THE UNIVERSITY OF CHICAGO

EXCURSION REFLECTED BROWNIAN MOTION AND LOEWNER EQUATIONS IN MULTIPLY CONNECTED DOMAINS

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
SHAWN DRENNING

CHICAGO, ILLINOIS
DECEMBER 2011

TABLE OF CONTENTS

ABSTRACT

Excursion reflected Brownian motion (ERBM) is a strong Markov process defined in a finitely connected domain $D \subset \mathbb{C}$ that behaves like a Brownian motion away from the boundary of D and picks a point according to harmonic measure from infinity to reflect from every time it hits a boundary component. We give a new construction of ERBM using its conformal invariance and show how the Poisson kernel and Green's function for ERBM can be used to construct conformal maps into certain canonical classes of finitely connected domains. One important reason for studying ERBM is the hope that it will be a useful tool in the study of SLE in multiply connected domains. To this end, we show how the Poisson kernel for ERBM can be used to derive a Loewner equation for simple curves growing in a certain class of finitely connected domains.

ACKNOWLEDGEMENTS

I would like to thank my advisor Greg Lawler for suggesting this line of research and for many useful conversations pertaining to it. Without him, this would not have been possible.

I would like to thank my parents for all of their support and encouragement throughout my life. I would like to thank all of the wonderful friends I have made while at Chicago for making my time here enjoyable. Finally, I would like to thank Kristen for all of her love and support and for being there every day.

CHAPTER 1

INTRODUCTION

1.1 Motivation and Results

Oded Schramm [21] introduced a one parameter family of random processes now called *Schramm-Loewner evolution* (SLE) as a proposed scaling limit for many discrete models arising in statistical mechanics which were expected to be conformally invariant in the limit. Since then, SLE has been extensively studied and has proven to be an important tool in providing mathematical rigor to a number of predictions in statistical mechanics. The definition of SLE in simply connected domains uses a classical result of Charles Loewner [17]. Namely, if $\gamma(t) : (0, \infty) \to \mathbb{H}$ is a simple curve parametrized such that $\gamma(t)$ has half-plane capacity $a(t)$ and $g(t) : \mathbb{H} \backslash \gamma(t) \to \mathbb{H}$ is the unique conformal map satisfying $\lim_{z \to \infty} g_t(z) - z = 0$, then $g_t(z)$ satisfies the initial value problem

$$\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1.1}$$

where U_t is a real-valued function called the *driving function.* The differential equation in (1.1) is one example of a *Loewner equation.* If (1.1) is solved with $\dot{a}(t) = 2$ and $U_t = \kappa B_t$, where B_t is a standard one-dimensional Brownian motion, then the random family of maps g_t is *generated by a curve* [20] in the sense that the domain of g_t is equal to the unbounded component of $\mathbb{H} \backslash \gamma(t)$ for a random family of curves $\gamma(t)$. If $0 \leq \kappa \leq 4$, then $\gamma(t)$ is almost surely a simple curve. When $\kappa > 4$, $\gamma(t)$ almost surely has self-intersections and when

$\kappa \geq 8$ it is almost surely a space-filling curve. Chordal SLE_κ in $\mathbb{H}$ from 0 to ∞ is defined to be the random family of curves $\gamma(t)$.

It is natural to ask whether an SLE process can be defined in multiply connected domains $D \subset \mathbb{H}$ in an analogous way. That is, is it possible to find an analog of (1.1) and an appropriate driving function so that the solution of the resulting initial value problem is generated by a curve with the properties one would expect of SLE_κ? An added difficulty of the multiply connected case is that not all n-connected domains are conformally equivalent. In the simply connected case, Schramm was able to show that any stochastic process that satisfies the *domain Markov property* and is conformally invariant must come from (1.1) with U_t a Brownian motion. Part of what makes this work is that $\mathbb{H}$ with a simple curve removed is conformally equivalent to $\mathbb{H}$. In the multiply connected case, requiring that SLE have the domain Markov property and be conformally invariant is not enough to uniquely determine the driving function. Bauer and Friedrich ([4],[5], [6]) defined a candidate for SLE in multiply connected domains by solving a Loewner equation for a curve growing in a multiply connected domain. They did not determine the "right" driving function for the process to be SLE, but they were able to narrow down the possible choices. In separate work ([22],[23]), Zhan took a similar approach. He showed that, in the case of the annulus, if in addition to satisfying the domain Markov property and conformal invariance, SLE is also assumed to be reversible, then the driving function is uniquely determined.

Recent work of Lawler [15] takes another approach to defining SLE in multiply connected domains. His definition is motivated by work [11] of Lawler, Schramm, and Werner. They showed that if $D \subset \mathbb{H}$ is a simply connected domain such that $\mathbb{H} \backslash D$ is bounded and $\mathbb{H}$ and D agree in a neighborhood of 0, then there is a local martingale M_t with the property that SLE in D is SLE in $\mathbb{H}$ weighted by M_t. In [16], Lawler and Werner showed that this local martingale is related to the *Brownian loop measure*. This led Lawler to suggest in [14] that SLE in a multiply connected domain could be defined by using the Brownian loop measure to specify its Radon-Nikodym derivative with respect to SLE in a simply connected domain. In [15], Lawler defines SLE in multiply connected domains in this way and in the case of the annulus, shows that the resulting process agrees with the one found by Zhan in [23]. Even though Lawler does not use a Loewner equation in a multiply connected domain to define SLE, the analysis of a Loewner equation in a multiply connected domain is still an important aspect of his work. The goal of this paper is to better understand the Loewner equations appearing in work on SLE in multiply connected domains.

The study of Loewner equations in multiply connected domains is not new and goes as far back as 1950 in work by Komatu [10]. These equations frequently feature special functions that make the calculation work, but are introduced without motivation. In most cases, we expect such functions can be given a probabilistic interpretation. For instance, the special function in (1.1) is $(z,x) \mapsto \frac{1}{z-x}$ and the probabilistic interpretation is that the imaginary part of $\frac{1}{z-x}$ is equal to $-\pi H_{\mathbb{H}}(z,x)$, where $H_{\mathbb{H}}(z,\cdot)$ is the Poisson kernel for Brownian motion in $\mathbb{H}$. We call a domain $D \subset \mathbb{H}$ a *chordal standard domain* if it is obtained by removing a finite number of horizontal line segments from the upper half plane. In this paper, we study a strong Markov process, *excursion reflected Brownian motion* (ERBM), whose Poisson kernel can be used to prove a Loewner equation for a simple curve growing in a chordal standard domain. Excursion reflected Brownian motion in a simply connected domain is just Brownian motion and, in this case, the Loewner equation we get is just (1.1)

Roughly speaking, if $D \subset \mathbb{C}$ is a domain with n "holes," ERBM is a strong Markov process that has the distribution of a Brownian motion away from ∂D and picks a point according to *harmonic measure from* ∞ to reflect from every time it hits ∂D. To understand the behavior of ERBM, we consider the case that $D = \mathbb{C}\backslash\mathbb{D}$. In this case, every time ERBM hits $\partial\mathbb{D}$, it picks a point uniformly on $\partial\mathbb{D}$ to reflect from. ERBM has what Walsh ([1], pg. 37) has called a "roundhouse singularity" in a neighborhood of $\mathbb{D}$. That is, in any neighborhood of a time that it hits $\partial\mathbb{D}$, it will hit $\partial\mathbb{D}$ uncountably many times and jump randomly from point to point on $\partial\mathbb{D}$. Finally, an important property of ERBM is that it is conformally invariant. This will be clear once we more precisely define what it means to "pick a point according to harmonic measure from ∞ to reflect from."

The existence of ERBM follows from more general work of Fukushima and Tanaka in [9]. Their work uses the theory of Dirichlet forms and does not take advantage of the conformal invariance of ERBM. An alternative construction making explicit use of the conformal invariance of ERBM was proposed by Lawler in [13]. He proposed that ERBM could be defined in any domain with "one hole" by first constructing the process in $\mathbb{C}\backslash\mathbb{D}$ using excursion theory and then defining it in any domain conformally equivalent to $\mathbb{C}\backslash\mathbb{D}$ via conformal invariance. To define ERBM in a domain with "n holes," multiple copies of the process defined in a domain with "one hole" can be pieced together. We take this basic approach and give a new construction of ERBM.

A function is ER-harmonic if it satisfies the mean value property with respect to ERBM. More precisely, a function u is ER-harmonic if it is harmonic on D and, for any curve η surrounding a boundary component A of D,

$$u(A) = \int_\eta u(z) \frac{H_{\partial U}(A, z)}{\mathcal{E}_U(A, \eta)} \, |dz| \, ,$$

where $\dfrac{H_{\partial U}(A_i, z)}{\mathcal{E}_U(A_i, \eta)}$ is the density for the distribution of the first time ERBM started at A hits η. It turns out that a harmonic function u on D that is constant on each connected component of ∂D is ER-harmonic if and only if it is the imaginary part of a holomorphic function on D. For this reason, the study of ER-harmonic functions is a useful tool in the study of conformal maps into certain classes of finitely connected domains.

Two important ER-harmonic functions are the Poisson kernel $H_D^{ER}(z, w)$ and Green's function $G_D^{ER}(z, w)$ for ERBM. In order to define these functions, it is necessary to choose at least one boundary component of D at which to kill the ERBM. Once this is done, the definitions and many of the properties of the Poisson kernel and Green's function for ERBM are similar to those for usual Brownian motion. The Poisson kernel for ERBM was first considered by Lawler in [13] as a way of understanding a classical theorem [2] of complex analysis stating that any n-connected domain $D \subset \mathbb{C}$ is conformally equivalent to a chordal standard domain. He sketched a proof showing that the imaginary part of any such map is equal to a real multiple of the Poisson kernel for ERBM. We give a complete proof here. Furthermore, we use the Green's function for ERBM to prove two other classical conformal mapping theorems.

We finish this section by stating our main result. This result was conjectured by Lawler in [13]. Let D be a chordal standard domain, $\gamma : (0, \infty) \to D$ be a simple curve with $\gamma(0) = 0$, $D_t = D \backslash \gamma(0, t)$, and $b(t)$ be the *excursion reflected half-plane capacity* of $\gamma(0, t)$. Excursion reflected half-plane capacity is a generalization of half-plane capacity to multiply connected domains and parameterizing our curves so the excursion reflected half-plane capacity is a differentiable function of time is the most convenient parametrization.

Theorem 1.1. *For each t, there is a unique conformal map $h_t : D_t \to h_t(D_t)$ such that $h_t(D_t)$ is a chordal standard domain and $\lim_{z \to \infty} h_t(z) - z = 0$. Furthermore, this map satisfies the initial value problem*

$$\dot{h}(t)(z) = -\dot{b}(t)\, \mathcal{H}^{ER}_{h_t(D_t)}\left(h_t(z), \tilde{U}_t\right), \quad h_0(z) = z,$$

where $\tilde{U}_t = h_t(\gamma(t))$ and $\mathcal{H}^{ER}_{h_t(D_t)}\left(\cdot, \tilde{U}_t\right)$ is a conformal map with imaginary part a real multiple of $H^{ER}_D\left(\cdot, \tilde{U}_t\right)$.

If $D = \mathbb{H}$, then this theorem is just a restatement of (1.1).

1.2 Outline of the Paper

Chapter 2 sets notation and contains necessary background material. In particular, we outline the proof of the chordal Loewner equation for a simple curve growing in $\mathbb{H}$ as in [12]. The basic structure of the proof of the analogous result for chordal standard domains is the same and uses some of the same preliminary results. We conclude the chapter by proving that if γ is a curve in $\mathbb{H}$ satisfying $\gamma(0) = 0$ and F is a locally real conformal transformation, then the time derivative of $\text{hcap}(F(\gamma[0,t]))$ at 0 is equal to $F'(0)\dot{a}(0)$, where $a(t) = \text{hcap}(\gamma[0,t])$. This result is stated in [12], but the proof is not given. We actually prove a slightly stronger result, finding the error term as well.

In Chapter 3 we define and construct ERBM in finitely connected domains $D \subset \mathbb{C}$. First, we construct the process in $\mathbb{C}\backslash\mathbb{D}$ by explicitly defining a transition kernel for ERBM in terms of the transition kernels for Brownian motion and reflected Brownian motion and then using general theory to show that there actually is a strong Markov process with this transition kernel. Finally, we check that the strong Markov process we obtain satisfies our definition of ERBM. Our construction is motivated by a similar construction of Walsh's Brownian motion in [3]. Once we have ERBM in $\mathbb{C}\backslash\mathbb{D}$, we define ERBM in any domain conformally equivalent to $\mathbb{C}\backslash\mathbb{D}$ via conformal invariance. In Chapter 3.4 we show how countably many independent ERBMs in 1-connected domains can be pieced together to construct an ERBM in an n-connected domain. ERBM in D induces a discrete time Markov chain on the connected components of the boundary of D, which we discuss in Chapter 3.5. This chain was observed by Lawler in [13] and appears implicitly in classical work on conformal mapping of multiply connected domains. We conclude the chapter with a brief discussion of ER-harmonic functions. We prove a maximal principle for ER-harmonic functions and show how ERBM can be used to construct ER-harmonic functions.

Chapter 4 introduces the Poisson kernel $H_D^{ER}(z,w)$ for ERBM and proves some of its basic properties. We gather a number of estimates for $H_D^{ER}(z,w)$ that are used in Chapter

6 and show how $H_D^{ER}(\cdot,w)$ can be used to construct conformal maps. More precisely, in [13] Lawler proved that if the level sets of $H_D^{ER}(\cdot,w)$ are Jordan curves, then $H_D^{ER}(\cdot,w)$ is the imaginary part of a conformal map into a chordal standard domain. In Chapter 4.3 we complete this proof by showing that the derivative of $H_D^{ER}(\cdot,w)$ always has full rank and that this implies the level sets of $H_D^{ER}(\cdot,w)$ are Jordan curves.

Chapter 5 introduces the Green's function $G_D^{ER}(z,w)$ for ERBM in D and proves some of its basic properties. In Chapter 5.2 we use the theory of Green's function for ERBM to prove two formulas from Chapter 3 necessary to show our construction of ERBM is well-defined. We conclude the chapter by showing how $G_D^{ER}(z,\cdot)$ can be used to construct conformal maps into circularly-slit annuli.

In Chapter 6 we prove Theorem 1.1. We start by defining $\mathcal{H}^{ER}(\cdot,x)$, the *complex Poisson kernel* for ERBM, for any finitely-connected domain $D\subset\mathbb{H}$ with $\mathbb{R}\subset\partial D$. The complex Poisson kernel for ERBM, initially considered by Lawler in [13], has the property that for any $x\in\mathbb{R}$, $\mathcal{H}_D^{ER}(\cdot,x)$ is a conformal map into a chordal standard domain with imaginary part equal to $\pi H_D^{ER}(\cdot,x)$. We use $\mathcal{H}_D^{ER}(\cdot,x)$ to show that there is a unique conformal map φ_D from D into a chordal standard domain satisfying $\lim_{z\to\infty}\varphi_D(z)-z=0$. The map h_t in Theorem 1.1 is equal to $\varphi_{g_t(D_t)}\circ g_t$, where g_t is as in (1.1). There are two main steps to the proof of Theorem 1.1. First we prove the result at $t=0$. This proof is similar in spirit to the proof of the analogous result for g_t. The second main step is to show that $\tilde{U}_t=h_t(\gamma(t))$ is a well-defined continuous function. We do this by combining the analogous fact for $U_t=g_t(\gamma(t))$ with derivative estimates for $\varphi_D(x)$ restricted to $\mathbb{R}$. The key observation is that $\varphi_D'(x)=\pi H_D^{ER}(\infty,x)$, where $H_D^{ER}(\infty,x)$ is the "normal derivative" of $H_D^{ER}(\cdot,x)$ at ∞. This allows us to use appropriate Poisson kernel estimates to provide the necessary estimates for $\varphi_D'(x)$. Since $g_t(D_t)$ varies with t, we need Poisson kernel estimates that are uniform over certain classes of domains.

CHAPTER 2

BACKGROUND MATERIAL

2.1 Some Notation

We denote the unit disk in $\mathbb{C}$ centered at the origin by $\mathbb{D}$ and the upper half-plane by $\mathbb{H}$. We let $\mathcal{Y}_n$ consist of all subdomains of $\mathbb{C}$ with n "holes." More precisely, let $\mathcal{Y}_n$ consist of all connected domains of the form

$$D = \mathbb{C} \backslash \left[A_0 \cup A_1 \cup \cdots \cup A_n \right],$$

where $A_0, A_1, \ldots, A_n$ are closed disjoint subsets of $\mathbb{C}$ such that A_i is simply connected, bounded, and larger than a single point for $1 \leq i \leq n$ (we allow A_0 to be empty) and $\mathbb{C} \backslash A_0$ is simply connected. We denote $\displaystyle\bigcup_{i=0}^{\infty} \mathcal{Y}_i$ by $\mathcal{Y}$.

We call $A \subset \mathbb{H}$ a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \backslash A$ is simply connected and denote the set of all compact $\mathbb{H}$-hulls by $\mathcal{Q}$. We denote the subsets of $\mathcal{Y}$ and $\mathcal{Y}_n$ consisting of domains such that A_0 is the union of $\mathbb{C} \backslash \mathbb{H}$ and a compact $\mathbb{H}$-hull by $\mathcal{Y}^*$ and $\mathcal{Y}_n^*$ respectively. We call $D \in \mathcal{Y}$ a *chordal standard domain* if D is the upper half-plane with a finite number of horizontal line segments removed. We denote the set of chordal standard domains and n-connected chordal standard domains by $\mathcal{CY}$ and $\mathcal{CY}_n$ respectively. If D is a domain with $\mathbb{C} \backslash A_0 = \mathbb{H}$, then if $r > 0$, we let

$$D^r = \left\{ z \in D : |z| \geq r \right\}.$$

We also let

$$H^r = \{z \in \mathbb{H} : |z| \leq r\}$$

and

$$\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}.$$

We denote the open annulus centered at $x \in \mathbb{R}$ with inner radius r and outer radius R by $A_{r,R}(x)$ and $A_{r,R}(x) \cap \mathbb{H}$ by $A^+_{r,R}(x)$. We write $A_{r,R}$ and $A^+_{r,R}$ for $A_{r,R}(0)$ and $A^+_{r,R}(0)$ respectively. Finally, we denote the open ball of radius r centered at z by $B_r(z)$ and, if $x \in \mathbb{R}$, $B_r(x) \cap \mathbb{H}$ by $B^+_r(x)$.

If A is a subset of $\mathbb{C}$, we let the radius of A, denoted $\text{rad}(A)$, be the infimum over all $r > 0$ such that $A \subset r\mathbb{D}$ and the diameter of A, denoted $\text{diam}(A)$, be the supremum over all $x, y \in A$ of $|x - y|$.

We will use c to denote a real constant that is allowed to change from one line to the next. We write $f(z) \sim g(z)$ as $z \to a$ if $\lim_{z\to a} \frac{f(z)}{g(z)} = 1$ and $f(z) \asymp g(z)$ as $z \to a$ if $f(z) = O(|g(z)|)$ and $g(z) = O(|f(z)|)$ as $z \to a$.

2.2 Poisson Kernel for Brownian Motion

Let $D \in \mathcal{Y}$ and let τ_D be the first time that a Brownian motion B_t leaves D. If ∂D has at least one regular point for Brownian motion, then for each $z \in D$, the distribution of B_{τ_D} defines a measure $\text{hm}_D(z, \cdot)$ on ∂D (with the σ-algebra generated by Borel subsets of ∂D) called *harmonic measure in D from z*. We say ∂D is *locally analytic* at $w \in \partial D$ if ∂D is an analytic curve in a neighborhood of w. If ∂D is locally analytic at w, then in a neighborhood of w, $\text{hm}_D(z, \cdot)$ is absolutely continuous with respect to arc length and the density of $\text{hm}_D(z, \cdot)$ at w with respect to arc length is called the *Poisson kernel for Brownian motion* and is denoted $H_D(z, w)$.

Some of the domains we consider will have two-sided boundary points (a recurring example is when D is a chordal standard domain). If w is a two-sided boundary point, we should really think of it as being two distinct boundary points, w^+ and w^-. In such cases, by abuse of notation, we will sometimes write $H_D(z, w)$ when we should consider $H_D(z, w^+)$ and $H_D(z, w^-)$ separately.

Harmonic measure is conformally invariant. That is, if $f : D \to D'$ is a conformal map, then

$$\mathrm{hm}_D(z, V) = \mathrm{hm}_{D'}(f(z), f(V)).$$

Using this, we see that if ∂D is locally analytic at w and $\partial D'$ is locally analytic at $f(w)$, then

$$H_{D'}(f(z), f(w)) = \left|f'(w)\right|^{-1} H_D(z, w). \tag{2.1}$$

It is well-known that

$$H_{\mathbb{H}}(x + iy, x') = \frac{1}{\pi} \frac{y}{(x - x')^2 + y^2}. \tag{2.2}$$

Using (2.2), it is straightforward to compute that if $|x| > \epsilon$, then $H_{\mathbb{H}}(\cdot, x)$ restricted to the boundary of $B_\epsilon^+(0)$ is bounded above by

$$\max\left\{\frac{\epsilon}{\pi(x - \epsilon)^2}, \frac{\epsilon}{\pi(x + \epsilon)^2}\right\}. \tag{2.3}$$

Using (2.1), it is sometimes possible to explicitly compute the Poisson kernel for a simply connected domain. Often though, having an estimate is good enough. One particularly important estimate ([12], pg. 50) that we will use extensively is that if $|z| \geq 2\epsilon$, then

$$H_{H^\epsilon}\left(z, \epsilon e^{i\theta}\right) = 2H_{\mathbb{H}}(z, 0)\sin\theta\left[1 + O\left(\frac{\epsilon}{|z|}\right)\right], \tag{2.4}$$

as $\frac{\epsilon}{|z|} \to 0$. In particular, if we fix a radius $r > 0$, then the $O\left(\frac{\epsilon}{|z|}\right)$ term can be replaced with an $O(\epsilon)$ term that is uniform over all z with $|z| > r$.

Using (2.1), (2.4), and the map $z \mapsto \frac{-1}{z}$, we see that if $|z| < r/2$, then

$$
\begin{aligned}
H_{B_r^+(0)}\left(z, re^{i\theta}\right) &= \frac{1}{r^2} H_{H^{1/r}}\left(\frac{-1}{z}, \frac{-e^{-i\theta}}{r}\right) \\
&= \frac{2}{r^2} H_{\mathbb{H}}\left(\frac{-1}{z}, 0\right) \sin(\theta)\left[1 + O\left(\frac{|z|}{r}\right)\right] \\
&= \frac{2\,\mathrm{Im}[z]}{r^2} \sin(\theta)\left[1 + O\left(\frac{|z|}{r}\right)\right], \qquad (2.5)
\end{aligned}
$$

as $\frac{|z|}{r} \to 0$. In particular for fixed r, the probability that a Brownian motion started at z exits $B_r^+(0)$ on $|z| = r$ is comparable to $\frac{\mathrm{Im}[z]}{r}$ as $z \to 0$.

The function $H_D(\cdot, w)$ can be characterized up to a positive multiplicative constant as the unique positive harmonic function on D that is "equal to" the Dirac delta function at w on ∂D.

Proposition 2.1. *Let $D \in \mathcal{Y}$ be such that ∂D is locally analytic at $w \in \partial D$. Then $H_D(\cdot, w)$ is up to a real constant multiple the unique positive harmonic function on D that satisfies $H_D(z, w) \to 0$ as $z \to w'$ for any $w' \in \partial D$ not equal to w.*

Next, we prove an estimate analogous to (2.4) for $D \in \mathcal{Y}^*$. A useful observation [13] that we will use in the proof of this estimate is that if $D_2 \subset D_1$ and D_1 and D_2 agree in a neighborhood of $w \in \partial D_1$, then

$$
H_{D_2}(z, w) = H_{D_1}(z, w) - \mathbf{E}^z\left[H_{D_1}\left(B_{\tau_{D_2}}, w\right)\right]. \qquad (2.6)
$$

Lemma 2.2. *Let $D \in \mathcal{Y}^*$ be such that ∂D is locally analytic and ∂D and $\partial \mathbb{H}$ agree in a neighborhood of* 0*. If $|z| > 2\epsilon$, then*

$$
H_{D^\epsilon}\left(z, \epsilon e^{i\theta}\right) = 2 \sin\theta \; H_D(z, 0)\left[1 + O(\epsilon)\right],
$$

where for any $r > 0$, $O(\epsilon)$ is uniform over all z with $|z| > r$.

Proof. Using (2.6), we have

$$H_{D^\epsilon}\left(z, \epsilon e^{i\theta}\right) = H_{H^\epsilon}\left(z, \epsilon e^{i\theta}\right) - \mathbf{E}^z\left[H_{H^\epsilon}\left(B_{\tau_{D_\epsilon}}, \epsilon e^{i\theta}\right)\right]. \tag{2.7}$$

We can rewrite $\mathbf{E}^z\left[H_{H^\epsilon}\left(B_{\tau_{D_\epsilon}}, \epsilon e^{i\theta}\right)\right]$ as (where by convention $H_{D^\epsilon}(z,w) = 0$ if $w \notin \partial D^\epsilon$)

$$\begin{aligned}
&\sum_{i=0}^{n} \int_{\partial A_i} H_{H^\epsilon}\left(w, \epsilon e^{i\theta}\right) H_{D^\epsilon}(z,w) \quad |dw| \\
=&2\sin\theta \sum_{i=0}^{n} \int_{\partial A_i} H_{\mathbb{H}}(w,0)\left[1 + O(\epsilon)\right] H_{D^\epsilon}(z,w) \quad |dw| \\
=&2\sin\theta \left[\sum_{i=0}^{n} \int_{\partial A_i} H_{\mathbb{H}}(w,0)\, H_{D^\epsilon}(z,w) \quad |dw|\right]\left[1 + O(\epsilon)\right].
\end{aligned} \tag{2.8}$$

Applying (2.6) again, we have

$$H_{D^\epsilon}(z,w) = H_D(z,w) - \mathbf{E}^z\left[H_D\left(B_{\tau_{D^\epsilon}}, w\right)\right]. \tag{2.9}$$

The probability that a Brownian motion in D started at z leaves D^ϵ on $\partial B_\epsilon^+(0)$ is less than the probability that a Brownian motion in H^ϵ does the same thing, which is $O(\epsilon)$ by (2.4). If R is such that $B_R^+(0) \subset D$, then in order for a Brownian motion in D started on $\partial B_\epsilon^+(0)$ to not exit D on $\mathbb{R}$, it has to leave $B_R^+(0)$ before hitting $\mathbb{R}$. By (2.5), the probability of this event is $O(\epsilon)$ and hence, the probability that a Brownian motion in D started on $\partial B_\epsilon^+(0)$ does not exit D^ϵ on $\mathbb{R}$ is $O(\epsilon)$. It follows that

$$\mathbf{E}^z\left[H_D\left(B_{\tau_{D^\epsilon}}, w\right)\right] = O\left(\epsilon^2\right). \tag{2.10}$$

Substituting (2.9) into (2.8) and using (2.10), we see that (2.8) is equal to

$$
\begin{aligned}
&2\sin\theta\left[\sum_{i=0}^{n}\int_{\partial A_i} H_{\mathbb{H}}(w,0)\, H_D(z,w) \quad |dw|\right][1+O(\epsilon)] \\
=&2\sin\theta\ \mathbf{E}^z\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right][1+O(\epsilon)].
\end{aligned}
$$

Combining this with (2.4) and (2.6), (2.7) becomes

$$
H_{D^\epsilon}\left(z,\epsilon e^{i\theta}\right) = 2\sin\theta\ H_D(z,0)[1+O(\epsilon)]. \tag{2.11}
$$

□

We will need an estimate for the derivative (in the second variable) of the Poisson kernel.

Lemma 2.3. *Let $D \in \mathcal{Y}^*$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$ and $f_z(x) := H_D(z,x)$. If $x' \in \mathbb{R}$ and $r < |z-x'|$ is such that $B_r^+(x') \subset D$, then $|f_z'(x')| \le \frac{4}{\pi r^2}$.*

Proof. Let $z = x+iy$. If $D = \mathbb{H}$, then, using (2.2), we can compute

$$
|f_z'(x')| = \left|\frac{2y(x-x')}{\pi (z-x')^4}\right| < \frac{2}{\pi (z-x')^2}. \tag{2.12}
$$

Recall that by (2.6) we have

$$
f_z(x) = H_{\mathbb{H}}(z,x) - \sum_{i=0}^{n}\int_{\partial A_i} H_{\mathbb{H}}(z,x)\, d\,\mathrm{hm}_D(z,x).
$$

Using bounded convergence, the result for general D follows from (2.12).

□

2.3 Some Brownian Measures

Let $D \in \mathcal{Y}$. If ∂D is locally analytic at w, then the *boundary Poisson kernel* is defined by

$$H_{\partial D}(w, z) = \frac{d}{dn} H_D(w, z),$$

where n is the inward pointing normal at w. If w is a two-sided boundary point, then we have two distinct boundary Poisson kernels, $H_{\partial D}\left(w^+, z\right)$ and $H_{\partial D}\left(w^-, z\right)$. In such cases, by abuse of notation, we will sometimes write $H_{\partial D}(w, z)$ when we should consider $H_{\partial D}\left(w^+, z\right)$ and $H_{\partial D}\left(w^-, z\right)$ separately. If f is a conformal map and $\partial f(D)$ is locally analytic at $f(w)$ and $f(z)$, then

$$H_{\partial D}(w, z) = \left|f'(w)\right| \left|f'(z)\right| H_{\partial f(D)}(f(w), f(z)). \tag{2.13}$$

If D is as in Lemma 2.2, then using Lemma 2.2, we have that

$$H_{\partial D^\epsilon}\left(z, \epsilon e^{i\theta}\right) = 2 \sin\theta \; H_{\partial D}(z, 0)\left[1 + O(\epsilon)\right], \tag{2.14}$$

where for any $r > 0$, $O(\epsilon)$ is uniform over all $z \in \partial D$ with $|z| > r$.

The definition of excursion reflected Brownian motion uses excursion measure. Excursion measure is usually defined as a measure on paths between two boundary points of D. Since we will only be interested in the norm of this measure, the definition we give of excursion measure is the norm of excursion measure as defined elsewhere ([12], [13]).

Definition 2.4. Suppose $D \subset \mathbb{C}$ is a domain with locally analytic boundary and V and V' are disjoint arcs in ∂D. Then

$$\mathcal{E}_D\left(V, V'\right) := \int_V \int_{V'} H_{\partial D}(z, w)\, |dz|\, |dw|$$

is called *excursion measure*. Excursion measure normalized to have total mass one is called *normalized excursion measure* and is denoted $\overline{\mathcal{E}}_D(V,\cdot)$.

Using (2.13), we can check that $\mathcal{E}_D$ is conformally invariant. This allows us to define $\mathcal{E}_D\left(V,V'\right)$ even if D does not have locally analytic boundary. We will often write $\mathcal{E}_D(A,V)$ for $\mathcal{E}_D(\partial A,V)$ and $\overline{\mathcal{E}}_D(A,V)$ for $\overline{\mathcal{E}}_D(\partial A,V)$. We will also write $H_{\partial D}(A,z)$ as shorthand for the quantity $\int_{\partial A} H_{\partial D}(z,w)\,|dz|$. Using (2.13), we see that if $f:D\to D'$ is a conformal map, then

$$H_{\partial D}(A,z)=H_{\partial f(D)}\left(f(A),f(z)\right)\left|f'(z)\right|.$$

As a result, it is possible to define $H_{\partial D}(A,z)$ even if A does not have locally analytic boundary.

We conclude by giving some well-known [13] results that we will need about the Brownian bubble measure. In what follows, let $D\in\mathcal{Y}_n^*$ be such that ∂D is locally analytic and $x\in\partial D\cap\mathbb{R}$. Define the *Brownian boundary bubble measure at x of bubbles leaving D* by

$$\Gamma(D;x)=\Gamma(D;x\mid\mathbb{H},x)=\pi\int_{\partial D}H_{\partial D}(x,z)\,H_{\mathbb{H}}(z,x)\,|dz|. \tag{2.15}$$

We claim

$$\pi H_D(z,0)=\pi H_{\mathbb{H}}(z,0)-\operatorname{Im}[z]\,\Gamma(D;0)\left[1+O(|z|)\right], \tag{2.16}$$

as $z\to0$. To see this, let $z=x+iy$ and observe that using (2.6) we have

$$\begin{aligned}\pi H_{\mathbb{H}}(z,0)-\pi H_D(z,0)=&\pi\mathbf{E}^z\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]\\ =&\pi\left(\mathbf{E}^z\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]-\pi\mathbf{E}^{iy}\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]\right)\\ &+\mathbf{E}^{iy}\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]\\ =&\pi\left(\mathbf{E}^z\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]-\mathbf{E}^{iy}\left[H_{\mathbb{H}}\left(B_{\tau_D},0\right)\right]\right)\\ &+\pi\int_{\partial D}H_{\mathbb{H}}(w,0)\,H_D(iy,w)\quad|dw|.\end{aligned} \tag{2.17}$$

Using (2.5), we see that the first term in (2.17) is equal to $O(|z|)\operatorname{Im}[z]$, as $z \to 0$. Using the expansion

$$H_D(iy, w) = yH_{\partial D}(0, w) + O\left(y^2\right),$$

as $y \to 0$, and (2.15), we see that the second term in (2.17) is equal to

$$\operatorname{Im}[z]\left[\Gamma(D; 0) + O(y)\right],$$

as $z \to 0$, from which (2.16) follows.

2.4 Green's Function for Brownian Motion

In what follows, let $D \in \mathcal{Y}$ be such that ∂D has at least one regular point for Brownian motion. In this setting, it is possible to define a (a.s. finite) Green's function for Brownian motion $G_D(z, w)$ (see, for instance, [12]). By convention, we scale $G_D(z, \cdot)$ so that it is the density for the occupation time of Brownian motion. As a result, what we mean by G_D may differ by a factor of π from what appears elsewhere.

It is well-known that $G_D(z, w) = G_D(w, z)$ and that $G_D(z, \cdot)$ can be characterized as the unique harmonic function on $D\setminus\{z\}$ such that $G_D(z, w) \to 0$ as $w \to \partial D$ and

$$G_D(z, w) = \frac{-\log|z - w|}{\pi} + O(1), \tag{2.18}$$

as $z \to w$. Another property of $G_D(z, w)$ is that it is conformally invariant. That is, if $f : D \to D'$ is a conformal map, then $G_{f(D)}(f(z), f(w)) = G_D(z, w)$. Finally, it is well-known that

$$G_{r\mathbb{D}}(0, z) = -\frac{\log r - \log|z|}{\pi} \tag{2.19}$$

and

$$G_{\mathbb{H}}(x+iy,i)=\frac{1}{2\pi}\log\frac{x^2+(y+1)^2}{x^2+(y-1)^2}. \tag{2.20}$$

Another fact that we will use is that the normal derivative of $G_D(z,\cdot)$ at $w\in\partial D$ is equal to $2H_D(z,w)$. While we will not prove this fact, we will prove the following lemma that is used in the proof and that we will use when we prove a similar statement for the Green's function for ERBM.

Lemma 2.5.

$$\int_0^{\pi} G_{\mathbb{H}}\left(e^{i\theta},\epsilon i\right)\sin\theta\ d\theta=\epsilon+O\left(\epsilon^2\right),$$

as $\epsilon\to 0$.

Proof. Using the Taylor series expansion for $\log 1+x$, the conformal invariance of G_D, and (2.20), we see that

$$\begin{aligned}\int_0^{\pi} G_{\mathbb{H}}\left(e^{i\theta},\epsilon i\right)\sin\theta\ d\theta&=\frac{1}{2\pi}\int_0^{\pi}\log\left(\frac{\cos^2\theta+(\sin\theta+\epsilon)^2}{\cos^2\theta+(\sin\theta-\epsilon)^2}\right)\sin\theta\ d\theta\\&=\frac{1}{2\pi}\int_0^{\pi}\left[4\epsilon\sin\theta+O\left(\epsilon^2\right)\right]\sin\theta\ d\theta\\&=\epsilon+O\left(\epsilon^2\right),\end{aligned}$$

as $\epsilon\to 0$. □

Finally, if $D_2\subset D_1$, it is easy to check that

$$G_{D_2}(z,w)=G_{D_1}(z,w)-\mathbf{E}^z\left[G_{D_1}\left(B_{\tau_{D_2}},w\right)\right]. \tag{2.21}$$

2.5 Chordal Loewner Equation

In this section, we outline the proof of the chordal Loewner equation for a simple curve growing in the upper half-plane as presented in [12]. Our purpose is both to motivate the proof we give of the analogous result in non-simply connected domains and to gather some preliminary results needed for that proof.

The Loewner equation we are interested in is a differential equation governing the behavior of the conformal map that maps the upper half-plane with a simple curve removed onto the upper half-plane. The next proposition shows that there is a unique such conformal map with a specified asymptotic at infinity.

Proposition 2.6. *If $A \in \mathcal{Q}$, then there is a unique conformal map $g_A : \mathbb{H} \backslash A \to \mathbb{H}$ such that*

$$\lim_{z \to \infty} \left[g_A(z) - z \right] = 0.$$

In particular, g_A has an expansion near infinity of the form

$$g_A(z) = z + \frac{a_1}{z} + O\left(|z|^{-2} \right), \quad z \to \infty.$$

The constant a_1 is called the *half-plane capacity (from infinity)* for A and is denoted $\operatorname{hcap}(A)$. There are several different ways to compute $\operatorname{hcap}(A)$.

Proposition 2.7. *Suppose $A \in \mathcal{Q}$, B_t is a Brownian motion in $\mathbb{H}$, and τ is the first time that B_t leaves $\mathbb{H} \backslash A$. Then for all $z \in \mathbb{H} \backslash A$,*

$$\operatorname{Im}\left[z - g_A(z) \right] = \mathbf{E}^z \left[\operatorname{Im}\left[B_\tau \right] \right].$$

Also, $\operatorname{hcap}(A)$ *is equal to each of the following.*

1. $\lim_{y \to \infty} y \mathbf{E}^{iy} \left[\operatorname{Im}\left[B_\tau \right] \right]$.
2. $\frac{2r}{\pi} \int_0^\pi \mathbf{E}^{re^{i\theta}} \left[\operatorname{Im}\left[B_\tau \right] \right] \sin\theta \, d\theta$ *for any $r > 0$ such that* $\operatorname{rad}(A) < r$.

We give one more interpretation of $\operatorname{hcap}(A)$.

Proposition 2.8. *Let $A \in \mathcal{Q}$ and $D = \mathbb{H} \backslash A$ and for any Borel subset V of ∂A, define*

$$\mu_A(V) = \lim_{y \to \infty} y \, \mathrm{hm}_D(iy, V).$$

The following statements hold.

1. *If* $\mathrm{rad}(A) < R$, *then* $\mu_A(V) = \frac{2R}{\pi} \int_0^\pi \mathrm{hm}_D\left(Re^{i\theta}, V\right) \sin\theta \, d\theta$.

2. μ_A *is a measure.*

3. $\mathrm{hcap}(A) = \int \mathrm{Im}[z] \, d\mu_A(z)$.

Proof. Using (2.4) and the strong Markov property for Brownian motion, it is straightforward to verify the first statement. Using monotone convergence, it is easy to see that the first statement implies the second. Using Proposition 2.7, it is a straightforward exercise in measure theory to prove the third statement. □

Using Proposition 2.7, we can find a uniform bound on the difference between $g_A(z)$ and $z + \frac{\mathrm{hcap}(A)}{z}$ in terms of $\mathrm{hcap}(A)$ and $\mathrm{rad}(A)$.

Proposition 2.9. *There is a $c < \infty$ such that for all $A \in \mathcal{Q}$ and $|z| \geq 2\,\mathrm{rad}(A)$,*

$$\left| z - g_A(z) + \frac{\mathrm{hcap}(A)}{z} \right| \leq c \frac{\mathrm{hcap}(A)\,\mathrm{rad}(A)}{|z|^2}.$$

This result can be interpreted as a proof of the chordal Loewner equation at $t = 0$. That is, if we think of A as being the trace of a simple curve at a small time t, this result shows that the time derivative of g_A at 0 is equal to the time derivative of $\mathrm{hcap}(A)$ at 0 divided by z.

In what follows, let $\gamma : [0, \infty) \to \mathbb{C}$ be a simple curve such that $\gamma(0) \in \mathbb{R}$ and $\gamma(0, \infty) \subset \mathbb{H}$. Let $a(t) = \mathrm{hcap}(\gamma_t)$ and assume (reparametrizing if necessary) that $a(t)$ is C^1. For each $t \geq 0$, let $\gamma_t := \gamma[0, t]$ and $g_t : \mathbb{H} \backslash \gamma_t \to \mathbb{H}$ be the unique conformal transformation satisfying $\lim_{z\to\infty} g_t(z) - z = 0$. For each $s > 0$, let $\gamma^s(t) = g_s(\gamma(s+t))$ and $g_{s,t} = g_{\gamma^s_{t-s}}$. Observe that $g_t = g_{s,t} \circ g_s$. The next proposition shows that g_t maps the "tip" of γ_t to a unique $U_t \in \mathbb{R}$ and that the resulting function $t \mapsto U_t$ is continuous.

Proposition 2.10. *For all $t > 0$, there is a unique $U_t \in \mathbb{R}$ such that*

$$\lim_{z\to\gamma(t)} g_t(z) = U_t,$$

where the limit is taken over $z \in \mathbb{H}\backslash\gamma_t$. Furthermore,

$$U_t = \lim_{s\to t^-} g_s(\gamma(t))$$

and $t \mapsto U_t$ is continuous.

The main tool in the proof of Proposition 2.10 is the following technical lemma.

Lemma 2.11. *There exists a constant $c < \infty$ such that if $0 \le s < t \le t_0 < \infty$, then*

$$\operatorname{diam}\left[g_s(\gamma(s,t))\right] \le c\sqrt{\operatorname{diam}(\gamma[0,t_0])\operatorname{osc}(\gamma, t-s, t_0)}$$

and

$$\|g_s - g_t\|_\infty \le c\sqrt{\operatorname{diam}(\gamma[0,t_0])\operatorname{osc}(\gamma, t-s, t_0)},$$

where

$$\operatorname{osc}(\gamma,\delta,t_0) = \sup\{|\gamma(s) - \gamma(t)| : 0 \le s < t \le t_0; |t-s| < \delta\}$$

and $g_s - g_t$ is considered as a function on $\mathbb{H}\backslash\gamma_t$.

The main tool in proving Lemma 2.11 is the Beurling estimate, which we will not discuss here. The proof also needs the following useful lemma.

Lemma 2.12. *Let $A \in \mathcal{Q}$ and $\operatorname{rad}(A) = r$. Then for all $x > r$,*

$$x \leq g_A(x) \leq x + \frac{r^2}{x}$$

and for all $x < -r$,

$$x + \frac{r^2}{x} \leq g_A(x) \leq x.$$

Furthermore, if $z \in \mathbb{H} \backslash A$, then

$$|g_A(z) - z| \leq 3 \operatorname{rad}(A).$$

We also need the following lemma.

Lemma 2.13. *Suppose $u : [0, t_0) \to \mathbb{C}$ is a continuous function such that the right derivative*

$$u'_+(t) = \lim_{\epsilon \to 0^+} \frac{u(t+\epsilon) - u(t)}{\epsilon}$$

exists for all $t \in [0, t_0)$ and is a continuous function. Then $u'(t) = u'_+(t)$ for all $t \in (0, t_0)$.

We state and discuss the proof of the chordal Loewner equation.

Theorem 2.14. *For all $z \in \mathbb{H} \backslash \gamma_{t_0}$ and $0 \leq t \leq t_0$, $g_t(z)$ is a solution to the initial value problem*

$$\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{2.22}$$

An important observation that we will return to when we prove the analogous result for multiply connected domains is that the imaginary part of $\frac{\dot{a}(t)}{g_t(z) - U_t}$ is equal to $-\pi \dot{a}(t) H_{\mathbb{H}}(z, U_t)$. Using the Schwarz reflection principle, we can show that (2.22) holds for $x \in \mathbb{R}$ as well. The idea behind the proof of Theorem 2.14 is to apply Proposition 2.9 to

$$g_{s+\epsilon}(z) - g_s(z) = g_{s,s+\epsilon}(g_s(z)) - g_s(z)$$

to conclude that $g_s(z)$ has a right derivative equal to $\frac{\dot{a}(s)}{g_s(z)-U_s}$ and then to apply Proposition 2.10 and Lemma 2.13.

If $t \mapsto U_t$ is a continuous function and $t \mapsto a(t)$ is an increasing C^1 function, then a converse to Theorem 2.14 holds. More precisely, for each $t \geq 0$ it is possible to find a $K_t \in \mathcal{Q}$ and conformal map $g_t : K_t \to \mathbb{H}$ such that

$$\dot{g}_t(z) = \frac{\dot{a}(t)}{z - U_t}, \quad g_0(z) = z.$$

A family of maps g_t arising in this way is called a *generalized Loewner Chain* with *driving function* U_t. While we will not make use of this important fact, we will need some facts about generalized Loewner chains.

It can be checked that if g_t is a generalized Loewner chain, then for all $z \in \mathbb{H} \backslash K_t$,

$$g_t'(z) = \exp\left\{ - \int_0^t \frac{\dot{a}(s)\, ds}{(g_s(z) - U_s)^2} \right\}. \tag{2.23}$$

Using this, we can derive an estimate for the spatial derivative of g_t restricted to the real line.

Lemma 2.15. *Let g_t be a generalized Loewner chain and let $r_t > 0$ be such that $\gamma_t \subset B_{r_t}(\gamma(0))$. Then there is a $c > 0$ such that*

$$1 - \frac{c r_t^2}{x^2} \leq g_t'(x) \leq 1,$$

for $x \in \mathbb{R}$ with $|x| > 3r_t$.

Proof. We will prove the result for $x > 3r_t$; the $x < -3r_t$ case is the same. Using Lemma 2.12, we see that

$$x \leq g_t(x) \leq x + \frac{r_t^2}{x}$$

and

$$r_t \leq g_t(r_t) \leq r_t + \frac{r_t^2}{r_t} = 2r_t,$$

and as a result,

$$g_t(x) - g_t(r_t) \geq x - 2r_t.$$

Since $g_s(x) - g_s(r_t)$ is a decreasing function of s (this can be shown using Theorem 2.14), we conclude that

$$g_s(x) - U_s \geq g_s(x) - g_s(r_t) \geq g_t(x) - g_t(r_t) \geq x - 2r_t,$$

for all $0 \leq s \leq t$. Using (2.23), it follows that

$$\exp\left\{-\frac{a(t)}{(x-2r_t)^2}\right\} \leq g_t'(x) \leq 1.$$

Since $a(t) \leq r_t^2$, we conclude that

$$\exp\left\{-\frac{r_t^2}{(x-2r_t)^2}\right\} \leq g_t'(x) \leq 1,$$

from which the result follows. □

It is not hard to check that the constant in Lemma 2.15 is less than or equal to 9.

We conclude the section by looking at the effect of applying a locally real conformal transformation on the time derivative of the half-plane capacity of a continuously increasing $K_t \in \mathcal{Q}$. We will need the following well-known result, which provides bounds for the derivatives of harmonic functions.

Lemma 2.16. *Let u be a real-valued harmonic function on a domain $D \subset \mathbb{C}$. For each*

$k \in \mathbb{N}$, there is a $c(k) > 0$ such that if $j \leq k$ is a non-negative integer, then

$$\left|\partial_x^j \partial_y^{k-j} u(z)\right| \leq c(k) \operatorname{dist}(z, \partial D)^{-k} \|u\|_\infty .$$

The proof of Proposition 2.17 is due to Greg Lawler.

Proposition 2.17. *Let $F : B_R(0) \to \mathbb{C}$ be a conformal transformation that maps reals to reals, $A \in \mathcal{Q}$, and $r = \operatorname{rad}(A)$. Then*

$$\operatorname{hcap}(F(A)) = F'(0)^2 \operatorname{hcap}(A) \left[1 + O\left(\frac{r}{R}\right)\right], \quad r \to 0,$$

where $O\left(\frac{r}{R}\right)$ is independent of F and A.

Proof. We start by reducing to the case of $R = 1$ and $F'(0) = 1$. Define

$$\hat{F}(z) = \frac{F(zR)}{RF'(0)}, \qquad \hat{A} = R^{-1}A$$

and assume that the result holds when $R = 1$ and $F'(0) = 1$. We then have

$$\begin{aligned} \operatorname{hcap}[F(A)] &= R^2 F'(0)^2 \operatorname{hcap}\left[\hat{F}\left(\hat{A}\right)\right] \\ &= R^2 F'(0)^2 \operatorname{hcap}\left(\hat{A}\right)\left[1 + O\left(\frac{r}{R}\right)\right] \\ &= F'(0)^2 \operatorname{hcap}(A)\left[1 + O\left(\frac{r}{R}\right)\right] \end{aligned}$$

By considering a translate of F, we may assume $F(0) = 0$. Hence, for the remainder of the proof, we assume $R = 1$, $F'(0) = 1$, and $F(0) = 0$.

We claim that

$$\operatorname{Im}[F(z)] = \operatorname{Im}[z]\left[1 + O(|z|)\right], \tag{2.24}$$

as $z \to 0$. Using the Koebe distortion theorem, we can find a uniform upper bound (independent of F) on $\left|F'(z)\right|$ over all $z \in \mathbb{C}$ such that $|z| \leq 3/4$. If follows by the Cauchy integral formula that there is a uniform upper bound for $\left|F''(z)\right|$ over all z such that $|z| < 1/2$. As a result, there is a $c > 0$ such that

$$\left|F'(z) - 1\right| \leq c\,|z|,$$

for all z such that $|z| < 1/2$. Writing $F = u + iv$, this implies that if $|z| < 1/2$, then

$$\left|\frac{\partial v}{\partial y}(z) - 1\right| \leq \left|\left(\frac{\partial v}{\partial y}(z) - 1\right) - i\frac{\partial u}{\partial y}(z)\right| = \left|F'(z) - 1\right| \leq c\,|z|.$$

Hence, writing $z = x + iy$, we see that

$$|\mathrm{Im}\, F(z) - y| = \left|\int_0^y \left(\frac{\partial v}{\partial y}(x + iy') - 1\right)\, dy'\right| \leq cy\,|z|,$$

from which (2.24) follows.

Define a measure $F \circ \mu_A$ on $\partial F(A)$ by

$$F \circ \mu_A(V) = \mu_A\left(F^{-1}(V)\right),$$

where μ_A is as in Proposition 2.8. Using Proposition 2.8 and (2.24), we have

$$\begin{aligned}
\mathrm{hcap}(A) &= \int \mathrm{Im}\,[z]\, d\mu_A(z) \\
&= \int \mathrm{Im}\left[F^{-1}(w)\right] d\left(F \circ \mu_A\right)(w) \\
&= [1 + O(r)] \int \mathrm{Im}\,[w]\, d\left(F \circ \mu_A\right)(w).
\end{aligned}$$

Since we also have

$$\operatorname{hcap}[F(A)] = \int \operatorname{Im}[w]\, d\mu_{F(A)}(w),$$

to complete the proof, it is enough to show that

$$\mu_{F(A)}(V) = F \circ \mu_A(V)\,[1 + O(r)],$$

where $O(r)$ is independent of V, F, and A.

Let $D = \mathbb{H}\backslash A$ and for any Borel subset V of ∂A, define

$$\operatorname{hm}_D(V) = \max\{\operatorname{hm}_D(z, V) : |z| = 2r\}.$$

We claim that

$$\operatorname{hm}_D(z, V) \asymp \frac{\operatorname{Im}[z]}{r} \operatorname{hm}_D(V), \tag{2.25}$$

for any $z \in D$ such that $|z| = 2r$. Using (2.4), we see that there is a constant $c > 0$ such that if u is a positive harmonic function on $\mathbb{H}\backslash B_r^+(0)$ and z_0 is the location of the maximum value of u restricted to $|z| = 2r$, then $\operatorname{Im}[z_0] > cr =: s(r)$. Let $\tilde{r} = \min\{r, s(r)\}$ and

$$K = \{z \in \mathbb{H} : |z| = 2r; \operatorname{Im}[z] \geq \tilde{r}/4\}.$$

Since $\frac{\operatorname{Im}[z]}{r}$ is bounded from below on K, it follows that there is a $c > 0$ such that for any $z \in K$

$$\operatorname{hm}_D(z, V) \leq \operatorname{hm}_D(V) \leq c\frac{\operatorname{Im}[z]}{r} \operatorname{hm}_D(V). \tag{2.26}$$

Fix $z \notin K$ such that $|z| = 2r$ and assume without loss of generality that $z \in B^+_{\frac{3\tilde{r}}{4}}(2r)$. Using (2.5), we can show that the location of the maximum value of any positive harmonic function on $B_r^+(2r)$ restricted to $\partial B^+_{\frac{3\tilde{r}}{4}}(2r)$ is bounded uniformly away from $\mathbb{R}$. As a result, we can use the Harnack inequality to find a constant $c > 0$ such that if $\tilde{w}$ is the unique point in $\partial B^+_{\frac{3\tilde{r}}{4}}(2r) \cap B^+_{2r}(0)$, then for all $w \in \partial B^+_{\frac{3\tilde{r}}{4}}(2r)$

$$\mathrm{hm}_D(w, V) < c\,\mathrm{hm}_D(\tilde{w}, V) \leq c\,\mathrm{hm}_D(V).$$

Since (2.5) implies that the probability a Brownian motion started at z does not exit $B^+_{\frac{3\tilde{r}}{4}}(2r)$ on $\mathbb{R}$ is $O\left(\frac{\mathrm{Im}[z]}{r}\right)$, it follows that there is a $c > 0$ such that

$$\mathrm{hm}_D(z, V) \leq c\frac{\mathrm{Im}[z]}{r}\mathrm{hm}_D(V).$$

Combined with (2.26), this gives the first half of (2.25).

Since the maximum of $\mathrm{hm}_D(z, V)$ restricted to $|z| = 2r$ occurs on K and $\frac{\mathrm{Im}[z]}{r}$ is bounded from above on K, using the Harnack inequality, we can find constants $c, C > 0$ such that if $z \in K$, then

$$\mathrm{hm}_D(z, V) \geq C\,\mathrm{hm}_D(V) \geq c\frac{\mathrm{Im}[z]}{r}\mathrm{hm}_D(V). \tag{2.27}$$

Fix $z \notin K$ such that $|z| = 2r$ and assume without loss of generality that $z \in B^+_{\frac{3\tilde{r}}{4}}(2r)$. Using the Harnack inequality, we can find constants $c, C > 0$ such that if $w \in \partial B^+_{\frac{3\tilde{r}}{4}}(2r)$ satisfies $\mathrm{Im}[w] > \tilde{r}/4$, then

$$\mathrm{hm}_D(w, V) \geq C\,\mathrm{hm}_D(\tilde{w}, V) \geq c\,\mathrm{hm}_D(V). \tag{2.28}$$

Using (2.5), we can find a $c > 0$ such that the probability a Brownian motion started at z leaves $B^+_{\frac{3\tilde{r}}{4}}(2r)$ in $\{w \in \mathbb{H} : \mathrm{Im}[w] > \tilde{r}/4\}$ is greater than $c\frac{\mathrm{Im}[z]}{r}$. Combining this with (2.28), we see that there is a $c > 0$ such that

$$\mathrm{hm}_D(z, V) \geq c\frac{\mathrm{Im}[z]}{r}\mathrm{hm}_D(V).$$

Combined with (2.27), this gives the second half of (2.25).

Let $D_1 = D \cap \mathbb{D}$ and V be a Borel subset of ∂A. If $z \in \mathbb{H}$ is such that $|z| = 2r$, we claim that

$$\mathrm{hm}_D(z, V) = \mathrm{hm}_{D_1}(z, V)\left[1 + O\left(r^2\right)\right], \quad r \to 0. \tag{2.29}$$

Using (2.5) and (2.4), we see that the probability that a Brownian motion started at z reaches $\{w \in \mathbb{H} : |w| = 1\}$ and then returns to $\{w \in \mathbb{H} : |w| = 2r\}$ without leaving $\mathbb{H}$ is $O(r \,\mathrm{Im}[z])$. Combining this with (2.25), we see that there is a $c > 0$ such that

$$\begin{aligned}\mathrm{hm}_D(z, V) - \mathrm{hm}_{D_1}(z, V) &\leq cr \,\mathrm{Im}[z]\, \mathrm{hm}_D(V) \\ &\leq cr^2\, \mathrm{hm}_D(z, V),\end{aligned}$$

which implies (2.29).

The conformal invariance of harmonic measure implies

$$\mathrm{hm}_{D_1}(z, V) = \mathrm{hm}_{F(D_1)}(F(z), F(V)). \tag{2.30}$$

The growth theorem for schlicht functions implies that $F(A)$ is contained in a disk centered at the origin with radius comparable to r and the Koebe 1/4 theorem implies that $F(D_1)$ contains a disk centered at the origin with radius 1/4. As a result, an argument similar to the one used to prove (2.29) implies

$$\mathrm{hm}_{F(D_1)}(F(z), F(V)) = \mathrm{hm}_{\mathbb{H}\setminus F(A)}(F(z), F(V))\left[1 + O\left(r^2\right)\right], \quad r \to 0. \tag{2.31}$$

We claim that for all $z \in \mathbb{H}$ such that $|z| = 2r$ we have

$$\mathrm{hm}_{\mathbb{H}\setminus F(A)}(F(z), F(V)) = \mathrm{hm}_{\mathbb{H}\setminus F(A)}(z, F(V))\left[1 + O(r)\right], \quad r \to 0. \tag{2.32}$$

We saw in the proof of (2.24) that there is a uniform bound for $\left|F''(z)\right|$ restricted to

$$\{z \in \mathbb{C} : |z| < 1/2\}.$$

As a result, there is a $c_F > 0$ such that for all $z \in \mathbb{H}$ with $|z| < 1/2$ we have

$$|F(z) - z| < c_F |z|^2. \tag{2.33}$$

It follows that there is a $r_1 < \infty$ such that if $r < r_1$ and $z \in \mathbb{H}$ is such that $|z| = 2r$, then $B_{r/2}(z) \cap F(A) = \emptyset$ and $F(z) \in B_{r/64}(z)$. Using (2.24), we see that there is a $r_2 < \infty$ such that if $r < r_2$ and $z \in \mathbb{H}$ is such that $|z| = 2r$, then $\mathrm{Im}[F(z) - z] < \mathrm{Im}[z]$.

Suppose $r < \min\{r_1, r_2, 1/4\}$ and let $u(z) = \mathrm{hm}_{\mathbb{H}\backslash F(A)}(z, F(V))$. To prove (2.32), we consider the case that $\mathrm{Im}[z] > r/32$ and $\mathrm{Im}[z] \le r/32$ separately. In the first case, since $B_{r/32}(z) \subset \mathbb{H}\backslash F(A)$, the Harnack inequality implies that there is a $c > 0$ such that $u(w) < cu(z)$ for all $w \in B_{r/48}(z)$. As a result, Lemma 2.16 implies that there is a $c > 0$ such that the partial derivatives of u are bounded in absolute value by $\frac{cu(z)}{r}$ on $B_{r/32}(z)$. Since $|F(z) - z| < r/32$, (2.32) follows from (2.33).

In the second case, assume without loss of generality that $z \in B^+_{r/16}(2r)$ and let $h_\theta(z) = H_{B^+_{r/2}(2r)}\left(z, \frac{r}{2}e^{i\theta}\right)$. By the Schwarz reflection principle, $h_\theta(z)$ can be extended to a harmonic function $v(z)$ on $B_{r/2}(2r)$. Using (2.5), we can find a $c > 0$ such that

$$|v(z)| \le c\frac{\sin\theta}{r}, \tag{2.34}$$

for all $z \in B_{r/4}(2r)$. As a result, Lemma 2.16 implies that there is a $c > 0$ such that $\left|v_{xy}(z)\right| < c\frac{\sin\theta}{r^3}$ for all $z \in B_{r/8}(2r)$. Let $z = x_1 + iy_1 \in B^+_{r/16}(2r)$ be such that $|z| = 2r$ and $F(z) = x_2 + iy_2$. Since $F(z) \in B^+_{r/8}(2r)$ and $v_x = 0$ on $\mathbb{R}$, we see that there is a $c > 0$ such that $|v_x(z)| < c\frac{y_2\sin\theta}{r^3}$ on the line segment $[x_2 + iy_2, x_1 + iy_2]$. Since by (2.33) the length of this line segment is $O\left(r^2\right)$ and by assumption $\operatorname{Im}\left[F\left(z\right)-z\right]<\operatorname{Im}\left[z\right]$, it follows that there is a $c>0$ such that

$$\left|v\left(F\left(z\right)\right)-v\left(x_1+iy_2\right)\right|<\frac{c\operatorname{Im}\left[z\right]\sin\theta}{r}, \tag{2.35}$$

for all $z\in B^+_{r/16}\left(2r\right)$. Using Lemma 2.16 and (2.34) it follows that there is a $c>0$ such that $\left|v_y\left(z\right)\right|<c\frac{\sin\theta}{r^2}$ for all $z\in B_{r/8}\left(2r\right)$. Since (2.24) implies that $\left|y_2-y_1\right|=y_1O\left(r\right)$, it follows that

$$\left|v\left(z\right)-v\left(x_1+iy_2\right)\right|<\frac{c\operatorname{Im}\left[z\right]\sin\theta}{r},$$

for all $z\in B^+_{r/16}\left(2r\right)$. Combining this with (2.35), we conclude that there is a $c>0$ such that

$$\left|v\left(F\left(z\right)\right)-v\left(z\right)\right|<\frac{c\operatorname{Im}\left[z\right]\sin\theta}{r},$$

for all $z\in B^+_{r/16}\left(2r\right)$. Combining this with (2.5), we have

$$\begin{aligned}\left|u\left(z\right)-u\left(F\left(z\right)\right)\right|&=\left|\frac{r}{2}\int_0^{\pi}\left[h_\theta\left(z\right)-h_\theta\left(F\left(z\right)\right)\right]u\left(\frac{re^{i\theta}}{2}\right)\,d\theta\right|\\&\leq cr\left|\frac{r}{2}\int_0^{\pi}\frac{\operatorname{Im}\left[z\right]\sin\theta}{r^2}u\left(\frac{re^{i\theta}}{2}\right)\,d\theta\right|\\&\leq cru\left(z\right),\end{aligned}$$

which proves (2.32) for $z\in B^+_{r/16}\left(2r\right)$.

Using Proposition 2.8, (2.29), (2.30), (2.31), and (2.32), we have

$$\begin{aligned}
F\circ\mu_A(V) &= \mu_A\left(F^{-1}(V)\right)\\
&= \frac{4r}{\pi}\int_0^\pi \mathrm{hm}_D\left(2re^{i\theta},F^{-1}(V)\right)\sin\theta\,d\theta\\
&= \frac{4r}{\pi}\left[\int_0^\pi \mathrm{hm}_{D_1}\left(2re^{i\theta},F^{-1}(V)\right)\sin\theta\,d\theta\right]\left[1+O\left(r^2\right)\right]\\
&= \frac{4r}{\pi}\left[\int_0^\pi \mathrm{hm}_{F(D_1)}\left(F\left(2re^{i\theta}\right),V\right)\sin\theta\,d\theta\right]\left[1+O\left(r^2\right)\right]\\
&= \frac{4r}{\pi}\left[\int_0^\pi \mathrm{hm}_{\mathbb{H}\setminus F(A)}\left(F\left(2re^{i\theta}\right),V\right)\sin\theta\,d\theta\right]\left[1+O\left(r^2\right)\right]\\
&= \frac{4r}{\pi}\left[\int_0^\pi \mathrm{hm}_{\mathbb{H}\setminus F(A)}\left(2re^{i\theta},V\right)\sin\theta\,d\theta\right]\left[1+O(r)\right]\\
&= \mu_{F(A)}(V)\left[1+O(r)\right],
\end{aligned}$$

which completes the proof. □

The reason for proving Proposition 2.18 is that it implies the following corollary, which we will need and is widely used in the analysis of SLE.

Corollary 2.18. *Let F be as in Proposition 2.17 and γ be as in Theorem 2.14. Then*

$$\lim_{t\to 0^+}\frac{\mathrm{hcap}\left(F\left(\gamma_t\right)\right)}{t} = F'(0)^2\,\dot{a}(0). \tag{2.36}$$

In particular, the limit in (2.36) *exists if and only if* $\dot{a}(0)$ *exists.*

Proof. Let R be as in Proposition 2.17 and $r_t = \mathrm{diam}(\gamma_t)$. By Proposition 2.17, we have

$$\mathrm{hcap}\left(F\left(\gamma_t\right)\right) = F'(0)^2\,\mathrm{hcap}\left(\gamma_t\right)\left[1+O\left(\frac{r_t}{R}\right)\right].$$

Dividing both sides of this equation by t and taking the limit as $t\to 0$, the result follows. □

CHAPTER 3

EXCURSION REFLECTED BROWNIAN MOTION

3.1 Definition

We start this chapter by giving a precise definition of excursion reflected Brownian motion in $D \in \mathcal{Y}$. Later we will see that for any $D \in \mathcal{Y}$, there is a unique process satisfying the conditions of our definition.

The Jordan curve theorem says that any Jordan curve η separates $\mathbb{C}$ into exactly two connected components. We will call the bounded connected component the *interior* of η and the unbounded connected component the *exterior* of η. If $A \subset \mathbb{C}$ is in the interior of η, we will say η *surrounds* A.

Definition 3.1. Let $E = D \cup \{A_1, \ldots, A_n\}$ be equipped with the quotient topology and let $E_\partial = E \cup \{A_0\}$ be the one-point compactification of E. A stochastic process B_D^{ER} with state space E_∂ is called an *excursion reflected Brownian motion* (ERBM) if it satisfies the following properties.

1. B_D^{ER} has the strong Markov property.

2. If we start the process at $z \in D$ and let

$$T = \inf \left\{ t : B_D^{ER}(t) \in \partial D \right\},$$

then for $0 \le t \le T$, $B_D^{ER}(t)$ is a Brownian motion in D killed at ∂D.

3. Let $\eta_1, \ldots, \eta_n$ be pairwise disjoint smooth Jordan curves in D such that η_i surrounds A_i and does not surround A_j for $j \neq i$. If

$$\sigma = \inf \left\{ t : B_D^{ER}(t) \in \eta_i \right\},$$

then $B_D^{ER}(\sigma)$ has the distribution of $\overline{\mathcal{E}}_{U_i}(A_i, \cdot)$, where U_i is the region bounded by ∂A_i and η_i.

4. B_D^{ER} is conformally invariant (this will be made more precise in Proposition 3.8) and the radial part of $B_{\mathbb{C}\setminus\mathbb{D}}^{ER}$ has the same distribution as the radial part of a reflected Brownian motion in $\mathbb{C}\setminus\mathbb{D}$.

We will often refer to ERBM in D or E when we really mean the process with the enlarged state space E_∂.

3.2 Excursion Reflected Brownian Motion in $\mathbb{C}\setminus\mathbb{D}$

The first step in constructing ERBM is to construct it in $E = \mathbb{C}\setminus\mathbb{D} \cup \left\{\overline{\mathbb{D}}\right\}$. We will mimic the construction of Walsh's Brownian motion given in [3]. The idea of the construction is that if a process exists that satisfies Definition 3.1, we can determine what its transition semigroup must be. Once we know what its transition semigroup must be, we use general theory to show that there actually is a process with that semigroup. Finally, once we have the process, we check that it actually satisfies Definition 3.1. For the remainder of this section, let $A_0 = \overline{\mathbb{D}}$.

Since the radial part of ERBM has the same distribution as the radial part of reflected Brownian motion, to describe the semigroup for ERBM, we need the Feller-Dynkin semigroup for reflected Brownian motion. There is much in the literature on reflected Brownian motion and it is possible to define it in very general domains. However, in $\mathbb{C}\setminus\mathbb{D}$ it is possible to give a simple construction. Let B_1 and B_2 be independent one-dimensional Brownian motions and define reflected Brownian motion in $\mathbb{H}$ to be the process $B_1 + i\left|B_2\right|$. We can then define reflected Brownian motion in $\mathbb{C}\backslash\mathbb{D}$ to be the image of reflected Brownian motion in $\mathbb{H}$ under the map $z \mapsto e^{-iz}$ with the appropriate time change (taking this approach, it is still necessary to check that the resulting process is Feller-Dynkin).

Proposition 3.2. *Let T_t^+ be the Feller-Dynkin semigroup for reflected Brownian motion in $\mathbb{C}\backslash\mathbb{D}$ and T_t^0 be the Feller-Dynkin semigroup for reflected Brownian motion killed when it hits $\mathbb{D}$. If $f \in C_0(E)$, define an operator P_t by*

$$P_t f(r,\theta) = T_t^+ \overline{f}(r,\theta) + T_t^0\left(f - \overline{f}\right)(r,\theta), \tag{3.1}$$

where $\overline{f}(r,\theta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(r,\theta)\,d\theta$. If there is a Feller-Dynkin process $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ in $\mathbb{C}\backslash\mathbb{D}$ satisfying Definition 3.1, its semigroup is P_t.

Proof. Let B_t and R_t be respectively Brownian motion and reflected Brownian motion in $\mathbb{C}\backslash\mathbb{D}$ and let $\mathbf{E}$, $\mathbf{E}_1$, and $\mathbf{E}_2$ be the expectations with respect to the probability measures induced by $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$, B_t, and R_t respectively. Let Ω be the underlying probability space $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$, B_t, and R_t are defined on and let τ be the first time $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ hits $\mathbb{D}$. With respect to the appropriate filtration, τ is a stopping time [19]. Finally, let

$$A_t = \{\omega \in \Omega : \tau \leq t\}.$$

By abuse of a notation, we will also denote the set of ω such that R_s has hit $\mathbb{D}$ by time t and the set of ω such that B_s has hit $\mathbb{D}$ by time t by A_t. Using (2) of Definition 3.1, we have that $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ has the distribution of a Brownian motion up until time τ. Using (3) and (4) of Definition 3.1, we have that on the set A_t, the angular part of $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ is uniformly distributed and the radial part is that of a reflected Brownian motion. Combining these facts, we have that if $f \in C_0(E)$, then

$$\begin{aligned} P_t f(x) &= \mathbf{E}^x\left[f\left(B^{ER}_{\mathbb{C}\setminus\mathbb{D}}(t)\right)\right] \\ &= \mathbf{E}^x\left[\mathbf{1}_{A_t} f\left(B^{ER}_{\mathbb{C}\setminus\mathbb{D}}(t)\right)\right] + \mathbf{E}^x\left[\mathbf{1}_{A_t^c} f\left(B^{ER}_{\mathbb{C}\setminus\mathbb{D}}(t)\right)\right] \\ &= \mathbf{E}^x_2\left[\overline{f}(R_t)\right] + \mathbf{E}^x_1\left[f(B_t)\right] - \mathbf{E}^x_2\left[\mathbf{1}_{A_t^c}\overline{f}(R_t)\right] - \mathbf{E}^x_1\left[\mathbf{1}_{A_t} f(B_t)\right] \\ &= T^+_t \overline{f}(x) + T^0_t f(x) - \mathbf{E}^x_1\left[\mathbf{1}_{A_t^c}\overline{f}(B_t)\right] - \mathbf{E}^x_1\left[\mathbf{1}_{A_t}\overline{f}(B_t)\right] \\ &= T^+_t \overline{f}(r,\theta) + T^0_t\left(f - \overline{f}\right)(r,\theta). \end{aligned}$$

In the second to last equality, we use the fact that by convention we define $\overline{f}(\mathbb{D})$ to be $f(\mathbb{D})$. □

Next we show that P_t, as defined in Proposition 3.1, is a Feller-Dynkin semigroup. We will continue to use the setup given in the beginning of the proof of Proposition 3.1.

Proposition 3.3. *P_t is a Feller-Dynkin semigroup on $C_0(E)$. That is,*

1. *$P_t : C_0(E) \to C_0(E)$*
2. *If $f \in C_0(E)$ and $0 \le f \le 1$, then $0 \le P_t f \le 1$.*
3. *P_0 is the identity on $C_0(E)$ and $P_t P_s = P_{t+s}$*
4. *$\lim_{t\to 0} \|P_t f - f\|_\infty = 0$ for all $f \in C_0(E)$.*

Proof. If $f \in C_0(E)$, then $\overline{f}$ is also in $C_0(E)$. We have

$$\begin{aligned} \left|P_t f(r,\theta) - P_t f\left(r',\theta'\right)\right| \le& \left|T^+_t \overline{f}(r,\theta) - T^+_t \overline{f}\left(r',\theta'\right)\right| \\ &+ \left|T^0_t f(r,\theta) - T^0_t f\left(r',\theta'\right)\right| \\ &+ \left|T^0_t \overline{f}(r,\theta) - T^0_t \overline{f}\left(r',\theta'\right)\right|. \end{aligned}$$

The fact that $P_t f$ is continuous follows from the fact that $T_t^0 f$, $T_t^0 \overline{f}$, and $T_t^+ \overline{f}$ are all continuous. Since both f and $\overline{f}$ vanish at infinity, so does $P_t f$. This proves (i).

From the proof of Proposition 3.1 we have

$$P_t f(x) = \mathbf{E}_2^x \left[\mathbf{1}_{A_t} \overline{f}(R_t) \right] + \mathbf{E}_1^x \left[\mathbf{1}_{A_t^c} f(B_t) \right].$$

If $0 \leq f \leq 1$, then

$$\begin{aligned} 0 &\leq \mathbf{E}_2^x \left[\mathbf{1}_{A_t} \overline{f}(R_t) \right] + \mathbf{E}_1^x \left[\mathbf{1}_{A_t^c} f(B_t) \right] \\ &\leq \mathbf{E}_2^x \left[\mathbf{1}_{A_t} \right] + \mathbf{E}_1^x \left[\mathbf{1}_{A_t^c} \right] \\ &= 1, \end{aligned}$$

from which (ii) follows.

It is clear that P_0 is the identity on $C_0(E)$. Observe that

$$\begin{aligned} \overline{P_s f}(r,\theta) &= \overline{T_s^+ \overline{f}}(r,\theta) + \overline{T_s^0 f}(r,\theta) - \overline{T_s^0 \overline{f}}(r,\theta) \\ &= T_s^+ \overline{f}(r,\theta) + \int_0^{2\pi} T_s^0 f(r,\theta)\, d\theta - T_s^0 \overline{f}(r,\theta) \\ &= T_s^+ \overline{f}(r,\theta) + T_s^0 \overline{f}(r,\theta) - T_s^0 \overline{f}(r,\theta) \\ &= T_s^+ \overline{f}(r,\theta) \end{aligned}$$

and thus,

$$P_s f(r,\theta) - \overline{P_s f}(r,\theta) = T_s^0 \left(f - \overline{f} \right)(r,\theta).$$

Using these two facts and the fact that (iii) holds for T_t^+ and T_t^0, we have

$$\begin{aligned} P_t P_s f(r,\theta) &= T_t^+ \overline{P_s f}(r,\theta) + T_t^0 \left(P_s f(r,\theta) - \overline{P_s f}(r,\theta)\right) \\ &= T_t^+ T_s^+ \overline{f}(r,\theta) + T_t^0 T_s^0 \left(f - \overline{f}\right)(r,\theta) \\ &= T_{t+s}^+ \overline{f}(r,\theta) + T_{t+s}^0 \left(f - \overline{f}\right)(r,\theta) \\ &= P_{t+s} f(r,\theta), \end{aligned}$$

from which (iii) follows.

Since (i)-(iii) hold, by (say) Lemma III.6.7 of [19], to prove (iv) it is enough to show that for all $f \in C_0(E)$ and $z \in E$ we have

$$\lim_{t \to 0} P_t f(z) = f(z).$$

Since T_t^+ and T_t^0 satisfy (i)-(iv), we have

$$\begin{aligned} \lim_{t \to 0} P_t f(z) &= \lim_{t \to 0} T_t^+ \overline{f}(z) + \lim_{t \to 0} T_t^0 \left(f - \overline{f}\right)(z) \\ &= \overline{f}(z) + \left(f(z) - \overline{f}(z)\right) \\ &= f(z), \end{aligned}$$

which proves (iv). $\square$

Using (say) Theorem III.7.1 of [19], given any measure μ on E, we can define a unique Feller-Dynkin process

$$B_{\mathbb{C}\backslash\mathbb{D}}^{ER} := \left(\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geq 0\}, \{B_{\mathbb{C}\backslash\mathbb{D}}^{ER}(t) : t \geq 0\}, \mathbf{P}^{\mu}\right) \tag{3.2}$$

with semigroup P_t. Furthermore, the filtration $\mathscr{F}_t$ is independent of the measure μ and X has the strong Markov property with respect to $\mathscr{F}_t$. We denote the angular and radial parts of $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ at time t by θ_t and R_t respectively.

Next we check that the process $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ defined in (3.2) satisfies Definition 3.1.

Proposition 3.4. *$B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ has the distribution of a Brownian motion up until the first time it hits ∂D.*

Proof. This follows immediately from (3.1). □

Proposition 3.5. *R_t has the same distribution as the radial part of a reflected Brownian motion in $\mathbb{C}\backslash\mathbb{D}$.*

Proof. We mimic the proof of Lemma 2.2 in [3]. Let $g \in C_0([1,\infty))$ and define $f \in C_0(E)$ by $f(r,\theta) = g(r)$. Observe that $\overline{f} = f$ and $f(X_t) = g(R_t)$. If S is any $\mathscr{F}_t$-stopping time, then

$$\begin{aligned}
\mathbf{E}^{\mu}\left[g\left(R_{S+t}\right)|\mathscr{F}_S\right] &= \mathbf{E}^{\mu}\left[f\left(X_{S+t}\right)|\mathscr{F}_S\right] \\
&= P_t f\left(X_S\right) \\
&= T_t^{+}\overline{f}\left(R_S,\theta_S\right) + T_t^{0}\left(f-\overline{f}\right)\left(R_S,\theta_S\right) \\
&= T_t^{+} f\left(R_S,\theta_S\right) \\
&= R_t^{+} g\left(R_S\right),
\end{aligned}$$

where R_t^{+} is the semi-group for the radial part of reflected Brownian motion in $\mathbb{C}\backslash\mathbb{D}$. The result follows. □

Proposition 3.6. *Let η be a smooth Jordan curve surrounding $\mathbb{D}$, U be the region bounded by η and $\partial\mathbb{D}$, and τ be the first time $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ hits η. If V is a smooth arc in η, then*

$$\alpha := \mathbf{P}^{\mathbb{D}}\left\{B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(\tau) \in V\right\} = \overline{\mathcal{E}}_U\left(\mathbb{D},V\right).$$

Proof. Let C_ϵ be the circle of radius ϵ centered at the origin. Since it is clear from (3.1) that $B^{ER}_{\mathbb{C}\setminus\mathbb{D}}$ is rotationally invariant, the result follows in the case that $\eta = C_\epsilon$.

Let $p(z)$ be the probability that a Brownian motion started at z exits U on η. For small enough ϵ, C_ϵ is in the interior of η. For such an ϵ, using the strong Markov property for ERBM and Proposition 3.4, we see that

$$\begin{aligned}
2\pi\alpha &= \int_{C_\epsilon}\left[\int_V H_U(z,w)\,|dw| + (1-p(z))\,\alpha\right]|dz| \\
&= 2\pi\alpha + \int_{C_\epsilon}\left[\int_V H_U(z,w)\,|dw| - p(z)\,\alpha\right]|dz| \\
&= 2\pi\alpha + \int_{C_\epsilon}\left[\int_V H_U(z,w)\,|dw| - \alpha\int_\eta H_U(z,w)\,|dw|\right]|dz|\,.
\end{aligned}$$

As a result, for small enough ϵ, we have

$$\alpha = \frac{\int_{C_\epsilon}\int_V H_U(z,w)\,|dw|\,|dz|}{\int_{C_\epsilon}\int_\eta H_U(z,w)\,|dw|\,|dz|}.$$

Since the derivative of $H_U(\cdot,w)$ is bounded in a neighborhood of $\mathbb{D}$, using the dominated convergence theorem, we see that

$$\begin{aligned}
\alpha &= \lim_{\epsilon\to 0}\frac{\int_{C_\epsilon}\int_V \frac{H_U(z,w)}{\epsilon}\,|dw|\,|dz|}{\int_{C_\epsilon}\int_\eta \frac{H_U(z,w)}{\epsilon}\,|dw|\,|dz|} \\
&= \frac{\int_{C_\epsilon}\int_V H_{\partial U}(z,w)\,|dw|\,|dz|}{\int_{C_\epsilon}\int_\eta H_{\partial U}(z,w)\,|dw|\,|dz|} \\
&= \frac{\mathcal{E}_U(\mathbb{D},V)}{\mathcal{E}_U(\mathbb{D},\eta)}.
\end{aligned}$$

□

Proposition 3.7. *There is a unique process Feller-Dynkin process with state space* $E = \mathbb{C}\setminus D\cup\{\mathbb{D}\}$ *satisfying Definition 3.1. Furthermore, this process can be defined so as to have continuous sample paths in the topology of* E.

Proof. Propositions 3.4, 3.5, and 3.6 combine to show that the process defined in (3.2) satisfies Definition 3.1. The uniqueness statement follows from Proposition 3.1. By construction, $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ is an R-process. Combining this with Propositions 3.4 and 3.5, and the fact that Brownian motion and reflected Brownian motion have continuous sample paths, it is easy to check that the sample paths of $B^{ER}_{\mathbb{C}\backslash\mathbb{D}}$ are continuous. □

3.3 Excursion Reflected Brownian Motion in Conformal Annuli

Let A be any compact, connected subset of $\mathbb{C}$ larger than a single point and

$$f : \mathbb{C}\backslash\mathbb{D} \to \mathbb{C}\backslash A$$

be a conformal map sending ∞ to ∞. It is a straightforward exercise to verify that f is unique up to an initial rotation. Let σ_t be the $\mathscr{F}_t$ stopping time given by

$$\int_0^{\sigma_t} \left| f' \left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}} (s) \right) \right|^2 ds = t$$

and define

$$B^{ER}_{\mathbb{C}\backslash A} (t) = f \left(B^{ER}_{\mathbb{C}\backslash D} (\sigma_t) \right)$$

and $\tilde{\mathscr{F}}_t = \mathscr{F}_{\sigma_t}$. We define ERBM in $\mathbb{C}\backslash A$ to be the process

$$B^{ER}_{\mathbb{C}\backslash A} := \left(\Omega, \mathscr{F}, \left\{ \tilde{\mathscr{F}}_t : t \geq 0 \right\}, \left\{ B^{ER}_{\mathbb{C}\backslash A} \right\}, \{\mathbf{P}^x\} \right).$$

Since $B^{ER}_{\mathbb{C}\backslash D}$ is rotationally invariant and f is unique up to an initial rotation, it is clear that the distribution of $B^{ER}_{\mathbb{C}\backslash A}$ does not depend on f. It is also not hard to check that the strong Markov property is preserved (see the discussion on pg. 277 of [19]). To ensure that

$B^{ER}_{\mathbb{C}\backslash A}(t)$ exists for all $t < \infty$, we need to verify that

$$\int_0^\infty \left| f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds = \infty \quad \text{a.s..} \tag{3.3}$$

In order for $B^{ER}_{\mathbb{C}\backslash A}(t)$ not to have a limit as $t \to \infty$, we need to verify that for all $t < \infty$,

$$\int_0^t \left| f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds < \infty \quad \text{a.s..} \tag{3.4}$$

We temporarily put these considerations aside.

Proposition 3.8. *Suppose $f : \mathbb{C}\backslash\mathbb{D} \to D_1$ and $g : D_1 \to D_2$ are conformal maps. Then the process*

$$B^{ER}_{D_2}(t) = B^{ER}_{D_1}(\sigma_t),$$

where

$$\int_0^{\sigma_t} \left| g'\left(B^{ER}_{D_1}(s)\right)\right|^2 ds = t$$

is an ERBM in D_2.

Proof. Let σ_r satisfy

$$\int_0^{\sigma_r} \left| g'\left(f\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right) f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds = r$$

and define a map $T : [0, \sigma_r] \to [0, \infty)$ by

$$t \mapsto \int_0^t \left| f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds. \tag{3.5}$$

It is straightforward to verify that T is a bijection (we use (3.4) here) onto $[0, T(\sigma_r)]$ with derivative $\left|f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2$. Using the change of variables formula, we have

$$\begin{aligned} r &= \int_0^{\sigma_r}\left|g'\left(f\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right)f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds \\ &= \int_0^{\sigma_r}\left|g'\left(B^{ER}_{D_1}(T(s))\right)f'\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(s)\right)\right|^2 ds \\ &= \int_0^{T(\sigma_r)}\left|g'\left(B^{ER}_{D_1}(s)\right)\right|^2 ds. \end{aligned}$$

As a result, $B^{ER}_{D_2}(r) = g\left(B^{ER}_{D_1}(T(\sigma_r))\right) = g\left(f\left(B^{ER}_{\mathbb{C}\backslash\mathbb{D}}(\sigma_r)\right)\right)$ and thus, the process in D_2 defined by g is the same as the process defined by $g \circ f$. The result follows. □

Proposition 3.9. *The process $B^{ER}_{\mathbb{C}\backslash A}$ satisfies Definition 3.1.*

Proof. The 1st and 4th property have already been discussed. The 2nd property follows from the conformal invariance of Brownian motion and the 3rd property follows from the conformal invariance of excursion measure. □

If K is a closed subset of $\mathbb{C}\backslash A$ it makes sense to discuss ERBM in $\mathbb{C}\backslash A$ killed at K. Most often we will do this when K is a simple, closed curve η surrounding A and refer to the corresponding process as ERBM in D, where D is the region bounded by η and ∂A.

3.4 Excursion Reflected Brownian Motion in Finitely Connected Domains

Let $D \in \mathcal{Y}_n$ and η_i, for $1 \leq i \leq n$, be as in Definition 3.1. Denote the domain bounded by η_i and ∂A_i by U_i. We will now define a process B^{ER}_D in D satisfying Definition 3.1. Intuitively, we define $B^{ER}_D(t)$ pathwise to be a Brownian motion up until the first time it hits an A_i, then let it be an ERBM in U_i until it hits η_i, then let it be a Brownian motion until it hits another A_i and so on. Adding rigor to this intuition is not hard, but is notationally cumbersome. For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots$ let $B^{(j)}_{U_i}$ be an ERBM in U_i

started at A_i and B_j be a Brownian motion in $\mathbb{C}$ started at the origin. We can construct these processes on the same probability space Ω so that they are all independent. Let $z \in E = D \cup \{A_0, \ldots, A_n\}$ and define

$$B_D^{ER}(t) = \begin{cases} z & \text{if } t = 0 \\ A_0 & \text{if } t > \tau \\ B_D^{ER}(\sigma_j) + B_i(t - \sigma_i) & \text{if } \sigma_j < t \le \tau_j \\ B_{U_i}^{(j)}(t - \tau_j) \text{ where } B_D^{ER}(\tau_j) = A_i & \text{if } \tau_j < t \le \sigma_{j+1} \end{cases}$$

where

$$\begin{aligned} \tau &= \inf\left\{t : B_D^{ER}(t) \in A_0\right\}, \\ \sigma_1 &= 0, \\ \sigma_j &= \inf\left\{t \ge \tau_{j-1} : B_D^{ER}(t) \in \eta_i\right\} \text{ for } j \ge 2, \\ \tau_j &= \inf\left\{t \ge \sigma_j : B_D^{ER}(t) \in A_i \text{ for some } i\right\}. \end{aligned}$$

It is not hard to check that the distribution of $B_D^{ER}(t)$ does not depend on the choice of η_i and that $B_D^{ER}(t)$ satisfies Definition 3.1.

3.5 A Markov Chain Associated with ERBM

Let h_j be the unique bounded harmonic function on D that is equal to 1 on ∂A_j and 0 on ∂A_i for $i \ne j$ (note that $h_j(z)$ is the probability that a Brownian motion started at z exits D at A_j). ERBM on D induces a discrete time Markov chain X with state space $\{A_0, \ldots A_n\}$ (see [13] pg. 37). The probability that the chain moves from A_i to A_j is equal to the probability that A_j is the first boundary component of D that B_D^{ER} started at A_i hits after the first time it hits η_i. That is, the chain has transition probabilities $p_{00} = 1$

and

$$p_{ij} = \int_{\eta_i} h_j(z) \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta_i)} |dz|,$$

for $i \neq 0$. This Markov chain is not entirely satisfactory since it is highly dependent on the particular choice of $\eta_1, \ldots, \eta_n$. However, this chain does induce another chain Y with transition probabilities

$$q_{ij} = \frac{p_{ij}}{1 - p_{ii}},$$

for $i \neq j$. Y is obtained from X by erasing all of the loops and it is not hard to see that its transition probabilities are independent of the choice of $\eta_1, \ldots, \eta_n$. Since $q_{j0} > 0$ for all $1 \leq j \leq n$, the eigenvalues of the transition matrix, $\mathbf{Q}$, for Y restricted to $A_1, \ldots, A_n$ have absolute value strictly less than one and, using standard results from Markov chain theory, we have that the Green's matrix

$$\mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \cdots + \mathbf{Q}^n + \cdots = (\mathbf{I} - \mathbf{Q})^{-1}. \tag{3.6}$$

is well-defined.

3.6 Excursion Reflected Harmonic Functions

Definition 3.10. A function

$$v : E \to \mathbb{R}$$

is called *ER-harmonic* if it satisfies

1. v is continuous on E and is harmonic when restricted to D

2. For $1 \leq i \leq n$, if η is a Jordan curve surrounding A_i, then

$$v(A_i) = \int_{\eta} v(z) \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)} |dz|, \tag{3.7}$$

where U_i is the region bounded by η and ∂A_i.

If it is clear what is meant, we will sometimes speak of the ER-harmonicity of a function with domain D rather than E. By an ER-harmonic function on $D - \{z\}$ or $D - \{A_i\}$ we mean a function that satisfies Definition 3.10 except that (2) is not necessarily satisfied for curves surrounding z and A_i respectively.

The following is a useful criterion for a function to be ER-harmonic.

Lemma 3.11. *Let η and η' be smooth Jordan curves surrounding A_j and not surrounding A_i for $i \neq j$. Then for any harmonic function v on D we have*

1.

$$\int_{\eta} \frac{d}{dn}\, v(z)\,|dz| = \int_{\eta'} \frac{d}{dn}\, v(z)\,|dz|,$$

where n is the outward pointing normal

2.

$$\int_{\eta} v(z)\, H_{\partial U_j}(A_j, z)\,|dz| = v(A_j)\,\mathcal{E}_{U_j}(A_j, \eta) + \int_{\eta} \frac{d}{dn}\, v(z)\,|dz|,$$

where U_j is the region bounded by η and ∂A_j. In particular, if v is continuous on E, then v is ER-harmonic if and only if for each i there is an η_i surrounding A_i with

$$\int_{\eta_i} \frac{d}{dn}\, v(z)\,|dz| = 0.$$

Proof. See [13] pg. 17. □

As with harmonic functions, if we specify suitable boundary conditions, there is a unique ER-harmonic function with these boundary conditions. The key to proving this uniqueness is a maximal principle for ER-harmonic functions.

Lemma 3.12 (Maximal principle for ER-harmonic functions)**.** *Let $v : E \cup \partial A_0 \to \mathbb{R}$ be a bounded, continuous function that is ER-harmonic when restricted to E. Then*

1. *The maximum of value of v is equal to the maximum value of v restricted to ∂A_0.*

2. *If there is a $z \in E$ such that v attains its maximum at z, then v is constant.*

Proof. It is clear that (2) implies (1), so it is enough to prove (2). Let z be a point where v attains its maximum. If $z \in D$, then by the strong maximal principle for harmonic functions [8], v is constant. If $z = A_i$, then using (3.7) it is clear there is some $z' \in D$ where v also attains its maximum and thus, v is constant. □

Proposition 3.13. *Suppose that ∂A_0 has at least one regular point for Brownian motion and let $F : \partial A_0 \to \mathbb{R}$ be a bounded, measurable function. Define*

$$v : \overline{D} \to \mathbb{R}$$

by

$$v(z) = \mathbf{E}^z \left[F\left(B_D^{ER}(\tau_D) \right) \right],$$

where τ_D is the first time an ERBM hits A_0. Then v is a bounded ER-harmonic function on D that is continuous at all regular points of ∂A_0 at which F is continuous. Furthermore, if every point of ∂A_0 is regular and F is continuous, then v is the unique ER-harmonic function that is equal to F on ∂A_0.

Proof. It is clear from the fact that F is bounded that v is also bounded. The proof that v is harmonic and continuous at the regular points of A_0 at which F is continuous is similar to the proof of the corresponding result for Brownian motion (see [18]). The fact that (3.7) holds follows from the strong Markov property for ERBM and (3) of Definition 3.1. The uniqueness statement follows from a straightforward application of Lemma 3.12. □

CHAPTER 4

THE POISSON KERNEL FOR ERBM

4.1 Definition and Basic Properties

Throughout this chapter let $D \in \mathcal{Y}_n$ be such that $A_0 \neq \emptyset$ and let

$$\tau_D = \inf \left\{ t \in \mathbb{R}^+ : B_D^{ER}(t) \in \partial A_0 \right\}.$$

The distribution of $B_D^{ER}(\tau_D)$ defines a measure $\mathrm{hm}_D^{ER}(z, \cdot)$ on ∂A_0 (with the σ-algebra generated by Borel subsets of ∂A_0) that we call *ER-harmonic measure in D from z*. Using the analogous result for harmonic measure and the construction of ERBM, it is easy to check that if ∂D is locally analytic at w, then $\mathrm{hm}_D^{ER}(z, \cdot)$ is absolutely continuous with respect to arc length in a neighborhood of w. The density of $\mathrm{hm}_D^{ER}(z, \cdot)$ at w with respect to arc length is called the *Poisson kernel* for ERBM and is denoted $H_D^{ER}(z, w)$.

If $\gamma : (-\delta, \delta) \to \partial A_0$, $\gamma(0) = w$ is an analytic curve, then we can explicitly define a version of $H_D^{ER}(z, w)$ by

$$H_D^{ER}(z, w) = \lim_{\epsilon \to 0} \frac{\mathrm{hm}_D^{ER}(z, \gamma(-\epsilon, \epsilon))}{\int_{-\epsilon}^{\epsilon} |\gamma'(x)| \, dx}. \tag{4.1}$$

It is clear that this definition is independent of γ. In what follows, when we refer to $H_D^{ER}(z, w)$, we will mean the version given by (4.1).

An analog of (2.1) holds for $H_D^{ER}(z, w)$.

Proposition 4.1. *If $f : D \to D'$ is a conformal transformation such that ∂D is locally analytic at w and $\partial D'$ is locally analytic at $f(w)$, then*

$$H_{D'}^{ER}(f(z), f(w)) = |f'(w)|^{-1} H_D^{ER}(z, w).$$

Proof. Since ERBM is conformally invariant, $\mathrm{hm}_D^{ER}(z, \cdot)$ is conformally invariant. Combining this with the change of variables formula, the result follows. □

Recall that $h_i(z)$ is the unique bounded harmonic function on D that is 1 on ∂A_i and 0 on ∂A_j for $j \neq i$. If V is a Borel subset of ∂A_0, then using the strong Markov property for ERBM, we see that

$$\mathrm{hm}_D^{ER}(z, V) = \mathrm{hm}_D(z, V) + \sum_{i=1}^{n} h_i(z)\, \mathrm{hm}_D^{ER}(A_i, V).$$

Combining this with (4.1), we see that

$$H_D^{ER}(z, w) = H_D(z, w) + \sum_{i=1}^{n} h_i(z)\, H_D^{ER}(A_i, w). \tag{4.2}$$

Using (4.2), it is sometimes possible to explicitly compute $H_D^{ER}(z, w)$. We do this calculation in the case that D is an annulus.

Proposition 4.2. *If $r > 1$ and $A_{e^{-r},1}$ is the annulus with $\partial A_0 = \partial \mathbb{D}$ and $\partial A_1 = \partial\left(e^{-r}\mathbb{D}\right)$, then*

$$H_{A_{e^{-r},1}}^{ER}\left(e^{i(x+iy)}, 1\right) = \frac{-\log|z|}{2\pi r} + \sum_{k\in\mathbb{Z}} \frac{\sin\left(\frac{\pi y}{r}\right)}{2r\left[\cosh\left(\frac{\pi(x+2\pi k)}{r}\right) - \cos\left(\frac{\pi y}{r}\right)\right]}.$$

Proof. Let

$$H_r = \{x + iy : x, y \in \mathbb{R}; 0 < y < r\}$$

and observe that the map $f : H_r \to A_{e^{-r},1}$ given by $f(z) = e^{iz}$ is surjective and locally conformal. Since $g(z) = e^{\frac{\pi}{r}z}$ is a conformal map from H_r to $\mathbb{H}$, a calculation using (2.1) and (2.2) shows that

$$H_{H_r}(x+iy, 0) = \frac{\sin\left(\frac{\pi y}{r}\right)}{2r\left[\cosh\left(\frac{\pi x}{r}\right) - \cos\left(\frac{\pi y}{r}\right)\right]}. \tag{4.3}$$

Combining (4.3) with a generalization of (2.1) (and using the fact that the relevant infinite sums converge absolutely), we see that

$$\begin{aligned} H_{A_{e^{-r},1}}\left(e^{i(x+iy)}, 1\right) &= \sum_{k\in\mathbb{Z}} H_{H_r}((x+2\pi k)+iy, 0) \\ &= \sum_{k\in\mathbb{Z}} \frac{\sin\left(\frac{\pi y}{r}\right)}{2r\left[\cosh\left(\frac{\pi(x+2\pi k)}{r}\right) - \cos\left(\frac{\pi y}{r}\right)\right]}. \end{aligned} \tag{4.4}$$

It is well-known that $h_1(z) = \frac{-\log|z|}{r}$ and easy to compute using the rotational invariance of ERBM that $H^{ER}_{A_{e^{-r},1}}(A_1, w) = \frac{1}{2\pi}$. Combining these two facts with (4.2) and (4.4), the result follows. □

4.2 Some Poisson Kernel Estimates

In this section, we gather some estimates for the Poisson kernel for ERBM that we will need in Chapter 6.

It is possible to compute $H^{ER}_D(A_i, \cdot)$ in terms of the boundary Poisson kernel and excursion measure. To do this, we will need an analog of (2.6) for $H^{ER}_D(z, \cdot)$. Namely, if $D_2 \subset D_1$ are domains in $\mathcal{Y}$ such that ∂D_2 and ∂D_1 agree in a neighborhood of x, the strong Markov property for ERBM gives

$$H^{ER}_{D_2}(w, x) = H^{ER}_{D_1}(w, x) - \mathbf{E}^w\left[H^{ER}_{D_1}\left(B_{\tau_{D_2}}, x\right)\right]. \tag{4.5}$$

Let τ^i be the first time an ERBM hits a boundary component of D other than ∂A_i. The distribution of $B_D^{ER}\left(\tau^i\right)$ defines a measure on ∂D. The next lemma computes the density of this measure restricted to ∂A_0. In Lemma 4.3 and Proposition 4.4, we suppose that D has locally analytic boundary.

Lemma 4.3. *If $1 \leq i \leq n$, then*

$$T_i\left(w\right) := \frac{H_{\partial D}\left(A_i, w\right)}{\sum_{j \neq i} \mathcal{E}_D\left(A_i, A_j\right)}$$

is a density for the distribution of $B_D^{ER}\left(\tau^i\right)$ restricted to ∂A_0.

Proof. Let η_i be a smooth Jordan curve surrounding A_i and not surrounding A_j for $j \neq i$ and let U_i be the region bounded by ∂A_i and η_i. Using (1), (2), and (3) of Definition 3.1 and the definition of p_{ii}, we see that

$$\tilde{T}_i\left(w\right) := \frac{\int_{\eta_i} H_D\left(z, w\right) H_{\partial U_i}\left(A_i, z\right)|dz|}{\left(1 - p_{ii}\right)\mathcal{E}_{U_i}\left(A_i, \eta_i\right)},$$

is a density for the distribution of $B_D^{ER}\left(\tau^i\right)$ restricted to ∂A_0. The strong Markov property for ERBM implies

$$\int_{\eta_i} H_D\left(z, w\right) H_{\partial U_i}\left(A_i, z\right)|dz| = H_{\partial D}\left(A_i, w\right). \tag{4.6}$$

Using (4.6), we have

$$\begin{aligned}
(1-p_{ii})\,\mathcal{E}_{U_i}(A_i,\eta_i) &= \int_{\eta_i}(1-h_i(z))\,H_{\partial U_i}(A_i,z)\,|dz| \\
&= \int_{\eta_i}\left(\sum_{j\neq i} h_j(z)\right) H_{\partial U_i}(A_i,z)\,|dz| \\
&= \sum_{j\neq i}\int_{\eta_i}\int_{A_j} H_D(z,w)\,H_{\partial U_i}(A_i,z)\,|dw|\,|dz| \\
&= \sum_{j\neq i}\int_{A_j}\int_{\eta_i} H_D(z,w)\,H_{\partial U_i}(A_i,z)\,|dz|\,|dw| \\
&= \sum_{j\neq i}\int_{A_j} H_{\partial D}(A_i,w)\,|dw| \\
&= \sum_{j\neq i}\mathcal{E}_D\left(A_i,A_j\right)
\end{aligned}$$

The result follows. □

The following proposition follows immediately.

Proposition 4.4. *Fix $w\in\partial A_0$ and let $\mathbf{H}$ be the $n\times 1$ vector with ith component equal to $H_D^{ER}(A_i,w)$, $\mathbf{T}$ be the $n\times 1$ vector with ith component equal to $T_i(w)$, and $\mathbf{Q}$ be as in Section 3.5. Then*

$$\mathbf{H}=\mathbf{T}+\mathbf{QT}+\mathbf{Q}^2\mathbf{T}+\ldots=(\mathbf{I}-\mathbf{Q})^{-1}\,\mathbf{T}.$$

Next we prove the analog of Lemma 2.2 for $H_D^{ER}(z,\cdot)$.

Proposition 4.5. *Let $D\in\mathcal{Y}^*$ be a domain with piecewise analytic boundary and such that ∂D and $\partial\mathbb{H}$ agree in a neighborhood of 0. If $|z|>2\epsilon$, then*

$$H_{D^\epsilon}^{ER}\left(z,\epsilon e^{i\theta}\right)=2H_D^{ER}(z,0)\sin\theta\,[1+O(\epsilon)]\,,\quad \epsilon\to 0. \tag{4.7}$$

Furthermore, for any $r>0$, $O(\epsilon)$ is uniform over all $z\in\mathbb{H}$ such that $|z|>r$.

Proof. For $1 \leq i \leq n$, let $h_i^\epsilon(z)$ be the unique bounded harmonic function on D^ϵ that is 1 on ∂A_i and 0 on the other boundary components of D^ϵ and let q_{ij}^ϵ be the probability that the Markov chain induced by $B_{D^\epsilon}^{ER}$ moves from A_i to A_j. Let $T_i^\epsilon(w)$ be the density introduced in Lemma 4.3 for $B_{D^\epsilon}^{ER}$, $\mathbf{T}^\epsilon$ be the $n \times 1$ vector with ith component $T_i^\epsilon\left(\epsilon e^{i\theta}\right)$, $\mathbf{Q}^\epsilon$ be the matrix with ij entry q_{ij}^ϵ, and $\mathbf{H}^\epsilon$ be the $n \times 1$ vector with ith component $H_{D^\epsilon}^{ER}\left(A_i, \epsilon e^{i\theta}\right)$. Proposition 4.4 implies

$$\mathbf{H}^\epsilon = (\mathbf{I} - \mathbf{Q}^\epsilon)^{-1} \mathbf{T}^\epsilon. \tag{4.8}$$

Let $\mathbf{Q} = \mathbf{Q}^0$ and $\mathbf{H} = \mathbf{H}^0$. Using (2.9) and (2.10) we see

$$h_i(z) = h_i^\epsilon(z) + O\left(\epsilon^2\right), \tag{4.9}$$

where $O(\epsilon)$ is uniform over all $z \in \mathbb{H}$ such that $|z| > r$. It follows that $\mathbf{Q}^\epsilon = \mathbf{Q} + O\left(\epsilon^2\right)$. Since inversion of matrices is a smooth operation (and in particular, Lipschitz), we conclude

$$(\mathbf{I} - \mathbf{Q}^\epsilon)^{-1} = (\mathbf{I} - \mathbf{Q})^{-1} + O\left(\epsilon^2\right). \tag{4.10}$$

Substituting (4.10) into (4.8) and using (2.14), we obtain

$$\begin{aligned}
\mathbf{H}^\epsilon &= (\mathbf{I} - \mathbf{Q}^\epsilon)^{-1} \mathbf{T}^\epsilon \\
&= 2\sin\theta \left[(\mathbf{I} - \mathbf{Q})^{-1} + O\left(\epsilon^2\right)\right] \mathbf{T}\left[1 + O(\epsilon)\right] \\
&= 2\sin\theta\, (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{T}\left[1 + O(\epsilon)\right] \\
&= 2\sin\theta\, \mathbf{H}\left[1 + O(\epsilon)\right].
\end{aligned} \tag{4.11}$$

Finally, using (4.9), (4.11), and Lemma 2.2, we see

$$\begin{aligned} H_{D^\epsilon}^{ER}\left(z, \epsilon e^{i\theta}\right) &= H_{D^\epsilon}\left(z, \epsilon e^{i\theta}\right) + \sum_{i=1}^{n} h_i^\epsilon(z)\, H_{D^\epsilon}^{ER}\left(A_i, \epsilon e^{i\theta}\right) \\ &= 2\sin\theta \left[H_D(z,0) + \sum_{i=1}^{n} h_i^\epsilon(z)\, H_D^{ER}(A_i, 0)\right][1 + O(\epsilon)] \\ &= 2\sin\theta \left[H_D(z,0) + \sum_{i=1}^{n} h_i(z)\, H_D^{ER}(A_i, 0)\right][1 + O(\epsilon)] \\ &= 2\sin\theta\ H_D^{ER}(z,0)\,[1 + O(\epsilon)]\,. \end{aligned}$$

□

For any $0 < r < R$, there are bounds for $H_D^{ER}(z,0)$ and $H_D^{ER}(z,x)$, for $|x| > 2R$, that are uniform over all $D \in \mathcal{Y}^*$ that agree with $\mathbb{H}$ outside of $A_{r,R}$.

Lemma 4.6. *Let $D \in \mathcal{Y}_n^*$ be such that $B_r^+(0) \subset D$ and $z \in D$ for all $z \notin B_R^+(0)$. Then there are constants $c_r, c_R < \infty$ such that for each $1 \le i \le n$, $H_D^{ER}(A_i, 0) \le c_r$ and, if $|x| \ge 2R$, $H_D^{ER}(A_i, x) \le c_R x^{-2}$.*

Proof. Fix j and let σ_1 be the first time B_D^{ER} started at A_j hits $\partial B_{r/2}^+(0)$ and, for $k > 1$, let σ_k be the first time after σ_{k-1} that B_D^{ER} has hit an A_i with $i \ge 1$ and then returned to $\partial B_{r/2}^+(0)$. Let $p_n(\theta)$ be the density for the distribution of $B_D^{ER}(\sigma_n)$ conditioned on $\sigma_n < \infty$ and q_n be the probability that $\sigma_n < \infty$. Using the strong Markov property for

B_D^{ER} and (2.2), we see

$$H_D^{ER}(A_j,0) = \sum_{n=1}^{\infty} q_n \left[\frac{r}{2} \int_0^{\pi} H_D\left((r/2)\, e^{i\theta}, 0 \right) p_n(\theta)\ d\theta \right] \tag{4.12}$$
$$\leq \sum_{n=1}^{\infty} q_n \left[\frac{r}{2} \int_0^{\pi} H_{\mathbb{H}}\left((r/2)\, e^{i\theta}, 0 \right) p_n(\theta)\ d\theta \right]$$
$$\leq \sum_{n=1}^{\infty} q_n \left[\frac{r}{2} \int_0^{\pi} \frac{2}{\pi r} p_n(\theta)\ d\theta \right]$$
$$= \frac{2}{\pi r} \sum_{n=1}^{\infty} q_n.$$

To complete the proof, it is enough to show that $\sum_{n=1}^{\infty} q_n$ is less than infinity. If $\sigma_n < \infty$, in order for σ_{n+1} to be less than infinity, a Brownian motion started on $\partial B_{r/2}^+(0)$ will have to hit $\partial B_R^+(0)$ before it hits the real line. It is easy to verify that there is a $p < 1$ uniformly bounding the probability of this event. It follows that $q_n \leq p^{n-1} q_1$ and hence, $\sum_{n=1}^{\infty} q_n \leq \frac{q_1}{1-p}$. This proves the first statement.

Observe that (2.3) implies $H_{\mathbb{H}}\left(\frac{3R}{2} e^{i\theta}, x \right) < \frac{24R}{\pi x^2}$ for all θ and x such that $|x| > 2R$. Using this fact, the proof of the second statement is similar to the proof of the first.

□

Lemma 4.7. *Let $D \in \mathcal{Y}_n^*$ and suppose that $B_r^+(0) \subset D$. Then there is a constant $c_r > 0$ such that $H_D^{ER}(z,0) \leq c_r$ for all z with $|z| > r$. If $w \in D$ for all $w \in \mathbb{H}$ such that $\mathrm{Im}[w] < r'$, then there is a constant $c_{r'} > 0$ such that $H_D(z,x) < c_{r'}$ for all z with $\mathrm{Im}[z] > r'$ and $x \in \mathbb{R}$.*

Proof. An ERBM started at $z \in D$ with $|z| > r$ has to hit $\partial B_r^+(0)$ before it can hit 0. As a result, the strong Markov property for ERBM implies that to prove the first statement, it is enough to find a bound for $H_D^{ER}(\cdot, 0)$ restricted to $\partial B_r^+(0)$. Since

$$H_D^{ER}\left(re^{i\theta}, 0 \right) = H_D\left(re^{i\theta}, 0 \right) + \sum_{j=1}^{n} h_j\left(re^{i\theta} \right) H_D^{ER}(A_j, 0),$$

the necessary bound follows from Lemma 4.6 and the fact that

$$H_D\left(re^{i\theta},0\right) \le H_{\mathbb{H}}\left(re^{i\theta},0\right) \le \frac{1}{\pi r}.$$

The proof of the second statement is similar. □

There is an analog to Lemma 2.3 for ERBM.

Lemma 4.8. *Let $D \in \mathcal{Y}^*$ and define $f_z(x) := H_D^{ER}(z,x)$. There is a $c > 0$ such that if $x \in \mathbb{R}$ and $r < |z-x|$ are such that $B_r^+(x) \subset D$, then $\left|f_z'(x)\right| \le cr^{-2}$.*

Proof. We may assume without loss of generality that $x = 0$. Define $f_i(x)$ to be equal to $H_D^{ER}(A_i,x)$. Since

$$f_z(x) = H_D(z,x) + \sum_{i=1}^{n} h_i(z)\, f_i(x),$$

using Lemma 2.3, to complete the proof it is enough to show there is a $c > 0$ such that $f_i'(0) < cr^{-2}$ for all $1 \le i \le n$. Using (4.12) (and the notation preceding it), we have

$$f_i(x) = \sum_{n=1}^{\infty} q_n \left[\frac{r}{2}\int_0^{\pi} H_D\left((r/2)\, e^{i\theta},0\right) p_n(\theta)\ d\theta\right].$$

Differentiating both sides of this equation and using the bounded convergence theorem, Lemma 2.3, and the computation following (4.12), the result follows. □

The next lemma gives an estimate on the effect on the Poisson kernel of removing a compact $\mathbb{H}$-hull from a domain D.

Lemma 4.9. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$ and suppose that there are real constants $0 < r < R$ such that $w \in D$ for all $w \notin A_{r,R}$ and a constant r' such that $w \in D$ for all $w \in \mathbb{H}$ with $\mathrm{Im}[w] < r'$. Let A be a compact $\mathbb{H}$-hull contained in $B_{r/2}^+(0)$. If*

$|x| > \mathrm{rad}\,(A) + \sqrt{\mathrm{rad}\,(A)}$ *and* $|z| > r$*, then there is a* $c > 0$ *depending only on* r*,* r'*, and* R *such that*

$$H_D^{ER}(z,x) - H_{D\backslash A}^{ER}(z,x) \leq cH_D^{ER}(z,0)\,\mathrm{rad}\,(A).$$

Furthermore, if $|x| > 2R$*, there is a* $c > 0$ *depending only on* r *and* R *such that*

$$H_D^{ER}(z,x) - H_{D\backslash A}^{ER}(z,x) \leq cH_D^{ER}(z,0)\,x^{-2}\,\mathrm{rad}\,(A)^2.$$

Proof. Using (4.5), we see that

$$H_D^{ER}(z,x) - H_{D\backslash A}^{ER}(z,x) = \mathbf{E}^z\left[H_D^{ER}\left(B_{\tau_{D\backslash A}}^{ER}, x\right)\right].$$

Let $\epsilon = \mathrm{rad}\,(A)$. We can bound $\mathbf{E}^z\left[H_D^{ER}\left(B_{\tau_{D\backslash A}}^{ER}, x\right)\right]$ by the probability that an ERBM started at z hits $\partial B_\epsilon^+(0)$ before leaving D multiplied by the maximum value of $H_D^{ER}(\cdot, x)$ restricted to $\partial B_\epsilon^+(0)$. Proposition 4.5 implies the probability an ERBM started at z hits $\partial B_\epsilon^+(0)$ before leaving D is

$$4\epsilon H_D^{ER}(z,0)\left[1 + O(\epsilon)\right]. \tag{4.13}$$

Next, recall that

$$H_D^{ER}\left(\epsilon e^{i\theta}, x\right) = H_D\left(\epsilon e^{i\theta}, x\right) + \sum_{i=1}^{n} h_i\left(\epsilon e^{i\theta}\right) H_D^{ER}(A_i, x). \tag{4.14}$$

Since $H_D\left(\epsilon e^{i\theta}, x\right) < H_{\mathbb{H}}\left(\epsilon e^{i\theta}, x\right)$, (2.3) implies $H_D\left(\epsilon e^{i\theta}, x\right)$ is uniformly bounded for $|x| > \epsilon + \sqrt{\epsilon}$ and less than a constant depending only on R multiplied by ϵx^{-2} for $|x| > 2R$. The remark following (2.5) implies $\sum_{i=1}^{n} h_i\left(\epsilon e^{i\theta}\right) = O(\epsilon)$ as $\epsilon \to 0$. Lemma 4.7 implies $H_D^{ER}(A_i, x)$ is bounded by a constant depending only on r' for all x and Lemma 4.6 implies $H_D^{ER}(A_i, x)$ is bounded by a constant depending only on R multiplied by x^{-2} for $|x| > 2R$.

Combining these facts with (4.13) and (4.14), the results follow. □

4.3 Conformal Mapping Using $H_D^{ER}(\cdot, w)$

Recall that a domain is called a chordal standard domain if it obtained by removing a finite number of horizontal line segments from the upper half-plane. It is a classical theorem of complex analysis [2] that every $D \in \mathcal{Y}_n$ is conformally equivalent to a chordal standard domain. Furthermore, this equivalence is unique up to a scaling and real translation. This section is devoted to using $H_D^{ER}(\cdot, w)$ to give a probabilistic proof of this fact. Our proof is based on the sketch of a proof given in [13]. In what follows, assume that ∂A_0 is locally analytic at $w \in \partial A_0$.

There is an analytic characterization of $H_D^{ER}(\cdot, w)$.

Proposition 4.10. $H_D^{ER}(\cdot, w)$ *is up to a real constant multiple the unique positive ER-harmonic function that satisfies* $H_D^{ER}(z, w) \to 0$ *as* $z \to w'$ *for any* $w' \in \partial A_0$ *not equal to* w.

Proof. Using (4.2), we see that $H_D^{ER}(\cdot, w)$ is harmonic on D. If V is a Borel subset of ∂A_0, then it follows from the strong Markov property for ERBM and (3) of Definition 3.1 that $\mathrm{hm}_D^{ER}(\cdot, V)$ is ER-harmonic. As a result, if γ is as in (4.1) and η and U_i are as in Definition 3.10, then

$$\begin{aligned}
H_D^{ER}(A_i, w) &= \lim_{\epsilon \to 0} \frac{\mathrm{hm}_D^{ER}(A_i, \gamma(-\epsilon, \epsilon))}{\int_{-\epsilon}^{\epsilon} |\gamma'(x)| \, dx} \\
&= \lim_{\epsilon \to 0} \int_{\eta} \frac{\mathrm{hm}_D^{ER}(z, \gamma(-\epsilon, \epsilon))}{\int_{-\epsilon}^{\epsilon} |\gamma'(x)| \, dx} \cdot \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)} \, |dz| \\
&= \int_{\eta_i} H_D^{ER}(z, w) \cdot \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)} \, |dz|,
\end{aligned}$$

where the last equality follows from the Harnack inequality and dominated convergence. This proves that $H_D^{ER}(\cdot, w)$ is ER-harmonic. It is clear from (4.2) and Proposition 2.1 that $H_D^{ER}(\cdot, w)$ has the required asymptotics at ∂A_0.

Suppose f is another positive ER-harmonic function that satisfies $f(z) \to 0$ as $z \to w'$ for any $w' \in \partial A_0$ not equal to w. The function

$$g(z) := f(z) - \sum_{i=1}^{n} h_i(z) f(A_i)$$

is a harmonic function with $g(A_i) = 0$ for each $1 \leq i \leq n$ that has the same boundary conditions as f at ∂A_0. It follows from Proposition 2.1 that there is a $c > 0$ such that $g(z) = cH_D(z, w)$. As a result, (4.2) implies $f(z) - cH_D^{ER}(z, w)$ is an ER-harmonic function that is 0 on A_0 and thus, by the maximal principle for ER-harmonic functions, $f(z) = cH_D^{ER}(z, w)$ for all $z \in D$. □

A useful fact about $H_D^{ER}(\cdot, w)$ that is not always true for $H_D(\cdot, w)$ when D is multiply connected is that $H_D^{ER}(\cdot, w)$ has no critical values (that is, its derivative always has full rank). This result will be crucial later when we prove that the level sets of $H_D^{ER}(\cdot, w)$ are Jordan curves. First we need two lemmas.

Lemma 4.11. *For each $r > 0$, the set*

$$V_r = \left\{ z \in D : H_D^{ER}(z, w) \leq r \right\}$$

is connected.

Proof. Using Proposition 4.1, we may assume without loss of generality that $\mathbb{C} \backslash A_0 = \mathbb{D}$. Let U_r consist of all $z \in D$ such that $H_D^{ER}(z, w) < r$ and that there is a path contained in V_r from z to $\partial A_0 - \{w\}$. If $z_1, z_2 \in U_r$, then there are curves γ_i, for $i = 1, 2$, in V_r connecting z_i to $\partial \mathbb{D}$. By staying "close" to $\partial \mathbb{D}$, we can find a path γ_3 in V_r connecting γ_1

and γ_2. It follows that U_r is path connected. It is straightforward to verify that U_r is open and that $H_D^{ER}(z,w) = r$ on $\partial U_r \cap D$.

Observe that $H_D^{ER}(\cdot,w)$ restricted to $D\backslash\overline{U_r}$ is an ER-harmonic function with boundary value always greater than or equal to r. By restricting to a domain where $H_D^{ER}(\cdot,w)$ is bounded and using Proposition 3.13, it is not hard to see that this implies that if $z \notin U_r$, then $H_D^{ER}(z,w) \geq r$. If there were a $z \notin \overline{U_r}$ with $H_D^{ER}(z,w) = r$, then, by the maximal principle for ER-harmonic functions, we would have that $H_D^{ER}(u,w) = r$ for all u in a neighborhood of z. Since a harmonic function on a connected domain that is constant on a non-empty open set is constant everywhere, this would imply $H_D^{ER}(z,w) = r$ for all $z \in D$. This is a contradiction, so we conclude that $V_r = \overline{U_r}$. Since U_r is connected, it follows that V_r is as well. □

Lemma 4.12. *If $f : D \to \mathbb{C}$ is a holomorphic function such that $f(z) = (z-a)^n g(z)$ for a holomorphic function g that is non-zero in a neighborhood of a, then there exists a conformal map h defined in a neighborhood of a such that $f \circ h^{-1}(z) = (z-a)^n$.*

Proof. Since $g(z) \neq 0$ in a neighborhood of a, we can define a branch of $\sqrt[n]{g(z)}$ in a neighborhood of a. A straightforward application of the argument principle shows that $h(z) := (z-a)\sqrt[n]{g(z)} + a$ is injective in a neighborhood of a and thus is a conformal map onto its image. It is easy to check that $f \circ h^{-1}(z) = (z-a)^n$. □

Proposition 4.13. *For all $z \in D$, the derivative of $H_D^{ER}(\cdot,w)$ at z has full rank.*

Proof. Suppose there were an $a \in D$ such that the derivative of $v(\cdot) := H_D^{ER}(\cdot,w)$ at a were zero. Let $r = v(a)$ and U_r be as in Lemma 4.11. Since v is harmonic, we can find a holomorphic function f defined in a neighborhood of a with imaginary part equal to v. Let n be the order of the zero of $f(z) - f(a)$ at a. Lemma 4.12 implies that there is a conformal map h defined on a neighborhood of a such that $f \circ h^{-1}(z) = (z-a)^n + f(a)$.

The set of points where v is equal to r is equal to the image under h^{-1} of the zero set of $\mathrm{Im}\left[(z-a)^n\right]$. As a result, for small enough ϵ, the set of points where v is equal to r separates $B_\epsilon(a)$ into $2n$ ordered connected components which alternate between being subsets of U_r and $D\backslash U_r$.

Let x and y be points in distinct connected components of $U_r \cap B_\epsilon(a)$. Since U_r is an open, connected set, we can find a path in U_r connecting x and y. By assumption, this path cannot be contained in $B_\epsilon(a)$ and, therefore, a subset of it is a path $\gamma_1 : [s,t] \to D\backslash B_\epsilon(a)$ connecting the connected components of $U_r \cap B_\epsilon(a)$ containing x and y respectively. Let γ_2 be a path in $U_r \cap B_\epsilon(a)$ connecting a to $\gamma_1(s)$, γ_3 be a path in $U_r \cap B_\epsilon(a)$ connecting $\gamma_1(t)$ to a, and γ be the concatenation γ_2, γ_1, and γ_3. Observe that γ is a Jordan curve and that $v \leq r$ on γ. As a result, Proposition 3.13 implies that $v \leq r$ on the interior of γ. However, this is a contradiction because the interior of γ contains one of the arcs of $\partial B_\epsilon(a)$ connecting $\gamma(s)$ and $\gamma(t)$ and both these arcs contain points where $v(z) > r$. The result follows. □

Next we prove that the level sets of $H_D^{ER}(\cdot, w)$ are Jordan curves.

Proposition 4.14. *If r is a positive real number, then*

$$\gamma_r := \{w\} \cup \left\{z : H_D^{ER}(z,w) = r\right\}$$

is a Jordan curve.[1] *Furthermore, γ_r separates D into two connected components,*

$$\left\{z : H_D^{ER}(z,w) < r\right\} \text{ and } \left\{z : H_D^{ER}(z,w) > r\right\}.$$

Proof. We will only consider the case where $r \neq H_D^{ER}(A_i, w)$ for any i. The other case is similar. Using Proposition 4.1, we may assume that $\partial A_0 = \mathbb{R}$ and $w = 0$. Proposition 4.13

1. If $r = H_D^{ER}(A_i, w)$ for some i, then in order for this to make sense, we have to work in the space E, not D.

and basic facts from differential topology imply

$$K_r := \left\{ z \in D : H_D^{ER}(z, w) = r \right\}$$

is a one-dimensional smooth real manifold. As a result, each connected component of K_r is diffeomorphic to either a circle or $\mathbb{R}$. Using Proposition 3.13, we see that the latter is not possible as it would imply $H_D^{ER}(\cdot, w)$ is constant on an open subset of D. As a result, it is not difficult to see that adding the point w to any connected component of K_r yields a Jordan curve γ_r.

Using (4.2), (2.6), and (say) (2.5), we can show that $H_D(z, 0) \sim H_{\mathbb{H}}(z, 0)$ as $z \to 0$. Combining this with (2.2), we see that

$$\lim_{x \to 0} H_D^{ER}\left(x + ax^2 i, 0\right) = \frac{a}{\pi},$$

for any $a > 0$. It follows that either for all $a > \pi r$ there is an $\delta_a > 0$ such that the interior of γ_r contains $t + t^2 i$ for all $|t| < \delta_a$ or for all $a > \pi r$ there is an $\delta_a > 0$ such that the exterior of γ_r contains $t + t^2 i$ for all $|t| < \delta_a$. In the latter case, it is easy to see that $H_D^{ER}(\cdot, w)$ is bounded on the interior of γ_r, and thus, using Proposition 3.13, is equal to r on the interior of γ_r. Since this is not possible, the former case must hold. In this case, if K_r has two distinct Jordan curves γ_r and γ_r' in it, then it is clear one of them must be contained in the interior of the other. An argument similar to the one in the proof of Lemma 4.11 shows that this is not possible. As a result, K_r has exactly one connected component and the result follows. □

We now have all of the tools necessary to show that $H_D^{ER}(\cdot, w)$ is the imaginary part of a conformal map into a chordal standard domain. The idea of the proof is to use Proposition 4.10 to show that if there is such a map, its imaginary part must be $H_D^{ER}(\cdot, w)$ and to use Proposition 4.14 to show that the map actually exists.

Theorem 4.15. *Let $D \in \mathcal{Y}_n$ and suppose ∂A_0 is a smooth Jordan curve (in the topology of E) such that there is no Jordan curve in D with A_0 in its interior. If $w \in \partial A_0$, then there is a $D' \in \mathcal{CY}_n$ and conformal map $f : D \to D'$ with $f(w) = \infty$. Furthermore, if g is another such map, then there are real constants r, x such that $g = rf + x$.*

Proof. Using Lemma 3.11 and (say) Proposition 13.3.5 of [7], we see that a harmonic function h that is continuous on E is the imaginary part of a holomorphic function if and only if it is ER-harmonic. It follows that if $D' \in \mathcal{CY}_n$ and $f : D \to D'$ is a conformal map with $f(w) = \infty$, then the imaginary part of f is a positive ER-harmonic function such that $f(z) \to 0$ as $z \to w'$ for any $w' \neq w$. By Proposition 4.10, this implies that the imaginary part of f is a real constant multiple of $H_D^{ER}(\cdot, w)$. Combining this with the fact the imaginary part of a holomorphic function determines the real part up to a real additive constant, we obtain the uniqueness statement.

As noted above, $v(\cdot) := H_D^{ER}(\cdot, w)$ is the imaginary part of a holomorphic function $f = u + iv$. Furthermore, u is defined up to a real additive constant by

$$u(z) = u(z_0) + \int_\gamma \frac{d}{dn}\, v(z)\, |dz|, \tag{4.15}$$

where γ is a smooth curve connecting z_0 and z and the normal derivative is chosen with the correct sign. To complete the proof, we need to show that f is injective and $f(D) \in \mathcal{CY}_n$.

Proposition 4.14 implies that the sign of $\frac{d}{dn}\, v(z)$ is constant on γ_r. As a result, $u(\gamma_r(t))$ is increasing (when the appropriate parametrization of γ_r is chosen). In fact, $u(\gamma_r(t))$ is strictly increasing since otherwise f would be constant on a segment of a curve (and hence everywhere constant). It follows that if $r \neq v(A_i)$ for any i, then f is injective on γ_r. If $r = v(A_i)$ and z and w are two points on γ_r, it is not hard to see that we can still find a curve connecting v and w on which the sign of $\frac{d}{dn} v(z)$ is constant. Arguing as before, it follows that f is injective on γ_r.

Let $w_\epsilon \in \partial A_0$ be distance ϵ away from w in the counterclockwise direction and n_ϵ be the inward pointing normal at w_ϵ. Using (2.2) and (2.1), we can check that $\frac{d}{dn_\epsilon} v\left(w_\epsilon\right) \sim \frac{1}{\pi\epsilon^2}$ as $\epsilon \to 0$. Since γ_r is tangent to ∂A_0 at w, it follows that

$$\left|\frac{d}{dn} v\left(\gamma_r\left(t_\epsilon\right)\right) \sim \frac{1}{\pi\epsilon^2}\right|, \quad \epsilon \to 0,$$

where t_ϵ is such that $\gamma_r\left(t_\epsilon\right)$ is distance ϵ from w. It follows from (4.15) that $\left|u\left(z\right)\right| \to \infty$ as z approaches w along γ_r and hence, $f\left(D\right) \in \mathcal{CY}_n$. $\square$

CHAPTER 5

THE GREEN'S FUNCTION FOR ERBM

5.1 Definition and Basic Properties

In this chapter, let $D \in \mathcal{Y}$ be such that it is possible to define a Green's function $G_D(z,w)$ for Brownian motion. Recall that we normalize $G_D(z,\cdot)$ so that it is a density for the expected amount of time a Brownian motion started at z spends in a set before exiting D.

Definition 5.1.

$$G_D^{ER}(z,\cdot) : E \to \mathbb{R}$$

is a *Green's function for ERBM* if for any Borel subset $V \subset D$

$$\mu_z(V) := \mathbf{E}^z\left[\int_0^{\tau_D} \mathbf{1}_V\left(B_D^{ER}(t)\right) dt\right] = \int_V G_D^{ER}(z,w)\, dw, \tag{5.1}$$

where $\tau_D = \inf\left\{t : B_D^{ER}(t) \in \partial A_0\right\}$.

Using the definition of ERBM and the analogous fact for Brownian motion, it is easy to prove that the probability that ERBM started at z is in a set of Lebesgue measure zero at some fixed time is 0. Combining this fact with Fubini's theorem, we see that if V has Lebesgue measure zero, then

$$\mu_z(V) = \int_0^{\tau_D} \mathbf{P}^z\left\{B_D^{ER}(t) \in V\right\} dt = 0.$$

As a result, we can define $G_D^{ER}(z,\cdot)$ as a Radon-Nikodym derivative. Furthermore, we have

$$G_D^{ER}(z,w)=\lim_{\epsilon\to 0}\frac{\mu_z(B(w,\epsilon))}{m(B(w,\epsilon))} \tag{5.2}$$

is a Green's function for ERBM, where m is Lebesgue measure. A priori, there is no reason the Green's function as defined cannot be infinite on a set of positive measure. This potential issue will be resolved by Proposition 5.2 and (5.8).

We have only given a probabilistic definition of $G_D^{ER}(z,\cdot)$ and our definition is unique only as an element of $L^1(D)$. It is also possible to give an analytic characterization of $G_D^{ER}(z,\cdot)$. More specifically, we will prove that there is a version of $G_D^{ER}(z,\cdot)$ that is the unique ER-harmonic function on $D-\{z\}$ satisfying certain boundary conditions (that depend on whether or not z is equal to some A_i). In particular, this will allow us to talk about "the" Green's function for ERBM rather than "a" Green's function. We start by proving an analog of (4.2) for $G_D^{ER}(z,\cdot)$.

Proposition 5.2.

$$G_D(z,w)+\sum_{i=1}^{n}h_i(z)\,G_D^{ER}(A_i,w)$$

is a version of $G_D^{ER}(z,\cdot)$.

Proof. This follows easily using the strong Markov property for ERBM and the fact that up until the first time it hits ∂D, ERBM has the distribution of a Brownian motion. □

As we expect, $G_D^{ER}(z,\cdot)$ is conformally invariant. To prove this we need a lemma.

Lemma 5.3. *If* $g\in L^1(D)$*, then for all Borel* $V\subset D$ *we have*

$$\mathbf{E}^z\left[\int_0^{\tau_D}\mathbf{1}_V\left(B_D^{ER}(t)\right)g\left(B_D^{ER}(t)\right)dt\right]=\int_V G_D^{ER}(z,w)\,g(w)\,dw.$$

Proof. By considering its real and imaginary parts separately, we may assume that g is a real-valued function. By considering its positive and negative parts separately, we may assume that g is a non-negative.

The result is easy to verify when g is a simple function. If g is not simple, we can approximate g from below by simple functions and use monotone convergence. □

Proposition 5.4. *If $f : D \to D'$ is a conformal map, then*

$$G_D^{ER}\left(f^{-1}(z), f^{-1}(\cdot)\right)$$

is a version of $G_{D'}^{ER}(z, \cdot)$.

Proof. It is enough to show that $G_D^{ER}\left(f^{-1}(z), f^{-1}(\cdot)\right)$ satisfies (5.1) for all open subsets of D'. Let V' be an open subset of D' and $V = f^{-1}(V')$. Using Lemma 5.3 and the change of variables formula, we have

$$\begin{aligned} \int_{V'} G_D^{ER}\left(z, f^{-1}(w)\right) dw &= \int_V G_D^{ER}(z, w) \left|f'(w)\right|^2 dw \\ &= \mathbf{E}^z\left[\int_0^{\tau_D} \left|\mathbf{1}_V\left(B_D^{ER}(t)\right) f'\left(B_D^{ER}(t)\right)\right|^2 dt\right]. \end{aligned} \tag{5.3}$$

Let

$$u(t) = \int_0^t \left|f'\left(B_D^{ER}(s)\right)\right|^2 ds. \tag{5.4}$$

Substituting $u^{-1}(r)$ for t and using the conformal invariance of ERBM, we see that (5.3) is equal to

$$\mathbf{E}^z\left[\int_0^{\tau_{D'}} \mathbf{1}_{V'}\left(B_D^{ER}(t)\right) dt\right], \tag{5.5}$$

which completes the proof. □

In the proof of Proposition 5.4, observe that we can only conclude that (5.3) is equal to (5.5) if (5.4) is almost surely finite for all $t < \infty$. This will be addressed when we prove

(3.4).

In order to prove $G_D^{ER}(z,\cdot)$ is ER-harmonic, we will need to be able to compute $G_{A_{1,r}}(z,\cdot)$. Using Proposition 5.2, to do this, it is enough to compute $G_{A_{1,r}}^{ER}(A_1,\cdot)$.

Lemma 5.5. *Let $A_{1,r} \in \mathcal{Y}_1$ be the annulus with $A_1 = \overline{\mathbb{D}}$ and $\partial A_0 = \partial B_r(0)$ for some $r > 1$ and B_t be a Brownian motion in $r\mathbb{D}$. If V is a Borel set bounded away from A_1, then*

$$\mathbf{E}^{A_1}\left[\int_0^{\tau_{A_{1,r}}} \mathbf{1}_V\left(B_{A_{1,r}}^{ER}(t)\right)dt\right] = \mathbf{E}^0\left[\int_0^{\tau_{r\mathbb{D}}} \mathbf{1}_V(B_t)\right]dt,$$

where $\tau_{A_{1,r}}$ and $\tau_{r\mathbb{D}}$ are respectively the first time $B_{A_{1,r}}^{ER}$ leaves $A_{1,r}$ and B_t leaves $r\mathbb{D}$. Furthermore, we have

$$G_{A_{1,r}}^{ER}(A_1,z) = \frac{-\log|z| + \log r}{\pi}.$$

Proof. Since V is bounded away from $\overline{D}$, there exists an $\epsilon > 0$ such that V is contained in the region bounded by the circle

$$C_\epsilon = \{z \in \mathbb{C} : |z| = 1 + \epsilon\}$$

and the outer boundary of $A_{1,r}$. Let $\sigma_1 = 0$, τ_j be the first time after σ_j that $B_{A_{1,r}}^{ER}$ hits C_ϵ, and σ_j for $j > 1$ be the first time after τ_{j-1} that $B_{A_{1,r}}^{ER}$ hits A_1. Similarly, let $\sigma_1' = 0$, τ_j' be the first time after σ_j' that B_t hits C_ϵ, and σ_j' for $j > 1$ be the first time after τ_{j-1}' that B_t hits the circle of radius 1, and $\sigma_1' = 0$. It follows from the strong Markov property for ERBM and (3) of Definition 3.1 that given that $\tau_j < \infty$, the distribution of $B_{A_{1,r}}^{ER}(\tau_j)$ is uniform on $C_{1+\epsilon}$. It is an easy exercise to check that given that $\tau_j' < \infty$, the distribution of B_{τ_j} is uniform on $C_{1+\epsilon}$. Using these two facts, the strong Markov property for ERBM, and the fact that an ERBM has the distribution of a Brownian motion up until the first time it hits the boundary of $A_{1,r}$, we see that

$$\mathbf{E}^{B^{ER}_{A_{1,r}}(\tau_j)}\left[\int_{\tau_j}^{\sigma_{j+1}} \mathbf{1}_V\left(B^{ER}_{A_{1,r}}(t)\right)dt\right] = \mathbf{E}^{B_{\tau'_j}}\left[\int_{\tau'_j}^{\sigma'_{j+1}} \mathbf{1}_V\left(B_{r\mathbb{D}}(t)\right)dt\right].$$

Combined with the fact that

$$\mathbf{E}^{B^{ER}_{A_{1,r}}(\sigma_j)}\left[\int_{\sigma_j}^{\tau_j} \mathbf{1}_V\left(B^{ER}_{A_{1,r}}(t)\right)dt\right] = \mathbf{E}^{B_{\sigma'_j}}\left[\int_{\sigma'_j}^{\tau'_j} \mathbf{1}_V\left(B_{r\mathbb{D}}(t)\right)dt\right] = 0,$$

the first result follows.

Using the first part of the proposition, we see that

$$G^{ER}_{A_{1,r}}\left(A_1, z\right) = G_{r\mathbb{D}}\left(0, z\right).$$

Combining this with (2.19), the second part of the proposition follows. □

A quantity that will help us understand $G^{ER}_D(z, \cdot)$ is the density for the amount of time ERBM started at A_i spends in a set from the time it hits a curve η_i surrounding A_i until the time it hits ∂D again. The next lemma establishes the existence and some properties of this density.

Lemma 5.6. *For $i = 1, \ldots, n$, let η_i and U_i be as in Definition 3.1. The function*

$$T_i(w) := G^{ER}_{U_i}(A_i, w) + \int_{\eta_i} G_D(z, w)\frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)}|dz|,$$

where by convention we let $G^{ER}_{U_i}(A_i, w) = 0$ for $w \notin U_i$, has the following properties.

1. *$T_i(w)$ is a density for the expected amount of time ERBM started at A_i spends in a set up until τ_2, where τ_2 is as in Section 3.4*

2. *$T_i(w)$ is harmonic on $D\backslash\eta_i$*

3. *If* $i \neq j$*, then* $\frac{1}{2}\int_{\eta_j} \frac{d}{dn} T_i(w)\,|dw| = p_{ij}$*, where* n *is the outward-pointing normal and* p_{ij} *is as in Section 3.5*

4. *If* η_i' *is a smooth curve in the interior of* U_i *that is homotopic to* η_i*, then*

$$\frac{1}{2}\int_{\eta_i'} \frac{d}{dn} T_i(w)\,|dw| = p_{ii} - 1.$$

Proof. It is clear using the strong Markov property for ERBM, the fact that ERBM has the distribution of a Brownian motion up until the first time it hits ∂D, and (3) of Definition 3.1 that the first statement holds.

Denote the second summand in the definition of $T_i(w)$ by $S_i(w)$. If $w \notin \eta_i$ and ϵ is small enough such that $B(w,\epsilon)$ does not intersect η_i, then using Fubini's theorem and the fact that $G_D(z,\cdot)$ is harmonic, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} S_i\left(w+\epsilon e^{i\theta}\right) d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left[\int_{\eta_i} G_D\left(z, w+\epsilon e^{i\theta}\right)\frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)}\,|dz|\right] d\theta \\
&= \int_{\eta_i} \frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)}\left[\frac{1}{2\pi}\int_0^{2\pi} G_D\left(z, w+\epsilon e^{i\theta}\right) d\theta\right]|dz| \\
&= \int_{\eta_i} \frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)} G_D(z,w)\,|dz| \\
&= S_i(w).
\end{aligned}$$

This shows that $S_i(w)$ satisfies the spherical mean value property at w and, thus, is harmonic on $D\backslash\eta_i$. It follows that to finish the proof of the second statement, we just have to show that $G_{U_i}^{ER}(A_i,\cdot)$ is harmonic away from η_i. Let $f_i : A_{1,r_i} \to U_i$ be a conformal map mapping the outer boundary of A_{1,r_i} to the outer boundary of U_i. Using Proposition 5.4 and Lemma 5.5, we see that

$$G_{U_i}^{ER}(A_i,w) = G_{D_{r_i}}^{ER}(A_1, f_i(w)) = \frac{-\log|f_i(w)| + \log r_i}{\pi}. \tag{5.6}$$

Since $\log|z|$ is harmonic and precomposing a harmonic function with a conformal map yields a harmonic function, $G_{U_i}^{ER}(A_i, \cdot)$ is harmonic away from η_i.

The proof of the third statement uses the fact that if z is in the exterior of η_j, then

$$\int_{\eta_j} \frac{d}{dn} G_D(z, w)\,|dw| = 2h_j(z)\,. \tag{5.7}$$

In the case that ∂A_j is a smooth Jordan curve, this is true because the normal derivative of $G_D(z, w)$ is $2H_D(z, w)$ on ∂A_j and the integral of the normal derivative of a harmonic function is the same over any two homotopic curves. If the boundary of A_j is not a smooth Jordan curve, we can map D conformally to a region where the image of ∂A_j is a smooth Jordan curve [7] and use the conformal invariance of the Green's function, the change of variables formula, the fact that conformal maps preserve angles and the result in the case that ∂A_j is a smooth Jordan curve. If $i \neq j$, using Fubini's theorem, the dominated convergence theorem, and (5.7), we have

$$\begin{aligned}
\int_{\eta_j} \frac{d}{dn}\, T_i(w)\,|dw| &= \int_{\eta_j} \frac{d}{dn} \int_{\eta_i} G_D(z, w)\, \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)}\,|dz|\,|dw| \\
&= \int_{\eta_i} \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)} \int_{\eta_j} \frac{d}{dn}\, G_D(z, w)\,|dw|\,|dz| \\
&= 2 \int_{\eta_i} \frac{H_{\partial U_i}(A_i, z)}{\mathcal{E}_{U_i}(A_i, \eta)} h_j(z)\,|dz| \\
&= 2p_{ij}
\end{aligned}$$

The proof of the fourth statement is similar to the proof of the third statement and will rely on calculating $\int_{\eta_i'} \frac{d}{dn} G_D(z, w)\,|dw|$. If z is a point in the interior of η_i' and η_i'' is a smooth Jordan curve in the interior of η_i' such that z is in the exterior of η_i'', then by setting up the appropriate contour integral and using Green's theorem, it is not hard to see that (with the normals appropriately oriented)

$$\int_{\eta_i'} \frac{d}{dn} G_D(z,w)\,|dw| = \int_{\eta_i''} \frac{d}{dn} G_D(z,w)\,|dw| + \int_{B_\epsilon(z)} \frac{d}{dn} G_D(z,w)\,|dw|.$$

Using (5.7) and the fact that $G_D(z,w) = -\frac{\log|z-w|}{\pi} + g_z(w)$, where g_z is harmonic on D, we have

$$\begin{aligned}
\int_{\eta_i'} \frac{d}{dn} G_D(z,w)\,|dw| &= \int_{\eta_i''} \frac{d}{dn} G_D(z,w)\,|dw| + \int_{B_\epsilon(z)} \frac{d}{dn} G_D(z,w)\,|dw| \\
&= 2h_i(z) - \int_{B(z,\epsilon)} \frac{d}{dn} \frac{\log|z-w|}{\pi}\,|dw| \\
&= 2(h_i(z) - 1).
\end{aligned}$$

Using this and arguing as in the proof of the third statement, we have

$$\begin{aligned}
\int_{\eta_i'} \frac{d}{dn} T_i(w) &= \int_{\eta_i'} \frac{d}{dn} \int_{\eta_i} G_D(z,w) \frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)}\,|dz|\,|dw| \\
&= \int_{\eta_i} \frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)} \int_{\eta_i'} \frac{d}{dn} G_D(z,w)\,|dw|\,|dz| \\
&= 2\int_{\eta_i} \frac{H_{\partial U_i}(A_i,z)}{\mathcal{E}_{U_i}(A_i,\eta)} (h_i(z) - 1)\,|dz| \\
&= 2(p_{ii} - 1).
\end{aligned}$$

□

We have all of the tools necessary to prove that $G_D^{ER}(z,\cdot)$ is ER-harmonic. In what follows, we continue to use the set up of the previous lemma.

Proposition 5.7. *There are versions of* $G_D^{ER}(\cdot,z)$ *and* $G_D^{ER}(z,\cdot)$ *that are ER-harmonic on* $D - \{z\}$.

Proof. Using Proposition 5.2, in order to show that there is a harmonic version of $G_D^{ER}(z,\cdot)$, it is enough to show that there is a harmonic version of $G_D^{ER}(A_i,\cdot)$ for each $1 \le i \le n$. Let $\mathbf{T}$ be the vector function with ith component $T_i(w)$ and let $\mathbf{D}$ be the diagonal matrix with ii entry $\frac{1}{1-p_{ii}}$. Using the strong Markov property for ERBM and Lemma 5.6, we see that $\mathbf{DT}$ is the vector function whose ith component is the density for the expected amount of time ERBM started at A_i spends in a set up until the first time it hits an A_j with $j \neq i$. Using (3.6) and the strong Markov property for ERBM, we see that the ith component of

$$\mathbf{DT} + \mathbf{QDT} + \mathbf{Q}^2\mathbf{DT} + \ldots = (\mathbf{I} - \mathbf{Q})^{-1}\,\mathbf{DT} \tag{5.8}$$

is a version of $G_D^{ER}(A_i,\cdot)$. Since $T_i(\cdot)$ is harmonic away from each η_i, it follows that there is a version of $G_D^{ER}(A_i,\cdot)$ that is harmonic away from each η_i. By choosing different η_i's and repeating this procedure, we can get a version of $G_D^{ER}(A_i,\cdot)$ that is harmonic away from a sequence of Jordan curves $\eta_1',\ldots,\eta_n'$ which are disjoint from each η_i. Finally, since any two versions of $G_D^{ER}(A_i,\cdot)$ are equal almost everywhere, we can find a version of $G_D^{ER}(A_i,\cdot)$ that is harmonic everywhere.

Using (3) and (4) of Lemma 5.6 and the fact that the ith component of (5.8) is a version of $G_D^{ER}(A_i,\cdot)$, we see that

$$\int_{\eta_j} \frac{d}{dn}\, G_D^{ER}(A_i,w)\,|dw| = \begin{cases} 0 & \text{if } j \neq i \\ -2 & \text{if } j = i \end{cases}. \tag{5.9}$$

It is easy to see using its definition and (5.6) that each $T_i(\cdot)$, and thus each $G_D^{ER}(A_i,\cdot)$, can be extended to a continuous function on E. Combining this with Lemma 3.11, we have that that there is a version of $G_D^{ER}(A_i,\cdot)$ that is ER-harmonic on $D - \{A_i\}$. Finally, using (5.7), (5.9), and Lemma 3.11, it follows that the version of $G_D^{ER}(z,\cdot)$ defined in Proposition 5.2 is ER-harmonic on $D - \{z\}$.

An argument similar to the one showing that $H_D^{ER}(\cdot, z)$ is ER-harmonic shows that $G_D^{ER}(\cdot, z)$ is ER-harmonic. □

We can now give an analytic characterization of $G_D^{ER}(z, \cdot)$.

Proposition 5.8. *If $z \in D$, then $G_D^{ER}(z, \cdot)$ is the unique ER-harmonic function on $D - \{z\}$ satisfying*

- $G_D^{ER}(z, w) = -\frac{\log|z-w|}{\pi} + O(1)$ *as* $w \to z$
- $G_D^{ER}(z, w) \to 0$ *as* $w \to w'$ *for any* $w' \in \partial A_0$.

Furthermore, $G_D^{ER}(A_i, \cdot)$ is the unique ER-harmonic function on $D - \{A_i\}$ that is equal to $G_D^{ER}(A_i, A_i)$ on ∂A_i and 0 on ∂A_0.

Proof. If $z \in D$, the asymptotics for $G_D^{ER}(z, \cdot)$ at the boundary are clear and the asymptotic at z follows from Proposition 5.2 and the corresponding result for $G_D(z, \cdot)$. The uniqueness follows from a proof similar to the corresponding result for $G_D(z, \cdot)$ (see [12], pg. 54). The second statement follows from an extension of Proposition 3.13. □

In what follows, when we write $G_D^{ER}(z, \cdot)$ or $G_D^{ER}(\cdot, w)$ we will mean a version that is ER-harmonic.

Corollary 5.9. *$G_D^{ER}(z, w) = G_D^{ER}(w, z)$ for all $z, w \in E$.*

Proof. $G_D^{ER}(\cdot, z)$ satisfies the conditions of Proposition 5.8 and thus, is the same function as $G_D^{ER}(z, \cdot)$. □

We conclude this section by proving that the normal derivative on ∂A_0 of the Green's function for ERBM is a multiple of the Poisson kernel for ERBM. We need a lemma.

Lemma 5.10. *Let $D \in \mathcal{Y}_n^*$ be a domain such that ∂D and $\partial \mathbb{H}$ agree in a neighborhood of 0 and fix $R > 0$ such that $B_R^+(0) \subset D$. Then if $\epsilon < r < R$,*

$$\int_0^\pi G_D^{ER}\left(re^{i\theta}, \epsilon i\right) \sin\theta \; d\theta = \epsilon \left(\frac{1}{r} + O(r)\right),$$

as $\epsilon, r \to 0$.

Proof. Using Proposition 5.2 and (2.21), we see that

$$G_D^{ER}\left(re^{i\theta}, \epsilon i\right) = G_{\mathbb{H}}\left(re^{i\theta}, \epsilon i\right) - \mathbf{E}^{re^{i\theta}}\left[G_{\mathbb{H}}\left(B_{\tau_D}, \epsilon i\right)\right] + \sum_{i=1}^n h_i\left(re^{i\theta}\right) G_D^{ER}\left(A_i, \epsilon i\right).$$

As a result, using Lemma 2.5, to complete the proof, it is enough to show that

$$\sum_{i=1}^n h_i\left(re^{i\theta}\right) G_D^{ER}\left(A_i, \epsilon i\right) - \mathbf{E}^{re^{i\theta}}\left[G_{\mathbb{H}}\left(B_{\tau_D}, \epsilon i\right)\right] = O\left(r\epsilon\right), \quad \epsilon, r \to 0,$$

where $O(r\epsilon)$ is uniform over $\theta \in (0, \pi)$.

It is easy to check using Lemma 5.6, (2.21), (2.20), and (5.8) that $G_D^{ER}(A_i, \epsilon i) = O(\epsilon)$. Using (2.5), it follows that for $1 \leq i \leq n$, $h_i\left(re^{i\theta}\right)$ is $O(r)$. Since the probabilty a Brownian motion started at $re^{i\theta}$ does not exit D on $\mathbb{R}$ is $O(r)$, using (2.20), it follows that $\mathbf{E}^{re^{i\theta}}\left[G_{\mathbb{H}}\left(B_{\tau_D}, \epsilon i\right)\right]$ is $O(r\epsilon)$. □

Proposition 5.11. *Let $D \in \mathcal{Y}_n$ be such that ∂D is locally analytic at $x \in \partial A_0$. Then the (inner) normal derivative of $G_D^{ER}(z, \cdot)$ at x is $2H_D^{ER}(z, x)$.*

Proof. We start by proving the result when $D \in \mathcal{Y}_n$ is a domain such that $\mathbb{C} \backslash A_0 = \mathbb{H}$ and

$x = 0$. Fix R such that $B_R^+(0) \subset D$. Using Proposition 4.5, we have that if $r < R$, then

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{G_D^{ER}(z, \epsilon i)}{\epsilon} &= \lim_{\epsilon \to 0} \frac{r \int_0^\pi H_{D^r}\left(z, re^{i\theta}\right) G_D^{ER}\left(re^{i\theta}, \epsilon i\right)\, d\theta}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{2r H_D^{ER}(z, 0) \left[\int_0^\pi G_D^{ER}\left(re^{i\theta}, \epsilon i\right) \sin\theta\, d\theta\right] [1 + O(r)]}{\epsilon} \\
&= 2H_D^{ER}(z, 0)\,[1 + O(r)].
\end{aligned}$$

Taking the limit as $r \to 0$, the result follows.

For arbitrary D, let $f : D \to D'$ be a conformal map such that $D' \in \mathcal{Y}_n$ is such that $\mathbb{C} \backslash A_0 = \mathbb{H}$ and x is mapped to 0. Using the Schwarz reflection principle, $f(z)$ can be extended to a map conformal in a neighborhood of x and $G_D^{ER}(z, \cdot)$ can be extended to a function harmonic in a neighborhood of x. Combining this with the fact that conformal maps preserve angles, the chain rule, (2.1), the conformal invariance of $G_D^{ER}(z, \cdot)$, and the result for D', the result follows.

□

5.2 Proofs of formulas (3.3) and (3.4)

The theory of Green's functions for ERBM can be used to prove formulas (3.3) and (3.4). We start with a lemma.

Lemma 5.12. *Let $A_{1,r} \in \mathcal{Y}_1$ be as in Lemma 5.5 and $\tau = \inf\left\{t : B_{A_{1,r}}^{ER}(t) \in \partial A_0\right\}$. If $f : A_{1,r} \to D$ is a conformal map and D is bounded, then*

$$\mathbf{E}^z\left[\int_0^\tau \left|f'\left(B_{A_{1,r}}^{ER}(s)\right)\right|^2 ds\right] < \infty.$$

Proof. Using Lemma 5.3, we have that for sufficiently small ϵ

$$\begin{aligned}\mathbf{E}\left[\int_0^\tau \left|f'\left(B_{A_{1,r}}^{ER}(s)\right)\right|^2 ds\right] &= \int_{A_{1,r}} G_{A_{1,r}}^{ER}(z,w)\left|f'(w)\right|^2 dw \\ &= \int_{B_\epsilon(z)} G_{A_{1,r}}^{ER}(z,w)\left|f'(w)\right|^2 dw \\ &\quad + \int_{A_{1,r}\setminus B_\epsilon(z)} G_{A_{1,r}}^{ER}(z,w)\left|f'(w)\right|^2 dw.\end{aligned}$$

Since $\left|f'(r)\right|$ is bounded on $B_\epsilon(z)$ and the Green's function for ERBM is integrable, the first integral in the sum is finite. Since $G_{A_{1,r}}^{ER}(z,\cdot)$ is bounded on $A_{1,r}\setminus B(z,\epsilon)$ and $\int_{A_{1,r}} \left|f'(w)\right|^2 dw$ is equal to the area of D (by a straightforward change of variables), the second integral in the sum is also bounded. $\square$

Proposition 5.13. *Let $f : \mathbb{C}\setminus\mathbb{D} \to D$ be a conformal map sending ∞ to ∞. Then a.s. we have*

$$\int_0^t \left|f'\left(B_{\mathbb{C}\setminus\mathbb{D}}^{ER}(s)\right)\right|^2 ds < \infty \tag{5.10}$$

and

$$\int_0^\infty \left|f'\left(B_{\mathbb{C}\setminus\mathbb{D}}^{ER}(s)\right)\right|^2 ds = \infty. \tag{5.11}$$

Proof. For fixed t, let W be the set of ω in the underlying probability space such that the left hand side of (5.10) is infinite and for each $n \in \mathbb{N}$, let W_n be the set of ω such that B_D^{ER} has not left $A_{1,n}$ by time t. By Lemma 5.12, the measure of $W_n \cap W$ is zero. It follows that for almost every $\omega \in W$, the path of $B_{\mathbb{C}\setminus\mathbb{D}}^{ER}$ up to time t is unbounded. It is easy to see from the definition of ERBM that this implies that W has measure 0.

It is easy to see that $\left|f'\right|$ is bounded below on the set

$$\{z \in \mathbb{C} : |z| > 2\}.$$

Since the set of t such that $\left|B_{\mathbb{C}\setminus\mathbb{D}}^{ER}(t)\right| > 2$ has infinite measure, (5.11) follows. $\square$

Proposition 5.13 clarifies the implicit use of (5.10) and its analogs. The reader can verify that the proof of Proposition 5.13 does not rely on any of the results that used (5.10). For instance, in the proof of Proposition 5.4 we used the fact that a.s. (5.10) holds for any finitely connected region D. Using the definition of ERBM, it is easy to see that to prove this, it is enough to prove it for any domain conformally equivalent to $\mathbb{C}\backslash\mathbb{D}$. Notice, however, that the only property of $\mathbb{C}\backslash\mathbb{D}$ we used in the proof of Lemma 5.12 was that $G_{A_{1,r}}^{ER}(z,\cdot)$ is bounded away from z and integrable in a neighborhood of z. Once we know Proposition 5.13 holds, the proof of Proposition 5.4 works for $D = \mathbb{C}\backslash\mathbb{D}$ and we can use Proposition 5.4 and Proposition 5.2 to conclude that $G_D^{ER}(z,\cdot)$ is bounded away from z and integrable in a neighborhood of z for any region D conformally equivalent to $\mathbb{C}\backslash\mathbb{D}$. This allows us to prove an analog of Proposition 5.13 for any conformal annulus, which is what we needed.

5.3 Conformal Mapping Using $G_D^{ER}(z,\cdot)$

Similar to how we used $H_D^{ER}(\cdot,w)$ to find a domain in $\mathcal{CY}_n$ conformally equivalent to D, we can use $G_D^{ER}(z,\cdot)$ to find domains in other canonical classes of finitely connected domains conformally equivalent to D. We start by giving analogs of Lemma 4.11, Proposition 4.13, and Proposition 4.14. The proofs are similar and are omitted.

Lemma 5.14. *For each positive real number r, the set*

$$V_r = \left\{w : G_D^{ER}(z,w) \le r\right\}$$

is connected. Here z may be equal to $\{A_i\}$ for some $1 \le i \le n$. Furthermore, for each $z \in V_r$, there is a path contained in V_r starting at z and ending at a point in ∂A_0.

Proposition 5.15. *For all $w \in D$, the derivative of $G_D^{ER}(z,\cdot)$ at w has full rank. Here z may be equal to $\{A_i\}$ for some $1 \le i \le n$.*

Proposition 5.16. *If r is a positive real number, then*

$$\left\{w : G_D^{ER}(z,w) = r\right\}$$

is a Jordan curve γ_r.[1] *Furthermore, γ_r separates D into two connected components*

$$\left\{w : G_D^{ER}(z,w) < r\right\} \text{ and } \left\{w : G_D^{ER}(z,w) > r\right\}.$$

Recall that a domain is a bilateral standard domain if it is an annulus of outer radius 1 with a finite number of concentric arcs removed. It is a classical theorem of complex analysis [2] that any finitely connected domain is conformally equivalent to a bilateral standard domain. Using $G_D^{ER}(A_i,\cdot)$, we can give a probabilistic proof of this fact.

Theorem 5.17. *Let $D \in \mathcal{Y}_n$ and suppose that there is no Jordan curve in D with A_0 in its interior. If $u = \pi G_D^{ER}(A_i,\cdot)$, then there is a bilateral standard domain D' and a conformal map $f = e^{-(u+iv)}$ from D onto D'. Furthermore, if g is another conformal map from D onto a bilateral standard domain D'' and g maps ∂A_i onto the inner radius of D'' and ∂A_0 onto the outer radius of D'', then f and g differ by a rotation.*

Proof. Fix $z_0 \in D$ and let $v(z_0) = x \in \mathbb{R}$ and

$$v(z) = v(z_0) + \int_\gamma \frac{d}{dn} u(z)\,|dz|\,, \tag{5.12}$$

where γ is a smooth curve connecting z_0 and z and n is a normal vector. It follows from the Cauchy-Riemann equations that $u+iv$ is locally holomorphic. Using (5.9), we see that v is well-defined up to an integer multiple of 2π. It follows that $f = e^{-(u+iv)}$ is a well-defined holomorphic function on D.

1. If $r = G_D^{ER}(z,A_i)$ for some i, then in order for this to make sense, we have to work in the space E, not D.

By an extension of the maximal principle for ERBM, u attains its maximum on ∂A_i. Using this, we see that the image of f is contained in the annulus of inner radius $e^{-u(A_i)}$ and outer radius 1. By Proposition 5.16, $\{z \in D : u(z) = r\}$ is a Jordan curve γ_r. By (5.9),

$$\int_{\gamma_r} \frac{d}{dn} u(z) |dz| = -2\pi.$$

It follows that if $r \neq u(A_j)$ for any j, then f maps γ_r injectively onto the circle of radius e^{-r} and if $r = u(A_j)$ for some (or several) j, then f maps γ_r injectively onto the circle of radius e^{-r} with one (or several) arc(s) removed. Putting all of this together, we see that f is a conformal map onto a bilateral standard domain.

Suppose $g = e^{-(u+iv)}$ is another conformal map from D onto a bilateral standard domain D'' and g maps ∂A_i onto the inner radius of D'' and ∂A_0 onto the outer radius of D''. To prove the uniqueness statement of the theorem, it is enough to show that $u = \pi G_D^{ER}(A_i, \cdot)$. Observe that $-\log(g)$ is a locally holomorphic, multi-valued function well-defined up to an integer multiple of 2π. As a result, u is a well-defined harmonic function. Let η_j for $j \neq i$ be a Jordan curve surrounding A_j whose interior contains no point of A_k for $j \neq k$. On the interior of η_j, $u + iv$ is a well-defined holomorphic map and as a result,

$$\int_{\eta'_j} \frac{d}{dn} u(z) |dz| = 0$$

for any Jordan curve η'_j surrounding A_j and in the interior of η_j. We conclude by Lemma 3.11 that u is ER-harmonic on $D \backslash A_i$ and since it is equal to zero on ∂A_0, it must be a multiple of $G_D^{ER}(A_i, \cdot)$. Using (5.9), it is easy to see that the only multiple that will work is π. $\square$

Recall that a domain is a standard domain if it is the unit disk with a finite number of concentric arcs removed. Using $G_D^{ER}(z, \cdot)$ instead of $G_D^{ER}(A_i, \cdot)$, we can prove another classical conformal mapping theorem.

Theorem 5.18. *Let $D \in \mathcal{Y}_n$ and suppose that there is no Jordan curve in D with A_0 in its interior. If $u = \pi G_D^{ER}(z, \cdot)$, then there is a standard domain D' and a conformal map $f = e^{-(u+iv)}$ from D onto D'. Furthermore, if g is another conformal map from D onto a bilateral standard domain that sends z to 0, then f and g differ by a rotation.*

Proof. The proof is similar to that of Theorem 5.17 and is omitted. □

CHAPTER 6

A LOEWNER EQUATION FOR CHORDAL STANDARD DOMAINS

6.1 The Complex Poisson Kernel for ERBM

We want to prove the analog of Proposition 2.6 for finitely connected domains. We start with some preliminaries.

Definition 6.1. Let $D \in \mathcal{Y}^*$ and for each $x \in \mathbb{R} \cap \partial D$ define

$$H_D^{ER}(\infty, x) = \lim_{y \to \infty} y H_D^{ER}(x + iy, x).$$

When we write $H_D^{ER}(\infty, x)$, it is assumed that $x \in \partial D \cap \mathbb{R}$ even if it is not explicitly stated. $H_D^{ER}(\infty, x)$ can be interpreted as the normal derivative of $H_D^{ER}(\cdot, x)$ at ∞.

Let R be such that $z \in D$ for all $z \in \mathbb{H}$ with $|z| > R$. Using the strong Markov property for ERBM and (2.4), we have that if $|z| > 2R$, then

$$\begin{aligned} H_D^{ER}(z, x) &= R \int_0^\pi H_{HR}\left(z, Re^{i\theta}\right) H_D^{ER}\left(Re^{i\theta}, x\right) d\theta \\ &= 2RH_{\mathbb{H}}(z, 0) \left[\int_0^\pi H_D^{ER}\left(Re^{i\theta}, x\right) \sin\theta \, d\theta\right] \left[1 + O\left(\left|z^{-1}\right|\right)\right]. \end{aligned} \tag{6.1}$$

Combining (6.1) with (2.2), we get the following lemma.

Lemma 6.2. *Let $D \in \mathcal{Y}^*$ and R be such that $z \in D$ for all $z \in \mathbb{H}$ with $|z| > R$. Then*

$$H_D^{ER}(\infty, 0) = \frac{2R}{\pi} \int_0^\pi H_D^{ER}\left(Re^{i\theta}, 0\right) \sin\theta \, d\theta.$$

An analog of Proposition 4.1 holds for $H_D^{ER}(\infty, x)$.

Proposition 6.3. *Let $D \in \mathcal{Y}^*$ and suppose that f is a conformal map such that $f(D) \in \mathcal{Y}^*$ and $f(A_i)$ is bounded for $1 \leq i \leq n$. If*

$$f(z) = a_0 + a_1 z + O\left(|z|^{-1}\right), \quad z \to \infty,$$

then

$$H_D^{ER}(\infty, x) = \frac{|f'(x)|}{a} H_{f(D)}^{ER}(\infty, f(x)).$$

Proof. Using Proposition 4.1, we have

$$\begin{aligned}
H_D^{ER}(\infty, x) &= \lim_{y \to \infty} y H_D^{ER}(iy, x) \\
&= \lim_{y \to \infty} y H_{f(D)}^{ER}(f(iy), f(x)) \left|f'(x)\right| \\
&= \lim_{y \to \infty} ay H_{f(D)}^{ER}(iay + O(1), f(x)) \frac{\left|f'(x)\right|}{a} \\
&= \frac{\left|f'(x)\right|}{a} H_{f(D)}^{ER}(\infty, f(x)).
\end{aligned}$$

The last equality follows from (6.1) combined with (2.2). □

The function introduced in the next proposition is a key component in the proof of the analog of Proposition 2.6 for finitely connected domains.

Proposition 6.4. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C} \backslash \mathbb{H}$. Then there is a unique function $\mathcal{H}_D^{ER} : D \times \mathbb{R} \to \mathbb{C}$ satisfying the following.*

1. *For each $x \in \mathbb{R}$, $z \mapsto \mathcal{H}_D^{ER}(z, x)$ is a conformal map onto a chordal standard domain.*

2. *The imaginary part of $\mathcal{H}_D^{ER}(z, x)$ is $\pi H_D^{ER}(z, x)$.*

3. For each $x \in \mathbb{R}$,

$$\mathcal{H}_D^{ER}(z,x) = \frac{-\pi H_D^{ER}(\infty, x)}{z-x} + O\left(|z|^{-2}\right), \quad z \to \infty.$$

4. For each $x \in \mathbb{R}$, there is a constant $r(D,x) > 0$ such that

$$\mathcal{H}_D^{ER}(z,x) = \frac{-1}{z-x} + r(D,x) + O(|z-x|), \quad z \to x.$$

Proof. Theorem 4.15 implies that for each $x \in \mathbb{R}$ there is a conformal map $\mathcal{H}_D^{ER}(\cdot, x)$ with imaginary part $\pi H_D^{ER}(z,x)$ from D onto a chordal standard domain. Since a conformal map is uniquely determined up to a real translation by its imaginary part, the uniqueness of $\mathcal{H}_D^{ER}(\cdot, x)$ will follow once we prove its asymptotic at ∞. For the remainder of the proof, we will assume $x = 0$. The $x \neq 0$ case can be handled by considering $\mathcal{H}_{D-x}^{ER}(z-x, 0)$.

Let $R > 0$ be such that $z \in D$ for all $z \in \mathbb{H}$ with $|z| > R$. For any $z \in D$ with $|z| > 2R$, using (2.4) and Lemma 6.2, we see

$$\begin{aligned} H_D^{ER}(z,0) &= \frac{\operatorname{Im}[z]}{|z|^2}\left[\frac{2R}{\pi}\int_0^{\pi} H_D^{ER}\left(Re^{i\theta},0\right)\sin\theta \, d\theta\right]\left[1 + O\left(\frac{R}{|z|}\right)\right] \\ &= \frac{H_D^{ER}(\infty,0)\operatorname{Im}[z]}{|z|^2}\left[1 + O\left(\frac{R}{|z|}\right)\right]. \end{aligned}$$

As a result, if we let

$$f(z) = \mathcal{H}_D^{ER}(z,0) + \frac{H_D^{ER}(\infty,0)}{z},$$

then $\operatorname{Im}[f(z)] = O\left(|z|^{-2}\right)$ as $z \to \infty$. Combining this with Lemma 2.16, we see that

$f'(z) = O\left(|z|^{-3}\right)$ as $z \to \infty$ and since $f(\infty) = 0$, for $|x+iy| > 2R$ we have

$$\begin{aligned} |f(x+iy)| &= \left| \int_y^\infty f'\left(x+iy'\right)\, dy' \right| \\ &\leq \int_y^\infty \left| f'\left(x+iy'\right) \right|\, dy' \\ &= O\left(|x+iy|^{-2}\right). \end{aligned}$$

The third statement of the proposition follows.

Let $f(z) = \mathcal{H}_D^{ER}(z,0) + \frac{1}{z}$ and observe that to prove the fourth statement it is enough to show that $f(z) = f(0) + O(|z|)$ as $z \to 0$. This will follow by the Schwarz reflection principle if we can show that $\left|f'(z)\right|$ (and hence $f(z)$) is bounded in a neighborhood of 0. The Cauchy-Riemann equations imply that to show this, it is enough to show that the partial derivatives of $|\mathrm{Im}[f(z)]|$ are bounded in a neighborhood of 0. This will follow from Proposition 2.16 if we can show that $\mathrm{Im}[f(z)] = O(\mathrm{Im}[z])$ as $z \to 0$.

Observe that (4.2) and (2.16) imply that

$$\begin{aligned} \pi H_D^{ER}(z,0) &= \pi H_D(z,0) + \pi \sum_{i=1}^n h_i(z)\, H_D^{ER}(A_i,0) \\ &= \pi H_{\mathbb{H}}(z,0) - \mathrm{Im}[z]\, \Gamma(D;0)\left[1 + O(|z|)\right] + \pi \sum_{i=1}^n h_i(z)\, H_D^{ER}(A_i,0), \end{aligned}$$

as $z \to 0$. It follows that

$$\mathrm{Im}[f] = -\,\mathrm{Im}[z]\, \Gamma(D;0)\left[1 + O(|z|)\right] + \pi \sum_{i=1}^n h_i(z)\, H_D^{ER}(A_i,0), \tag{6.2}$$

as $z \to 0$. Since $h_i(z)$ is zero on $\mathbb{R}$, we can extend $h_i(z)$ to a function that is harmonic in a neighborhood of 0. As a result, letting $z = x + iy$, we can write

$$h_i(x+iy) = \frac{\partial h_i}{\partial y}(x)\, y + O\left(y^2\right),$$

as $y \to 0$. Substituting this into (6.2) and using the fact that $\frac{\partial h_i}{\partial y}(x)$ is bounded in a neighborhood of 0, the result follows.

□

We call the map $\mathcal{H}_D^{ER}$ the *complex Poisson kernel* for ERBM.

Proposition 6.5. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$ and denote the image of D under the map $z \mapsto \frac{-1}{z-x}$ by D_x^*. Then there is a unique conformal map φ_D satisfying*

$$\lim_{z\to\infty} \varphi_D(z) - z = 0$$

that maps D onto a chordal standard domain. Furthermore, for each $x \in \mathbb{R}$, we have

$$\varphi_D(z) = \mathcal{H}_{D_x^*}\left(\frac{-1}{z-x}, 0\right) + x - r\left(D_x^*, 0\right). \tag{6.3}$$

Proof. Using Proposition 6.4, it is straightforward to verify that the the map in (6.3) has the required properties. The uniqueness of φ_D is easy to check using Theorem 4.15. □

A quantity that will be of particular interest to us is $\varphi_D'(x)$ for $x \in \mathbb{R}$. In what follows, we continue to use the setup of Proposition 6.5.

Lemma 6.6. *φ_D can be extended to a map that is conformal in a neighborhood of any $x \in \mathbb{R}$. Furthermore, we have*

$$\varphi_D'(x) = \pi H_{D_x^*}^{ER}(\infty, 0). \tag{6.4}$$

Proof. The first statement follows from the Schwarz reflection principle. The formula for $\varphi'_D(x)$ can be computed from (6.3) using Proposition 6.4. □

An important fact is that $\varphi'_D(x) = \pi H_D^{ER}(\infty, x)$. This will follow from (6.4) if we can show that $H_D^{ER}(\infty, x) = H_{D_x^*}^{ER}(\infty, 0)$.

Lemma 6.7. *For any $x \in \mathbb{R}$, $H_D^{ER}(\infty, x) = H_{D_x^*}^{ER}(\infty, 0)$.*

Proof. Using Proposition 5.11, Corollary 5.9, and the conformal invariance of G_D^{ER}, we have

$$\begin{aligned}
H_{D_x^*}^{ER}(\infty, 0) &= \lim_{y\to\infty} y H_{D_x^*}^{ER}(iy, 0) \\
&= \lim_{y\to\infty}\lim_{\epsilon\to 0} \frac{y G_{D_x^*}^{ER}(iy, i\epsilon)}{2\epsilon} \\
&= \lim_{y\to\infty}\lim_{\epsilon\to 0} \frac{y G_D^{ER}\left(\frac{i}{y}+x, \frac{i}{\epsilon}+x\right)}{2\epsilon} \\
&= \lim_{y\to\infty}\lim_{\epsilon\to 0} \frac{y G_D^{ER}\left(\frac{i}{\epsilon}+x, \frac{i}{y}+x\right)}{2\epsilon} \\
&= \lim_{\epsilon\to 0}\lim_{y\to\infty} \frac{y G_D^{ER}\left(\frac{i}{\epsilon}+x, \frac{i}{y}+x\right)}{2\epsilon} \\
&= H_D^{ER}(\infty, x).
\end{aligned}$$

The interchange of limits in the second to last equality is justified by Proposition 5.2 and the fact that the interchange is valid when G_D^{ER} is replaced by G_D. □

Proposition 6.8. *For any $x \in \mathbb{R}$, $\varphi'_D(x) = \pi H_D^{ER}(\infty, x)$.*

Corollary 6.9. *If D is a chordal standard domain, then for any $x \in \mathbb{R}$, $\pi H_D^{ER}(\infty, x) = 1$.*

Proof. Since $\varphi_D(z) = z$ when D is a chordal standard domain, the result follows from Proposition 6.8. □

Using $\varphi_D(z)$, we can prove an analog of Proposition 2.6 for finitely connected domains.

Proposition 6.10. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$ and let $A \in \mathcal{Q}$ be such that $A \cap A_i = \emptyset$ for $1 \le i \le n$. Then there is a unique conformal map h_A^D satisfying*

$$\lim_{z\to\infty} h_A^D(z) - z = 0$$

that maps D onto a chordal standard domain.

Proof. Proposition 2.6 implies there exists a unique conformal map $g_A : \mathbb{H}\backslash A \to \mathbb{H}$ satisfying

$$\lim_{z\to\infty} g_A(z) - z = 0.$$

The map $h_A^D(z) := \varphi_{g_A(D\backslash A)} \circ g_A(z)$ satisfies the conditions of the proposition. The uniqueness of $h_A^D(z)$ is easy to check using Theorem 4.15. □

$h_A^D(z)$ has an expansion at infinity

$$h_A^D(z) = z + \frac{a_1}{z} + O\left(|z|^{-2}\right), \quad z \to \infty.$$

We call the constant a_1 the *excursion reflected half-plane capacity (from infinity)* for A in D and denote it $\mathrm{hcap}^{ER}(A)$. Since $\mathrm{hcap}^{ER}(A)$ depends not only on A, but also on the domain D, our notation is misleading, but it will usually be clear from context what domain we mean when we write $\mathrm{hcap}^{ER}(A)$. We continue to use the setup of Proposition 6.10.

Proposition 6.11. *Let τ be the smallest t such that $B_D^{ER}(t) \in \partial\mathbb{H} \cup A$. Then for all $z \in D\backslash A$, we have*

$$\mathrm{Im}\left[z - h_A^D(z)\right] = \mathbf{E}^z\left[\mathrm{Im}\left[B_D^{ER}(\tau)\right]\right].$$

Also, $\mathrm{hcap}^{ER}(A)$ *is equal to each of the following.*

1. $\lim_{y\to\infty} y\mathbf{E}^{iy}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]$

2. $\dfrac{2R}{\pi}\int_0^{\pi}\mathbf{E}^{Re^{i\theta}}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]\sin\theta\ d\theta$ *for any R such that $z\in D$ for all $z\in\mathbb{H}$ with $|z|>R$.*

Proof. Since $\operatorname{Im}\left[z-h_A^D(z)\right]$ is a bounded ER-harmonic function and the imaginary part of $h_A(z)$ is equal to 0 on $\partial\mathbb{H}\cup A$, the first statement follows from Proposition 3.13.

Using the first part of the proposition, we have

$$\begin{aligned}\lim_{y\to\infty} y\mathbf{E}^{iy}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right] &= \lim_{y\to\infty} y\operatorname{Im}\left[iy-h_A^D(iy)\right]\\ &= \lim_{y\to\infty} y\operatorname{Im}\left[iy-\left[iy+\frac{\operatorname{hcap}^{ER}(A)}{iy}+O\left(y^{-2}\right)\right]\right]\\ &= \operatorname{hcap}^{ER}(A).\end{aligned}$$

This proves the first equality for $\operatorname{hcap}^{ER}(A)$.

Using the first equality for $\operatorname{hcap}^{ER}(A)$ and (2.4), we have

$$\begin{aligned}\operatorname{hcap}^{ER}(A) &= \lim_{y\to\infty} y\mathbf{E}^{iy}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]\\ &= \lim_{y\to\infty} Ry\int_0^{\pi}\mathbf{E}^{Re^{i\theta}}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]H_{HR}\left(iy,Re^{i\theta}\right)\ d\theta\\ &= \lim_{y\to\infty} Ry\int_0^{\pi}\mathbf{E}^{Re^{i\theta}}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]\left[\frac{2}{\pi y}\sin\theta\left[1+O\left(y^{-1}\right)\right]\right]\ d\theta\\ &= \frac{2R}{\pi}\int_0^{\pi}\mathbf{E}^{Re^{i\theta}}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]\sin\theta\ d\theta.\end{aligned}$$

This proves the second equality for $\operatorname{hcap}^{ER}(A)$. □

Lemma 6.12. *Let $r=\operatorname{rad}(A)$ and τ be as in Proposition 6.11. Then*

$$\operatorname{hcap}^{ER}(A)=2rH_D^{ER}(\infty,0)\left[\int_0^{\pi}\mathbf{E}^{re^{i\theta}}\left[\operatorname{Im}\left[B_D^{ER}(\tau)\right]\right]\sin\theta\ d\theta\right]\left[1+O(r)\right],$$

as $r \to 0$.

Proof. Let R be such that $z \in D$ for all $z \in \mathbb{H}$ with $|z| > R$. Using Proposition 4.5, Proposition 6.11, and Lemma 6.2, we have

$$\begin{aligned}
\operatorname{hcap}^{ER}(A) =& \frac{2R}{\pi} \int_0^{\pi} \mathbf{E}^{Re^{i\theta_1}} \left[\operatorname{Im}\left[B_D^{ER}(\tau) \right] \right] \sin\theta_1 \, d\theta_1 \\
=& \frac{2Rr}{\pi} \int_0^{\pi} \int_0^{\pi} \mathbf{E}^{re^{i\theta_2}} \left[\operatorname{Im}\left[B_D^{ER}(\tau) \right] \right] H_{Dr}^{ER}\left(Re^{i\theta_1}, re^{i\theta_2} \right) \sin\theta_1 \, d\theta_2 \, d\theta_1 \\
=& \frac{2R}{\pi} \left[\int_0^{\pi} H_D^{ER}\left(Re^{i\theta_1}, 0 \right) \sin\theta_1 \, d\theta_1 \right] \\
& 2r \left[\int_0^{\pi} \mathbf{E}^{re^{i\theta_2}} \left[\operatorname{Im}\left[B_D^{ER}(\tau) \right] \right] \sin\theta_2 \, d\theta_2 \right] [1 + O(r)] \\
=& 2r H_D^{ER}(\infty, 0) \left[\int_0^{\pi} \mathbf{E}^{re^{i\theta_2}} \left[\operatorname{Im}\left[B_D^{ER}(\tau_2) \right] \right] \sin\theta_2 \, d\theta_2 \right] [1 + O(r)]
\end{aligned}$$

□

The next result gives a uniform bound on the difference between $h_A(z)$ and $z - \frac{\operatorname{hcap}^{ER}(A)\mathcal{H}_D^{ER}(z,0)}{\pi H_D^{ER}(\infty,0)}$ in terms of $\operatorname{hcap}(A)$ and $\operatorname{rad} A$. This can be interpreted as a proof of a Loewner equation for chordal standard domains at $t = 0$.

Proposition 6.13. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$. There is a constant $c < \infty$ that depends only on D and z such that for all $A \in \mathcal{Q}$ and $z \in D$ with $|z| \geq 2\operatorname{rad}(A)$*

$$\left| z - h_A^D(z) - \frac{\operatorname{hcap}^{ER}(A)\,\mathcal{H}_D^{ER}(z,0)}{\pi H_D^{ER}(\infty,0)} \right| \leq c \operatorname{hcap}^{ER}(A) \operatorname{rad}(A).$$

Proof. Let $r = \operatorname{rad}(A)$,

$$h(z) = z - h_A^D(z) - \frac{\operatorname{hcap}^{ER}(A)\,\mathcal{H}_D^{ER}(z,0)}{\pi H_D^{ER}(\infty,0)},$$

and

$$v(z) = \operatorname{Im}[h(z)] = \operatorname{Im}\left[z - h_A^D(z) \right] - \frac{\operatorname{hcap}^{ER}(A)\, H_D^{ER}(z,0)}{H_D^{ER}(\infty,0)}.$$

If $|z| > 2\,\mathrm{rad}\,(A)$, using Proposition 4.5, Proposition 6.11, and Lemma 6.12, we have

$$\begin{aligned}
\mathrm{Im}\left[z - h_A^D(z)\right] &= \mathbf{E}^z\left[\mathrm{Im}\left[B_D^{ER}(\tau)\right]\right] \\
&= r\int_0^{\pi} \mathbf{E}^{re^{i\theta}}\left[\mathrm{Im}\left[B_D^{ER}(\tau)\right]\right] H_{D^r}^{ER}\left(z, re^{i\theta}\right)\, d\theta \\
&= 2rH_D^{ER}(z,0)\left[\int_0^{\pi} \mathbf{E}^{re^{i\theta}}\left[\mathrm{Im}\left[B_D^{ER}(\tau)\right]\right]\sin\theta\, d\theta\right][1+O(r)] \\
&= \left[\frac{H_D^{ER}(z,0)\,\mathrm{hcap}^{ER}(A)}{H_D^{ER}(\infty,0)}\right][1+O(r)],
\end{aligned}$$

as $r \to 0$. It follows that there exists a $c > 0$ such that

$$|v(z)| \leq cH_D^{ER}(z,0)\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}\,(A)\,. \tag{6.5}$$

Let $R > r$ be such that $z \in D$ for all $z \in \mathbb{H}$ satisfying $|z| > R$, $\tilde{\gamma}$ be a curve from z to $i2R$ (that avoids $B_{2r}^{+}(0)$), and $M_{\tilde{\gamma}}$ be the maximum value of $H_D^{ER}(\cdot, 0)$ restricted to $\tilde{\gamma}$. Using Lemma 2.16 and (6.5), we see that there is a $c > 0$ such that the partial derivatives of v restricted to $\tilde{\gamma}$ are bounded in absolute value by

$$c\frac{M_{\tilde{\gamma}}\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}\,(A)}{d}, \tag{6.6}$$

where d is the distance from $\tilde{\gamma}$ to ∂D. It follows that

$$|h(z) - h(i2R)| < c\frac{lM_{\tilde{\gamma}}\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}\,(A)}{d}, \tag{6.7}$$

where l is the length of $\tilde{\gamma}$.

Using (2.2), it is easy to check that $H_D^{ER}(iy,0) = O\left(y^{-1}\right)$ as $iy \to \infty$. Using this fact along with an argument similar to the one used to obtain (6.6), we see that there is a $c > 0$

such that if $y \geq 2R$, then

$$\left|h'(iy)\right| \leq \frac{cH_D^{ER}(iy,0)\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}(A)}{y}. \tag{6.8}$$

Combining this with the fact that $h(iy) \to 0$ as $y \to \infty$, we have

$$\begin{aligned}
|h(i2R)| &= \left|\int_{2R}^{\infty} h'\left(iy'\right)\ dy'\right| \\
&\leq \int_{2R}^{\infty} \left|h'\left(iy'\right)\right|\ dy' \\
&\leq c\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}(A)\int_{2R}^{\infty}\frac{H_D^{ER}\left(iy',0\right)}{y'}\ dy' \\
&\leq \frac{c\,\mathrm{hcap}^{ER}(A)\,\mathrm{rad}(A)}{2R}.
\end{aligned}$$

Combining this with (6.7), the result follows. □

With a little more work, it is possible to show that the constant c can be chosen so as to depend only on D and the distance from z to ∂D. We can also get an improved bound for $z \in D$ with $|z| > 2R$.

6.2 The Chordal Loewner Equation in Chordal Standard Domains

In what follows, let D be a chordal standard domain and $\gamma : [0,\infty) \to D$ be a simple curve with $\gamma(0) \in \mathbb{R}$. Denote $\gamma[0,t]$ by γ_t and, for each $t \geq 0$, let $D_t := D\backslash\gamma_t$, $g_t : \mathbb{H}\backslash\gamma_t \to \mathbb{H}$ be the unique conformal transformation satisfying $\lim_{z\to\infty} g_t(z) - z = 0$, h_t be the unique conformal transformation satisfying $\lim_{z\to\infty} h_t(z) - z = 0$ that maps D_t onto a chordal standard domain, and φ_t be the unique map on $g_t(D_t)$ such that $h_t = \varphi_t \circ g_t$. For each $s > 0$, let $\gamma^s(t) = h_s(\gamma(s+t))$ and $h_{s,t} = h_{\gamma^s_{t-s}}^{h_s(D_s)}$. Observe that $h_t = h_{s,t} \circ h_s$.

Let $b(t) = \text{hcap}^{ER}(\gamma_t)$ and $a(t) = \text{hcap}(\gamma_t)$. Recall that h_t has an expansion

$$h_t(z) = z + \frac{b(t)}{z} + O\left(|z|^{-2}\right), \quad z \to \infty.$$

Reparametrizing if necessary, we may assume that $b(t)$ is C^1. A priori, we do not know that $\dot{a}(t)$ exists, but later we will show that, in fact, $\dot{a}(t)$ exists if and only if $\dot{b}(t)$ exists and give a formula relating the two quantities.

Let

$$\tilde{U}_t = \lim_{s \to t^-} h_s(\gamma(t)).$$

Assuming for the moment that $\tilde{U}_t$ is well-defined, we can state our main theorem.

Theorem 6.14. *For any $z \in D_t$, $h_t(z)$ satisfies the initial value problem*

$$\dot{h}_t(z) = -\dot{b}(t)\,\mathcal{H}^{ER}_{h_t(D_t)}\left(h_t(z), \tilde{U}_t\right), \quad h_0(z) = z.$$

The first main step in the proof of Theorem 6.14 is Proposition 6.13, which essentially establishes the theorem for $t = 0$. The second main step is proving that $\tilde{U}_t$ is a continuous function. We know that

$$U_t = \lim_{s \to t^-} g_s(\gamma(t)) \tag{6.9}$$

is a well-defined continuous function. To prove $\tilde{U}_t$ is continuous, the basic idea is to use the continuity of U_t along with estimates for $\varphi_t'(x)$. Proposition 6.8 implies that finding estimates for $\varphi_t'(x)$ is equivalent to finding estimates for $H^{ER}_{g_s(D_s)}(\infty, x)$. The following lemmas provide the necessary estimates.

Lemma 6.15. *Let $D \in \mathcal{Y}^*$ be such that there exist constants $0 < r < R$ such that $A_i \subset A^+_{r,R}$ for $1 \leq i \leq n$ and $x \in \partial A_0$ for all $x \in \mathbb{R}$ with $|x| > R$. Then there is a constant $C_{r,R} < \infty$ such that $H^{ER}_D(\infty, 0) < C_{r,R}$.*

Proof. Lemma 6.2 implies

$$H_D^{ER}(\infty, 0) = \frac{2R}{\pi} \int_0^{\pi} H_D^{ER}\left(Re^{i\theta}, 0\right) \sin\theta \, d\theta.$$

Since by Lemma 4.7 there is a uniform bound depending only on r for $H_D^{ER}\left(Re^{i\theta}, 0\right)$, the result follows. □

Lemma 6.16. *Let $D \in \mathcal{Y}$ be such that $A_0 = \mathbb{C}\backslash\mathbb{H}$ and such that $A_i \subset B_R^+(0)$ for $1 \leq i \leq n$. Then there is a $C_R < \infty$ such that for all $x \in \mathbb{R}$ with $|x| > 2R$ we have*

$$\left| H_D^{ER}(\infty, x) - \frac{1}{\pi} \right| < C_R x^{-2}.$$

Proof. Since

$$H_{\mathbb{H}}(\infty, x) := \lim_{y \to \infty} y H_{\mathbb{H}}(x + iy, x) = 1/\pi,$$

to complete the proof, it is enough to find a constant C_R such that

$$\left| H_{\mathbb{H}}(\infty, x) - H_D^{ER}(\infty, x) \right| < C_R x^{-2}.$$

Let τ be the first time a Brownian motion in $\mathbb{H}$ hits ∂H^R. Since B_D^{ER} has the distribution of a Brownian motion up until the first time it hits ∂D, we have

$$\begin{aligned} \left| H_{\mathbb{H}}(\infty, x) - H_D^{ER}(\infty, x) \right| \leq & \lim_{y \to \infty} y \mathbf{P}^{x+iy} \{|B_\tau| = R\} \\ & \sup_{\theta \in (0,\pi)} \left\{ \left| H_D^{ER}\left(Re^{i\theta}, x\right) + H_{\mathbb{H}}\left(Re^{i\theta}, x\right) \right| \right\}. \end{aligned}$$

Using (2.4), we see that $\lim_{y \to \infty} y \mathbf{P}^{x+iy} \{|B_\tau| = R\}$ is bounded by a constant depending only on R. Using (2.2), we see that there is a $c > 0$ depending only on R such that

$$\left| H_{\mathbb{H}}\left(Re^{i\theta}, x\right) \right| < c x^{-2}. \tag{6.10}$$

Finally, using (4.2) and (2.6), we have

$$\begin{aligned}
H_D^{ER}\left(Re^{i\theta},x\right) &= H_D\left(Re^{i\theta},x\right)+\sum_{i=1}^{n}h_i\left(Re^{i\theta}\right)H_D^{ER}\left(A_i,x\right)\\
&\leq H_{\mathbb{H}}\left(Re^{i\theta},x\right)+\sup_{1\leq i\leq n}H_D^{ER}\left(A_i,x\right).
\end{aligned}$$

As a result, (6.10) and Lemma 4.6 imply that there is a $c>0$ depending only on R such that

$$\left|H_D^{ER}\left(Re^{i\theta},x\right)\right|<cx^{-2}.$$

The result follows. □

Lemma 6.17. *Let $D\in\mathcal{Y}$ be such that $A_0=\mathbb{C}\backslash\mathbb{H}$ and suppose that there are constants $r'>0$ and $0<r<R$ such that $w\in D$ for all $w\in\mathbb{H}$ with $\operatorname{Im}[w]<r'$ and $A_i\subset A_{r,R}^{+}$ for $1\leq i\leq n$. If A is a compact $\mathbb{H}$-hull contained in $B_{r/2}^{+}(0)$ and $\epsilon=\operatorname{rad}(A)$, then there is a $c>0$ depending only on r, R, and r' such that*

$$H_D^{ER}(\infty,x)-H_{D\backslash A}^{ER}(\infty,x)<c\epsilon,$$

for all $x\in\mathbb{R}$ with $|x|>\epsilon+\sqrt{\epsilon}$. Furthermore, if $|x|>2R$, then there is a $c>0$ depending only on r, R, and r' such that

$$H_D^{ER}(\infty,x)-H_{D\backslash A}^{ER}(\infty,x)<\frac{c\epsilon^2}{x^2}.$$

Proof. Lemma 4.9 implies that there is a $c>0$ depending only on r, R, and r' such that if $|x|>\epsilon+\sqrt{\epsilon}$, then

$$H_D^{ER}(\infty,x)-H_{D\backslash A}^{ER}(\infty,x)\leq cH_D^{ER}(\infty,0)\,\epsilon.$$

Since, by Lemma 6.15, $H_D^{ER}(\infty, 0)$ is bounded by a constant depending only on r and R, the first statement follows. Using the second part of Lemma 4.9, the second statement follows similarly. $\square$

Using the Koebe distortion theorem, we can extend bounds for $\left|\varphi'(x)\right|$ on $\mathbb{R}$ to bounds for $\left|\varphi'(z)\right|$ restricted to a compact $\mathbb{H}$-hull.

Lemma 6.18. *Let $D \in \mathcal{Y}_*$, $A \in \mathcal{Q}$, and $x \in A \cap \mathbb{R}$. If $\delta > 0$ is such that* $\operatorname{dist}(A, A_i) > \delta$ *for $1 \leq i \leq n$ and* $\operatorname{dist}(A, z) > \delta$ *for all $z \in (\partial A_0) \setminus \mathbb{R}$, then there is a bound for $\left|\varphi_D'(z)\right|$ restricted to A that depends only on δ,* $\operatorname{rad}(A)$, *and $\varphi_D'(x)$.*

Proof. It is easy to see that we can find open balls $B_1, B_2, \ldots B_m$ satisfying the following.

1. B_i is a ball of radius $\delta/4$ centered at c_i and $A \subset \bigcup_{i=1}^{m} B_i$
2. For all $1 \leq i \leq m$, $B_\delta(c_i)$ does not intersect A or A_j for any $1 \leq j \leq n$
3. There is an upper bound for m depending only on δ and $\operatorname{rad}(A)$.

Using the Koebe distortion theorem, we can find a bound depending only on m and $\left|\varphi_D'(x)\right|$ for $\left|\varphi_s'(z)\right|$ restricted to $\bigcup_{i=1}^{m} B_i$. Since $A \subset \bigcup_{i=1}^{m} B_i$, the result follows. $\square$

In what follows, we once again let D and γ be as in Theorem 6.14 and fix $t_0 > 0$. In order to be able to apply the previous lemmas, we need to find estimates for the distance between $g_s(A_i)$ and $g_s(\gamma)$ that are uniform over all $0 \leq s \leq t_0$ and $1 \leq i \leq n$.

Lemma 6.19. *There exist positive constants $r_1 < R_1$, r_2, and d that depend only on D, γ, and t_0 such that for each $0 \leq s \leq t_0$ and $1 \leq i \leq n$ the following hold.*

1. $g_s(A_i) \subset A^+_{r_1,R_1}(U_s)$
2. $g_s(A_i) \subset \{z \in \mathbb{H} : \operatorname{Im}[z] > r_2\}$
3. *The distance between $g_s(\gamma)$ and $g_s(A_i)$ is greater than d*

4. If $i \neq j$, then the distance between $g_s(A_i)$ and $g_s\left(A_j\right)$ is greater than d.

Finally, there is a uniform bound on $\operatorname{diam}\left[g_s(\gamma(s,t))\right]$ *over all* $0 \leq s < t \leq t_0$.

Proof. Let $R_i(s) = \sup_{z \in A_i} |g_s(z) - U_s|$ and $r_i(s) = \inf_{z \in A_i} |g_s(z) - U_s|$. Lemma 2.11 and Proposition 2.10 imply that $R_i(s)$ and $r_i(s)$ are continuous functions of s. This proves the first statement. The proofs of the remaining statements are similar. □

We also need a lemma similar to Lemma 6.19 for h_s.

Lemma 6.20. *There exist constants $0 < r < R$ and r' depending only on D, γ, and t_0 such that for each $0 \leq s \leq t_0$ and $1 \leq i \leq n$ we have*

1. $h_s(A_i) \subset A^+_{r,R}(\varphi_s(U_s))$
2. $h_s(A_i) \subset \left\{z \in \mathbb{H} : \operatorname{Im}[z] > r'\right\}$.

Proof. Let r_1, R_1, and r_2 be as in Lemma 6.19. Using Lemma 6.15, we can find an upper bound M for $\left|\varphi_s'(U_s)\right|$ that depends only on r_1 and R_1. Using Lemma 6.2 and the (easy) fact that there is a positive lower bound for $H^{ER}_{g_s(D_s)}\left(2R_1 e^{i\theta}, 0\right)$ that depends only on θ and r_2, we see that there is a lower bound $m > 0$ for $\left|\varphi_s'(U_s)\right|$ that depends only on R_1 and r_2.

The Koebe 1/4 theorem implies that there is a constant $r > 0$ that depends only on r_1 and m such that $B_r^+(\varphi_s(U_s)) \subset \varphi_s(g_s(D_s))$. Since $h_s = \varphi_s \circ g_s$, it follows that $B_r^+(\varphi_s(U_s)) \subset h_s(D_s)$.

The Koebe distortion theorem implies that there is an upper bound that depends only on M, r_2, and R_1 for $\left|\varphi_s'(z)\right|$ restricted to the boundary of $B^+_{2R_1}(U_s)$. As a result, there is a constant $R > 0$ that depends only on M, r_2, and R_1 such that $\varphi_s\left(B^+_{2R_1}(U_s)\right) \subset B^+_R(\varphi_s(U_s))$. The first statement of the proposition follows.

Similarly, the Koebe distortion theorem implies that there is a lower bound greater than zero for $\left|\varphi_s'(x)\right|$ restricted to $[-2R_1, 2R_1]$ that depends only on m, r_2, and R_1. Combined with Lemma 6.16, this gives a lower bound for $\left|\varphi_s'(x)\right|$ restricted to $\mathbb{R}$. As a result, the Koebe 1/4 theorem implies that there is an $r' > 0$ depending only on m, r_2, and R_1 such that for each $x \in \mathbb{R}$, $B_{r'}^{+}(\varphi_s(x)) \subset \varphi_s(g_s(D_s))$. The second statement of the proposition follows. □

We have the tools to prove an analog of Lemma 2.11 for h_s.

Proposition 6.21. *There exists a constant $c < \infty$ that depends only on D, γ, and t_0 such that if $0 \leq s < t \leq t_0 < \infty$, then*

$$\operatorname{diam}\left[h_s(\gamma(s,t))\right] \leq c\sqrt{\operatorname{osc}(\gamma, t-s, t_0)}$$

and

$$\|h_s - h_t\|_\infty \leq c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)},$$

where

$$\operatorname{osc}(\gamma, \delta, t_0) = \sup\left\{|\gamma(s) - \gamma(t)| : 0 \leq s, t \leq t_0; |t-s| \leq \delta\right\}$$

and $h_s - h_t$ is considered as a function on D_t.

Proof. Let r_1, R_1, and d be as in Lemma 6.19. Lemma 6.15, combined with Proposition 6.8, shows that there is an upper bound for $\varphi_s'(U_s)$ that depends only on r_1 and R_1. As a result, Lemma 6.18 implies that there is a bound for $\left|\varphi_s'(z)\right|$ restricted to $g_s(\gamma_t)$ that depends only on r_1, R_1, d, and t_0. Since $h_s = \varphi_s \circ g_s$, the first statement of the proposition follows from Lemma 2.11.

Since $h_t = h_{s,t} \circ h_s$, to prove the second statement of the proposition it is enough to show that there is a $c < \infty$, depending only on D, γ, and t_0, such that

$$\left\|h_{s,t}(z) - z\right\|_\infty \leq c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}. \tag{6.11}$$

Let $f_{s,t} := g_{h_s(\gamma_t)}$, $\phi_{s,t}$ be the unique conformal map such that $h_{s,t} = \phi_{s,t} \circ f_{s,t}$, $d_{s,t} = \operatorname{diam}\left[h_s\left(\gamma\left(s,t\right)\right)\right]$, and r, R, and r' be as in Lemma 6.20. Note that by the Schwarz reflection principle, $\phi_{s,t}$ can be extended to a conformal map on

$$\mathbb{R} \cup \left\{z : z \text{ or } \overline{z} \text{ is in the image of } f_{s,t}\right\}.$$

For any $z \in \overline{\mathbb{H}} \backslash h_s\left(\gamma_t\right)$, Lemma 2.12 implies that

$$\left|f_{s,t}\left(z\right) - z\right| \leq 3d_{s,t}. \tag{6.12}$$

As a result, if $d_{s,t}$ is sufficiently small, then the image of $f_{s,t}$ restricted to $B_r^+\left(\varphi_s\left(U_s\right)\right)$ contains $B^+_{4\sqrt{d_{s,t}}}\left(\varphi_s\left(U_s\right)\right)$ and the image of $f_{s,t}$ restricted to $B_R^+\left(\varphi_s\left(U_s\right)\right)$ is contained in a half-disk centered at $\varphi_s\left(U_s\right)$ with radius depending only on $d_{s,t}$ and R. Since the first part of the proposition shows that $d_{s,t} \to 0$ as $|t-s| \to 0$, it follows that there is a $\delta > 0$ and constants $4\sqrt{d_{s,t}} < r_\delta < R < R_\delta$ such that if $|t-s| < \delta$, then $d_{s,t} < 1$ and the image of $f_{s,t}$ contains all $z \in \mathbb{H}$ in the complement of $A^+_{r_\delta, R_\delta}\left(\varphi_s\left(U_s\right)\right)$.

For the remainder of the proof, we assume that $|t-s| < \delta$ and let $c > 0$ be a (changing) constant that depends only on r, R, r', and δ. Lemma 6.15, combined with the Koebe distortion theorem, shows that there is an upper bound M_δ for $\phi'_{s,t}\left(x\right)$ restricted to

$$\left[\varphi_s\left(U_s\right) - \frac{10\sqrt{d_{s,t}}}{3}, \varphi_s\left(U_s\right) + \frac{10\sqrt{d_{s,t}}}{3}\right]$$

that depends only on r_δ and R_δ. Lemma 6.17 implies that if $x \in \mathbb{R}$ satisfies $\left|x - \varphi_s\left(U_s\right)\right| > d_{s,t} + \sqrt{d_{s,t}}$, then

$$H^{ER}_{h_s(D_s)}\left(\infty, x\right) - H^{ER}_{h_s(D_s)\backslash h_s(\gamma_t)}\left(\infty, x\right) < c d_{s,t}. \tag{6.13}$$

Lemma 2.15 implies that if $x \in \mathbb{R}$ satisfies $|x - \varphi_s(U_s)| > 3d_{s,t}$, then

$$1 - \frac{cd_{s,t}^2}{(x - \varphi_s(U_s))^2} \le f'_{s,t}(x) \le 1. \tag{6.14}$$

Since Lemma 2.12 implies that

$$f_{s,t}\left(\left[\varphi_s(U_s) - 3\sqrt{d_{s,t}}, \varphi_s(U_s) + 3\sqrt{d_{s,t}}\right]\right) \subset \left[\varphi_s(U_s) - \frac{10\sqrt{d_{s,t}}}{3}, \varphi_s(U_s) + \frac{10\sqrt{d_{s,t}}}{3}\right],$$

Corollary 6.9 and Proposition 6.3, combined with (6.13) and (6.14), show that for all $x \in \mathbb{R}$ such that $|x - \varphi_s(U_s)| > \frac{10\sqrt{d_{s,t}}}{3}$, we have

$$\left|\phi'_{s,t}(x) - 1\right| \le cd_{s,t}. \tag{6.15}$$

Using the second part of Lemma 6.17, we see that if $x \in \mathbb{R}$ satisfies $|x - \varphi_s(U_s)| > 2R$, then

$$H^{ER}_{h_s(D_s)}(\infty, x) - H^{ER}_{h_s(D_s)\setminus h_s(\gamma_t)}(\infty, x) < \frac{cd_{s,t}^2}{x^2}. \tag{6.16}$$

Since Lemma 2.12 implies that

$$f_{s,t}([\varphi_s(U_s) - 2R, \varphi_s(U_s) + 2R]) \subset [\varphi_s(U_s) - 3R, \varphi_s(U_s) + 3R],$$

using (6.13) and (6.16), we see that for all $x \in \mathbb{R}$ such that $|x - \varphi_s(U_s)| > 3R$ we have

$$\left|\phi'_{s,t}(x) - 1\right| \le \frac{cd_{s,t}^2}{x^2}. \tag{6.17}$$

Let $x \in \mathbb{R}$ and assume without loss of generality that $x > \varphi_s(U_s)$. Since

$$\lim_{y\to\infty}\left(\phi_{s,t}(y) - y\right) = 0,$$

using (6.15) and (6.17), we have

$$\begin{aligned}\left|\phi_{s,t}(x)-x\right| &= \lim_{y\to\infty}\left|\left(\phi_{s,t}(y)-y\right)-\left(\phi_{s,t}(x)-x\right)\right| \\ &= \left|\int_x^{\infty}\left(\phi'_{s,t}(y)-1\right)\,dy\right| \\ &\leq \left|4\sqrt{d_{s,t}}M_\delta+3Rcd_{s,t}+cd_{s,t}^2\int_{3R}^{\infty}\frac{dy}{y^2}\right| \\ &\leq c\sqrt{d_{s,t}}.\end{aligned}$$

Combined with the first statement of the proposition and (6.12), this proves (6.11) and hence the proposition in the special case that $|s-t|<\delta$. Since if $s<r<t$, then both $\operatorname{osc}(\gamma, r-s, t_0)$ and $\operatorname{osc}(\gamma, t-r, t_0)$ are less than $\operatorname{osc}(\gamma, t-s, t_0)$, the general case follows from the special case and the triangle inequality. □

Proposition 6.22. *For any $t>0$, there is a unique $\tilde{U}_t\in\mathbb{R}$ such that*

$$\lim_{z\to\gamma(t)} h_t(z)=\varphi_t(U_t)=\tilde{U}_t,$$

where the limit is taken over $z\in\mathbb{H}\backslash\gamma_t$. Furthermore,

$$\tilde{U}_t=\lim_{s\to t^-} h_s(\gamma(t))$$

and $t\mapsto\tilde{U}_t$ is a continuous map.

Proof. Using Proposition 6.21, the proof is similar to the analogous proof for g_t (see [12]).

□

The final ingredient in the proof of Theorem 6.14 is to show that for any $z\in D\backslash\gamma$ the map $t\mapsto\mathcal{H}^{ER}_{h_t(D_t)}\left(h_t(z),\tilde{U}_t\right)$ is continuous. We start by proving the analogous fact for $H^{ER}_{h_t(D_t)}\left(h_t(z),\tilde{U}_t\right)$.

Lemma 6.23. *Fix $z \in D$ and let t_0 be such that $z \notin \gamma_{t_0}$. Then there are constants $\delta > 0$ and $c > 0$ that depend only on γ, D, z, and t_0 such that if $0 < s < t < t_0$ and $t - s < \delta$, then*

$$\left| H^{ER}_{h_s(D_s)}\left(h_s(z), \tilde{U}_s\right) - H^{ER}_{h_t(D_t)}\left(h_t(z), \tilde{U}_t\right)\right| < c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}.$$

Proof. Throughout the proof, all constants will depend only on D, γ, z, and t_0. Let $r_{s,t} = \operatorname{osc}(\gamma, t-s, t_0)$, $d_{s,t}$ be as in the proof of Proposition 6.21, and

$$r_z = \min\left\{r, \inf\left\{\operatorname{dist}\left(h_s(z), \tilde{U}_s\right) : 0 \le s \le t_0\right\}\right\},$$

where r is as in Lemma 6.20. Using Proposition 6.21, it is easy to see that there is a $\delta_1 > 0$ such that if $0 < t - s < \delta_1$, then $r_{s,t} < 1$ and $h_s(\gamma(s,t)) \subset B^+_{r_z/2}\left(\tilde{U}_s\right)$.

Since Proposition 6.21 implies that

$$d_{s,t} + \sqrt{d_{s,t}} = O\left(\sqrt[4]{r_{s,t}}\right) \tag{6.18}$$

and $\left|h_{s,t}(x) - x\right| = O\left(\sqrt[4]{r_{s,t}}\right)$, it follows that

$$h_{s,t}\left(\tilde{U}_s + d_{s,t} + \sqrt{d_{s,t}}\right) - h_{s,t}\left(\tilde{U}_s - d_{s,t} - \sqrt{d_{s,t}}\right) = O\left(\sqrt[4]{r_{s,t}}\right).$$

As a result, there is a $\delta_2 > 0$ such that if $0 < t - s < \delta_2$, then $h_{s,t}(x) \in B^+_{r_z/2}\left(\tilde{U}_t\right)$ for all x such that $|x - \varphi_s(U_s)| \le d_{s,t} + \sqrt{d_{s,t}}$. Let $\delta = \min\{\delta_1, \delta_2\}$ and for the remainder of the proof, assume that $0 < t - s < \delta$ and $x = \tilde{U}_s + d_{s,t} + \sqrt{d_{s,t}}$.

Lemma 4.8 implies that

$$\left| H^{ER}_{h_t(D_t)}\left(h_t(z), \tilde{U}_t\right) - H^{ER}_{h_t(D_t)}\left(h_t(z), h_{s,t}(x)\right)\right| = O\left(\sqrt[4]{r_{s,t}}\right). \tag{6.19}$$

Using Proposition 4.1, we see that

$$\left|h'_{s,t}(x)\right| H^{ER}_{h_t(D_t)}\left(h_t(z), h_{s,t}(x)\right) = H^{ER}_{h_s(D_s)\setminus h_s(\gamma_t)}\left(h_s(z), x\right). \tag{6.20}$$

Using the chain rule, Proposition 6.8, and Proposition 6.3, we see that

$$h'_{s,t}(x) = \pi H^{ER}_{g_{h_s(\gamma_t)}(h_s(D_s))}\left(\infty, g_{h_s(\gamma_t)}(x)\right) g'_{h_s(\gamma_t)}(x) = \pi H^{ER}_{h_s(D_s)\setminus h_s(\gamma_t)}(\infty, x).$$

Since $\pi H^{ER}_{h_s(D_s)}(\infty, x) = 1$, Lemma 6.17 and Proposition 6.21 together imply $h'_{s,t}(x) = 1 + O\left(\sqrt{r_{s,t}}\right)$. Combining this with (6.20) and Lemma 4.7, we conclude that

$$H^{ER}_{h_s(D_s)\setminus h_s(\gamma_t)}\left(h_s(z), x\right) - H^{ER}_{h_t(D_t)}\left(h_t(z), h_{s,t}(x)\right) = O\left(\sqrt{r_{s,t}}\right). \tag{6.21}$$

Next, using Lemma 4.9, we see that

$$H^{ER}_{h_s(D_s)\setminus h_s(\gamma_t)}\left(h_s(z), x\right) - H^{ER}_{h_s(D_s)}\left(h_s(z), x\right) = O\left(\sqrt{r_{s,t}}\right). \tag{6.22}$$

Finally, using (6.18) and arguing as in (6.19), we see that

$$H^{ER}_{h_s(D_s)}\left(h_s(z), \tilde{U}_s\right) - H^{ER}_{h_s(D_s)}\left(h_s(z), x\right) = O\left(\sqrt[4]{r_{s,t}}\right). \tag{6.23}$$

Combining (6.19), (6.21), (6.22), and (6.23), the result follows. □

It is not hard to see that the proof of Lemma 6.23 can be modified to show that the constants c and δ can be chosen uniformly over all z in a compact set.

Lemma 6.24. *Fix $z \in D$ and let t_0 be such that $z \notin \gamma_{t_0}$. Then the map*

$$t \mapsto \mathcal{H}^{ER}_{h_t(D_t)}\left(h_t(z), \tilde{U}_t\right)$$

is a continuous function on $[0, t_0)$.

Proof. Assume without loss of generality that $\gamma(0) = 0$. Let

$$f_t(w) := \mathcal{H}^{ER}_{h_t(D_t)}\left(h_t(w), \tilde{U}_t\right) = u_t(w) + iv_t(w)$$

and if $s < t$, let $f_{s,t} := f_t - f_s$ and $v_{s,t} := v_t - v_s$. Let R be as in Lemma 6.20 and $\tilde{R} = \max\left\{\operatorname{diam}\left(\gamma_{t_0}\right), R\right\}$. Finally, let $\tilde{\gamma}$ be a path in D_{t_0} from z to $i2\tilde{R}$.

Lemma 6.23 and Lemma 2.16 imply that there is a $\delta_1 > 0$ and $c > 0$ such that if $|t-s| < \delta_1$, then the partial derivatives of $v_{s,t}$ restricted to $\tilde{\gamma}$ are bounded in absolute value by $\frac{c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}}{d}$, where $d = \operatorname{dist}\left(\tilde{\gamma}, \partial D_{t_0}\right)$. As a result, there is a constant $c > 0$ that depends only on γ, D, z and t_0 such that if $|t-s| < \delta_1$, then $\left|f'_{s,t}(w)\right| < c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}$ for all $w \in \tilde{\gamma}$. It follows that if $|t-s| < \delta_1$, then

$$\left|f_{s,t}(z) - f_{s,t}\left(i2\tilde{R}\right)\right| < cl\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}, \tag{6.24}$$

where l is the length of $\tilde{\gamma}$.

If $y > 2\tilde{R}$, (2.4) implies that there is a $c > 0$ such that the probability a Brownian motion in $\mathbb{H}$ started at y leaves $\mathbb{H}\backslash\tilde{R}\mathbb{D}$ on $\partial B^+_{\tilde{R}}(0)$ is less than $\frac{c\tilde{R}}{y}$. Lemma 6.23 implies that there is a $\delta_2 > 0$ and $c > 0$ such that if $|t-s| < \delta_2$, then the maximum value of $\left|v_{s,t}(z)\right|$ restricted to $\partial B^+_{\tilde{R}}(0)$ is less than $c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}$. It follows that there is a $c > 0$ such that if $|t-s| < \delta_2$, then

$$v_{s,t}(y) < \frac{c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}}{y}.$$

As a result, Lemma 2.16 implies that there is a $c > 0$ such that if $|t-s| < \delta_2$, then the partial derivatives of $v_{s,t}(iy)$ are bounded in absolute value by $\frac{c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}}{y^2}$. We conclude that there is a $c > 0$ such that if $|t-s| < \delta_2$, then

$$\left|f'_{s,t}(iy)\right| < \frac{c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}}{y^2}$$

for all $y > 2\tilde{R}$. Since $\lim_{y\to\infty} f_{s,t}(iy) = 0$, it follows that if $|t-s| < \delta_2$, then

$$\left|f_{s,t}\left(i2\tilde{R}\right)\right| \leq \int_{2\tilde{R}}^{\infty} \left|f'_{s,t}(iy)\right| \, dy \leq \frac{c\sqrt[4]{\operatorname{osc}(\gamma, t-s, t_0)}}{2\tilde{R}}. \tag{6.25}$$

Combining this with (6.24), the result follows. □

We have everything we need in order to prove Theorem 6.14.

Proof of Theorem 6.14. Let f be the conformal map such that $h_{s,s+\epsilon}(z) = f\left(z - \tilde{U}_s\right) + \tilde{U}_s$. Applying Proposition 6.13 to f, we see that for sufficiently small $\epsilon > 0$,

$$\begin{aligned} h_{s+\epsilon}(z) - h_s(z) =& h_{s,s+\epsilon}(h_s(z)) - h_s(z) \\ =& -(b(s+\epsilon) - b(s))\, \mathcal{H}^{ER}_{h_s(D_s)}\left(h_s(z), \tilde{U}_s\right) \\ &+ \operatorname{diam}\left[\gamma(s, s+\epsilon)\right]\left[b(s+\epsilon) - b(s)\right] O(1). \end{aligned}$$

Dividing this by ϵ and taking the limit as $\epsilon \to 0$, we see that $h_t(z)$ has right derivative at s equal to

$$-\dot{b}(s)\, \mathcal{H}^{ER}_{h_s(D_s)}\left(h_s(z), \tilde{U}_s\right).$$

Using Lemma 2.13 and Lemma 6.24, the result follows. □

Up until now we have assumed that the curve γ is parametrized such that $b(t)$ is C^1. While this was the most convenient parametrization to use when formulating and proving Theorem 6.14, in applications we will often start with a curve that is only assumed to be parametrized such that $a(t)$ is C^1. This will not pose a problem though because, as we now prove, the ER half-plane capacity is C^1 if and only if the usual half-plane capacity is. For the remainder of this section we will assume $D \in \mathcal{Y}^*$ and $\gamma(0) \in \partial D \cap \mathbb{R}$.

Lemma 6.25. *$\dot{b}(0)$ exists if and only if $\dot{a}(0)$ exists. If both quantities exist, then*

$$\dot{b}(0) = \pi H_D^{ER}(\infty, 0)\,\dot{a}(0).$$

In particular, if D is a chordal standard domain, then $\dot{a}(0) = \dot{b}(0)$.

Proof. Assume without loss of generality that $\gamma(0) = 0$ and $z \in D$ for all $z \in \mathbb{H}$ such that $|z| < 1$. Let $r_t = \operatorname{rad}(\gamma_t)$ and

$$\tilde{D}_t = \{z \in D_t : |z| < 1\}.$$

Define τ_1^t to be the first time a Brownian motion in $\mathbb{H}$ exits $\mathbb{H} \backslash \gamma_t$, τ_2^t to be the first time an ERBM in D exits D_t, $X_1^t = \operatorname{Im}\left[B_{\tau_1^t}\right]$, and $X_2^t = \operatorname{Im}\left[B_D^{ER}\left(\tau_2^t\right)\right]$. Finally, define

$$M_1(t) = \int_0^\pi \mathbf{E}^{r_t e^{i\theta}}\left[X_2^t\right] \sin\theta \, d\theta$$

and observe that Lemma 6.12 implies

$$b(t) = 2 r_t H_D^{ER}(\infty, 0)\, M_1(t)\left[1 + O(r_t)\right], \quad r_t \to 0. \tag{6.26}$$

As a result, $\dot{b}(t)$ exists if and only if

$$\lim_{t \to 0} \frac{r_t M_1(t)}{t} \tag{6.27}$$

exists.

Let E_t^z be the event that a Brownian motion started at $z \in \tilde{D}_t$ does not leave $\tilde{D}_t$ on

$\{z \in \mathbb{H} : |z| = 1\}$ and define

$$M_2(t) = \int_0^\pi \mathbf{E}^{r_t e^{i\theta}} \left[X_2^t ; E_t^{r_t e^{i\theta}} \right] \sin\theta \, d\theta.$$

We claim that the limit in (6.27) exists if and only if

$$\lim_{t\to 0} \frac{r_t M_2(t)}{t} \tag{6.28}$$

exists and if both limits exist, then they are equal. Observe that

$$\mathbf{E}^{r_t e^{i\theta}} \left[X_2^t \right] = \mathbf{E}^{r_t e^{i\theta}} \left[X_2^t ; E_t^{r_t e^{i\theta}} \right] + \int_0^\pi \mathbf{E}^{e^{i\theta_1}} \left[X_2^t \right] H_{\tilde{D}_t} \left(r_t e^{i\theta}, e^{i\theta_1} \right) \, d\theta_1 \tag{6.29}$$

and that, using Proposition 4.5,

$$\mathbf{E}^{e^{i\theta_1}} \left[X_2^t \right] = 2 r_t H_D^{ER} \left(e^{i\theta_1}, 0 \right) M_1(t) \left[1 + O(r_t) \right], \tag{6.30}$$

for all $r_t < 1/2$. It follows that if the limit in (6.27) exists, then

$$\lim_{t\to 0} \frac{\mathbf{E}^{e^{i\theta_1}} \left[X_2^t \right]}{t}$$

exists and is a continuous function of θ_1. In particular, there is an upper bound for

$$\frac{\mathbf{E}^{e^{i\theta_1}} \left[X_2^t \right]}{t}$$

that is uniform over all $0 \leq \theta_1 \leq \pi$ and t sufficiently small. Since the remark following (2.5) implies that

$$\int_0^\pi H_{\tilde{D}_t} \left(r_t e^{i\theta}, e^{i\theta_1} \right) \, d\theta_1$$

is comparable to $r_t \sin\theta$, it follows that

$$\lim_{t\to 0} \frac{\int_0^\pi H_{\tilde{D}_t}\left(r_t e^{i\theta}, e^{i\theta_1}\right) \mathbf{E}^{e^{i\theta_1}}\left[X_2^t\right]\, d\theta_1}{t} = 0. \tag{6.31}$$

Combining this with (6.29), it is easy to check that the limit in (6.28) exists and is equal to the limit in (6.27).

If the limit in (6.28) exists, then using (6.29), (6.30), Lemma 4.7, and the remark following (2.5), we see that there is a $c > 0$ independent of t such that

$$\begin{aligned} M_1(t) - M_2(t) &= \int_0^\pi \left[\int_0^\pi H_{\tilde{D}_t}\left(r_t e^{i\theta}, e^{i\theta_1}\right) \mathbf{E}^{e^{i\theta_1}}\left[X_2^t\right]\, d\theta_1\right] \sin\theta\, d\theta \\ &\leq c M_1(t)\, r_t, \end{aligned}$$

for all $r_t < 1/2$. Thus, for sufficiently small t, $M_1(t) \leq 2M_2(t)$ and as a result

$$\limsup_{t\to 0} \frac{r_t M_1}{t} \leq 2 \lim_{t\to 0} \frac{r_t M_2(t)}{t} < \infty.$$

Using this, we can argue as before to show that (6.31) holds, from which it is easy using (6.29) to show that the limit in (6.27) exists and is equal to the limit in (6.28).

Using (6.26) and our claim, it follows that $\dot{b}(0)$ exists if and only if the limit in (6.28) exists and in that case,

$$\dot{b}(0) = 2H_D^{ER}(\infty, 0) \lim_{t\to 0} \frac{r_t M_2(t)}{t}. \tag{6.32}$$

Using Proposition 2.7, a similar argument as the one used to prove the claim shows that

$\dot{a}(0)$ exists if and only if

$$\lim_{t\to 0}\frac{\int_0^\pi \mathbf{E}^{r_t e^{i\theta}}\left[X_1^t; E_t^{r_t e^{i\theta}}\right]\sin\theta\, d\theta}{t}$$

exists. Since Brownian motion and ERBM have the same distribution in $\tilde{D}_t$, this limit is the same as the one in (6.28). It follows that $\dot{a}(0)$ exists if and only if the limit in (6.28) exists and in that case,

$$\dot{a}(0) = \frac{2}{\pi}\lim_{t\to 0}\frac{r_t M_2(t)}{t}. \tag{6.33}$$

Combining (6.32) and (6.33), the result follows. □

Proposition 6.26. *$\dot{b}(t)$ exists if and only if $\dot{a}(t)$ exists. If both quantities exist, then*

$$\dot{b}(t) = \varphi_t'(U_t)^2\,\dot{a}(t).$$

Proof. Recall that $\gamma^s(t) = h_s(\gamma(s+t))$ and define $\alpha(t) = \mathrm{hcap}(\gamma^s(t))$ and $\beta(t) = \mathrm{hcap}^{ER}(\gamma^s(t))$. Since $\mathrm{hcap}^{ER}(\gamma^s(t)) = b(s+t) - b(t)$, $\dot{\beta}(0)$ exists if and only if $\dot{b}(t)$ exists and if they both exist, then they are equal. Lemma 6.25 implies that $\dot{\alpha}(0)$ exists if and only if $\dot{\beta}(0)$ exists and, in that case, they are equal. Finally, since $\gamma^s(t)$ is the image under φ_s of $g_s(\gamma(s+t))$, Corollary 2.18 implies that $\dot{\alpha}(0)$ exists if and only if $\dot{a}(s)$ exists and if they both exist, then $\dot{\alpha}(0) = \varphi_s'(U_s)^2\,\dot{a}(t)$. The result follows. □

REFERENCES

[1] *Temps locaux*, Astérisque, vol. 52, Société Mathématique de France, Paris, 1978, Exposés du Séminaire J. Azéma-M. Yor, Held at the Université Pierre et Marie Curie, Paris, 1976–1977, With an English summary. MR 509476 (81b:60042)

[2] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR 510197 (80c:30001)

[3] Martin Barlow, Jim Pitman, and Marc Yor, *On Walsh's Brownian motions*, Séminaire de Probabilités, XXIII, Lecture Notes in Math., vol. 1372, Springer, Berlin, 1989, pp. 275–293. MR MR1022917 (91a:60204)

[4] Robert O. Bauer and Roland M. Friedrich, *Stochastic Loewner evolution in multiply connected domains*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 8, 579–584. MR 2111355

[5] ______, *On radial stochastic Loewner evolution in multiply connected domains*, J. Funct. Anal. **237** (2006), no. 2, 565–588. MR 2230350 (2007d:60007)

[6] ______, *On chordal and bilateral SLE in multiply connected domains*, Math. Z. **258** (2008), no. 2, 241–265. MR 2357634 (2009b:60292)

[7] John B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995. MR MR1344449 (96i:30001)

[8] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998. MR MR1625845 (99e:35001)

[9] Masatoshi Fukushima and Hiroshi Tanaka, *Poisson point processes attached to symmetric diffusions*, Ann. Inst. H. Poincaré Probab. Statist. **41** (2005), no. 3, 419–459. MR 2139028 (2006d:60125)

[10] Yûsaku Komatu, *On conformal slit mapping of multiply-connected domains*, Proc. Japan Acad. **26** (1950), no. 7, 26–31. MR 0046437 (13,734c)

[11] Gregory Lawler, Oded Schramm, and Wendelin Werner, *Conformal restriction: the chordal case*, J. Amer. Math. Soc. **16** (2003), no. 4, 917–955 (electronic). MR 1992830 (2004g:60130)

[12] Gregory F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, vol. 114, American Mathematical Society, Providence, RI, 2005. MR MR2129588 (2006i:60003)

[13] ______, *The Laplacian-b random walk and the Schramm-Loewner evolution*, Illinois J. Math. **50** (2006), no. 1-4, 701–746 (electronic). MR MR2247843 (2007k:60261)

[14] ______, *Partition functions, loop measure, and versions of SLE*, J. Stat. Phys. **134** (2009), no. 5-6, 813–837. MR 2518970 (2010i:60232)

[15] ______, *Defining SLE in multiply connected domains with the Brownian loop measure*, preprint (2011).

[16] Gregory F. Lawler and Wendelin Werner, *The Brownian loop soup*, Probab. Theory Related Fields **128** (2004), no. 4, 565–588. MR 2045953 (2005f:60176)

[17] Karl Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann. **89** (1923), no. 1-2, 103–121. MR 1512136

[18] Peter Mörters and Yuval Peres, *Brownian motion*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010, With an appendix by Oded Schramm and Wendelin Werner. MR MR2604525

[19] L. C. G. Rogers and David Williams, *Diffusions, Markov processes, and martingales. Vol. 1*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000, Foundations, Reprint of the second (1994) edition. MR MR1796539 (2001g:60188)

[20] Steffen Rohde and Oded Schramm, *Basic properties of SLE*, Ann. of Math. (2) **161** (2005), no. 2, 883–924. MR 2153402 (2006f:60093)

[21] Oded Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. **118** (2000), 221–288. MR 1776084 (2001m:60227)

[22] Dapeng Zhan, *The scaling limits of planar LERW in finitely connected domains*, Ann. Probab. **36** (2008), no. 2, 467–529. MR 2393989 (2009d:60114)

[23] ______, *Reversibility of whole-plane SLE*, arXiv:1004.1865v2 (2010).