blanket assumption that all group actions are orientation preserving and all fixed sets of all subgroups are orientable. We also assume that all groups described as acting are nontrivial.

Our main new result is that this replacement property for fixed sets of the circle group when the normal representation is a multiple of 2 is true in considerable generality for arbitrary compact Lie groups (including, of course, all finite groups):

Theorem A: *Suppose that G is a compact Lie group acting locally linearly on a topological manifold W. Suppose that near the 1-skeleton of the fixed set F the G action can be identified with a complex G-bundle whose normal representation is a multiple of 2. Then the forgetful map*

$$S^G(W \text{ (rel } \partial, \text{ if desired)}) \to S(F)$$

is a split surjective map. In particular, for such actions, it is always possible to replace the fixed set by any simple homotopy equivalent (homology) manifold.

Remark 1. *If the replaced fixed set is a manifold, we can arrange for the new action to be locally linear and with the same normal representation.*

Remark 2. *This theorem also holds in the PL locally linear category.*

The meaning of the condition is as follows. If an action is locally linear, then, by definition, there is a representation ρ normal to the fixed set; in other words, the equivariant tangential germ data is linearized at a 0-skeleton. (It is only well-defined up to (stable) topological equivalence, of course.) The hypothesis we make is somewhat stronger than that ρ is equivalent to 2ρ' for some complex representation ρ'. Essentially the issue is something like orientability, a condition on the germ about a 1-skeleton: as we transport our identification of the normal structure with a representation as we move around the fixed set, do we obtain "nontrivial monodromy"? Our complexity hypothesis ensures that we do not. In Theorem 2, stated in section 2, this condition will be weakened.

In the smooth case, the structure group of the equivariant normal bundle reduces to a product of compact groups of type O, U, and Sp according to whether the irreducible summands of ρ' are of type **R, C,** and **H**. Only type **R** would allow nontrivial monodromy. However, of course, our results even for smooth G-manifolds will only produce topological G-manifolds.

In the special case of odd order abelian groups this theorem was proved in [CW2] making use, in part, of the constructions from [WY1]. At the time we had speculated that something like the above might be true. In the intervening time we verified many special cases by complicated ad hoc constructions. The general and simple proof[3] given in section 3 below came as a pleasant surprise to us – essentially it reduces the general case to that of unitary groups, but not by means of any fixed embedding of the group in question into the unitary group (as occurs, for instance, in the "holomorphic induction" proof of Bott periodicity). In this way, Theorem A is, in essence, deduced from the theorem of Browder et al referred to above.

In section 2, we will review some results and the methods of [CW2], and will give some new examples, in particular showing strong differences between the PL locally linear and topological locally linear categories. The proofs will be in section 3.

Section 4 will give an example of an interesting stratified product, namely that for n>1 there is a stratified space and a product map inducing a map:

$$\otimes (D^3 \cup \mathbf{C}^{2n}) : L_k(\pi) \to L_{k+4n}(\pi)$$

which is an isomorphism. Moreover, there is a straightforward extension where π is replaced by a general stratified space X. The space $D^3 \cup \mathbf{C}^{2n}$ arises in the proofs of our main theorem – the appearance of the D^3 essentially accounts for the role of the embedding theorem of Browder et al, and that this is an isomorphism explains why replacement of 2*complex representations can be reduced to (a proof of) that classical theorem.

For n=1, this looks similar to the classical periodicity, but actually we do not have such a space for n =1 (per se; there is an appropriate example of a 3-cell complex in [WY1], but it has rather different features). For n>1 the stratified space looks far from having a standard signature equal to 1 as in usual product formulae, in that its middle dimensional homology vanishes (indeed, it is a homotopy sphere), so the result is perhaps somewhat unexpected.

[3] However, we admit that the brevity of this paper is contributed to by our assumption that the interested reader will turn to [CW2] for the details that we do not discuss here.

We are happy to contribute this paper to the special issue of the Pure and Applied Quarterly in honor of Tom Farrell and Lowell Jones. As we shall see below, two ideas from their seminal paper [FJ] play a role in our story.

§2. Review of Previous Results and Methods.

As we mentioned in the introduction, the most classical approach to replacement theorems is by means of analysis of classifying spaces of neighborhoods. The earlier paper [CW2] developed a rather different perspective that showed that these problems can be intimately tied to issues regarding product formulae in stratified surgery. This perspective is the one we adopt here, although our approach to the relevant product formulae is entirely different and non-computational. For the convenience of the reader, in this section we summarize these ideas and their earlier application. We also take advantage of this opportunity to correct some computational errors made in the addendum to Theorem 0.1 and to Theorem 2.5 of that paper.

The main previous positive results regarding replacement of fixed point sets are the following:

Theorem 1 ([CW2]): *If G is a finite group of odd order acting on a manifold so that the small gap hypothesis holds (i.e., no stratum is codimension two in another), and so that the action is smoothable near the 1-skeleton, one can replace the fixed point sets. Moreover, the new action can be assumed to be on the original manifold.*

Note, though, that there is no requirement that the normal representation be twice a complex one. It is automatically complex, since the group is odd order – but replacement is true without the doubling. In contrast, for the group S^1, the doubling is necessary, as mentioned above.

We remark that if the strong gap hypothesis holds (i.e., that any stratum is less than ½ the dimension of any larger stratum that contains it), then the condition on monodromy is irrelevant, as we shall explain below.

Some cases of the following result were proved in [CW2]; it follows from the proof of Theorem A and a stabilization/destabilization procedure used in that paper and that we will immediately review.

Theorem 2: *If G is cyclic, the action satisfies the small gap hypothesis, and the normal representation = 2ρ' for a complex representation ρ', then replacement is possible. For general G, the same holds assuming in addition that the representation satisfies the strong gap hypothesis.*

This theorem trades the 1-skeleton hypothesis of Theorem A of the introduction for either cyclicity or a strong gap hypothesis. The latter implies (at the stratified homotopy level) trivial monodromy by a theorem of Browder [Sc][4]. It is for cyclic groups that the stabilization trick then gets used. As a result, Theorem 1 can also be strengthened for odd order cyclic groups to not require the 1-skeleton condition.

Replacement of the fixed set of a G-action is the same as splitting the map $S^G(M) \to S(F)$. By definition, strong replacement gives a splitting which is trivial when composed with the forgetful map $S^G(M) \to S(M)$. Consider the diagram:

$$\begin{array}{ccccc} S(M) & \leftarrow & S^G(M) & \rightarrow & S(F) \\ \downarrow\cong & & \downarrow\cong & & \downarrow= \\ S(M\times D(V), \text{rel } \partial) & \leftarrow & S^G(M\times D(V), \text{rel } \partial) & \rightarrow & S(F) \end{array}$$

The only arrows that requires explanation are the left two vertical ones. The leftmost vertical arrow is classical Siebenmann periodicity; the middle one is an isovariant analogue. Such maps were first constructed in [Y], and they exist in the generality needed here by [WY2]; that the diagram commutes is part of the general theory.

If one has replacement under the strong gap hypothesis, and a suitable representation V exists so that M×V now satisfies the strong gap hypothesis, then on the bottom line one has a splitting, and by periodicity, on the top line as well. (And, similarly for strong replacement.) By inspection, such representations exist for cyclic groups.

The statement here corrects the statement in the addendum to Theorem 0.1 in [CW2] asserting the above result for all abelian groups; presumably that is true, but our current methods only yield the full result for cyclic groups. We had miscalculated that in the representation theory of odd order groups it would always be possible to stabilize (via "periodicity representations") to achieve the strong gap hypothesis. Unfortunately, this is not even true for most representations of $G = \mathbf{Z_p} \times \mathbf{Z_p}$.

We note another difference between Theorem 1 and Theorem 2; in the latter, it is not possible to assume that the new action is on the original manifold. There are rigidity theorems.

Theorem 3.1: *Suppose that G is a group acting semi-freely and PL locally linearly on a manifold with simply connected fixed set F of codimension other than 2. Then it is possible to replace F by F'∈S(F) in such a way that the complement of F is unchanged, if and only if:*

*(i) G is connected and the normal representation is 2*complex, or F'=F*

[4] Recall our blanket assumption about orientability of all fixed sets.

or *(ii) G is finite, the Kervaire classes of F' in $H^{4i+2}(F; Z/2)$ all vanish*
or *(iii) G is finite with cyclic 2-sylow subgroup and the codimension is a multiple of 4*
or *(iv) G is finite with quaternionic 2-sylow subgroup and the codimension is a multiple of 8.*

This corrects the quaternionic statement in Theorem 2.5 of [CW2]. It is interesting to compare and contrast this with the analogous theorem for the topological locally linear category.

Theorem 3.2: *Suppose that G is a group acting semi-freely and topologically locally linearly on a manifold with simply connected fixed set F of codimension other than 2 and suppose that H*(F ; **Z**) has no two-torsion. Then it is possible to replace F by F'∈S(F) in such a way that the complement of F is unchanged, if and only if:*

*(i) G is connected and the normal representation is 2*complex, or F'=F*
or *(ii) G is finite with cyclic 2-sylow subgroup, and the codimension is a multiple of 4 or the Kervaire classes of F' in $H^{4i+2}(F; Z/2)$ all vanish,*
or *(iii) G is finite with quaternionic 2-sylow subgroup.*

We remark that strong replacement for semi-free PL locally linear actions is completely analyzed by Theorem 0.2 of [CW2].

Recall that normal invariants are classified by maps [F : G/Top], and have Kervaire characteristic classes $k^{4i+2} \in H^{4i+2}(F ; \mathbf{Z}_2)$ (see [RS]). The statements in Theorems 3.1 and 3.2 show that for even order groups there is a (very) partial rigidity that one doesn't have for odd order groups – but which does not go as far as the rigidity that is present for the positive dimensional case.

The main idea of [CW2] is to view the replacement problem as one of changing the base in a block bundle. That paper was written in the PL locally linear setting, but it is straightforward to rewrite the paper in the language of stratified surgery theory [We1] to cover the topological setting. Not surprisingly, the critical issues remain unchanged from the ad hoc approach used in [CW2] – but the contrasting Theorems 3.1 and 3.2 show that the resulting detailed calculations do indeed differ.

For definiteness, we employ the PL language.

The regular neighborhood of the fixed set can be viewed as a block bundle over F; we wish to make its boundary a block bundle over F'; equivariantly coning this structure and gluing it into the complement of the interior of the regular neighborhood of F, produces the new action. "Strong replacement," which involves knowing that the glued manifold is the original one, requires base change for the "bubble quotient," introduced in [CW2], but which will not be discussed here.

Notice that S(F) arises as an obstruction to base change: Given an element F' → F in S(F) the obstruction to changing the base of the identity block bundle F → F to F' is identified with the element that F' represents in S(F). Using this, the base change of ∂N/G → F to block fiber over F' is then the transfer of the element of S(F) to the total space of the boundary of the regular neighborhood ∂N/G → F. A geometric argument shows that transfers can be computed by knowing the (stratified homotopy type) over the 1-skeleton. (This argument uses a thickening $CP^2 \times F$ of F to ensure enough of a skeleton over each simplex of F. It is reminiscent of the transfer trick of [FJ] although used for a completely different reason.)

The results on replacement discussed above all are consequences of computing the products S(V)/G × on stratified L-groups. Under the trivial monodromy assumption, the transfer is identified with the result of transfer to a trivial bundle, i.e., the product. (A similar calculation for the bubble quotient yields the "strong replacement" wherein the manifold acted upon is unchanged, as is the action on the complement of the fixed set.)

The stratified L-groups used here are the ones introduced by Browder and Quinn [BQ] for transverse isovariant surgery. In [We1], it is shown that these L^{BQ} play a key role in the non-transverse theory as well; indeed, in that theory, they describe both the global surgery obstruction, and in a cosheaf form, the normal invariants.

When we are discussing finite group actions, away from the prime 2 all of the L^{BQ} that occur in stratified surgery naturally split; consequently, the isovariant structure set then splits into pieces that are concentrated on the components of the singularity set. This is because using **Q** as the coefficient ring in place of **Z**, permutation modules are all projective, so one can use the singular chain complexes to give a description of the relevant L-groups. On the other hand, for any X, an induction beginning with Ranicki's localization theorem [R] for Wall surgery obstruction groups, shows that $\mathbf{L}^{BQ}(X) \to \mathbf{L}^{BQ}(X ; \mathbf{Q})$ is a Z[1/2] equivalence.

At the prime 2, for odd order groups, Lueck and Madsen [LM] show that some odd multiple of S(V)/G bound (for the related purpose of showing a splitting of equivariant L-groups). This shows that there are splittings (for odd order groups) even integrally.

For the group S^1 acting on $V = \mathbf{C}^k$ by scalar multiplication, the product that one is computing is $\times\, CP^{k-1}$. It then clear why one has a dichotomy between replacement and rigidity depending on whether one has a CP^{odd}, which kills surgery obstructions, or a CP^{even}, which gives rise to periodicity isomorphisms on surgery groups. (For general spheres of representations, of course, one tends to neither kill all nor preserve all surgery obstructions, leading to a more varied range of phenomena in the replacement problem.)

Theorem 2 makes use of stratified spaces as the coboundaries for S(V)/G rather than merely quotients of G-manifolds. (See [DS] for a discussion of the difficulties in finding G-manifolds for G = $\mathbf{Z}_2$.) For abelian groups, [WY1] constructed explicit coboundaries for 2×any irreducible representation, and then combined these to deal with the general case. The complexity of the relevant coboundaries thus grew: there were $3^k - 1$ strata in the coboundary where k = ½ the complex dimension of the representation.

The original application of these stratified spaces was to the construction of isovariant periodicity maps $S^G(M) \to S^G(M \times D(V) \text{ rel } \partial)$ for suitable representations of G; these were applied above.

Unfortunately, these ad-hoc techniques did not suffice for general Lie groups. In [WY2] the authors used a much more efficient stratified coboundary for representation spheres (or better, an improved construction of periodicity spaces) using a key calculation at the core of [FJ] – the symmetric square construction.

What the argument directly shows, however, (see [WY2] for a detailed discussion of this technical point) is that a representation $V \oplus \varepsilon^4$ is a periodicity representation; for the periodicity problem, one can, after the fact remove the trivial summand ε^4, but for the replacement problem this cannot be done: one would only obtain the result that F' can be isovariantly embedded in $M \times D^4$, which was obvious anyway as F' embeds in $F \times D^4$. While we seemed no closer to proving that the general 2ρ is a "replacement representation," it then became a more pressing issue to try to prove such a replacement theorem. This will be accomplished in the following section.

§3. Proofs of theorems.

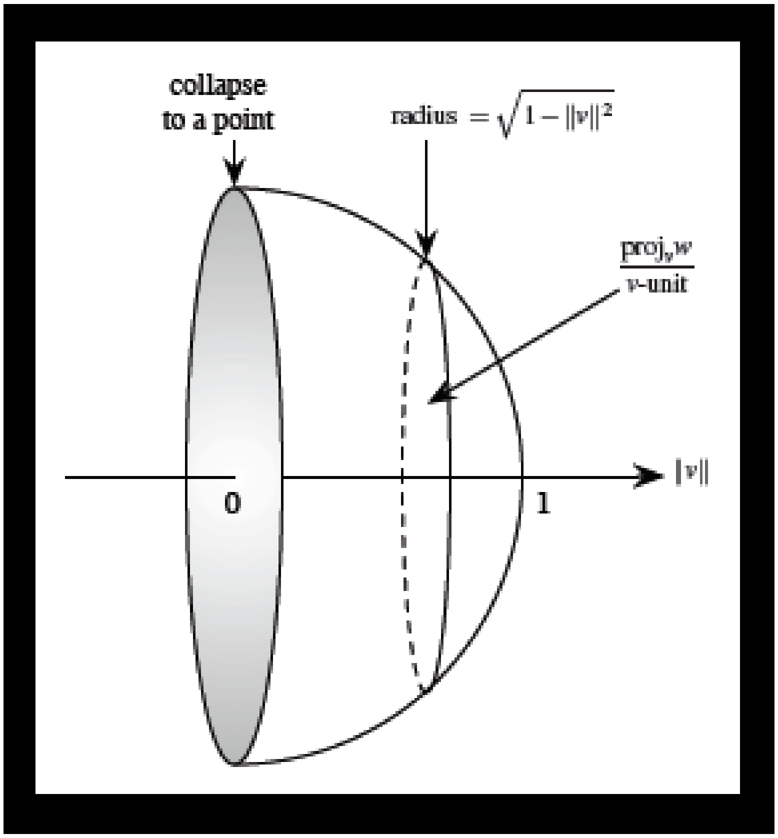
We begin with a geometrical observation (see [Da] for a much more complete analysis):

<u>Proposition</u>: $S(\mathbf{C}^{2n})/U(n) \cong D^3$ if n>1.

The action of U(n) here is twice the defining representation τ. The action on the sphere has two orbit types, depending on whether the pair of vectors (v,w) span a 1 or 2 dimensional subspace of $\mathbf{C}^n$. If they span a 1-dimensional subspace, then (v,w) together give a homogeneous coordinate for a point in the complex projective line that the subspace determines $CP^1 \cong \partial D^3$. (Note, the subspace they generate changes at will by the U(n) action. This element is unchanged.)

If v and w do not lie on a line, then one can decompose w as a sum of the orthogonal projection to v, and the projection to the orthogonal complement. The first gives a point on the sphere, and the second describes the radial coordinate of the ball. (The origin corresponds to where this distance = 1, so that the first coordinate is the 0 vector, and there is only a point rather than CP^1 of indeterminacy.)

The following figure describes the geometry of this identification:



We now continue with the proof of Theorem A by showing that the product × S(2ρ') is trivial as a map of (isovariant) surgery groups. However, using ρ' to embed G into U(n), it suffices to show that × S(2τ) = 0. Then, by looking at the quotient and using the previous proposition, the π-π theorem on vanishing of appropriate relative surgery groups completes the proof, at least if the dimensions of all of the strata in the quotient that touch the fixed set are of dimension at least five, so that we can apply surgery theory.

Note, though, that the dimension of the fixed set is at least 3 (for there to be any other manifold homotopy equivalent to the fixed set). The condition on ρ guarantees that the quotient of the sphere by the action is at least 2 dimensional (with the extreme case being 2×the defining representation for the circle), so there are no low dimensional complications.

Remark: This π-π argument gives a conceptual a-priori explanation for the replacement part of the Browder-Casson-Haefliger-Sullivan-Wall embedding theorem. The book [We1] contains three proofs of the full result.

Now we turn to the proofs of Remarks 1 and 2 in the introduction.

Remark 1 asserts that when the replaced fixed set is assumed to be a manifold, then one has local linearity of the replacement. This follows from a relative form of Theorem A.

The *manifolds* F' simple homotopy equivalent to F are the same as those simple homotopy equivalent to F^o (F punctured) rel ∂ by gluing in the "missing" disk.. Applying the relative version of Theorem A to M^o, and gluing back in the disk of M that we removed, gives an action on M' with fixed set F' that is locally linear near the center of this disk. This implies, in the topological category, local linearity by the topological homogeneity of homotopically stratified spaces, Corollary 1.3 of [Q].

Remark 2 regarding the PL locally linear category requires no explanation except for one low dimensional issue. Whereas the topological category uses stratified and controlled surgery, the PL case uses instead blocked surgery, whose formal obstruction theory is identical aside from the decorations for the L-groups. However, blocked surgery requires that all blocks have no low dimensional strata, which would naively restrict the (dimensions of strata in the) normal representations to which our theorem would apply.

However, the stratified spaces we consider have only strata of dimension at least 3, and thus the issue of blocked surgery only occurs on the 0 and 1 skeleta of the fixed sets. However, using Theorem 12.1 of [Wa1], we can arrange for any simple homotopy equivalence F' → F to be a PL homeomorphism near the 1 skeleton, and the blocked surgery thus starts with simplices of dimension 2, and thus with blocks of dimension at least 5, completing the argument for Theorem A. This, together with the stabilization/destabilization argument given in section 2, proves Theorem 2.

Despite the brevity of the proof, even for finite groups the result has quite interesting consequences. We will discuss first one such implication for product formualae and then return to the proofs of Theorems 3.1 and 3.2.

Denoting by τ the defining representation for Q_{2r}, we claim that the product, × $S(k\tau)/Q_{2r}$, is trivial on L^s if and only if k is even. Therefore one can (universally[5]) perform replacement of a fixed set with that normal representation in the PL locally linear category iff k is even.

If k is even, vanishing follows from Theorem A. The stabilization/destabilization argument (using the periodicity for 2τ) shows that the product for $S(k\tau)/Q_{2r}$ is the same as for $S(\tau)/Q_{2r}$ if k is odd. This last product was computed in [CS1].

This gives us almost enough information to prove Theorem 3.1. As before, the crux of the issue is the computation of the transfer. The case of the groups of S^1 and SU(2) boil down to complex and quaternionic projective space, and are obvious. For the finite group cases, it is more complicated because the products are sometimes neither identically 0 (replacement) nor an isomorphism (rigidity).

[5] that is, for all fixed sets.

The links of the fixed points of a semi-free actions are space forms. We note that all of our space forms represent 0 in the symmetric L-group $L^n(\mathbf{Z}G) \otimes \mathbf{Z}[1/2]$ (as mentioned earlier in our discussion of why L^{BQ} splits). Thus we can work localized at the prime 2, where the normal invariants are determined by characteristic classes. Moreover, we can therefore, following [Wa2], restrict our attention to the cover corresponding to the 2-sylow subgroup, which are either lens spaces or quaternionic space forms in our situation.

Using [HMTW] we can readily see that the product formula does not depend on the k-invariant of the space form (recall that the product only depends on the homotopy type, and space forms of a given dimension are classified by their k-invariants). This is because the product formula is determined by "κ-homomorphisms" applied to mod 2 reductions of characteristic classes that are a combination of Wu classes, and the Morgan-Sullivan L-class [MS]. The latter (reduced mod 2) is the square of the Wu class as well, so everything is determined by Wu classes. By the Wu formula for Wu classes, these are determined via the action of the Steenrod algebra on mod 2 cohomology, in any case a 2-local homotopy invariant. Since the k-invariants must be odd (by, for instance, the Borsuk-Ulam theorem), we are reduced to the unitypical case, which was the case discussed above (in the more difficult, quaternionic case).

This gives the vanishing of obstructions in dimensions a multiple of 4 in the cyclic case and the multiples of 8 in the quaternionic.

The analysis in the nontrivial cases now must take into account that the product of the Kervaire surgery problem (generating $L_2(e)$) with the relevant manifolds are non-zero, but that these (and their obvious descendents) are the only nontrivial products on L-theory with coefficients (see [MS]).

These tell us the homological part of the obstruction to base change in the block fibrations. There is, in general, a final surgery obstruction. It is at this point that we use the hypothesis of simple connectivity, and a variant the trick used in the proof of Remark 1 to complete the proof. (Since we are working in PL rather than Top, we cannot appeal to [Q].) We puncture the fixed set, and replace that incurring no final normal cobordism. At that point we cone the boundary, and obtain a new group action that is PL and locally linear except perhaps at the final cone point. However, the obstruction to locally linearity there is determined by an element of the (Tate cohomology of the) Whitehead group, see [We3] (or [CW3] if working up to concordance, which is adequate for us). However, since we are working throughout with simple homotopy equivalences, the vanishing is a tautology.

We are left with the topological locally linear Theorem 3.2. Since we are in a two stratum situation, it is simplest to note that the teardrop neighborhood theorem of [HTWW] directly applies, and shows that the relevant surgery vanishing is in L^{-f} (the negative L-group). A fortiori, calculation in L^p suffices. Vanishing here on $L_2(e)$ is well known (see [HMTW]) but to complete the map of spectra $\mathbf{L}(e) \to \mathbf{L}(Q_{2r})$ would require knowing vanishing for homotopy with coefficients, and in particular computing the product $L_3(e ;\mathbf{Z}_2) \rightarrow L_6(Q_{2^r}; ;\mathbf{Z}_2)$. However, if the fixed set F does not have torsion in its cohomology, the information about integral homotopy is enough and the proof is completed in the same way as Theorem 3.1.

We close with a question:

Problem: Is it true that for all semi-free topologically locally linear quaternionic group actions, that topological replacement is always possible?

Some Complementary Examples: Here are some phenomena which occur even for as simple a group as SU(2).

Recall that the irreducible representations of SU(2) can be described in terms of spaces of polynomials in two variables. We denote by ρ_d the irreducible complex representation of SU(2) acting on the space of degree d polynomials on a 2 dimensional vector space. It is of dimension d+1.

Recall that the maximal torus T of SU(2) is of dimension 1; it can be thought of as being given by diagonal matrices with diagonal entries (u, u^{-1}), $|u| = 1$. In that case, the monomials form a basis of eigenspaces for T. The Weyl group is $\mathbf{Z}_2$ with a representative element given by (x,y) goes to (-y,x) (of order 4 in SU(2)).

Notice that the nature of ρ_d is very different for d even and d odd. In the former case, the action is not effective: the center –I acts trivially, and the monomial $(xy)^{d/2}$ is fixed by all of T. In the latter case, the action is effective, and all isotropy groups are finite (i.e., the action on the unit sphere is *locally free*).

For any G-action, each subgroup H defines a closed stratum in M/G, namely $M^H/(NH/H)$: the image of the H fixed set in the quotient. For the d even case, setting H = T, we get a stratum which is a circle. Since crossing with a circle is injective on structure sets (see [Wa1, Sh]) aside from decorations, we see that there is *rigidity* (up to decorations) in this case.

Before proceeding, there are two comments worth making.

1. It is easy to see that if F' is h-cobordant to F (i.e., the decoration issue), then there is a G-action on a manifold homotopy equivalent to M with F' as fixed set. However, this construction does not produce a manifold equivariantly simple homotopy equivalent to the original action, and is therefore not a replacement in the sense of this paper. We will not consider the decoration aspect of the rigidity for these representations in this paper.

2. For $k\rho_d$, when k is even, Theorem A implies replacement. The stabilization/destabilization trick does not reduce the case of $k\rho_d$ for k odd to ρ_d, because

ρ_d does not satisfy the weak gap hypothesis (see e.g., [Y]). For instance, rigidity that stems from strata that are circles, does not automatically persist for odd multiples of these representations.

Now let us turn to the case of d = 2r+1. First of all, note that the fixed set of $\mathbf{Z}_d$ is an S^3 (= the unit sphere of the complex vector space spanned by x^d and y^d). The quotient of this fixed set by $N(\mathbf{Z}_d)/\mathbf{Z}_d$ is $\mathbf{R}P^2$. Crossing with $\mathbf{R}P^2$ preserves Kervaire invariants, so we see that at least it is necessary for the Kervaire cohomology classes to vanish for replacement to be possible.

On the other hand, we can fairly directly completely analyze the obstruction away from the prime 2 as follows. Since $S(\rho_d)$ is a locally free G manifold, the quotient $S(\rho_d)/SU(2)$ is an orbifold, and hence, a rational homology manifold. This will enable us to compute $\mathbf{L}^{BQ}(S(\rho_d)/SU(2))\otimes Z[1/2]$ as $\mathbf{L}_{2d-2}(e) \otimes Z[1/2]$.

The strata are in a 1-1 correspondence with (quotients) of fixed sets of various isotropy groups. Since $\rho_d(-I) = -I$, the unique element of order 2 in SU(2) fixes nothing, so no even order groups can have fixed points, and hence all isotropy must be odd order cyclic, i.e., lie up to conjugacy on T. Consequently, the strata are in a 1-1 correspondence with odd integers up to d. Aside from the number 1 which corresponds to the whole space, these fixed sets are spheres of even dimension complex vector spaces, acted upon by NC/C (the quotient of the normalizer of the cyclic group C by C). All gaps are at least 4 dimensional, and the fundamental group is $\mathbf{Z}_2$ acting orientation reversingly. The top piece is simply connected. This formally gives the desired result (as L^{BQ} is built up out the ordinary L-groups of the pure strata, and the L-groups of $\mathbf{Z}_2$ with nontrivial orientation character is 2-torsion).

Now we have to understand what crossing with this orbifold does on ordinary signatures. From the locally free SU(2) action, we see this quotient is a Q-quaternionic projective space of dimension 2(2r+1)-2 = 4r-4 =4(r-1). This has signature 1 or 0 depending on whether r is odd or even.

Consequently, away from the prime 2 one has rigidity for d = 3 mod 4 and no obstruction at all for d = 1 mod 4.

To summarise: for all irreducible representations one has Kervaire cohomology obstructions. For d even, there is rigidity (up to decoration). For d odd, it depends: for d =1 mod 4 there is no obstruction away from 2, but for d = 3 mod 4, there is rigidity away from 2. It is reasonable to conjecture, that for d = 3 mod 4 one actually has rigidity, but given how our obstructions at 2 and away from 2 come from such different sources, it seems that proving this could be difficult.

§4. A product that induces periodicity in surgery.

Finally, we return to discuss the periodicity isomorphism mentioned at the end of the introduction. The general setting for this, as introduced in [WY1], is a stratified space Z that contains a subset Σ with the property that $L^{BQ}(\Sigma \times X) = 0$ for any X. For our purposes, it is enough to consider the case were Σ is the singular set of Z, e.g. Z- Σ is a manifold. Then for any X there is a product:

$$\otimes Z : L^{BQ}(X) \to L^{BQ}(\mathrm{cl}(Z\text{-}\Sigma) \times X).$$

Here cl(Z-Σ) is a closure of the top pure stratum as a manifold with boundary. (Alternatively, one can easily work with the L-theory with compact supports.) The definition is trivial: One inverts the isomorphism

$$L^{BQ}(\mathrm{cl}(Z\text{-}\Sigma) \times X) \to L^{BQ}(Z \times X)$$

that immediately follows from the assumption that $L^{BQ}(\Sigma \times X) = 0$, and the obvious product construction:

$$\times Z : L^{BQ}(X) \to L^{BQ}(\mathrm{cl}(Z\text{-}\Sigma) \times X).$$

The basic example to which we applied this to was $Z = C\mathbf{P}^2 \cup D^3$, which ultimately gave rise to the map $L^{BQ}(X) \to L^{BQ}(X \times D^4, \text{rel } \partial)$ and a formal product construction of Siebenmann periodicity. The key property that $L^{BQ}(\Sigma \times X) = 0$ is the π-π theorem.

Now, let us take $Z = D^3 \cup \mathbf{C}^{2n}$. Here we view $D^3 = S^{4n-1}/U(n)$ as the quotient of the unit sphere of the biaxial unitary action by U(n). Then, we attach the $\mathbf{C}^{2n}$ by compactifying it by attaching its unit sphere at ∞ and then using the obvious quotient map.

Clearly this Z is suitable for the ⊗, again by the π-π theorem. And, our claim is that this product is an isomorphism (or, equivalently, just the ×Z map with target the L^{BQ}.)

From our perspective this is clear: Suppose that H ⊂ G is a subgroup, and ρ is a unitary representation of G so that ρ has no H-fixed vectors. By definition if ρ has strong replacement for G then ρ|H has strong replacement.

From the stratified point of view this is a transfer map associated to the stratified system of fiber bundles $L^{BQ}(D(\rho) \cup S(\rho)/G) \to L^{BQ}(D(\rho) \cup S(\rho)/H)$, associated to the bubble quotient.

Now we apply this to the diagonal subgroup $S^1 \subset U(n)$. For the subgroup, ignoring the stratification, we end up in $X \times \mathbf{C}P^{2n}$, and therefore the map on the U(n) level: $L^{BQ}(X) \to L^{BQ}(D^{4n} \times X \text{ re } \partial)$ must be an isomorphism, as we see by restriction.

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