A User's Guide to ϵ-δ proofs

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Before getting into the gritty details of how an ϵ-δ proof works, we should ask ourselves why we even bother with this ϵ-δ nonsense to begin with? It may seem that we exert a whole lot of unnecessary energy "proving" things that we already knew were true. For example, we all know that to evaluate the limit $\lim_{x\to 1} 3x+2$, we can just plug 1 into $3x+2$ to see that the limit is $3\cdot 1+2=5$. Since we already know $\lim_{x\to 1} 3x+2=5$, why should we bother proving it? The first answer answer we can give to this question, is to ask how we know our method for evaluating this limit is valid? That is, to evaluate the limit $\lim_{x\to 1} 3x+2$ why is it valid to just plug 1 into $3x+2$? Any attempt to give a (mathematically) satisfactory answer to this question will inevitably lead back to ϵ-δ (if you don't believe me, try to answer this question for yourself).

The second answer to this question is that it is good practice and will help us understand limits better. Calculus is the study of limits (they will appear in the definition of both the derivative and the integral), so in order to have a good understanding of calculus it is essential that we have a good understanding of limits. If you do not understand the ϵ-δ definition of a limit and you cannot do "simple" ϵ-δ proofs, you do not truly have a good understanding of limits (it may sound harsh, but unfortunately it is true).

Now that I have (hopefully) convinced you of the necessity of ϵ-δ, let's try to motivate the ϵ-δ definition. Intuitively, what we mean by $\lim_{x\to c} f(x)=L$ is that whenever x is "near" c, $f(x)$ is "near" L. Unfortunately, the word "near" does not have much mathematical meaning and any mathematical definition that uses it will not be very useful unless we are more precise about what we mean by "near." What we really mean by x is "near" c is that the distance between x and c is small or, in mathematical language, $|x-c|<\delta$ where δ is a small number. Similarly, what we mean by $f(x)$ is "near" L is that $|f(x)-L| < \epsilon$ where ϵ is a small number.[1] The ϵ-δ definition of a limit makes use of this mathematical interpretation of "near."

Definition 1. *Let f be a function defined everywhere on an interval containing c except possibly at c.*[2] *We say that $\lim_{x\to c} f(x) = L$ if for all $\epsilon > 0$, we can find $\delta > 0$ so that if x satisfies $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.*

Now let's attempt to unravel this definition. You should think of ϵ as how close we want $f(x)$ to be to L and δ as how close we need x to be to c in order for $f(x)$ to be within distance ϵ from L. With this in mind, the definition of a limit says that $\lim_{x\to c} f(x) = L$ if no matter how small of an ϵ you pick (that is, for any $\epsilon > 0$), you can find a δ so that if x is within distance δ of c (that is, $|x-c| < \delta$), then $f(x)$ is within distance ϵ of L (that is, $|f(x)-L| < \epsilon$). Notice that the definition of a limit does not require that $|f(x)-L| < \epsilon$ for all x satisfying $|x-c| < \delta$, but rather, that $|f(x)-L| < \epsilon$ for all x satisfying $0 < |x-c| < \delta$. This is so that $\lim_{x\to c} f(x)$ can have a value even when f is not defined at c (for instance $\lim_{x\to 1} \frac{x-1}{x-1} = 1$ even though $\frac{x-1}{x-1}$ is undefined at 1) and so that $\lim_{x\to c} f(x)$ can be L even if $f(c) \neq L$.

At this point we are ready to give an outline of how to prove that $\lim_{x\to c} f(x) = L$.

1. In order to show that $\lim_{x\to c} f(x) = L$ we need to show that for any $\epsilon > 0$, we can find a $\delta > 0$ so that if x satisfies $0 < |x-c| < \delta$ then $|f(x) - L| < \epsilon$. The first step in an ϵ-δ proof is to figure out what δ should be. That is, we need to come up with a formula for δ in terms of ϵ. This may require some work.

2. Once we have figured out what δ should be, we assume that x satisfies $0 < |x-c| < \delta$, and show that this implies that $|f(x)-L| < \epsilon$. In this part of the proof, we will often start with $|f(x) - L|$, do some algebra to put it in a form where we can use the hypothesis $0 < |x-c| < \delta$, and then use this hypothesis to show that what we have is less than ϵ. Precisely what we mean by this will be made clear in the examples that follow.

[1] Keep in mind that ϵ and δ are just symbols we use to represent small numbers. It is a convention to use ϵ and δ in the definition of a limit, but we could use other symbols, if we wanted to, so long as we are clear about what we mean.

[2] All we are saying here is that we have a function that is defined for all $x \neq c$ "near" c.

Example 1. Prove that $\lim_{x \to 2} 3x - 1 = 5$. We will follow the outline for an ϵ-δ proof given above.

1. In order to show that $\lim_{x \to 2} 3x - 1 = 5$, we need to show that for any $\epsilon > 0$, we can find a δ so that if x satisfies $0 < |x - 2| < \delta$, then $|(3x - 1) - 5| < \epsilon$. First we will figure out what δ should be in terms of ϵ. We see that

$$\begin{aligned} |(3x-1)-5| < \epsilon &\Leftrightarrow |3x-6| < \epsilon \text{ (by simplifying the equation)} \\ &\Leftrightarrow |3||x-2| < \epsilon \text{ (property of absolute value)} \\ &\Leftrightarrow 3|x-2| < \epsilon \text{ (since } |3| = 3) \\ &\Leftrightarrow |x-2| < \epsilon/3 \text{ (dividing both sides of inequality by 3)} \end{aligned}$$

   The above algebra shows us that if $|x-2| < \epsilon/3$ then $|(3x-1)-5| < \epsilon$. Therefore, we should let $\delta = \epsilon/3$.

2. Now that we know what δ should be, the rest is easy. Given an $\epsilon > 0$, we set $\delta = \epsilon/3$. Now if $0 < |x - 2| < \delta$ we have

$$\begin{aligned} |(3x-1)-5| &= |3x-6| \text{ (simplifying)} \\ &= |3||x-2| \text{ (property of absolute value)} \\ &= 3|x-2| \text{ (since } |3| = 3) \\ &< 3\delta \text{ (since } |x-2| < \delta) \\ &= 3\epsilon/3 \text{ (since } \delta = \epsilon/3 \\ &= \epsilon \end{aligned}$$

   So we have shown that for any $\epsilon > 0$, we can find a δ so that if $0 < |x - 2| < \delta$, then $|f(x) - 5| < \epsilon$. It now follows from the definition of a limit that $\lim_{x \to 2} 3x - 1 = 5$.

This is how your ϵ-δ proofs should look. Every step is justified and it is easy to follow my line of thought (notice the use of complete English sentences to explain my work). Finally, note that at no point did I assume that $|(3x - 1) - 5| < \epsilon$. If I did this, I would be assuming what I wanted to prove, which would be a grievous mathematical sin.