TRANSLATION SURFACES: DYNAMICS AND HODGE THEORY

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ABSTRACT. A translation surface is a multifaceted object that can be studied with the tools of dynamics, analysis, or algebraic geometry. Moduli spaces of translation surfaces exhibit equally rich features. This survey provides an introduction to the subject and describes some developments that make use of Hodge theory to establish algebraization and finiteness statements in moduli spaces of translation surfaces.

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1. Introduction

Riemann surfaces are mathematical jewels that shine brightly regardless of the angle from which we're looking at them. To Mumford's list[1] of algebraic curves, complex-analytic 1-manifolds, and constant curvature surfaces, one can add translation surfaces. Translation surfaces can be defined either as polygons in the plane glued according to simple rules, or as algebraic curves with an abelian differential. The relationship between these two points of view is highly transcendental and is at the heart of a fruitful exchange between the two theories.

The polygonal point of view arises naturally in low-dimensional dynamical systems such as billiards or interval exchange transformations. A productive way to understand the dynamics on an individual translation surface is to study the moduli spaces of all such surfaces and a natural dynamical system on it given by the action of the group $\mathbf{GL}_2(\mathbb{R})$. While the definition of the $\mathbf{GL}_2(\mathbb{R})$-action is far removed from algebraic geometry, its properties are closely related to Hodge theory and hence to the algebraic geometry of Riemann surfaces. This survey presents an overview of these ideas and connections.

Translation surfaces. To give on a compact Riemann surface X a holomorphic 1-form ω is the same as to give a collection of charts on X to $\mathbb{C}$, such that the transition maps are by translations; the charts are allowed to be ramified at finitely many points corresponding to the zeros of ω and are given locally by $z \mapsto \int_{z_0}^{z} \omega$. These special charts of (X, ω) have an echo in the moduli space $\Omega\mathcal{M}_g(\kappa)$ of genus g Riemann surfaces with holomorphic 1-form having zeros of multiplicities $\kappa = (k_1, \ldots, k_n)$. Indeed these moduli spaces are themselves locally modeled on complex vector spaces, with linear transition functions between charts that are called "period coordinates".

The action of $\mathbf{GL}_2(\mathbb{R})$ is locally in period coordinates given by a diagonal action on a product of copies of $\mathbb{C} \cong \mathbb{R}^2$, or explicitly and more globally in terms of the real and imaginary parts of the holomorphic 1-form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \operatorname{Re}\omega \\ \operatorname{Im}\omega \end{bmatrix} = \begin{bmatrix} a\operatorname{Re}\omega + b\operatorname{Im}\omega \\ c\operatorname{Re}\omega + d\operatorname{Im}\omega \end{bmatrix}$$

[1] [Mum75, Lecture 1]

The subgroup $\mathbf{SL}_2(\mathbb{R})$ preserves a natural Lebesgue-class probability measure constructed by Masur [Mas82] and Veech [Vee82]. The Hopf argument, a standard tool in ergodic theory, implies that already the diagonal subgroup acts ergodically. So, in the measure-theoretic sense most orbits are dense. The first $\mathbf{SL}_2(\mathbb{R})$-closed orbits, or equivalently translation surfaces (X, ω) whose stabilizers are lattices, were discovered by Veech [Vee89] who also established striking properties of the straight-line flow on such surfaces. After analogues of Ratner's measure and topological rigidity theorems [Rat91] were established by McMullen in genus 2 [McM07], Eskin, Mirzakhani, and Mohammadi [EMM15, EM18] proved:

Theorem 1 (Topological and measure rigidity)**.** *For any $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$, its $\mathbf{GL}_2(\mathbb{R})$-orbit closure $\mathcal{M} := \overline{\mathbf{GL}_2(\mathbb{R}) \cdot (X, \omega)}$ is locally in period coordinates a linear manifold.*
Furthermore, any ergodic $\mathbf{SL}_2(\mathbb{R})$-invariant probability measure is supported on a codimension 1 submanifold of such an orbit closure, and is of Lebesgue class on it.

McMullen [McM03] also discovered that in genus 2, interesting orbit closures parametrize Riemann surfaces whose Jacobians have real multiplication. Möller [Möl06a] proved that over the lowest-dimensional orbit closures, the zeros of the holomorphic 1-forms must map to torsion points on (a factor of) the Jacobian. These results were extended to all orbit closures in [Fil16a, Fil16b] and used to *characterize* orbit closures and hence prove that they have a purely algebro-geometric description:

Theorem 2 (Real multiplication, Torsion, and Algebraicity)**.** *Let $\mathcal{M}$ be an orbit closure as in Theorem 1. Then there exists a factor $\mathcal{F} \subset \mathcal{J}$ of the relative Jacobian over $\mathcal{M}$, and a subgroup $\mathcal{S}$ of the free abelian group on the zeros of the 1-forms, such that:*

real multiplication: *The factor $\mathcal{F}$ admits real multiplication by a totally real number field.*

torsion: *The Abel-Jacobi map, possibly twisted by real multiplication:*

$$AJ : \mathcal{S} \to \mathcal{F}$$

maps the subgroup $\mathcal{S}$ to a torsion subgroup of $\mathcal{F}$.

algebraicity: *These conditions, combined with a dimension bound, characterize the locus $\mathcal{M}$ inside the ambient moduli space.*

In particular, $\mathcal{M}$ is an algebraic subvariety defined over $\overline{\mathbb{Q}}$.

Note that the orbit closure might be the entire moduli space, so the factor $\mathcal{F}$ is nonempty but not necessarily proper, and the totally real number field might well be $\mathbb{Q}$. Similarly, the subgroup $\mathcal{S}$ might be trivial.

Typical and Atypical orbit closures. Because an orbit closure $\mathcal{M}$ is characterized in Theorem 2 by imposing certain algebro-geometric conditions, one can try to construct one by imposing those conditions and studying the locus where the conditions hold. This only yields an orbit closure if a further dimension bound is attained. In analogy with results in unlikely intersections and the Zilber–Pink conjectures, see [BKU21] in particular, we will call an orbit closure *typical* if its dimension agrees with the expected dimension by intersecting the Hodge loci, and *atypical* otherwise. It is natural to extend the notion of (a)typical to the relative situation of an orbit closure $\mathcal{N}$ containing another $\mathcal{M}$. In this language, one can state one of the main results in [EFW18]:

Theorem 3 (Finiteness of Atypical, Abundance of Typical)**.** *Every orbit closure $\mathcal{N}$ contains only finitely many maximal atypical suborbit closures. If an orbit closure $\mathcal{N}$ admits relatively typical suborbit closures, then those are dense in $\mathcal{N}$.*

In the statement, "maximal" means with respect to inclusion. Not all orbit closures admit typical suborbit closures, and those that do have a straightforward characterization of numerical invariants, see Theorem 5.1.7. Note that the formulation of Theorem 3 in terms of the typical/atypical dichotomy is not how the results were originally stated in [EFW18].

Algebraic Hulls. A key tool in the proof of Theorem 3 is the notion of algebraic hull of a dynamical system, and which can be applied, in particular, to orbit closures for the $\mathbf{GL}_2(\mathbb{R})$-action. It is feasible to compute the algebraic hull in this situation because of its rigidity properties established in [Fil16a], namely that measurable and "polynomial" algebraic hulls coincide. It also turns out that relative (a)typicality can be formulated in terms of the algebraic hull (see Theorem 5.4.5), and to establish finiteness of atypical it suffices to instead establish that algebraic hulls appropriately equidistribute.

These equidistribution results of the algebraic hull can in particular be used to establish further dynamical properties of orbit closures:

Theorem 4 (Monodromy and Lyapunov spectrum of square-tiled surfaces)**.** *For a square-tiled surface (X, ω) denote by $\mathcal{T}_{(X,\omega)}$ its orbit closure. Then for all square-tiled (X, ω) in a fixed stratum $\Omega\mathcal{M}_g(\kappa)$ and outside finitely many proper, atypical suborbit closures, the Zariski closure of monodromy over $\mathcal{T}_{(X,\omega)}$ is isomorphic to $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2g-2}(\mathbb{R})$, and the Lyapunov spectrum is simple.*

Square-tiled surfaces yield typical closed orbits and are a particularly well-studied class, see §4.5.4 for a discussion, and Theorem 5.4.7 for a proof of the above result.

Meromorphic strata; Compactifications; Classification. Although the bulk of the survey is dedicated to explaining the context and techniques behind the above results, we give a broader discussion of the geometry of translation surfaces and their moduli spaces. In particular, we give a brief overview of the recent work of Bainbridge, Chen, Gendron, Grushevsky, and Möller [BCG$^+$18, BCG$^+$19b] on compactifications of strata of holomorphic differentials, and the adjacent notion of meromorphic (and more generally multiscale) differentials.

These tools ought to be especially useful for the question of classifying or giving restrictions for orbit closures. We include an overview of some of the known techniques as well.

Bialgebraic geometry. With the benefit of hindsight, one can view the period coordinates on strata and the related finiteness questions within the emerging framework of "bialgebraic geometry", see [KUY18] for an introduction. Finiteness results analogous to Theorem 3 were established by Baldi, Klingler, and Ullmo [BKU21] in a broader Hodge-theoretic setting, and very closely related methods and results were developed by Bader, Fisher, Miller, and Stover [BFMS21] in the context of hyperbolic manifolds and their totally geodesic submanifolds. It is natural to wonder: can these methods and results be put into a common framework? For some questions related to the bialgebraic geometry of strata of translation surfaces, see the recent work of Klingler and Lerer [KL22].

Overview of this text

We have included a number of questions for further investigation, some quite precise and others more open-ended. These questions are included throughout the text, at places where they fit naturally with the narrative.

Translation surfaces. We start in Section 2 with the definition and some illustrative examples of translation surfaces. Examples are selected from the dynamics of billiards in polygons, and classical algebraic geometry. Our exposition is necessarily brief, as there are now many excellent treatments in the literature.

Moduli spaces. Strata parametrizing translation surfaces with a specified number and multiplicity of zeros are introduced in Section 3. One of their key features is the presence of period coordinates, described in §3.1. These coordinate systems endow strata with a locally linear structure and can be seen as a moduli space counterpart to the local flat geometry of a translation surface. An overview of known topological properties of strata is included.

We turn to the $\mathbf{GL}_2(\mathbb{R})$-action in §3.2 and illustrate its significance with some classical applications to the ergodic theory of billiards. Strata of *meromorphic* translation surfaces are described briefly in §3.3, complemented by a discussion of an example of Bakker–Mullane [BM20] of a closed, linear submanifold of a meromorphic stratum which is not algebraic. We end with an overview in §3.4 of algebro-geometric compactifications of strata.

Orbit closures. We introduce linear manifolds in §4.1 and describe the measure and topological rigidity results of Eskin, Mirzakhani, and Mohammadi. The discussion includes Example 4.1.10 of an orbit closure with self-intersection along a smaller-dimensional one.

Background in Hodge theory is provided in §4.2, followed by a discussion of Hodge-theoretic rigidity results in §4.3. These rigidity results provide analytic control of measurable equivariant bundles and establish their real-analyticity and compatibility with the Hodge structure. With these properties in hand, we describe in §4.4 a characterization of orbit closures in terms of real multiplication and torsion conditions on factors of Jacobians. Some examples are included in §4.5.

Finiteness results. We introduce the typical-atypical dichotomy in §5.1 and use this point of view to state in Section 5 optimal results on finiteness of atypical and abundance of typical orbit closures. While the abundance results are by now well-understood, to prove finiteness we need an additional tool: the algebraic hull, introduced in §5.2. After deducing, based on Hodge-theoretic rigidity, the necessary results on algebraic hulls in §5.3, we establish the finiteness theorems in §5.4.

Classification. We end this survey with a brief overview of classification questions in Section 6. Results based on Wright's cylinder deformation theorem are included in §6.1, while results that use arithmetic and algebro-geometric methods are presented in §6.2. We conclude with a discussion of algorithmic questions related to translation surfaces and their orbit closures in §6.3.

Further references. A number of recent developments are not included in this survey and we include some pointers to the literature. The dynamics of the horocycle flow is surveyed by Chaika–Weiss [CW22]. Moduli spaces of dilation surfaces, a generalization of translation surfaces, are discussed by Apisa, Bainbridge, and Wang in [ABW22]. For results on the dynamics of the relative foliation, see the work of Calsamiglia, Deroin, and Francaviglia [CDF15] and Winsor [Win22a].

The reader can choose from a number of excellent surveys devoted to closely related topics: on the ergodic theory of translation surfaces and their moduli spaces, see the surveys of Masur–Tabachnikov [MT02], Zorich [Zor06], and Forni–Matheus [FM14]; for an exposition of the work of Eskin–Mirzakhani–Mohammadi [EMM15, EM18] see the report of Quint [Qui16]; for Teichmüller curves, the surveys of Hubert–Schmidt [HS06] and McMullen [McM23]; for general orbit closures, see Wright's survey [Wri15b]; for an algebro-geometric point of view see Möller's report [Möl18] and Chen's lecture notes [Che17].

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2. Translation surfaces

There are many equivalent definitions of a translation surface, but we have to start somewhere and we will take it to mean: a pair (X, ω) consisting of a compact Riemann surface and a holomorphic 1-form. In practice it is useful to extend the notion and allow meromorphic 1-forms, as well as more general stable Riemann surfaces, and introduce marked points. We shall develop these notions in §3.3, but to start with, the reader unfamiliar with the subject can restrict to the compact, holomorphic case.

Outline of section. We introduce in §2.1 some standard constructions of translation surfaces, with a view towards examples of orbit closures in moduli spaces that we consider later on. The reader will find a more thorough introduction, with more examples and relations to various low-dimensional dynamical systems, in the surveys of Zorich [Zor06] and Masur–Tabachnikov [MT02]. A thorough treatment of the relations to quasiconformal maps, Teichmüller theory, and applications to dynamics can be found in the survey of Forni–Matheus [FM14].

Some elementary algebro-geometric constructions of translation surfaces are included in §2.2. Throughout this survey, the two parallel points of view of "flat" geometry and algebraic geometry will frequently interact.

2.1. Dynamics

2.1.1. Gluing polygons. The most immediate way to obtain a translation surface is to start with a collection of polygons $P_i \subset \mathbb{R}^2$ and specify a gluing of all edges such that the result is a compact surface, with two requirements: all glued edges are isometric via a translation in the plane, and the interiors of polygons are on opposite sides of the identified edges. One can relax the notion of polygon and allow, more generally, "abstract" surfaces with polygonal boundary that are immersed in $\mathbb{R}^2$.

Let $X := (\coprod P_i)/\sim$ be the resulting surface, with $Z_0 \subset X$ the images of vertices of the polygons. If we fix a basepoint $x_0 \in X \setminus Z_0$, we obtain a map from the universal cover to the plane $\widetilde{X \setminus Z_0} \to \mathbb{R}^2 \cong \mathbb{C}$, which endows the universal cover, and hence $X \setminus Z_0$, with the structure of a Riemann surface. It is also immediate to check that the holomorphic 1-form dz on $\mathbb{C}$ descends to $X \setminus Z_0$. A key point is that by standard removable singularities theorems in complex analysis, the complex-analytic structure extends to X and so does the holomorphic 1-form, denoted ω. Some of the points in Z_0 can result in zeros of ω, namely those for which the total sum of angles of the polygons is in $2\pi\mathbb{N}_{\geq 2}$. Note that the sum of angles around any point will be in $2\pi\mathbb{N}$, since the gluing is done by translations in the plane.

A classical example is to start with a parallelogram and identify opposite sides. This gives a torus, equipped with a complex structure and a nowhere vanishing holomorphic 1-form. Note that the sides of the parallelogram can be recovered, as points in $\mathbb{C} \cong \mathbb{R}^2$, by integrating the 1-form over topological cycles on the torus.

Here is an interesting variant of this construction that leads to a genus 2 surface. Take two parallelograms in $\mathbb{R}^2$, not necessarily isometric, and make cuts (aka slits) in each of them that *are* isometric. Glue the opposite edges of the parallelograms to obtain tori, and glue the tori along the "opposite" sides of the slit to obtain a closed surface. The holomorphic 1-form has two zeros, at the vertices of the slits. See Figure 2.1.2 for an illustration.

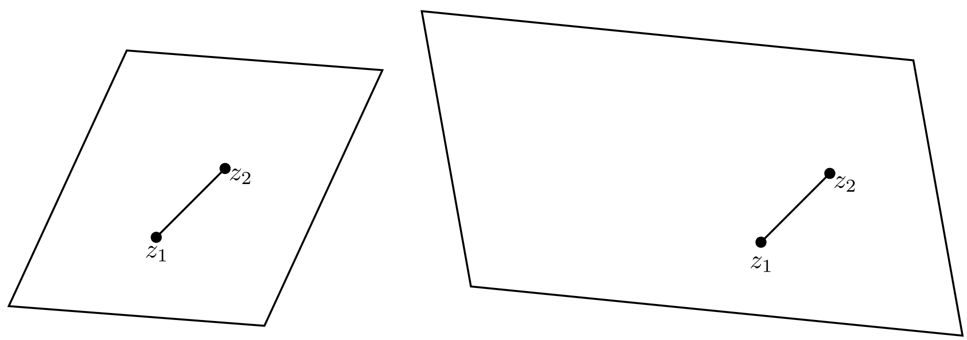

FIGURE 2.1.2. Two tori glued along parallel and isometric slits.

2.1.3. Billiards. The above construction arises naturally in low-dimensional dynamics. Start by considering a ball moving without friction on a billiard table in the shape of a polygon P. When the angles of P are *not* all rational multiples of π, little is known about this dynamical system. Even in the case of a triangle, it is expected but not known if a periodic trajectory exists (avoiding the vertices). We thus restrict to the case of a polygon with angles in $\mathbb{Q}\pi$.

2.1.4. Unfolding construction. It is now possible to reduce the dynamics to a straight line flows on a *compact* surface, with finitely many singularities. Concretely, let Φ be the surface obtained by gluing P and its mirror image along corresponding sides, and denote by $Z \subset \Phi$ the vertices. Note that Φ is homeomorphic to a sphere. The flat metric on Φ is induced by the Euclidean one on P, and we have a *holonomy representation* $\rho\colon \pi_1(\Phi \setminus Z) \to \mathbf{SO}_2(\mathbb{R})$, by parallel transport along paths. Since $\mathbf{SO}_2(\mathbb{R})$ is abelian, the representation factors through the first homology $\mathrm{H}_1(\Phi \setminus Z)$.

Let us also note that everything we said so far works even if the original P had irrational angles. Furthermore, we can allow P to be only immersed, i.e P can be an abstract 2-manifold with boundary, with marked points on the boundary, equipped with an immersion (i.e. local diffeomorphism) $P \to \mathbb{R}^2$

such that pieces of the boundary outside the marked points go to straight lines.

This construction, introduced in [FK36] and independently in [KZ75], is usually presented in more concrete terms by drawing successive reflections of P in its sides. The rational angles of P guarantee that only finitely many reflections are necessary, and this immediately relates to the preceding discussion in §2.1.1.

2.1.5. A finite cover. Returning to the holonomy representation $\rho\colon \mathrm{H}_1(\Phi\setminus Z) \to \mathbf{SO}_2(\mathbb{R})$, if the angles of P are rational multiples of π, then every generator maps to a rotation by a rational angle. Therefore the image of ρ is a finite subgroup $G_P \subset \mathbf{SO}_2(\mathbb{R})$ and we can pass to a finite G_P-cover Φ^h of Φ, with points $Z^h \subset \Phi^h$ branching over $Z \subset \Phi$, on which the holonomy representation is trivial. Geometrically, since Φ^h is also tessellated by P and its reflection, trivial holonomy is equivalent to the sum of angles around points in Z^h being an integral multiple of 2π.

To keep the notation more suggestive, we will denote by (X_P, ω_P) the Riemann surface obtained by completing Φ^h at the finitely many punctures, and equipping it with the induced holomorphic 1-form.

An illustrative example was analyzed by Veech [Vee89]:

2.1.6. Example (Regular polygons)**.** Fix $n \geq 3$ and consider the triangle with angles $\left(\frac{\pi}{2}, \frac{\pi}{n}, \frac{(n-2)\pi}{2n}\right)$, and we assume for simplicity that $n = 2g+1$ is odd. The surface Φ is a sphere, with three cone points p_2, p_n, p_{2n} with cone angles $\pi, \frac{2\pi}{n}, \frac{(n-2)\pi}{n}$. The holonomy cover $\Phi^h \to \Phi$ has degree $2n$, there are n preimages of p_2 ramified to order 2, there are 2 preimages of p_n ramified to order n, and there is one preimage of p_{2n} ramified to order $2n$. We see that the preimages of p_2, p_n in Φ^h are not cone points anymore, but the preimage of p_{2n} is a cone point with cone angle $(n-2)(2\pi)$. This implies that the genus of Φ^h is g, where $n = 2g+1$, and the holomorphic 1-form on Φ^h has a single zero of order $2g-2$ at the preimage of p_{2n}.

The geometric picture of the holonomy cover Φ^h is obtained by gluing the regular n-gon and its reflected copy, identifying parallel sides, see Figure 2.1.7.

2.1.8. Caution: unfolding and "sameness". The construction in §2.1.4, applied to the regular n-gon P, will yield a compact surface which is larger than the one more traditionally considered, i.e. the one obtained by gluing opposite sides of the n-gon when n is even, or parallel sides of the polygon and its reflected copy when n is odd. The billiard trajectories on a regular n-gon are most naturally described on the translation surface (X_P, ω_P).

The distinction arises when we ask what unfoldings of the polygon P in the plane are "the same". If we don't ask the marked sides to go to the marked sides under a translation, then the unfolding construction leads to the smaller surface just described.

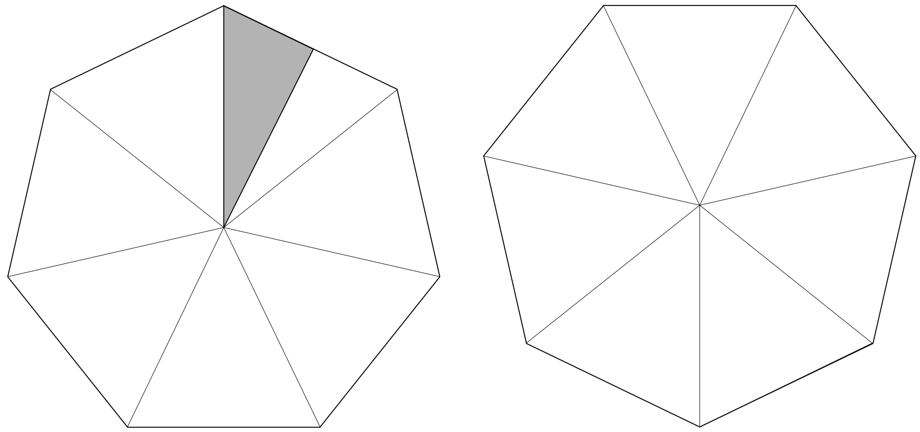

FIGURE 2.1.7. The unfolding of the triangle with angles $\frac{\pi}{2}, \frac{\pi}{7}, \frac{5\pi}{14}$ as in Example 2.1.6 and its holonomy cover obtained from the regular polygon.

2.1.9. Saddle connections and closed trajectories. A translation surface (X, ω) is equipped with a flat metric, with singular points at the zeros of ω, where the cone angle is $2\pi(k+1)$ is ω has a zero of order k. Globally one can write the metric as $\sqrt{-1}\omega \wedge \overline{\omega}$. We can therefore speak of geodesics, which we will refer to as straight lines or geodesics.

Two types of geodesics appear frequently: the closed geodesic, and saddle connections which by definition are geodesics that connect two singular points. In the case of translation surfaces obtained from billiards, closed geodesics are in bijection with closed billiard trajectories that avoid vertices of the polygon, and saddle connections are in bijection with trajectories that go between two vertices.

Observe that a closed geodesic comes in a 1-parameter family and sweeps out a cylinder; we will consider such closed geodesics as equivalent. We will say more about cylinders in §6.1, but for now let us note that a consequence of the measure classification results of Eskin, Mirzakhani, and Mohammadi is that on *every* translation surface, the number of equivalence classes of closed geodesics of length at most T is $\sim cT^2$, see [EM18, Thm. 1.8] for the precise statement. See also [Fil20, Thm. B] for an analogous counting result on K3 surfaces.

2.1.10. Dynamics of the billiard flow. For a fixed angle θ we can consider the transformation T_t^θ mapping a point $x \in X$ to one which is distance t away at angle θ. Because of the singularities, these transformations are defined away from a codimension 1 subset of X, but nonetheless on a common set of full measure Lebesgue they are well-defined and form a group. One can verify that in the case of translation surfaces coming from polygonal billiards with rational angles as in §2.1.3, this models the billiard flow on the table.

Two basic questions about this flow are whether there are dense trajectories, and whether the natural invariant measure is ergodic. One can ask (and answer) these questions in a stronger form: whether the system is minimal, i.e. every trajectory is dense, and whether the system is uniquely ergodic, i.e. there is only one invariant probability measure.

It turns out that $T^\theta_\bullet$ is minimal for any θ outside of a countable set: in fact removing all θ's in which there is a saddle connection suffices, see [MT02, Thm. 1.8]. Unique ergodicity is more delicate, and we will return to it in Theorem 3.2.8.

An example of *failure* of unique ergodicity, for an uncountable set of directions θ, is described in [MT02, §3.1]. The translation surface in question is built out of two tori, with an appropriate choice of slits, as in §2.1.1.

2.2. Algebraic geometry

2.2.1. Algebraic curves and bialgebraic structures. Recall that a compact Riemann surface X can also be viewed as an algebraic curve over $\mathbb{C}$, and a holomorphic 1-form ω is in this case also called an (abelian) differential. We can view the extra datum of the 1-form as giving a "bialgebraic structure" on X, in the sense of [KUY18, Def. 4.1]. Namely, we have a holonomy representation $\rho\colon \pi_1(X) \to \mathbb{C}$ (which factors through H_1) and an equivariant "developing" map $\widetilde{X} \xrightarrow{\mathrm{Dev}} \mathbb{C}$, where we view $\mathbb{C}$ as an affine algebraic curve. We will see an echo of this in the moduli space of translation surfaces, see §3.1.

Algebraic curves, and integrals of differentials over them, have been studied from the early days of dynamical systems. We include a classical illustration:

2.2.2. Example (Periods of elliptic curves)**.** Consider a particle moving in a 1-dimensional potential given by

$$V(q) = q^2(q-1)(q+2)$$

Its phase space consists of points with coordinates (q, p), with p denoting the velocity. The Hamiltonian is

$$H(q,p) = \frac{1}{2}p^2 + V(q)$$

and the phase curves are the level sets of $H(q,p)$. When these level sets are smooth, at energy E they are the real points of the elliptic curve $p^2 = 2(E - V(q))$.

The *period* of motion is given by the integral

$$2\int_{q_0}^{q_1} \frac{dq}{\sqrt{2(E-V(q))}}$$

where $V(q_0) = V(q_1) = E$ and there are no further solutions to this equation in (q_0, q_1). Note that this is an integral of an abelian differential on the elliptic curve. For certain level sets there are two intervals (q_0, q_1) and (q_0', q_1') at the same energy, but located in different wells of the potential.

On the complex points of the elliptic curve, the two cycles of integration are *homologous*, therefore the integrals are the same. We reach a classically known conclusion: in a quartic potential, the two periods of motion at the same energy agree.

This example also illustrates why hyperelliptic integrals and curves were extensively studied early on: they describe the motion of particles in polynomial potentials.

2.2.3. Example (Regular polygons)**.** We continue with Example 2.1.6, but now give an algebro-geometric description. Let as before $n = 2g+1$ be an odd natural number and let (X_P, ω_P) denote the translation surface constructed in Example 2.1.6, obtained by gluing a regular n-gon with its reflected copy along parallel sides. We will verify that the pair (X_P, ω_P) is isomorphic to (the completion of):

$$y^2 = x^n - 1 \quad \text{with} \quad \omega_P = \frac{dx}{y}.$$

In fact, the map $X_P := \Phi^h \to \Phi \cong \mathbb{P}^1$ is also immediately described, it coincides with $(x, y) \mapsto x^n$. Indeed, the hyperelliptic double cover $X_P \to \mathbb{P}^1$ given by $(x, y) \mapsto x$ is ramified over the n-th roots of unity, and ∞, each with ramification order 2, and the point $0 \in \mathbb{P}^1$ has two preimages. Now the map $x \mapsto x^n$ takes n-th roots of unity to 1, and is ramified of order n over $0, \infty$.

From this we immediately deduce that $X_P \to \mathbb{P}^1$ with $(x, y) \mapsto x^n$ is of degree $2n$ and ramified over $0, 1, \infty$ with respective sizes of preimages $2, n, 1$ and ramification orders $n, 2, 2n$. Geometrically, preimages of 0 correspond to the two centers of the two copies of the regular polygon, preimages of 1 correspond to the n midpoints of the sides, and preimages of ∞ correspond to preimages of the vertices of the regular n-gon (which all get identified by the gluing).

3. Moduli spaces

It is a remarkable fact that to study the dynamics on an individual translation surface, it is useful to understand the moduli space of all translation surfaces with prescribed combinatorial data. In this section we define the appropriate moduli spaces and structures on them that are of intrinsic interest and can also be used to study individual surfaces.

Outline of section. Period coordinates are a key feature of the moduli space of translation surfaces, echoing the flat geometry of an individual surface. We describe them in §3.1, followed by a description of the action of $\mathbf{GL}_2(\mathbb{R})$ in §3.2. This action plays an essential role in all that follows. We include a description of the geometry of strata of meromorphic differentials (i.e. translation surfaces of infinite area) in §3.3. These turn out to play an essential role in the construction of compactifications of holomorphic strata, described in §3.4.

3.1. Period coordinates

3.1.1. Setup. For this section, a translation surface will mean a pair (X, ω) consisting of a compact Riemann surface X and a holomorphic 1-form ω. Let $\kappa := (k_1, \dots, k_n)$ denote the multiplicities of zeros of ω, with $k_1 + \cdots + k_n = 2g - 2$ where g is the genus of X. It turns out that a useful moduli space for pairs (X, ω) is obtained if we freeze the vector κ. We will denote by cardinality of the set of zeros by $|\kappa| := n$.

3.1.2. The Hodge bundle and its stratification. Let $\mathcal{M}_g$ denote the moduli space of genus g Riemann surfaces and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the Hodge bundle, whose fiber over $X \in \mathcal{M}_g$ is the space of all holomorphic 1-forms on X. While $\Omega\mathcal{M}_g$ is a rank g vector bundle, it is stratified[2] by the algebraic subsets $\Omega\mathcal{M}_g(\kappa)$ of holomorphic 1-forms with zeros of multiplicities given by κ. We will refer to each $\Omega\mathcal{M}_g(\kappa)$ as a *stratum of translation surfaces* (its connected components are described in §3.1.16) and proceed to equip it with natural complex-analytic local charts, which in particular imply that it is a smooth orbifold. Before doing so, we need some topological preliminaries.

3.1.3. Relative homology. Let (S, Z) be a pair consisting of a compact genus g surface S with a finite set of points $Z = \{z_1, \dots, z_n\}$. The first integral homology of S is denoted $H_1(S; \mathbb{Z})$ and with the intersection product, it is isomorphic to $\mathbb{Z}^{2g}$ with its standard symplectic form. Let also $H_1(S, Z; \mathbb{Z})$ denote the group of cycles whose boundaries are in Z, i.e. a class $[\gamma] \in H_1(S, Z; \mathbb{Z})$ is represented by a collection of paths γ on S, with integral weights, such that their boundaries satisfy $\partial\gamma \subset Z$, with equivalence induced by 2-cycles in S. We have the fundamental short exact sequence, induced by the long exact sequence of the pair (S, Z):

$$0 \to \mathrm{H}_1(S; \mathbb{Z}) \hookrightarrow \mathrm{H}_1(S, Z; \mathbb{Z}) \twoheadrightarrow \tilde{\mathrm{H}}_0(Z; \mathbb{Z}) \to 0 \tag{3.1.4}$$

where $\tilde{\mathrm{H}}_0$ denotes the reduced 0-th homology of Z, i.e. any assignment of integral weights to points in $\mathbb{Z}$, of total weight 0.

3.1.5. Relative cohomology. We can dualize the short exact sequence in Eqn. (3.1.4) and take complex coefficients to obtain:

$$0 \to \tilde{\mathrm{H}}^0(Z; \mathbb{C}) \hookrightarrow \mathrm{H}^1(S, Z; \mathbb{C}) \xrightarrow{p} \mathrm{H}^1(S; \mathbb{C}) \to 0 \tag{3.1.6}$$

This is the basic object that leads to local charts on the stratum $\Omega\mathcal{M}_g(\kappa)$. Indeed, suppose (X, ω) is a translation surface and $Z \subset X$ is the set of zeros of ω. Then integration of ω along paths yields a cohomology class $[\omega] \in \mathrm{H}^1(X, Z; \mathbb{C})$. One can think of the cohomology class $[\omega]$ as an intrinsic

[2] One should include the "trivial" stratum of zero 1-forms, isomorphic to $\mathcal{M}_g$ itself.

way to encode the possible polygonal divisions of the translation surface: $[\omega]$ gives the answer, as a complex number, of any side of any polygon.

To describe the local structure of the moduli space of pairs (X, ω) we need:

3.1.7. Definition (Marked deformations)**.** For a translation surface (X_0, ω_0), a *marked deformation* consists of:

- a complex manifold B with distinguished basepoint $b_0 \in B$
- a fibration in Riemann surfaces $\mathcal{X} \to B$ with a holomorphic 1-form ω on $\mathcal{X}$; fiber over $b \in B$ denoted $\mathcal{X}_b$ and the restriction of ω to it denoted ω_b.
- an identification of the pair $(X_0, \omega_0) \cong (\mathcal{X}_{b_0}, \omega_{b_0})$
- sections $\sigma_1, \ldots, \sigma_n \colon B \to \mathcal{X}$ with disjoint images such that the zeros of ω_b coincide with $Z_b := \{\sigma_i(b)\}$.

A marked deformation $(\mathcal{X}, B)$ is called *universal* if any other marked deformation $(\mathcal{X}', B')$ is obtained, after possibly shrinking B', from a unique *classifying map* $B' \to B$ by pullback.

It is immediate that up to shrinking the base B, a universal marked deformation, if it exists, is unique up to unique isomorphism. With these preliminaries, one can verify the following result, established first by Veech [Vee90, Thm. 7.15]:

3.1.8. Theorem (Local structure of deformations)**.**

(i) *For a translation surface (X, ω) with zero set of ω denoted Z, a universal marked deformation exists and the base can be taken to be an open neighborhood $U_{[\omega]}$ of $[\omega]$ in $H^1(X, Z; \mathbb{C})$.*

(ii) *The classifying map of a family $(\mathcal{X}, B)$ is given by $b \mapsto [\omega_b]$ using a local smooth trivialization of the fibration and identification $H^1(X_{b_0}, Z_{b_0}) \xrightarrow{\sim} H^1(X_b; Z_b)$.*

(iii) *Furthermore, the classifying map from any sufficiently small neighborhood of $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$ to $U_{[\omega]}$ is a local biholomorphism.*

In the last statement, a "sufficiently small neighborhood" is to be understood in the sense of orbifolds/stacks.

3.1.9. The developing map. Fix now a basepoint $s := (X, \omega) \in \Omega\mathcal{M}_g(\kappa)$. We have the fundamental group $\pi_1(\Omega\mathcal{M}_g(\kappa), s)$ and the corresponding universal cover $\widetilde{\Omega\mathcal{M}}_g(\kappa)$. From the local description of the stratum in Theorem 3.1.8(iii) we obtain a locally biholomorphic map

$$\mathrm{Dev}_\mu \colon \widetilde{\Omega\mathcal{M}}_g(\kappa) \to \mathrm{H}^1(X, Z; \mathbb{C}) \tag{3.1.10}$$

We will refer to the map Dev as the *developing map* or alternatively as *period coordinates*. It is equivariant for a representation

$$\rho_\mu \colon \pi_1(\Omega\mathcal{M}_g(\kappa), s) \to \mathrm{Mod}(X, Z) \tag{3.1.11}$$

where $\mathrm{Mod}(X, Z)$ denotes the mapping class group of diffeomorphisms of X preserving the set Z. This mapping class group acts on cohomology via a linear representation

$$
\begin{aligned}
&L\colon \operatorname{Mod}(X,Z) \to \mathbf{Sp}\left(\mathrm{H}^1(X,Z)\right)\\
&\text{where } \mathbf{Sp}\left(\mathrm{H}^1(X,Z)\right) \cong \mathbf{Sp}(\mathrm{H}^1(X)) \ltimes \operatorname{Hom}\left(\mathrm{H}^1(X), \tilde{\mathrm{H}}^0(Z)\right)
\end{aligned}
\tag{3.1.12}
$$

where the last semidirect product structure comes from the short exact sequence in Eqn. (3.1.6) (with integer coefficients). Let us note, again, that in analogy with the case of curves discussed in §2.2.1, strata are therefore endowed with a bialgebraic structure in the sense of [KUY18, Def. 4.1].

From the short exact sequence Eqn. (3.1.6), we obtain on the stratum a short exact sequence of local systems that will be denoted as:

$$
0 \to \mathrm{W}_0 \hookrightarrow \mathrm{H}^1_{rel} \xrightarrow{p} \mathrm{H}^1 \to 0 \tag{3.1.13}
$$

The kernel of the map p is denotes by W_0 since it is the weight 0 piece of a mixed Hodge structure, see Definition 4.2.3 below.

3.1.14. Topology of the developing map. A series of natural questions arise about the above structures. First, one can ask what is the image of the representation in the mapping class group. This was answered by Calderon–Salter [CS20, Thm. A] who prove that the image of $\rho\colon \pi_1(\Omega\mathcal{M}_g(\kappa)) \to \operatorname{Mod}(X,Z)$ is surjective onto the mapping class subgroup that preserves a "framing" (ignoring hyperelliptic strata, see §3.1.16 below).

Next, since the image of the developing map is an open set, one can ask for its characterization. In the case of a maximal stratum, and at least after projecting to absolute cohomology, this was answered by Haupt [Hau20] who found a simple topological obstruction for a cohomology class to be in the image, coming from torus covers, and showed that's the only obstruction. Kapovich [Kap20] found an approach to this question based on Ratner's theorems, using that the image is open and invariant under a lattice in the corresponding Lie group. This was further extended by Bainbridge, Johnson, Judge, and Park [BJJP20] to all strata.

3.1.15. Question (Haupt for orbit closures)**.** Determine the image of the developing map in *relative* cohomology $\mathrm{H}^1(X,Z;\mathbb{C})$ for all strata. Similarly, for any orbit closure $\mathcal{M}$ (see Section 4 below) determine the image of the developing map restricted to $\mathcal{M}$, inside the vector space $T_{(X,\omega)}\mathcal{M}_{\mathbb{C}}$, for a basepoint $(X,\omega) \in \mathcal{M}$.

3.1.16. Connected components. Given a configuration of zeros κ, the question of what are the connected components of $\Omega\mathcal{M}_g(\kappa)$ was answered by Kontsevich and Zorich [KZ03, Thms. 1, 2]. The results can be summarized as follows (with $g \geq 4$):

- $\Omega\mathcal{M}_g(2g-2)$ and $\Omega\mathcal{M}_{2k+1}(2k,2k)$ each have three connected components: the hyperelliptic one, and two more distinguished by even/odd spin structures.
- $\Omega\mathcal{M}_{2k}(2k-1,2k-1)$ has two connected components, a hyperelliptic and a non-hyperelliptic one.

- When κ is divisible by 2, i.e. $k_i = 2k_i', \forall i$ then there are two connected components distinguished by spin structures.
- All other strata are connected.

In the remaining (low) genera, we have:

- $g = 3$: $\Omega\mathcal{M}_3(2,2)$ and $\Omega\mathcal{M}_3(4)$ each have two components, the hyperelliptic and the odd spin one. The remaining strata are connected.
- $g = 2$: there are two connected strata $\Omega\mathcal{M}_2(1,1)$ and $\Omega\mathcal{M}_2(2)$.

We recall one definition of the spin invariant for a finite (possibly with multiplicity) set of points $D \subset X$ such that there exists a holomorphic 1-form on X with divisor of zeros $2D$. Then one says that D is even/odd according to $\dim \mathrm{H}^0(X; \mathcal{O}_X(D)) \mod 2$; a classical theorem implies that this discrete quantity is locally constant in holomorphic families, hence defines an invariant of connected components. A topological definition can be found in [KZ03, §3.1].

3.1.17. On the topology of strata. Returning to strata themselves, one would like to know more about their topology. It has been speculated that perhaps strata are $K(\pi,1)$-spaces, i.e. the universal cover is contractible. This was verified in genus 3 by Looijenga–Mondello [LM14]. Chen in [Che19] considered the question of how far are strata from affine algebraic varieties, more broadly in the setting of meromorphic differentials. Let us also note that Zykoski in [Zyk22, Thm. 1.1] gives a finite simplicial complex which is homotopy equivalent to a stratum. The following observation might be useful in studying the geometry of minimal strata, i.e. those with one zero:

3.1.18. Question (Algebraic symplectic geometry of strata)**.** Suppose $\mathcal{M}$ is a connected component of $\Omega\mathcal{M}_g(2g-2)$, or more generally an orbit closure (see Section 4) with zero torsion corank. The symplectic pairing on H^1 induces a nondegenerate symplectic form on $\mathcal{M}$, which is moreover algebraic for the algebraic structure on $\mathcal{M}$, and equivariant for the $\mathbb{C}^\times$-action.

Is it possible to embed $\mathcal{M}$ into a "symplectic singularity" $\mathcal{M} \hookrightarrow \mathcal{M}^s$, in the sense of [Kal09, Def. 1.1] such that the scaling action is dilating in the sense of [Kal09, Def. 1.7]? If so, what does this tell us about the topology and geometry of minimal strata, and more generally of orbit closures with zero torsion corank?

3.2. The action of $\mathbf{GL}_2(\mathbb{R})$

3.2.1. Action on the complexification of a vector space. Suppose H is a real vector space and $\mathrm{H}_{\mathbb{C}} := \mathrm{H} \otimes_{\mathbb{R}} \mathbb{C}$ is its complexification. The group $\mathbf{GL}_2(\mathbb{R})$ acts naturally on $\mathbb{R}^2$; using the isomorphism $\mathbb{C} \cong \mathbb{R}^2$ we also obtain an action on $\mathrm{H}_{\mathbb{C}} \cong \mathrm{H} \otimes_{\mathbb{R}} \mathbb{R}^2$ as follows. A vector $\omega \in \mathrm{H}_{\mathbb{C}}$ decomposes into a real and imaginary part $\omega = \operatorname{Re}\omega + \sqrt{-1}\operatorname{Im}\omega$ and the action is explicitly:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \operatorname{Re}\omega \\ \operatorname{Im}\omega \end{bmatrix} = \begin{bmatrix} a\operatorname{Re}\omega + b\operatorname{Im}\omega \\ c\operatorname{Re}\omega + d\operatorname{Im}\omega \end{bmatrix} \tag{3.2.2}$$

Note that this action is $\mathbb{R}$-linear but *not* $\mathbb{C}$-linear, unless the matrix belongs to $\mathbb{C}^\times$, i.e. is of the form $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ with $a^2 + b^2 \neq 0$. So the group action, while real-analytic, is not biholomorphic and does not preserve holomorphic functions.

3.2.3. The action on a stratum. The above construction of a $\mathbf{GL}_2(\mathbb{R})$-action can be extended to a stratum of translation surfaces $\Omega\mathcal{M}_g(\kappa)$. The period coordinates from Eqn. (3.1.10) are equivariant for the $\mathbf{GL}_2(\mathbb{R})$-action on both sides, but it is worth emphasizing that just the existence of period coordinates does not guarantee that the action on the vector space lifts to the action on a stratum. Period coordinates only imply the existence of vector fields that satisfy the commutation relations of $\mathfrak{gl}_2(\mathbb{R})$, not that they also integrate to an action of the group.

Indeed, the action on a stratum is best seen in terms of the polygonal description of a translation surface from §2.1.1. Specifically, suppose $(X, \omega) = (\coprod P_i)/\sim$ is a polygonal presentation and $g \in \mathbf{GL}_2(\mathbb{R})$. Now g naturally acts on the plane $\mathbb{R}^2$ and we define the new surface $g \cdot (X, \omega) := (\coprod gP_i)/\sim$ to be the gluing of the polygons gP_i using the same combinatorial equivalence relation $\sim$. Observe that crucially, if two segments in the plane are isometric via a translation, they remain so after we apply to both of them an element of $\mathbf{GL}_2(\mathbb{R})$.

The above construction is highly transcendental from the point of view of algebraic geometry, i.e. the algebraic curve and differential $g(X, \omega)$ cannot be easily expressed in terms of (X, ω) using the standard tools of algebraic geometry. Instead, one has to compute the periods of ω and manipulate them. In particular, let us note that for a fixed $g \in \mathbf{GL}_2(\mathbb{R})$, the induced transformation of a stratum is *not* holomorphic if g is not in $\mathbb{C}^\times$.

It is also possible to give an alternative description of the $\mathbf{GL}_2(\mathbb{R})$-action, by acting directly on the real and imaginary parts of ω exactly as in Eqn. (3.2.2). A pleasant exercise is to verify that in local period coordinates, the vector field giving the action of $g_t := \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]$ is "maximally antiholomorphic": any holomorphic function invariant by it must be constant.

We will see nonetheless in §4.4 that a subtle interaction between algebraic geometry, *and* arithmetic, does come into play in the geometry of the $\mathbf{GL}_2(\mathbb{R})$-action.

3.2.4. Masur–Veech measure. The area of the translation surface (X, ω) can be computed cohomologically as $\frac{\sqrt{-1}}{2}[\omega] \cup [\overline{\omega}]$, so the image of the developing map lands in the open subset of H^1_{rel} where the self-intersection $\sqrt{-1}[\omega] \cup [\overline{\omega}]$ is strictly positive (after mapping H^1_{rel} to the absolute cohomology H^1). After rescaling by an element of $\mathbb{R}_{>0}$, we can always ensure that a translation surface has area 1 and we will denote by $\Omega\mathcal{M}_g(\kappa)^1$ the subset of surfaces thus normalized. Note that we have a natural diffeomorphism $\Omega\mathcal{M}_g(\kappa)^1 \times \mathbb{R}_{>0} \xrightarrow{\sim} \Omega\mathcal{M}_g(\kappa)$.

In the short exact sequence of Eqn. (3.1.13), the H^1-piece has a symplectic form while the W_0-piece is (virtually) trivial, so there is a natural monodromy and $\mathbf{SL}_2(\mathbb{R})$-invariant volume on H^1_{rel}. We can also radially induce a measure on the unit area surfaces, by assigning to $A \subset \Omega\mathcal{M}_g(\kappa)^1$ the volume of $A \times (0,1) \subset \Omega\mathcal{M}_g(\kappa)$.

The resulting measure on the stratum is called the *Masur–Veech* measure, and it is a fundamental result of Masur [Mas82, §5] and Veech [Vee86, Thm. 1], [Vee82, Thm. 1.1], that the measures are *finite*. With the normalizations implicit in the above construction, it becomes an interesting question to explicitly compute it. For instance, the Masur–Veech measure of $\Omega\mathcal{M}_1(0)$ is $\pi^2/3$ (see [Zor06, pg. 92]).

3.2.5. Subgroups of interest. Depending on the intended application, it is important to analyze a subgroup of $\mathbf{GL}_2(\mathbb{R})$. Traditionally one restricts to $\mathbf{SL}_2(\mathbb{R})$, since it preserves the Masur–Veech measure, and hence one can apply the tools of ergodic theory. The other important subgroups are

$$P\colon \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \quad A\colon \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \quad N\colon \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \text{ and } R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

The action of $g_t := \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right]$ is called the *Teichmüller geodesic flow*, and plays a key role in the entire story; see Theorem 3.2.8 below for an illustration. We will not discuss the dynamics of the unipotent subgroup N in this survey, but see the work of Chaika, Smillie, and Weiss [CSW20] for some recent developments.

Let us also note that one can also restrict the action to the connected component of the identity $\mathbf{GL}_2^+(\mathbb{R})$. The action of $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ agrees with that of complex conjugation on $\Omega\mathcal{M}_g(\kappa)$, viewed as an algebraic variety over $\mathbb{C}$. In particular, this induces a dichotomy on orbit closures (see Section 4) according to whether they are preserved by complex conjugation, or not. If they are preserved by complex conjugation, and so can be descended to varieties over $\mathbb{R}$, it is meaningful to ask:

3.2.6. Question (Real locus in orbit closures)**.** Describe the real-algebraic locus of an orbit closure, i.e. those translation surfaces (X,ω) which are isomorphic to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right](X,\omega)$, or equivalently isomorphic to $(\overline{X},\overline{\omega})$ where $\overline{\bullet}$ denotes complex conjugation.

3.2.7. An application: unique ergodicity. An illustration of the connection between the dynamics on an individual translation surface (X,ω) and that on its moduli space is provided by the following criterion due to Masur [Mas92, Thm. 1.1]:

3.2.8. Theorem (Recurrence implies unique ergodicity). *Suppose that the g_t-orbit of (X, ω) returns to a compact set $K \subset \Omega\mathcal{M}_g(\kappa)$ infinitely often in the future, i.e. there exist $t_i \to +\infty$ such that $g_{t_i}(X, \omega) \in K$.*
Then the horizontal foliation of ω is uniquely ergodic.

This statement can be viewed as a general principle applicable in many situations, and a version of it in the setting of K3 surfaces is established in [FT21, Thm. 4.3.1]. It also turns out that the assumption of recurrence to compact sets in Theorem 3.2.8 can be weakened to sufficiently slow divergence. This has been developed by Cheung–Eskin [CE07, Thm. 1.1] using techniques from flat geometry and extended by Treviño [Tre14, Thms. 2-4] to also cover infinite genus, finite-area translation surfaces, using techniques from Hodge theory. Besides Masur's original proof, other treatments can be found in [For02, Thm. 0.1] and [AF08, Thm. 1.1] which also establish estimates on the speed of convergence of ergodic averages, as well as [FM14, Thm. 59], and [McM20, Thm. 1.1].

One of the early striking applications of this criterion was obtained by Kerckhoff, Masur, and Smillie [KMS86, Thm. 2]:

3.2.9. Theorem (Recurrence for every surface). *For every translation surface (X, ω) and for Lebesgue-a.e. $\theta \in [0, 2\pi]$, the g_t-orbit of $R_\theta(X, \omega)$ is recurrent in the sense of Theorem 3.2.8.*
In particular, the horizontal foliation of $R_\theta(X, \omega)$ is uniquely ergodic, for Lebesgue-a.e. θ.

A crucial point of the above theorem is that it applies to *every* translation surface. In particular, it applies to a translation surface obtained from a billiard table with rational angles, and gives:

3.2.10. Corollary (Unique ergodicity of rational billiards). *Let P be a polygon in $\mathbb{R}^2$ with angles in $\mathbb{Q}\pi$. Then for Lebesgue-a.e. θ, the billiard flow on the unit tangent bundle of P in direction θ is uniquely ergodic.*

Note that for every θ, there is a natural billiard flow-invariant Lebesgue class measure supported on the appropriate subset of the unit tangent bundle.

For applications of this flavor, it has been desirable to obtain theorems that apply to *every* translation surface (X, ω). We will describe the most general such results, due to Eskin, Mirzakhani, and Mohammadi, in Section 4.

3.2.11. Irrational billiards. An observation of Katok and Zemljakov is that ergodicity, as well as minimality, is a property that holds on a G_δ-set in the parameter space of polygons with given number of sides. Corollary 3.2.10 shows that ergodicity holds at the rational-angled polygons, which are dense, hence a dense G_δ-set of polygons are ergodic. In particular, there are "many", in the Baire category sense, irrational polygons which are ergodic. The same holds for minimality, and was established much earlier in [KZ75].

3.3. Meromorphic strata

One may generalize the setup of a translation surface (X, ω) to the case when ω is allowed to have poles. In this case the area of the surface is infinite and a number of significant differences arise. The dynamics on the individual surface is, in a way, simpler: there is a "convex core" $C(X, \omega)$ that has finite area and contains all the bounded linear trajectories; all other linear trajectories escape to a pole. While the moduli spaces continue to have period coordinates, a natural $\mathbf{GL}_2(\mathbb{R})$-action, and volume forms, the analogue of Masur–Veech measure has infinite total mass. The $\mathbf{GL}_2(\mathbb{R})$-action can now have positive-dimensional stabilizers, and closed submanifolds that are $\mathbb{R}$-linear in period coordinates need not be algebraic. We include below some illustrative examples.

3.3.1. Setup. Let $\kappa = (k_1, \ldots, k_n)$ be a collection of integers and $\Omega\mathcal{M}_g(\kappa)$ denote the parameter space of pairs (X, ω) where X is a compact Riemann surface of genus g and ω is a meromorphic differential with zeros of order $k_i > 0$, poles of order $k_i < 0$, and marked points corresponding to $k_i = 0$; the relation $\sum k_i = 2g - 2$ must hold. For a meromorphic differential ω, we will denote by $(\omega)_{<0}$ the divisor of poles, by $(\omega)_{>0}$ the divisor of zeros, and by $(\omega)_{\geq 0}$ the divisor of zeros and the marked points.

Period coordinates on $\Omega\mathcal{M}_g(\kappa)$ are also available, and in this case they are valued in the relative cohomology group

$$\mathrm{H}^1_\circ(X \setminus (\omega)_{<0}, (\omega)_{\geq 0}; \mathbb{C})$$

where $X \setminus (\omega)_{<0}$ is the open Riemann surface with the poles of ω removed, and we are taking the cohomology relative to the finite subset $(\omega)_{\geq 0}$. The subscript $\circ$ denotes the codimension 1 subspace cut out by the condition that the sum of residues vanishes: it is given by pairing the cohomology against the 1-cycle that circles each pole exactly once clockwise. See also [BCG$^+$19a, Thm. 2.1] for the algebraic description of the same cohomology group. Note that the total dimension, in the presence of poles, is $2g + |\kappa| - 2$, as opposed to $2g + |\kappa| - 1$ in the holomorphic case.

3.3.2. Connected components. Boissy [Boi15] classified the connected components. Surprisingly, in genus 1 there can be an arbitrarily large number of connected components as $|\kappa|$ grows, while for genus $g \geq 2$, there are at most three connected components, just like in the case of holomorphic differentials treated by Kontsevich & Zorich [KZ03], see §3.1.16.

3.3.3. Geometry of meromorphic differentials. The results of [HKK17, §2.3] show that any (X, ω), where ω has at least one pole, has a canonical *convex core* $C(X, \omega) \subset X$, defined as the convex hull of the non-pole singularities. The complement of poles $X \setminus (\omega)_{<0}$ retracts to $C(X, \omega)$, and the boundary $\partial C(X, \omega)$ is a finite union of saddle connections. Any saddle connection or flat cylinder is contained in $C(X, \omega)$. The structure is analogous to the convex core of a geometrically finite hyperbolic surface (or manifold).

3.3.4. A description of $\Omega\mathcal{M}_1(2,-2)$. A more detailed study of the genus 1 case was done by Tahar, and we refer to [Tah18, §4] for proofs of the next results.

It is possible to exhibit any $(X,\omega) \in \Omega\mathcal{M}_1(2,-2)$ by gluing two half-planes in $\mathbb{C}$, with appropriate "wedges" removed or added. This naturally leads to a $\mathbf{GL}_2(\mathbb{R})$-invariant stratification with three strata. Note that the topological structure of the convex core $C(X,\omega)$ does not change under the action of $\mathbf{GL}_2(\mathbb{R})$.

Two strata are open subsets of $\Omega\mathcal{M}_1(2,-2)$ and homogeneous under the $\mathbf{GL}_2(\mathbb{R})$-action. In one such stratum, the convex core consists of a cylinder, in the other it consists of two non-collinear saddle connections. Finally, the real codimension 1 stratum consists of translation surfaces with convex core consisting of two collinear saddle connections.

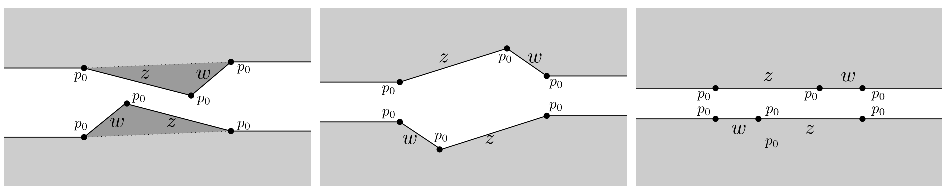

FIGURE 3.3.5. The three strata in $\Omega\mathcal{M}_1(2,-2)$, with pole p_{-1} at infinity. In the first stratum, the convex core is a cylinder (shaded). In the second, it consists of two nonparallel saddle connections. In the third, the two saddle connections are parallel.

Note that orbits in the real codimension 1 stratum have stabilizer in $\mathbf{GL}_2(\mathbb{R})$ conjugated to the subgroup $\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$ and the stratum is partitioned into a real 1-parameter family of $\mathbf{GL}_2(\mathbb{R})$-orbits. The invariant of an orbit is the ratio of lengths of the two collinear saddle connections, and the orbit itself is naturally identified with $\mathbb{C}^\times$.

Let $\mathcal{M}_{1,1}[2]$ denote the moduli space of genus 1 Riemann surfaces with one marked point, and a choice of nontrivial point of order 2 (with respect to the marked point). This is a degree 3 cover of $\mathcal{M}_{1,1}$, and a quotient by an involution of the the space with a full marking on $\mathbb{Z}/2$-homology. It has two cusps and one orbifold point of order 2, at the "square" torus. We have a forgetful map

$$\Omega\mathcal{M}_1(2,-2) \to \mathcal{M}_{1,1}[2]$$

that takes (X,ω) to the underlying Riemann surface of genus 1, with origin as the zero, and the pole as a non-trivial point of order 2. Indeed, the difference between the zero and pole of ω in the group structure is 2-torsion, since the divisor of zeros and poles of ω must induce the trivial line bundle, because this line bundle has a nontrivial section (ω itself) and has degree 0.

With respect to the stratification of $\Omega\mathcal{M}_1(2,-2)$ described above, we note that the real codimension 1 stratum maps to the locus of Riemann surfaces with a real structure, and for which the marked 2-torsion is also real; on $\mathcal{M}_{1,1}[2]$ this consists of one hyperbolic geodesics connecting one of the cusps with itself. This geodesic cuts $\mathcal{M}_{1,1}[2]$ into two components, one of which contains the orbifold point.

3.3.6. A non-algebraic linear manifold. It was observed by Bakker & Mullane [BM20] that strata of meromorphic differentials can contain $\mathbb{R}$-linear, but non-algebraic manifolds, in contrast to Theorem 4.4.10. Consider $\Omega\mathcal{M}_1(2,-2,0,0)$, the stratum of meromorphic 1-forms on a genus 1 surface with a zero and a pole of order 2, and two marked points (with all 4 points distinct). The natural forgetful map

$$\Omega\mathcal{M}_1(2,-2,0,0)\to\Omega\mathcal{M}_1(2,-2)$$

is algebraic and induces an algebraic structure on its fibers. Let us fix one elliptic curve with meromorphic differential (X_0,ω_0), for instance the one corresponding to $z=w=1$ in Figure 3.3.5, and call $p_0,p_{-1}\in X_0$ the zero and the pole. All our constructions will be equivariant for the $\mathbb{C}^\times$-action by scaling. Then the fibers of the forgetful map are identified with $(X_0\setminus\{p_0,p_{-1}\})^{(2)}$, the set of distinct pairs of points on X_0 avoiding p_0,p_{-1}. We now proceed to describe the linear structure.

Besides the two periods z,w, we also have $z_1,z_2\in\mathbb{C}$ as per Figure 3.3.7, subject to the constraints that ensure all points are distinct. Note that say when p_1 passes through one slit, we have $z_1\mapsto z_1+z$ or $z_1\mapsto z_1+w$ (depending on the slit) and similarly for p_2.

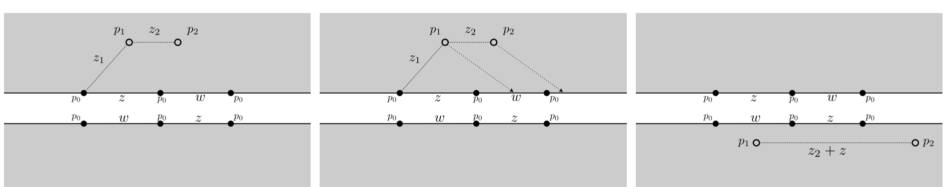

FIGURE 3.3.7. Left: a surface in $\Omega\mathcal{M}_1(2,-2,0,0)$. Middle: The two marked points move, with one of them crossing a branch cut. Right: One point crossed the branch cut, the marked period between them changed.

We now set $\mathcal{L}$ to be preimage in $\Omega\mathcal{M}_1(2,-2,0,0)$ of the locus given in period coordinates by the linear equations $z=w$ and $w=z_2$. We can intersect $\mathcal{L}$ with the algebraic locus where p_1 is fixed (and $\operatorname{Im} z_1\neq 0$), and the other two periods are fixed to $z=w=1$, and find that it will intersect this complex 1-dimensional set in countably many points that in period coordinates are equal to p_1+k with $k\in\mathbb{Z}\setminus\{0\}$.

3.4. Compactifications

We will use in this section the term (smooth) "algebraic curve" as a substitute for "compact Riemann surface", and more generally the language of algebraic geometry as it makes the discussion more streamlined.

3.4.1. Some context. The moduli space $\mathcal{M}_g$ of genus g algebraic curves carries a natural algebraic structure, i.e. it is covered by charts contained in some $\mathbb{C}^n$ and cut out by algebraic equations. This can rarely be made explicit, especially for large g, but nonetheless puts strong finiteness conditions on the geometry of $\mathcal{M}_g$. The space is not compact and a natural smooth compactification is available: the Deligne–Mumford compactification [DM69] denoted $\overline{\mathcal{M}}_g$. Besides smoothness, its other essential property is that the universal family family of curves over $\mathcal{M}_g$ extends to $\overline{\mathcal{M}_g}$; fibers over the boundary are semistable, of arithmetic genus g.

The universal family over $\overline{\mathcal{M}}_g$ allows one to study the compactified moduli space in the same way as the original space: by considering the geometry of the curves, instead of looking for explicit equations of the moduli space.

Analogous constructions for the strata $\Omega\mathcal{M}_g(\kappa)$ have been developed in [BCG$^+$18, BCG$^+$19b] and this section provides a brief survey of some of this work.

3.4.2. Warm-up: Marked stable curves. By definition, a *marked stable curve* (X, Z) consists of an algebraic curve X, with irreducible components smooth curves X_i so $X = \cup X_i$, and with finitely many marked and labeled points $Z \subset X$ that are distinct from the nodes of X. The *nodes* of $X = \cup X_i$ are the points of the components X_i where the map $\coprod X_i \to X$ is not injective. Finally, the stability requirement is that for every irreducible component we have

$$2 \cdot \operatorname{genus}(X_i) - 2 + \#(\text{points on } X_i) > 0$$

where points refers to nodes and marked points.

The *dual stable graph* Γ_X of a stable curve is defined to have vertices the set of irreducible components, with a vertex denoted $[X_i]$ and labeled by the genus $of X_i$, and with half-edges given by the nodes or marked points. The half-edges corresponding to opposite nodes are glued to give a full edge, the half-edges of marked points are labeled by the index of the corresponding marked point.

3.4.3. The Deligne–Mumford compactification. A more detailed discussion is in the foundational work of Deligne & Mumford [DM69, Thm. 5.2]. We will describe set-theoretically the strata $\mathcal{M}_\Gamma$ of the compactification $\overline{\mathcal{M}}_{g,n}$. Each possible dual stable graph Γ gives a stratum, with a point on that stratum described as follows. Each vertex v of Γ, with label g_v and valency $\deg v$, gives a genus g_v Riemann surface X_v with $\deg v$ marked points. The marked points corresponding to nodes are glued accordingly to give the stable curve $X = \cup X_v$ with the remaining marked points.

The open stratum $\mathcal{M}_{g,n}$ corresponds to one vertex of genus g, with n half edges coming out of it. The codimension of any stratum is the number of full edges of the corresponding dual stable graph. Up to finite automorphism groups we have

$$\mathcal{M}_\Gamma \approx \prod \mathcal{M}_{g_v, \deg v}$$

For example, connected trivalent graphs with $3g-3$ edges, or equivalently $2g-2$ vertices, parametrize the deepest points of $\overline{\mathcal{M}}_g$.

3.4.4. Incidence variety compactification. To proceed we define a *stable differential* ω on a stable curve X to be the datum of meromorphic differentials ω_i on each irreducible component X_i, with poles only allowed at the nodes of X_i, such that the poles are simple and residues at opposite nodes add up to zero. The bundle of holomorphic differentials over the moduli space $\mathcal{M}_g$ is denoted $\Omega\mathcal{M}_g$, and it extends over the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ as a rank g vector bundle denoted $\Omega\overline{\mathcal{M}}_g$ and parametrizing stable differentials. It pulls back naturally to $\overline{\mathcal{M}}_{g,n}$ and denoted $\Omega\overline{\mathcal{M}}_{g,n}$.

For a given configuration of zeros $\kappa = (k_1, \ldots, k_n)$ and stratum $\Omega\mathcal{M}_g(\kappa)$, there is a natural map assigning to a differential its divisor of zeros:

$$\begin{aligned}
&\Omega\mathcal{M}_g(\kappa) \to \Omega\mathcal{M}_{g,n} \\
&\quad (X,\omega) \mapsto (X,\omega,z_1,\ldots,z_n) \text{ where } (\omega) = \sum k_i z_i
\end{aligned}$$

The *incidence variety compactification* $\Omega\overline{\mathcal{M}}_{g,n}^{inc}(\kappa)$ is defined to be the closure of the image of this (injective) map. The main result of [BCG+18, Thm. 1.3] is a characterization of the stable differentials in the compactification, to which we now proceed.

3.4.5. Definition (Twisted κ-differentials)**.** Let $(X, z_1, \ldots, z_n)$ be a stable curve $X = \cup X_i$ with n marked and labeled points. For $\kappa = (k_1, \ldots, k_n)$, a *twisted κ-differential* is:

(i) A meromorphic differential ω_i on X_i, with zeros and poles allowed only at the nodes and marked points, and furthermore required to satisfy $\operatorname{ord}_{z_j} \omega_i = k_j$.

(ii) At opposite nodes q_1, q_2 on components X_{i_1}, X_{i_2} require the pole or vanishing orders to satisfy:

$$\operatorname{ord}_{q_1} \omega_{i_1} + \operatorname{ord}_{q_2} \omega_{i_2} = -2.$$

If furthermore $\operatorname{ord}_{q_1} \omega_{i_1} = \operatorname{ord}_{q_2} \omega_{i_2} = -1$, then require also the residues to satisfy:

$$\operatorname{Res}_{q_1} \omega_{i_1} + \operatorname{Res}_{q_2} \omega_{i_2} = 0.$$

The extra datum on the dual graph Γ_X is a partial order $\succeq$, such that any two elements are comparable but in general $[X_i] \succeq [X_j]$ and $[X_i] \preceq [X_j]$ does not imply $[X_i] = [X_j]$, in other words the partial order is not strict. Such a partial order is equivalent to a real-valued function on the vertices, which we will always assume takes values in $\mathbb{Z}_{\leq 0}$ and the different levels are ordered by this function; for $l \in \mathbb{Z}_{\leq 0}$ we will denote by $X_{>l}$ and X_l the subsets of X whose irreducible components are at level strictly above l, and level l respectively. Edges will be called *horizontal* and *vertical* according to how they go between levels, and the enhancement to a partial order on a graph Γ will be denoted $\overline{\Gamma}$.

Given a twisted κ-differential on a stable curve X, a partial order on Γ_X is called *compatible* if, for opposite nodes $q_j \in X_{i_j}, j = 1, 2$ we have

$$[X_{i_1}] \succeq [X_{i_2}] \iff \operatorname{ord}_{q_1} \omega_{i_1} \geq \operatorname{ord}_{q_2} \omega_{i_2}.$$

Note that the last condition is equivalent to $\operatorname{ord}_{q_1} \omega_{i_1} \geq -1$. Furthermore, we impose the following *Global Residue Condition*: for every level l, and for every connected component X' of $X_{>l}$ that does not have a prescribed pole, we have:

$$\sum_{\text{node } q_i \in X_{>l} \cap X_l} \operatorname{Res}_{q_i^-} \omega_{X_i} = 0.$$

For a node $q \in X_i \cap X_j$ we denote by q^- the node that lives on the component at the lower level. The condition of not having a prescribed pole is vacuous if all $k_i \geq 0$, and otherwise it means for no $z_i \in X'$ we have $k_i < 0$.

We ca now state a characterization of points that belong to the incidence variety compactification [BCG+18, Thm. 1.3]:

3.4.6. Theorem (Characterization of limits)**.** *A marked stable curve (X, Z) with stable differential ω belongs to $\Omega\overline{\mathcal{M}}^{inc}_{g,|\kappa|}(\kappa)$ with $|Z| = |\kappa|$ if and only if:*

- *There exists a level graph structure on the dual graph Γ_X, such that the top vertices of $\overline{\Gamma}_X$ corresponds to components where $\omega \not\equiv 0$.*
- *There exists a twisted κ-differential η on X, compatible with $\overline{\Gamma}_X$, such that on the top components we have $\omega = \eta$.*

It turns out that the incidence compactification $\Omega\overline{\mathcal{M}}^{inc}_{g,n}(\kappa)$ is not smooth, even in the orbifold sense. To address this, the authors of [BCG+19b] have introduced a larger moduli space, which has a lot of desirable properties. We outline the structures and results, referring to the original text for the details.

3.4.7. Definition (Multiscale differential)**.** A *multiscale differential* of type κ on $(X, Z) \in \overline{\mathcal{M}}_{g,n}$ is the datum of:

(i) An *enhanced level structure* on $\overline{\Gamma}_X$.
(ii) A twisted κ-differential ω on (X, Z) compatible with the enhanced level structure.
(iii) A prong-matching condition for every pair of opposite nodes on X.

We refer to [BCG+19b, §2] for the precise definitions of these notions. The main results are Thms. 1.2-1.4 in loc. cit.:

3.4.8. Theorem (Moduli space of multiscale differentials)**.** *There exists a complex-analytic orbifold $\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ parametrizing a universal family of multiscale differentials, with the following additional properties:*

(i) *The stratum $\Omega\mathcal{M}_{g,n}(\kappa)$ is open and dense.*
(ii) *The boundary is a simple normal crossing divisor.*
(iii) *The space $\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ admits a free $\mathbb{C}^{\times}$-action whose quotient is compact.*

3.4.9. Theorem ($\mathbf{GL}_2(\mathbb{R})$ action on bordification)**.** *There exists a real-oriented blowup of $\Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ denoted $\Xi\hat{\mathcal{M}}_{g,n}(\kappa)$, which is an orbifold with corners parametrizing a universal family of real multiscale differentials, with the following additional properties:*

(i) *The map $\Xi\hat{\mathcal{M}}_{g,n}(\kappa) \to \Xi\overline{\mathcal{M}}_{g,n}(\kappa)$ is proper and the fibers are isomorphic to $(\mathbb{R}/\mathbb{Z})^N$ over a multiscale differential with $N+1$ levels.*
(ii) *The action of $\mathbf{GL}_2(\mathbb{R})$ on the open subset $\Omega\mathcal{M}_{g,n}(\kappa)$ extends continuously to $\Xi\hat{\mathcal{M}}_{g,n}(\kappa)$.*

Let us note that while we have restricted our discussion to start with holomorphic differentials $\Omega\mathcal{M}_g(\kappa)$, the authors of [BCG+19b] allow more general meromorphic strata as in §3.3.

4. Orbit closures

Outline of section. In this section we describe a series of rigidity properties of orbit closures of the $\mathbf{GL}_2(\mathbb{R})$-action. These turn out to be orbifolds with interesting geometric and arithmetic properties. Their measure-theoretic and topological properties are outlined in §4.1. In Example 4.1.10 we illustrate a situation where the orbit closure has self-intersections when immersed in the ambient stratum. After some preliminaries from Hodge theory in §4.2, we describe in §4.3 complex-analytic rigidity features of orbit closures. One application of these rigidity properties is to prove that orbit closures have natural algebraic structures, and can in fact be characterized by arithmetic properties of the Jacobian varieties of the underlying Riemann surfaces. This is explained in §4.4. Further consequences of the Hodge-theoretic rigidity results are contained in Section 5, which describes finiteness results for orbit closures. We end with an overview of some examples of orbit closures in §4.5, including some linear manifolds which are not orbit closures but are of independent interest.

4.1. Measure and Topological Rigidity

In this section we describe the measure-theoretic and topological rigidity results obtained by Eskin, Mirzakhani, and Mohammadi [EMM15, EM18]. These results were motivated and inspired by Ratner's rigidity theorems for unipotent flows, which established Raghunathan's conjectures [Rat91]. The unipotent flow on strata has substantially more involved behavior than on homogeneous spaces (see the constructions of Chaika–Smillie–Weiss [CSW20]), so the techniques are rather different and originate in the work of Benoist–Quint [BQ11].

4.1.1. Setup. Fix a stratum $\Omega\mathcal{M}_g(\kappa)$ of translation surfaces and recall from §3.1.9 that on its universal cover we have period coordinates, equivalently a developing map which is a local biholomorphism:

$$\mathrm{Dev}\colon \widetilde{\Omega\mathcal{M}}_g(\kappa) \to \mathrm{H}^1(X_0, Z_0; \mathbb{C})$$

for some reference translation surface (X_0, ω_0).

4.1.2. Definition (Linear Immersed Submanifold)**.** A *linear immersed submanifold* of $\Omega\mathcal{M}_g(\kappa)$ is a manifold $\mathcal{M}^a$ together with a proper immersion $\iota\colon \mathcal{M}^a \to \Omega\mathcal{M}_g(\kappa)$, such that for any sufficiently small open set $U \subset \mathcal{M}^a$, the following holds: take the image $\iota(U) \subset \Omega\mathcal{M}_g(\kappa)$ and lift it to the universal cover as $\widetilde{\iota(U)} \subset \widetilde{\Omega\mathcal{M}}_g(\kappa)$, then the image under Dev inside $\mathrm{H}^1(X_0, Z_0; \mathbb{C})$ is an open set inside a linear subspace. We will denote the image of the immersion ι by $\mathcal{M} := \iota(\mathcal{M}^a) \subset \Omega\mathcal{M}_g(\kappa)$ and frequently omit $\mathcal{M}^a$ and ι from the notation.

For a subfield $k \subset \mathbb{C}$, if the local linear equations of the charts $\iota(U)$ can be taken with coefficients in k, we will say that $\mathcal{M}$ is *k-linear*. For convenience, we will say *k-linear submanifold* instead of k-linear immersed submanifold.

We will typically be interested in linear submanifolds that are at least $\mathbb{R}$-linear, so we make that assumption from now on. Note that in this case, the submanifold is invariant under the action of $\mathbf{GL}_2(\mathbb{R})$. A converse is provided by an observation of Kontsevich: any complex submanifold of $\Omega\mathcal{M}_g(\kappa)$ that is invariant under $\mathbf{GL}_2(\mathbb{R})$ must be $\mathbb{R}$-linear (see [Möl08, Prop. 1.2] and assume irreducibility).

4.1.3. The basic exact sequence. When discussing linear submanifolds, we will refer only to $\mathcal{M} \subset \Omega\mathcal{M}_g(\kappa)$ and omit from notation the "abstract" manifold $\mathcal{M}^a$ that maps to $\mathcal{M}$. However, most objects are naturally defined on $\mathcal{M}^a$. Of these, the most important ones are the local systems that describe the tangent space of $\mathcal{M}$. Specifically, let $T\mathcal{M}$ denote the tangent bundle of $\mathcal{M}$, viewed as a vector bundle on $\mathcal{M}^a$. Since charts of $\mathcal{M}^a$ are locally cut out by linear equations, the bundle $T\mathcal{M}$ is a local subsystem of H^1_{rel}. Then in analogy with the short exact sequence in Eqn. (3.1.13), we have one on the tangent space of $\mathcal{M}$:

$$0 \to \mathrm{W}_0(T\mathcal{M}) \hookrightarrow T\mathcal{M} \overset{p}{\twoheadrightarrow} \mathrm{H}^1(T\mathcal{M}) \to 0 \tag{4.1.4}$$

Note that the local system W_0 on the entire stratum has finite monodromy and can be trivialized on a finite cover that labels the marked points. Therefore, the same is true of $\mathrm{W}_0 T\mathcal{M}$, and in particular it carries a monodromy-invariant positive-definite inner product.

4.1.5. Cylinder and Torsion Corank. The two basic numerical invariants of an orbit closure $\mathcal{M}$ are its *rank* (or *cylinder rank*), introduced by Wright [Wri15a, Def. 1.11] and defined to be $\frac{1}{2}\dim \mathrm{H}^1(T\mathcal{M})$, as well as its *torsion corank* defined to be $\dim \mathrm{W}_0(T\mathcal{M})$; we will refer to $\dim \mathrm{W}_0/\dim \mathrm{W}_0(T\mathcal{M})$ as the *torsion rank* of $\mathcal{M}$. We will see in Theorem 4.4.7 the relation between W_0 and torsion, and the connection between cylinders and $\mathrm{H}^1(T\mathcal{M})$ in §6.1.

The cylinder rank is always an integer, as follows from the next basic result regarding the tangent space $T\mathcal{M}$ that was proved by Avila, Eskin, and Möller [AEM17, Thm. 1.4-1.5]:

4.1.6. Theorem ($T\mathcal{M}$ is symplectic)**.** *For an $\mathbb{R}$-linear manifold $\mathcal{M}$ admitting an $\mathbf{SL}_2(\mathbb{R})$-invariant probability measure, the symplectic form obtained from the topological cup product is nondegenerate on $\mathrm{H}^1(T\mathcal{M})$.*

4.1.7. Volume normalizations. For a linear manifold $\mathcal{M} \subset \Omega\mathcal{M}_g(\kappa)$, denote by $\mathcal{M}^1 \subset \mathcal{M}$ the subset of area 1 translation surfaces, so the natural map $\mathcal{M}^1 \times \mathbb{R}_{>0} \to \mathcal{M}$ is a bijection. Note also that $\mathbf{SL}_2(\mathbb{R})$ preserves $\mathcal{M}^1$. The natural Lebesgue measures on $\mathcal{M}$ and $\mathcal{M}^1$ are constructed in charts, in analogy with Masur–Veech measure from §3.2.4. A key input is that $T\mathcal{M}$ is symplectic, in the sense of Theorem 4.1.6.

We can now state the main results of Eskin, Mirzakhani, and Mohammadi. For convenience we will state them on the subset of area 1 surfaces, and in particular $\Omega\mathcal{M}_g(\kappa)^1$ denotes the subset of the stratum of surfaces normalized in this way.

4.1.8. Theorem (Measure and Topological Rigidity)**.** *Let $P \subset \mathbf{SL}_2(\mathbb{R})$ denote the upper-triangular matrices and fix a stratum $\Omega\mathcal{M}_g(\kappa)^1$.*

Measure rigidity: [EM18, Thm. 1.4] *For any P-ergodic invariant probability measure μ on the stratum, there exists an $\mathbb{R}$-linear immersed submanifold $\mathcal{M}$ such that μ is the Lebesgue measure on $\mathcal{M}^1$ described in §4.1.7. In particular μ is $\mathbf{SL}_2(\mathbb{R})$-invariant.*

Topological rigidity: [EMM15, Thm. 2.1] *For any (X, ω) in the stratum, there exists an $\mathbb{R}$-linear immersed submanifold $\mathcal{M}$ which is its P-orbit closure, i.e. $\mathcal{M}^1 = \overline{P \cdot (X, \omega)}$. Furthermore, $\mathcal{M}^1$ admits a finite $\mathbf{SL}_2(\mathbb{R})$-invariant measure.*

Equidistribution: [EMM15, Thm. 2.3] *The space of P-invariant ergodic probability measures on the stratum is sequentially compact for the weak-* topology.*

Isolation: [EMM15, Thm. 2.3] *For any sequence of linear immersed submanifolds $\mathcal{M}_i$ admitting a finite P-invariant measure μ_i, after passing to a subsequence still denoted $\{\mathcal{M}_i, \mu_i\}$, there exists another linear immersed submanifold $\mathcal{M}$, with finite measure μ, and $i_0 \geq 1$ such that for $i \geq i_0$ we have $\mathcal{M}_i \subset \mathcal{M}$ and $\mu_i \rightharpoonup^* \mu$.*

4.1.9. Remark (On rigidity)**.**

(i) The equidistribution theorem has its name justified by the following reformulation: any sequence of P-invariant ergodic probability measures converges weakly along a subsequence to another such.

(ii) The statements for the upper triangular group P imply the analogous ones for $\mathbf{SL}_2(\mathbb{R})$. Part of the result is that any finite P-invariant measure is also $\mathbf{SL}_2(\mathbb{R})$-invariant, since Lebesgue measure on an $\mathbb{R}$-linear submanifold is $\mathbf{SL}_2(\mathbb{R})$-invariant. However, since P is amenable, one can more easily construct P-invariant probability measures.

We end with a cautionary example that illustrates the necessity of allowing self-intersections.

4.1.10. Example (Self-intersections of an orbit closure)**.** Denote by $\mathcal{M} \subset \Omega\mathcal{M}_3(2,2)$ the locus of surfaces that are unramified $(\mathbb{Z}/2)$-covers of surfaces in $\Omega\mathcal{M}_2(2)$. Note that we have an "abstract" finite cover $\mathcal{M}^a \to \Omega\mathcal{M}_2(2)$ and a map $\mathcal{M}^a \to \mathcal{M} \subset \Omega\mathcal{M}_3(2,2)$. We will next verify that inside $\Omega\mathcal{M}_3(2,2)$, the invariant subvariety $\mathcal{M}$ has self-intersections along a locus $\mathcal{R} \subset \mathcal{M}$ that we now describe in more detail.

Specifically, let $\Omega\mathcal{M}_1(0)$ denote the translation surfaces of genus 1 with one marked point. Set $\mathcal{R}$ to be a finite (unramified) cover of $\Omega\mathcal{M}_1(0)$, consisting of translation surfaces $(X,\omega) \in \Omega\mathcal{M}_3(2,2)$ with an action of the permutation group on 3 elements $\mathbb{S}_3$, such that the cyclic subgroup of $\mathbb{S}_3$ fixes the zeros of ω and transpositions exchange the zeros. Note that $\mathcal{R}$ is contained in $\mathcal{M}$, but that elements $(X,\omega) \in \mathcal{R}$ map to some $(X',\omega') \in \Omega\mathcal{M}_2(2)$ in three distinct ways, one for each transposition in $\mathbb{S}_3$.

Concretely, the tangent space $T_{(X,\omega)}\Omega\mathcal{M}_3(2,2) = \mathrm{H}^1_{rel}(X, Z_\omega)$ has an action of $\mathbf{S}_3$ and using this action splits as $\mathrm{H}^1(X) \oplus \tilde{\mathrm{H}}^0(Z_\omega)$, where $Z_\omega \subset X$ are the two zeros of ω. We claim, and will verify shortly, that $\mathrm{H}^1(X)$ as an $\mathbb{S}_3$-representation consists of the trivial representation with multiplicity two, and the unique 2-dimensional representation also with multiplicity two. In fact we have $\mathrm{H}^1(X) = \mathrm{H}^1(X/\mathbb{S}_3) \oplus \mathrm{T}_X$ where $X/\mathbb{S}_3$ is the torus with a marked point, and $\mathrm{T_X} = \mathrm{V_X} \otimes \mathrm{M_X}$, where V_X is a rank 2, weight 1 Hodge structure on which $\mathbb{S}_3$ acts trivially, and M_X is a 2-dimensional vector space on which $\mathbb{S}_3$ acts in its unique irreducible of dimension 2.

The three branches of $\mathcal{M}$ that pass through $(X,\omega) \in \mathcal{R}$ are parametrized by the choice of transposition $\sigma \in \mathbb{S}_3$ and have as their tangent spaces:

$$T_{(X,\omega),\sigma}\mathcal{M} = \mathrm{H}^1(X/\mathbb{S}^3) \oplus [\mathrm{V}_X \otimes (\mathrm{M}_X^\sigma)] = \mathrm{H}^1(X/\sigma)$$

where M_X^σ denotes the σ-fixed line inside M_X. Note that the first factor $\mathrm{H}^1(X/\mathbb{S}^3)$ is simply the tangent space of $\mathcal{R}$ at (X,ω).

It remains to verify the assertion about the decomposition of $\mathrm{H}^1(X)$ into $\mathbb{S}_3$-representations. This can be seen from the Chevalley–Weil formula (see e.g. [Ara22]) or more elementarily as follows. Triangulate X as $X^{(0)} \coprod X^{(1)} \coprod X^{(2)}$ where the 0-dimensional piece $X^{(0)}$ consists of the two zeros. The action of $\mathbb{S}_3$ on $X^{(1)}$ and $X^{(2)}$ is free, so the characters on the corresponding groups that compute the homology are $\frac{x_1}{6}\chi_{reg}$ and $\frac{x_2}{6}\chi_{reg}$, where $x_i = \#X^{(i)}$ and χ_{reg} denotes the character of the regular representation (note that $6 = \#\mathbb{S}_3$ so that the ranks match). Finally on the zero-dimensional group the character is the sum of the trivial and sign representations $\mathbf{1} + \chi_{sg}$. We know that after taking homology, H^0 and H^2 are 1-dimensional and $\mathbb{S}_3$ acts trivially on them, so it follows that we must subtract the corresponding pieces from the character on H^1:

$$\begin{aligned}\chi_{\mathrm{H}^1(X)} &= \frac{x_1}{6}\chi_{reg} - \left(\frac{x_2}{6}\chi_{reg} - \mathbf{1}\right) - \chi_{sg}\\ &= \frac{x_1 - x_2}{6}\chi_{reg} + \mathbf{1} - \chi_{sg}\\ &= \chi_{reg} + \mathbf{1} - \chi_{sg}\\ &= 2\left(\mathbf{1} + \chi_\rho\right)\end{aligned}$$

where we have used that $x_1 - x_2 = 2 \cdot \mathrm{genus}(X) = 6$ since $2 - x_1 + x_2$ is the Euler characteristic of X, that for any finite group G the Peter–Weyl theorem yields $\chi_{reg} = \sum_{\mathrm{irrep}\ \xi} \dim \xi \cdot \chi_\xi$, and ρ denotes the unique irreducible of $\mathbb{S}_3$ of dimension 2.

4.2. Hodge Theory

This section provides the background in Hodge theory necessary to describe, and establish, the results in §4.3. The information contained in the cohomology groups H^1 and H^1_{rel} is equivalent to that contained in their duals, the homology groups $\check{\mathrm{H}}_1$ and $\check{\mathrm{H}}_{1,rel}$. For many constructions, the homology groups provide a more convenient geometric interpretation, for instance the Jacobian (see §4.2.7) is most easily described using homology. Speaking informally, one can think of the structures in cohomology as encoding the tangent space data to an orbit closure inside a stratum, while structures in homology describe the linear equations cutting out the orbit closure inside a stratum.

4.2.1. Setup. Let $k \subset \mathbb{R}$ be a subring, for instance $\mathbb{Z}, \mathbb{Q}$, or a number field, and let H be a free k-module of finite rank. We will denote by H_R the extension of scalars from k to any ring R containing k.

4.2.2. Definition (Pure Hodge structure, Polarization)**.** A weight n Hodge structure on H is a decomposition of the complexification

$$H_{\mathbb{C}} = H^{n,0} \oplus \cdots H^{p,q} \oplus \cdots H^{0,n} \quad p + q = n$$

such that $H^{p,q} = \overline{H^{q,p}}$ for all p, q with $p + q = n$.

The *Hodge filtration* is defined by $F^pH := \oplus_{i \geq p} H^{i,j}$ and it is decreasing, i.e. $F^pH \supseteq F^{p+1}H$. The *Hodge decomposition* can be reconstituted by $H^{p,q} = F^pH \cap \overline{F^qH}$. The *Weil operator* is defined on $H_{\mathbb{C}}$ first by $Cx := \sqrt{-1}^{p-q}$ if $x \in H^{p,q}$, and it descends to H since $C\overline{x} = \overline{Cx}$.

A *polarization* of the Hodge structure is a k-bilinear form $I(-,-)$ on H such that the bilinear form $Q(x,y) := I(x,Cy)$ is symmetric and positive-definite on $H_{\mathbb{R}}$.

Note that since $C^2 = (-1)^n$, it follows that I is $(-1)^n$-symmetric.

4.2.3. Definition (Mixed Hodge structure)**.** A *mixed Hodge structure* on H is the data of an increasing filtration $W_\bullet H$ on H, called the *weight filtration*, and a decreasing filtration $F^\bullet H$ on $H_{\mathbb{C}}$, called the *Hodge filtration*, such that for any n, the quotient $\operatorname{gr}^W_n H := W_n H/W_{n-1}H$ with its induced Hodge filtration is a pure weight n Hodge structure.

For a ring $R \subset \mathbb{C}$ containing k, we will say that the mixed Hodge structure is R-split if H_R with its induced mixed Hodge structure is isomorphic to the direct sum $\oplus_{n\in\mathbb{Z}} \operatorname{gr}^W_n H_R$.

4.2.4. Example (Compact Riemann surfaces)**.** Let X be a compact Riemann surface of genus g. Then its integral cohomology $H := H^1(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $2g$ and admits a pure weight 1 polarized Hodge structure.

The complexification $H_{\mathbb{C}}$ has the Hodge decomposition

$$H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$$

with $H^{1,0}$ spanned by cohomology classes of holomorphic 1-forms on X. Indeed, a holomorphic 1-form on X defines a cohomology class by integrating it along 1-cycles. The space $H^{0,1}$ is defined as the complex-conjugate of $H^{0,1}$. The symplectic form given by cup product $I(x,y) := x \cup y$ has a positivity property for a holomorphic 1-form ω, namely $\sqrt{-1}[\omega] \cup [\overline{\omega}] > 0$. The reason is that cup product on 1-forms is given by integration on X and in local coordinates on X, if $\omega = f(z)dz$ then $\sqrt{-1}\omega \wedge \overline{\omega} = 2|f(z)|^2 dx \wedge dy$.

Let us also note that it is possible to describe the entire Hodge filtration using complex-analytic objects. The first term F^1H is just the space of holomorphic 1-forms, while $F^0H = H_{\mathbb{C}}$ can also be described as follows. Fix a point $x \in X$ and let Z_x denote the (infinite countable dimension) vector space of meromorphic 1-forms on X with a pole allowed at x only. Since there's at most one pole, and the sum of all residues must vanish, such a meromorphic 1-form has vanishing residue at x. Let also B_x denote the vector space of all meromorphic 1-forms on X that arise as differentials of meromorphic functions on X, with poles allowed only at x. There is a natural map $Z_x \to H_{\mathbb{C}}$, since a meromorphic 1-form on X with vanishing residues also gives a cohomology class. The kernel consists precisely of B_x, and the map can be verified to be surjective.

4.2.5. Example (Marked points on Riemann surfaces)**.** Let now X be a compact Riemann surface and $S \subset X$ a finite set of points. Let us describe the mixed Hodge structure on $\mathrm{H}^1_{rel} := \mathrm{H}^1(X,S;\mathbb{Z})$ with nontrivial pieces in weights 0 and 1. Using the short exact sequence in Eqn. (3.1.6) but with integral coefficients, we set $W_0\mathrm{H}^1_{rel} := \tilde{\mathrm{H}}^0(S)$ and W_1 to be all of H^1_{rel}. The only non-trivial piece of the Hodge filtration is $F^1\mathrm{H}^1_{rel}$ and again this is defined as the image of the holomorphic 1-forms on X, since these give not just absolute, but also relative cohomology classes by integration.

Note that the extra data in the mixed Hodge structure is that of a lift of F^1H from absolute to relative cohomology.

4.2.6. Linear algebra operations on Hodge structures. The standard operations on vector spaces or free $\mathbb{Z}$-modules, such as duality, tensor product, and Hom-spaces, are defined in the natural way for Hodge structures. Perhaps the shortest way to define these operations is by viewing a Hodge structure on a real vector space as a representation of the abelian $\mathbb{R}$-algebraic group $\mathbf{S} := \operatorname{Res}^{\mathbb{C}}_{\mathbb{R}} \mathbf{G}_m$ which is called the Deligne torus; it's $\mathbb{R}$-points are identified with $\mathbb{C}^{\times}$. We also extend the notion of a pure Hodge structure of weight n to allow arbitrary integral indexes with $p+q=n$, and allow the weight n to be negative as well.

Duals and tensor products of mixed Hodge structures are defined in the natural way. Note that mixed Hodge structures form an abelian category, a nontrivial fact since for example filtered vector spaces do not form one. For a (mixed) Hodge structure H, its dual will be denoted by $\check{H}$; its weights are the *negatives* of those of the original. Let us finally note that a mixed Hodge structure is always $\mathbb{R}$-split, and we will see an example below in Eqn. (4.2.12). For an $\mathbb{R}$-split mixed Hodge structure, we have a decomposition $H_{\mathbb{C}} = \oplus H^{p,q}$ such that $W_nH = \oplus_{p+q\leq n} H^{p,q}$ and $F^pH = \oplus_{i\geq p} H^{p,q}$.

4.2.7. Jacobians. We now specialize to weight 1 and describe a geometric interpretation of the above linear-algebraic data. Suppose that H carries a weight 1 Hodge structure over $\mathbb{Z}$, and we denote by $H_{\mathbb{Z}}$ the corresponding $\mathbb{Z}$-module of rank $2g$. Denote by $\check{H}$ the dual Hodge structure, which has weight -1 and Hodge decomposition

$$\check{H}_{\mathbb{C}} = \check{H}^{0,-1} \oplus \check{H}^{-1,0} \quad \check{H}^{p,q} := \left(H^{-p,-q}\right)^*$$

The nontrivial piece of the Hodge filtration is $F^0\check{H} = \check{H}^{0,-1}$, which can also be described as

$$\check{H}^{0,-1} \cong \{\xi\colon H \to \mathbb{C}\colon \xi|_{H^{1,0}} \equiv 0\}.$$

The *Jacobian* associated to this Hodge structure is defined to be the compact complex torus

$$\operatorname{Jac}(H) := F^0\check{H}\backslash \check{H}_{\mathbb{C}}/\check{H}_{\mathbb{Z}} \cong \operatorname{Hom}(H^{1,0},\mathbb{C})\Big/\operatorname{Hom}(H_{\mathbb{Z}},\mathbb{Z}) \tag{4.2.8}$$

which has complex dimension g. When H admits a polarization, the torus $\operatorname{Jac}(H)$ can be holomorphically embedded in a projective space and is an abelian variety.

This somewhat roundabout definition, via the dual Hodge structure, has the advantage that it is more readily connected to geometry. Indeed, when $H = \mathrm{H}^1(X)$ for a Riemann surface X and $x_0 \in X$ is a basepoint, we can holomorphically map X to $\operatorname{Jac}(H)$ by taking a point $x \in X$ to the functional $\int_{\gamma(x_0,x)}\colon H^{1,0} \to \mathbb{C}$ where $\gamma(x_0,x)$ is some path connecting x and x_0. If we replace the path $\gamma(x_0, x)$ by another one, their difference is a closed cycle $[\delta] \in \mathrm{H}_1(X;\mathbb{Z})$ and hence we must quotient by this ambiguity. Note that all the information of the geometry of the Jacobian is contained in the embedding of the lattice $\mathrm{H}_1(X;\mathbb{Z})$ into $\mathrm{Hom}\left(H^{1,0};\mathbb{C}\right)$.

4.2.9. Points on Jacobians as extensions. Suppose now that H_{rel} carries an integral mixed Hodge structure of weights 0 and 1, with weight 1 quotient denoted H. Suppose that the weight 0 part has rank r and the weight 1 part has rank $2g$. We will show that such mixed Hodge structures are in bijection with a collection of r points on the torus $\mathrm{Jac}(H)$. For convenience we will work with the dual mixed Hodge structure $\check{H}_{rel}$, which suffices since applying duality twice returns the initial structure.

We have the short exact sequence, where for brevity we set $\check{P} := \mathrm{gr}_0^W\left(\check{H}_{rel}\right)$:

$$0 \leftarrow \check{P} \twoheadleftarrow \check{H}_{rel} \hookleftarrow \check{H} \leftarrow 0 \tag{4.2.10}$$

and where $\check{H} \cong \mathrm{gr}_{-1}^W \check{H}_{rel}$ is the weight -1 Hodge structure dual to the weight 1 Hodge structure H. Note also that in our situation $\check{P} \cong (W_0 H)^*$.

1		$\times$	
0	$\bigcirc$	$\otimes$	$\times$
-1		$\bigcirc$	
	-1	0	1

1		g	
0	g	r	g
-1		g	
	-1	0	1

FIGURE 4.2.11. Location of Hodge numbers and ranks. Crosses denote the support of H_{rel} and circles that of $\check{H}_{rel}$; they are exchanged by duality.

We also have the nontrivial piece of the Hodge filtration $F^0\check{H}_{rel} \subset \check{H}_{rel,\mathbb{C}}$, which induces on $\check{H}$ a weight -1 Hodge structure by $F^0\check{H} := \check{H} \cap F^0\check{H}_{rel}$.

Let us check that $\check{H}_{rel}$ is $\mathbb{R}$-split, in the sense of Definition 4.2.3. Set $\check{H}_{rel}^{0,0} := F^0\check{H}_{rel} \cap \overline{F^0\check{H}_{rel}}$, which is a rank r real subspace of $\check{H}_{rel}$. We then have a natural splitting of the short exact sequence in Eqn. (4.2.10) over $\mathbb{R}$ by setting

$$\sigma_{\mathbb{R}} \colon \check{P}_{\mathbb{R}} \xrightarrow{\sim} \check{H}_{rel}^{0,0}. \tag{4.2.12}$$

While this map is not guaranteed to be defined over the corresponding $\mathbb{Z}$-modules, we can pick an arbitrary

$$\sigma_{\mathbb{Z}} \colon \check{P}_{\mathbb{Z}} \hookrightarrow \check{H}_{rel,\mathbb{Z}} \tag{4.2.13}$$

such that composing with the projection back to $\check{P}$ yields the identity map. Since $\sigma_{\mathbb{R}}$ has the same property, we find that their difference must land in $\check{H}_{\mathbb{R}}$:

$$\kappa := \sigma_{\mathbb{Z}} - \sigma_{\mathbb{R}} \colon \check{P}_{\mathbb{Z}} \to \check{H}_{\mathbb{R}}$$

Now the map $\sigma_{\mathbb{Z}}$ was not canonical, but only well-defined by the addition of an arbitrary element in $\operatorname{Hom}(\check{P}_{\mathbb{Z}}, \check{H}_{\mathbb{Z}})$, so we have a well-defined class

$$[\kappa] \in \operatorname{Hom}(\check{P}_{\mathbb{Z}}, \check{H}_{\mathbb{R}}/\check{H}_{\mathbb{Z}})$$

Note that the target abelian group is a compact torus of real dimension $2g$, and it can be alternatively described using the isomorphism

$$\check{H}_{\mathbb{R}} \cong \check{H}_{\mathbb{C}}/F^0\check{H}$$

and we see that it is by definition isomorphic to the Jacobian of the original Hodge structure H.

To summarize, we have obtained (see also [Car80, §3]):

4.2.14. Proposition (Extensions classified by points)**.** *Let H be a pure weight 1 Hodge structure over $\mathbb{Z}$, and let P be a free $\mathbb{Z}$-module viewed as a trivial weight 0 Hodge structure. Then we have an isomorphism*

$$\operatorname{Ext}^1_{MHS}(H, P) \cong \operatorname{Hom}_{\mathbb{Z}}(\check{P}, \operatorname{Jac}(H))$$

where $\operatorname{Ext}^1_{MHS}$ denotes the (group of) extensions in the category of mixed Hodge structures over $\mathbb{Z}$, and $\check{P} := \operatorname{Hom}(P, \mathbb{Z})$ is the dual $\mathbb{Z}$-module.

If we select a basis of $\check{P}$, say with cardinality r, the right-hand side above can be identified with a choice of r points on $\operatorname{Jac}(H)$.

4.2.15. Torsion and $\mathbb{Q}$-splittings. It is natural to ask what happens if the image of $\check{P}$ above lands in the torsion subgroup of $\operatorname{Jac}(H)$. By following the above constructions, one can verify that this is equivalent to the existence of a map $\sigma_{\mathbb{Q}}$ analogous to $\sigma_{\mathbb{Z}}$ in Eqn. (4.2.13), but all groups with scalars extended to $\mathbb{Q}$, such that $\sigma_{\mathbb{Q}}$ after tensoring with $\mathbb{R}$ becomes an isomorphism between $\check{P}_{\mathbb{R}}$ and $\check{H}^{0,0}_{rel}$. An explicit example, in the case of elliptic curves, is worked out in [Fil16b, Ex. 3.8].

4.2.16. Endomorphisms. Given a complex torus, say obtained as a Jacobian $\operatorname{Jac}(H) = \check{H}_{\mathbb{C}}/\left(F^0\check{H} + \check{H}_{\mathbb{Z}}\right)$, an *endomorphism* is a holomorphic map T from $\operatorname{Jac}(H)$ to itself that is also a group homomorphism. Such a holomorphic map can be lifted to the universal cover of $\operatorname{Jac}(H)$, which is naturally identified with the complex vector space $\check{H}_{\mathbb{C}}/F^0\check{H}$. It will preserve the image of the lattice $\check{H}_{\mathbb{Z}}$, and conversely any linear map of the complex vector space, that also preserves the lattice, will lead to an endomorphism.

The simplest endomorphisms are the ones that any abelian group has, namely for any $n \in \mathbb{Z}$ we have $x \mapsto n \cdot x$. If we have a $\mathbb{Z}$-splitting compatible with the Hodge structures $H = H_1 \oplus H_2$, then we can act by pairs of integers (n_1, n_2) individually. For example $(0, 1)$ will correspond to projecting to the second factor. More generally, if there is a $\mathbb{Q}$-splitting, then a subring of $\mathbb{Z} \oplus \mathbb{Z}$ will act by endomorphisms. One refers to the Hodge structures in the splitting as *factors* of the original, and similarly for their corresponding Jacobians.

We will work exclusively with polarized Hodge structures, so the complex tori are also abelian varieties. The endomorphism rings of $\mathrm{Jac}(H)$ will be denoted by $\mathbf{End}_{\mathbb{Z}}(H)$ and $\mathbf{End}_{\mathbb{Q}}(H)$ respectively, depending on whether we take integral of rational coefficients. Since the polarization is given by a symplectic form denoted $I(x,y)$ on $H_{\mathbb{Z}}$, we will call an endomorphism T *symmetric* if $I(Tx,y) = I(x,Ty)$, and will typically restrict to the subalgebra of symmetric endomorphisms. The general endomorphism algebra $\mathbf{End}_{\mathbb{Q}}(\mathrm{Jac}(H))$ is a semisimple $\mathbb{Q}$-algebra, see [BL04, 5.3.7-8], and Ch. 5 of loc. cit for more on endomorphism rings of abelian varieties.

4.2.17. Real multiplication. We now specialize to the case of interest in the analysis of linear immersed submanifolds that arise as $\mathbf{GL}_2(\mathbb{R})$-orbit closures. Specifically, we assume that we have a totally real number field k, of degree d over $\mathbb{Q}$, embedded in the rational endomorphism algebra of a polarized Hodge structure of weight 1 on the free $\mathbb{Z}$-module H. By passing to a rational factor of H (see [BL04, 5.3.7]), we can and will assume that $k = \mathbf{End}_{\mathbb{Q}}(\mathrm{Jac}(H))$.

We can thus view $H_{\mathbb{Q}}$ as a vector space over k, say of rank r, so $\dim_{\mathbb{Q}} H_{\mathbb{Q}} = r \cdot d$. Then the real vector space $H_{\mathbb{R}} = H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ splits as

$$H_{\mathbb{R}} = H_{\iota_1} \oplus \cdots \oplus H_{\iota_d}$$

where $\{\iota_j\}$ are the distinct embeddings $\iota_j \colon k \hookrightarrow \mathbb{R}$. Indeed, recall that we have an isomorphism of $\mathbb{R}$-algebras $k \otimes_{\mathbb{Q}} \mathbb{R} \cong \oplus_{\iota_j} \mathbb{R}$, i.e. after extension of scalars to $\mathbb{R}$, k becomes isomorphic as an algebra to a product of d copies of $\mathbb{R}$. The subspace $H_{\iota_j} \subset H_{\mathbb{R}}$ can be characterized by the property that the action of k on it is via its embedding ι_j, namely

$$v \in H_{\iota_j} \iff \forall a \in k, \quad a \cdot v = \iota_k(a) v$$

where $a\dot{v}$ denotes the action of the element a on the vector v.

A vector $v \in H^{1,0}_{\iota_j}$ will be called an *eigenform* for real multiplication. When necessary, we will also emphasize that it is an eigenform for the particular embedding ι_j.

An interesting example of real multiplication is described in §4.5.13.

4.2.18. Real multiplication and orders. Keeping the assumptions and notations as above, we now consider the integral constraints on real multiplication. The algebra $\mathcal{O} := \mathbf{End}_{\mathbb{Z}}(\mathrm{Jac}(H))$ is a subring of k, with a unit, of $\mathbb{Z}$-rank equal to d, so it is (by definition) an *order* in k. The simplest order is the ring of integers $\mathcal{O}_k \subset k$, and any order is contained in $\mathcal{O}_k$. In the analogy between number fields and algebraic curves, the ring of integers corresponds to a smooth model of the curve, while an order corresponds to a singular model.

The structure of a $\mathcal{O}$-module on $H_{\mathbb{Z}}$ can be more complicated to describe, but once $\mathcal{O}$ is fixed, there are only finitely many possibilities up to isomorphism.

4.2.19. The Hodge metrics. Returning to Example 4.2.4, the first cohomology of a compact Riemann surface X admits a polarized weight 1 Hodge structure:

$$\mathrm{H}^1(X;\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

The polarization, coming from cup product in cohomology or equivalently integration of differential forms, induces natural positive-definite inner products on the real and complex cohomology, that we will refer to as the "Hodge metrics".

In the relative case, as in Example 4.2.5, for a finite set of points $Z \subset X$, the cohomology group $\mathrm{H}^1_{rel}(X, Z)$ admits a mixed Hodge structure in the sense of Definition 4.2.3. It is $\mathbb{R}$-*split*, see Eqn. (4.2.12), so over $\mathbb{R}$ we have a natural isomorphism

$$\mathrm{H}^1_{rel}(X, Z;\mathbb{R}) \cong \mathrm{H}^1(X;\mathbb{R}) \oplus \tilde{\mathrm{H}}^0(Z;\mathbb{R}).$$

We can equip the first summand with its Hodge metric, and the second one with almost any natural metric, for instance descending the metric from $\mathrm{H}^0(Z;\mathbb{R})$ for which every point is of norm 1 and orthogonal to the others. We will continue to refer to this construction as the Hodge metric on H^1_{rel}.

4.2.20. Variations of Hodge structure, weight 1. Let us now describe what happens when we allow the Riemann surface X to vary holomorphically, say over a complex manifold B. The cohomology groups $\mathrm{H}^1(X;\mathbb{C})$ form a vector bundle denoted H^1 equipped with a flat connection called the *Gauss–Manin connection* and denoted ∇^{GM}. A local flat frame can be given by fixing a basis of the integral cohomology $\mathrm{H}^1(X;\mathbb{Z})$ at one basepoint, and moving in to nearby fibers continuously. This characterizes ∇^{GM} uniquely.

The subbundle $\mathcal{H}^{1,0}$ with fibers $H^{1,0}(X) \subset \mathrm{H}^1(X;\mathbb{C})$ varies holomorphically, but it is typically *not flat*. Note that the complex-conjugate fibers $H^{0,1}(X)$ vary anti-holomorphically, but the quotient $\mathrm{H}^1(X;\mathbb{C})/H^{1,0}(X)$ does have a holomorphic structure and is naturally in bijection with $H^{0,1}(X)$; the corresponding bundle is denoted $\mathcal{H}^{0,1} := H^1/\mathcal{H}^{1,0}$.

Differentiating $\mathcal{H}^{1,0}$ by the Gauss–Manin connection, and taking the quotient by $\mathcal{H}^{1,0}$, yields a holomorphic, fiberwise linear, map of bundles called the *second fundamental form*:

$$\sigma\colon \mathcal{H}^{1,0} \to \mathcal{H}^{0,1} \otimes \Omega^1_B$$

where Ω^1_B is the holomorphic cotangent bundle of the base B.

4.2.21. Variations of Hodge structure, higher weight. The above discussion generalizes to Hodge structures of higher weight. The Gauss–Manin connection is defined in the same manner on the bundle H, and the holomorphic subbundles are given by the Hodge filtration $\mathcal{F}^pH \subset H$, so the Hodge bundles are

$$\mathcal{H}^{p,q} := \mathcal{F}^pH/\mathcal{F}^{p-1}H$$

and are typically neither subbundles nor quotients of H. The second fundamental form generalized to the maps

$$\sigma_p \colon \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_B$$

using the essential *Griffiths transversality property* that $\nabla^{GM}(\mathcal{F}^p H) \subset \mathcal{F}^{p-1}H$. A crucial calculation (see e.g. [Gri84, Ch. II, Prop. 4] [Fil16a, Prop. 4.12]) expresses the *curvature* of the Hodge bundles with the Hodge metric in terms of the second fundamental forms:

$$\Omega_{\mathcal{H}^{p,q}} = \sigma_p^\dagger \wedge \sigma_p + \sigma_{p+1} \wedge \sigma_{p+1}^\dagger \tag{4.2.22}$$

where $\dagger$ denotes the adjoint for the metric given by the intersection pairing; this metric agrees, up to sign, with the Hodge metric on $\mathcal{H}^{p,q}$.

A useful consequence of the above curvature formula is that it implies the curvature of the "rightmost" bundle $\mathcal{H}^{0,n}$ is nonpositive. Bundles with nonpositive curvature tend not to have global sections, because the norm of a section is a *plurisubharmonic* function. In particular it satisfies the maximum principle so must be constant when the base B is compact.

4.3. Hodge-theoretic rigidity

4.3.1. Setup. We now return to the setting of the $\mathbf{GL}_2(\mathbb{R})$-action on a stratum $\Omega\mathcal{M}_g(\kappa)$, fix an orbit closure $\mathcal{M}$ with $\mathbf{SL}_2(\mathbb{R})$-invariant probability measure μ on $\mathcal{M}^1$. Our goal is to describe some of the results of [Fil16a] that relate dynamics with the Hodge theory of the Riemann surfaces parametrized by $\mathcal{M}$. In fact, some of the results are proved without knowledge of the orbit closure $\mathcal{M}$ and make use only of the invariant probability measure μ.

4.3.2. The Kontsevich–Zorich cocycle. The cohomology groups $\mathrm{H}^1(X;\mathbb{C})$ and $\mathrm{H}^1_{rel}(X;\mathbb{C})$ give local systems over $\Omega\mathcal{M}_g(\kappa)$, that will be denoted H^1 and H^1_{rel} respectively. The action of $\mathbf{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_g(\kappa)$ induces a parallel transport map, say if $g \in \mathbf{SL}_2(\mathbb{R})$ and (X,ω) is a translation surface with $g \cdot (X,\omega) = (X',\omega')$, then we have a map $\mathrm{H}^1(X;\mathbb{C}) \to \mathrm{H}^1(X',\mathbb{C})$. Note that $\mathbf{SL}_2(\mathbb{R})$ is not simply-connected, but there is a natural trivialization for the action of the maximal compact $\mathbf{SO}_2(\mathbb{R})$, since $k(X,\omega) \cong (X,\lambda_k\omega)$ where $k \in \mathbf{SO}_2(\mathbb{R})$ and $\lambda_k \in \mathbb{C}$ is the corresponding unit norm complex number. So the action is well-defined, and the local systems descend to the quotient $\Omega\mathcal{M}_g(\kappa)^1/\,\mathbf{SO}_2(\mathbb{R}) \cong \Omega\mathcal{M}_g(\kappa)/\mathbb{C}^\times$. This space no longer has an action of $\mathbf{SL}_2(\mathbb{R})$, but instead is foliated by the quotient orbits, which are isomorphic to the hyperbolic plane $\mathbb{H} \cong \mathbf{SL}_2(\mathbb{R})/\,\mathbf{SO}_2(\mathbb{R})$. We will refer to these hyperbolic planes as *Teichmüller discs*.

The Hodge structures on $\mathrm{H}^1(X;\mathbb{C})$ and $\mathrm{H}^1_{rel}(X;\mathbb{C})$ also descend to the quotient space, and restricted to each leaf $\mathbb{H}$ yield a variation of (mixed) Hodge structures.

4.3.3. Invariant subbundles. We will be interested in subbundles of H^1, or those obtained from H^1 by standard operations of linear algebra (duals, tensor products, quotients). For one such local system E, and an $\mathbf{SL}_2(\mathbb{R})$-invariant ergodic probability measure μ, a *measurable* $\mathbf{SL}_2(\mathbb{R})$*-invariant subbundle* is the data of a measurable family of subspaces $S'(x) \subset E(x)$, defined for μ-a.e. $x \in \Omega\mathcal{M}_g(\kappa)$, such that $gS'(x) = S'(gx)$ for any $g \in \mathbf{SL}_2(\mathbb{R})$.

4.3.4. Theorem (Hodge compatibility, pure case)**.** *Let μ be an $\mathbf{SL}_2(\mathbb{R})$-invariant probability measure on $\Omega\mathcal{M}_g(\kappa)$, and let S be a measurable $\mathbf{SL}_2(\mathbb{R})$-invariant subbundle of the local system E obtained from H^1 by standard linear algebra operations.*

Then, denoting by $C(x)$ the Weil operator of the Hodge structure on $E(x)$ (see Definition 4.2.2), the measurable subbundle with fibers $C(x)S(x)$ is also $\mathbf{SL}_2(\mathbb{R})$-invariant. As a consequence, the Hodge-orthogonal $S^{\perp}$ of S is also an $\mathbf{SL}_2(\mathbb{R})$-invariant subbundle, and the fibers $S(x)$ and $S^{\perp}(x)$ admit Hodge decompositions, compatibly with that on $E(x)$.

This result is a restatement of [Fil16a, Thm. 1.1], which additionally describes a semisimple decomposition of E, compatible with the underlying Hodge structures, from which any $\mathbf{SL}_2(\mathbb{R})$-invariant subbundle can be obtained.

4.3.5. Proof outline. The first step in the proof is to obtain from an invariant subbundle $\mathrm{S} \subset \mathrm{E}$ an invariant section of some other subbundle TE constructed by linear-algebraic operations from E. For instance, this could be the coordinates of the subbundle in an exterior power representation, so we'd take $\Lambda^k\mathrm{S} \in \Lambda^k\mathrm{E} =: \mathrm{TE}$ where $k = \operatorname{rk}\mathrm{S}$, or a projector onto S inside $\mathbf{End}(\mathrm{E})$.

The proof then proceeds to show that an $\mathbf{SL}_2(\mathbb{R})$-invariant section ϕ of a bundle equipped with a polarized Hodge structure must be invariant by the Weil operator. Equivalently, each (p,q)-component $\phi^{p,q}$ of the section must also be $\mathbf{SL}_2(\mathbb{R})$-invariant. In Hodge theory, statements of this flavor go under the name "Theorem of the Fixed Part", see [Fil16a, Thm. 6.3].

The proof of this last statement proceeds by induction, starting with the "rightmost" component $\phi^{0,n}$. The formula for the curvature of the Hodge bundle as calculated in Eqn. (4.2.22) shows that $\log\left\|\phi^{0,n}\right\|$ is subharmonic when restricted to each Teichmüller disc. The geometric interpretation of subharmonicity is that on a finite area disc, the value at the center is bounded above by the average over the disc. In general, there are plenty of subharmonic functions on a single copy of $\mathbb{H}$, but using the finite invariant measure μ and an appropriate version of the ergodic theorem, one can show that the value at the center of a.e. disc is bounded by the average of the function for the measure μ, thus concluding that $\log\left\|\phi^{0,n}\right\|$ is constant. A local calculation in differential geometry implies that $\phi^{0,n}$ is flat, and then the argument is repeated by starting with the flat section $\phi - \phi^{0,n}$ and its rightmost Hodge component $\phi^{1,n-1}$.

We have omitted some technical points in the above outline and refer to [Fil16b, §5] for the details.

4.3.6. The mixed case. We now return to the basic exact sequence of Eqn. (3.1.13) and recall that H^1_{rel} and H^1 are the local systems of relative and absolute cohomology. Fix a measurable $\mathbf{SL}_2(\mathbb{R})$-invariant subbundle $\mathrm{S}_{rel} \subset \mathrm{H}^1_{rel}$ with projection to H^1 denoted $\mathrm{S} := p(\mathrm{S}_{rel})$, and with kernel denoted $W_0\mathrm{S}$. Recall that by Theorem 4.3.4, the subbundle S carries μ-a.e. an induced pure weight 1 Hodge structure from H^1.

4.3.7. Theorem (Hodge compatibility, mixed case)**.** *With notation as above, assume the bundle* S *admits no* $\mathbf{SL}_2(\mathbb{R})$*-invariant section.*

Then the subbundle S_{rel} *carries a mixed Hodge structure, induced compatibly from* H^1_{rel}*. Namely, setting* $\mathrm{S}^{1,0}_{rel} := \mathrm{S}_{rel,\mathbb{C}} \cap (\mathcal{F}^1\mathrm{H}^1_{rel})$*, we have on a set of full* μ*-measure that*

$$\mathrm{S}_{rel,\mathbb{C}} \cong W_0\mathrm{S} \bigoplus \left(\mathrm{S}^{1,0}_{rel} \oplus \overline{\mathrm{S}^{1,0}_{rel}}\right)$$

and the last two terms map to S *compatibly with its pure weight* 1 *Hodge structure.*

A version of this result was proved in [Fil16b, Thm. 4.2], but for a special class of invariant subbundles S coming from the tangent space $T\mathcal{M}$ of an orbit closure. In fact the more general statement above holds, although I do not know how to construct other $\mathbf{SL}_2(\mathbb{R})$-invariant subbundles.

4.3.8. Proof outline. To establish Theorem 4.3.7, we quotient first all the bundles by $W_0\mathrm{S}$ and so can assume that $W_0\mathrm{S} = \{0\}$, i.e. p is an isomorphism between S_{rel} and S. Let

$$\sigma_{\mathrm{S}} \colon \mathrm{S} \to \mathrm{S}_{rel} \subset \mathrm{H}^1_{rel}$$

be the inverse map. Since S carries a pure weight 1 Hodge structure, and and we saw in Example 4.2.5 that mixed Hodge structures of weight 1 are $\mathbb{R}$-split (see §4.2.9 and Eqn. (4.2.12)), we also have an $\mathbb{R}$-splitting

$$\sigma_{\mathbb{R}} \colon \mathrm{S} \to p^{-1}(\mathrm{S}) \subset \mathrm{H}^1_{rel}.$$

The difference $\phi := \sigma_{\mathrm{S}} - \sigma_{\mathbb{R}}$ is a map from S to H^1_{rel} which, when composed with p, yields the zero map, so we can view it as a map

$$\phi \colon \mathrm{S} \to \mathrm{W}_0.$$

On a finite cover of $\Omega\mathcal{M}_g(\kappa)$ where the marked points are trivialized, the piece W_0 can be assumed trivial, i.e. isomorphic to some $\mathbb{R}^k$ with a trivial weight 0 Hodge structure. Each coordinate of the map ϕ can then be verified to be holomorphic on a Teichmüller disc, after identifying $\mathrm{S}_{\mathbb{R}} \cong \mathrm{S}_{\mathbb{C}}/\mathrm{S}^{1,0} =: \mathcal{S}^{0,1}$. We can again use the negative curvature property of $\mathcal{S}^{0,1}$ and find that $\log \|\phi\|$ is subharmonic, and prove that it must be $\mathbf{SL}_2(\mathbb{R})$-invariant as in the proof of Theorem 4.3.4 outlined in §4.3.5. By assumption, the section vanishes and so S_{rel} carries a compatible mixed Hodge structure. □

We have again omitted several technical points in the above outline. The most important adjustment is that to complete the argument, one must introduce a modified Hodge norm on H^1, see [Fil16b, §4.4] and [EMM15, §7.2].

4.3.9. Remark (On splittings)**.**

(i) For simplicity, we have stated both Theorem 4.3.4 and Theorem 4.3.7 under the assumption that the subbundles in question are real. Analogous results hold also for complex subbundles, with the appropriate notion of complex Hodge structure.

(ii) The assumption that S has no invariant sections can fail only in the case of the Forni subspace of H^1, by definition the $\mathbf{SL}_2(\mathbb{R})$-invariant piece on which parallel transport is by isometries for the Hodge metric. See [AEM17, §1] for more about the Forni subspace.

The Hodge-theoretic rigidity properties established in Theorem 4.3.4 and Theorem 4.3.7 are also used to strongly constrain *measurable* $\mathbf{GL}_2(\mathbb{R})$-invariant bundles. We return to these questions in §5.3, where these tools turn out to be useful in establishing finiteness results for orbit closures.

4.4. Algebraicity: Real Multiplication and Torsion

We can now collect concrete consequences for orbit closures from the abstract results regarding bundles and their compatibility with Hodge structures. The main results are Theorem 4.4.2, which shows that a factor of the Jacobian has real multiplication, and Theorem 4.4.7, which shows that certain combinations of the zeros (or marked points) have to be torsion on the corresponding factor of the Jacobian. Together, these results are used to establish that orbit closures are (quasi-projective) algebraic subvarieties of a stratum.

Let us note that in the case of Teichmüller curves, which are the lowest-dimensional orbit closures and automatically algebraic, the real multiplication and torsion results were first obtained by Möller [Möl06b, Möl06a]. The proofs below are based on different principles and are used to *characterize* algebro-geometrically the orbit closures.

4.4.1. Setup. We now proceed to apply Theorem 4.3.4 and Theorem 4.3.7 on the compatibility of the $\mathbf{SL}_2(\mathbb{R})$-invariant bundles with the Hodge structure to the particular case of the tangent bundles $T\mathcal{M}$ and $p(T\mathcal{M})$ of an orbit closure $\mathcal{M}$. As before, we fix $\mathcal{M}$ and its invariant probability measure μ.

Let $k \subset \mathbb{R}$ be the smallest field over which $\mathcal{M}$ is k-linear in the sense of Definition 4.1.2; it is a number field by [Wri14, Thm. 1.1]. In the case of absolute cohomology, we obtain (see [Fil16a, Thm. 1.6]):

4.4.2. Theorem (Real multiplication)**.** *Every $(X, \omega) \in \mathcal{M}$ has a factor of the Jacobian that admits real multiplication by k, with ω as an eigenform. In particular, k is a totally real number field.*
Furthermore, the factors with real multiplication vary holomorphically over $\mathcal{M}$.

In the case of Teichmüller curves (see §4.5.3), this result was first established by Möller [Möl06b, Thm. 2.7].

4.4.3. Proof outline. The real multiplication result is a direct consequence of Theorem 4.3.4 applied to the local system $\mathrm{H}^1(T\mathcal{M}) := p(T\mathcal{M})$ and its Galois conjugates, see Eqn. (4.1.4) for its definition. First, since $\mathrm{H}^1(T\mathcal{M})$ is a local system defined over the number field k, we can act on it by the Galois group of $\mathbb{Q}$ by viewing it inside the local system with all algebraic entries $\mathrm{H}^1(T\mathcal{M}) \subset \mathrm{H}^1_{\overline{\mathbb{Q}}}$. We obtain a finite collection of local systems that we will denote H^1_ι that are indexed by embeddings $\iota\colon k \to \mathbb{C}$, with one distinguished embedding $\iota_0\colon k \to \mathbb{R}$ corresponding to $\mathrm{H}^1(T\mathcal{M})$.

Now Theorem 4.3.4 implies that each H^1_ι underlies a weight 1 variation of Hodge structures, and that they are all pairwise Hodge-orthogonal. By definition, see §4.2.17, this yields the real multiplication result. $\square$

4.4.4. Twisted torsion. To state the result in the relative case, we need some notation. Let $\mathcal{J}\mathcal{M} \to \mathcal{M}$ denote the bundle of factors of the Jacobian that admit real multiplication by Theorem 4.4.2, and let $\mathcal{O} \subset k$ be the largest order which acts on every fiber. Consider also the local system $\mathrm{H}^1_{\mathcal{M}} := \oplus_\iota \mathrm{H}^1_\iota$, and taking its preimage under p we have a short exact sequence of local systems underlying a variation of mixed Hodge structure:

$$0 \to \mathrm{W}_0 \to \mathrm{H}^1_{\mathcal{M},rel} \to \mathrm{H}^1_{\mathcal{M}} \to 0.$$

analogous to Eqn. (4.1.4). Note that in weight 1 we have $\mathrm{H}^1(T\mathcal{M}) \subseteq \mathrm{H}^1_{\mathcal{M}}$ with equality if and only if k equals $\mathbb{Q}$. Similarly, in weight 0 we have $\mathrm{W}_0(T\mathcal{M}) \subseteq \mathrm{W}_0$ with equality if and only if the torsion corank of $\mathcal{M}$ is equal to the number of marked points.

Recall that $\check{\mathrm{W}}_0 := \mathrm{Hom}(\mathrm{W}_0, \mathbb{Z})$ is the dual local system. Now $\mathrm{W}_0(T\mathcal{M}) = \mathrm{W}_{0,k} \cap T\mathcal{M}$ is a local system defined after extension of scalars to k, and dualizing we will consider $\mathrm{W}_0(T\mathcal{M})^\perp_k$, which consists of those functionals that vanish on $\mathrm{W}_0(T\mathcal{M})$. To shorten notation, we will set

$$\Lambda := \mathrm{W}_0(T\mathcal{M})^\perp_k \cap \check{W}_{0,\mathcal{O}}$$

to be the intersection with the $\mathcal{O}$-lattice.

Note that $\dim_k \mathrm{W}_0(T\mathcal{M})^\perp_k = n - t$ where n is the number of marked points and t is the torsion corank of $\mathcal{M}$ (see §4.1.5). Geometrically, denoting by $Z \subset X$ the marked points, since $\mathrm{W}_0 = \tilde{\mathrm{H}}^0(Z)$ is the reduced cohomology, it follows that $\check{\mathrm{W}}_0 = \tilde{\mathrm{H}}_0(Z)$ is the reduced homology, i.e. it consists of $\mathbb{Z}$-linear combinations of marked points, with coefficients adding up to 0. Then Λ is a free $\mathbb{Z}$-module of rank $(n-t) \cdot \deg_{\mathbb{Q}} k$ and a projective module over $\mathcal{O}$ of rank $(n-t)$; it consists of $\mathcal{O}$-linear combinations of marked points with coefficients adding up to 0.

4.4.5. Twisted Abel–Jacobi map. The usual Abel–Jacobi map assigns to any $\mathbb{Z}$-linear combination of points $[\delta]$ on a Riemann surface X a point on the abelian variety $\mathrm{Jac}(X)$ by choosing a 1-chain $[\gamma]$ with $\partial[\gamma] = [\delta]$ and mapping

$[\delta]$ to the functional of integration along γ, i.e. $\int_\gamma \in \operatorname{Hom}\left(\mathrm{H}^{1,0}(X);\mathbb{C}\right)$. For the finite set $Z \subset X$ of marked points we thus have the map

$$AJ\colon \tilde{\mathrm{H}}_0\left(Z;\mathbb{Z}\right) \to \operatorname{Jac}(X).$$

We can project to the factor $\mathcal{JM}(X)$ admitting real multiplication by the order $\mathcal{O}$, then extend scalars to $\mathcal{O}$, and extend the Abel–Jacobi map equivariantly for the $\mathcal{O}$-action:

$$AJ_{\mathcal{O}}\colon \tilde{\mathrm{H}}_0\left(Z;\mathcal{O}\right) \to \mathcal{JM}(X). \tag{4.4.6}$$

We will refer to it as the *twisted Abel–Jacobi map*. Using it, we can now state the key result constraining the mixed Hodge structure on $\mathrm{H}^1_{rel}(X,Z)$ (see [Fil16b, Thm. 1.3]):

4.4.7. Theorem (Twisted torsion)**.** *For every $(X,\omega)\in\mathcal{M}$, the restriction of the twisted Abel–Jacobi map to the submodule Λ:*

$$AJ_{\mathcal{O}}\colon \Lambda \to \mathcal{JM}(X)$$

lands in the torsion subgroup of the abelian variety. In particular, a finite index subgroup of Λ maps to the origin.

In the case of Teichmüller curves (see §4.5.3), this result was first established by Möller [Möl06a, Thm. 3.3].

4.4.8. Proof outline. The above result is a consequence of Theorem 4.3.7, applied to the tangent bundle $T\mathcal{M} \subset \mathrm{H}^1_{rel}$ and to its Galois conjugates $T\mathcal{M}_\iota$. Indeed, the theorem implies that the sequence

$$0 \to \mathrm{W}_{0,k}/\mathrm{W}_0(T\mathcal{M}) \to p^{-1}\left(\mathrm{H}^1(T\mathcal{M})\right)/W_0(T\mathcal{M}) \xrightarrow{p} \mathrm{H}^1(T\mathcal{M}) \to 0$$

is a k-split variation of mixed Hodge structures. A splitting of local systems over k is provided by $T\mathcal{M}/\mathrm{W}_0(T\mathcal{M})$, and the content of Theorem 4.3.7 is that this splitting is compatible with the mixed Hodge structures.

The same argument applies to the Galois conjugates of $T\mathcal{M}$, with analogous short exact sequences. Following through the analogue of Proposition 4.2.14, but in the presence of real multiplication on the Jacobian yields Theorem 4.4.7. □

4.4.9. Putting the pure and mixed together. Recall that by Theorem 4.1.8, the orbit closure $\mathcal{M}$ is a complex manifold, locally in period coordinates described by k-linear equations. We now proceed to explain the various algebraic restrictions placed by the real multiplication and torsion results. The reader can refer to the diagram Eqn. (5.1.12) for some of the spaces that appear in the discussion below (with the difference that the diagram uses instead of $\mathcal{O},\mathcal{S},\Lambda'$ the scalars r,d,t).

Theorem 4.4.2 implies that there exists an order $\mathcal{O}\subset k$, and a splitting $\mathcal{S}$ on $\mathrm{H}^1_{\mathbb{Q}}$ with appropriate $\mathcal{O}$-module structure on one of the factors, with the following property. Denote by $\mathcal{EA}_{\mathcal{O},\mathcal{S}} \subset \mathcal{A}_g$ the subvariety of the moduli space of dimension g principally polarized abelian varieties with a splitting

$\mathcal{S}$ and real multiplication by $\mathcal{O}$ of prescribed type. Then under the Torelli map $\Omega\mathcal{M}_g(\kappa) \to \mathcal{A}_g$, the image of $\mathcal{M}$ is contained in $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S}}$.

Now let $\mathcal{A}_{g,n} \to \mathcal{A}_g$ be the universal family of n-tuples of marked points (not necessarily distinct, but labeled) on the abelian variety, and denote its restriction to $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S}}$ by $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S},n}$. Pass to a finite cover of $\Omega\mathcal{M}_g(\kappa)$ on which the marked points are also labeled, and let $\Omega\mathcal{M}_{\mathcal{O},\mathcal{S}} \subset \Omega\mathcal{M}_g(\kappa)$ be the algebraic subset that's the preimage of $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S}}$ under the Torelli map. Then the twisted torsion Theorem 4.4.7 implies that there exists a submodule $\Lambda' \subset \tilde{\mathrm{H}}_0(Z;\mathcal{O})$ (recall that this local system is globally trivialized on the finite cover), such that under the augmented Torelli map $\Omega\mathcal{M}_{\mathcal{O},\mathcal{S}} \to \mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S},n}$, $\mathcal{M}$ is contained in the algebraic locus where Λ' maps to the origin of the abelian variety. Denote the corresponding algebraic loci by $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S},\Lambda'} \subset \mathcal{A}_{g,n}$ and its preimage $\Omega\mathcal{M}_{\mathcal{O},\mathcal{S},\Lambda'}$, so the two main results imply that $\mathcal{M} \subset \Omega\mathcal{M}_{\mathcal{O},\mathcal{S},\Lambda'}$.

Finally, we need to impose the eigenform condition so we set $\Omega\mathcal{A}_{g,n} \to \mathcal{A}_{g,n}$ to be the bundle of holomorphic 1-forms on the parametrized abelian varieties. Over $\mathcal{E}\mathcal{A}_{\mathcal{O},\mathcal{S},\Lambda'}$, we have the further algebraic sublocus where the corresponding 1-form is an eigenform of real multiplication, denoted $\mathcal{E}\Omega\mathcal{A}_{\mathcal{O},\mathcal{S},\Lambda'}$. Let $\mathcal{E}\mathcal{M} \subset \Omega\mathcal{M}_g(\kappa)$ be the preimage of $\Omega\mathcal{A}_{\mathcal{O},\mathcal{S},\Lambda'}$ in the stratum.

So Theorem 4.4.2 and Theorem 4.4.7 together imply that $\mathcal{M} \subset \mathcal{E}\mathcal{M}$. A local calculation with period coordinates shows that $\mathcal{M}$ must coincide with an irreducible component of $\mathcal{E}\mathcal{M}$, so we conclude (see [Fil16b, Thm. 1.1]):

4.4.10. Theorem (Algebraicity)**.** *An orbit closure $\mathcal{M}$ is an algebraic subvariety of $\Omega\mathcal{M}_g(\kappa)$, defined over $\overline{\mathbb{Q}}$. Its algebraic Galois conjugates are also orbit closures.*

4.4.11. Remark (On algebraicity)**.**

(i) The analysis in §4.4.9 shows that *if $\mathcal{M}$ exists*, then it agrees with the algebraic locus described by real multiplication and twisted torsion. In general, one can make a choice order $\mathcal{O}$, splitting $\mathcal{S}$, and lattice Λ', but the locus $\mathcal{E}\mathcal{M}$ obtained in this manner will typically have lower dimension than required for it to be $\mathbf{GL}_2(\mathbb{R})$-invariant. We will return to the calculation of the expected dimension of $\mathcal{E}\mathcal{M}$ in §5.1 and compare it with that of $\mathcal{M}$. Orbit closures for which these dimensions agree will be called *typical*, and otherwise *atypical*.

(ii) A theorem of Möller [Möl05, Thm. 5.4] shows that the action of the Galois group of $\mathbb{Q}$ is already faithful on orbit closures of square-tiled (aka origami) translation surfaces. See §4.5.4 for a discussion of this class of examples.

We end with a few questions regarding the above constructions.

4.4.12. Question (Galois structure of the field of linear definition)**.** Let k be the field of affine definition of an orbit closure $\mathcal{M}$. For the following questions, one can consider more generally k-linear immersed submanifold in a stratum of (possibly meromorphic) differentials, possibly of order higher than 1.

Can there be examples of $\mathcal{M}$ with k which is not cyclotomic, or contained in a cyclotomic, extension of $\mathbb{Q}$? Can there be examples when k is not Galois over $\mathbb{Q}$?

Note that there is an abundance of examples with k a quadratic extension of $\mathbb{Q}$, discovered by Calta [Cal04] and McMullen [McM03]. These arise by a construction that generalizes to (relatively) typical examples, as described in Theorem 5.1.7 below. Arbitrary cyclotomic fields occur in the Veech [Vee89] and Bouw–Möller examples [BM10], see §4.5.9 below.

4.5. Examples

The list of examples below is far from exhaustive, we have selected only some (of many) representative examples and refer the reader to the original papers for more details.

4.5.1. Setup. Let us first remark that a standard technique in ergodic theory, the Hopf argument, implies that the action of the diagonal subgroup $g_t \subset \mathbf{GL}_2(\mathbb{R})$ is ergodic on any connected component of a stratum. Therefore, a "generic" orbit closure of $\mathbf{GL}_2(\mathbb{R})$ is dense.

For most classification questions, one is only interested in *primitive* orbit closures, i.e. ones that are not obtained from an orbit closure in lower genus by taking a (possibly ramified) covering construction of the underlying translation surfaces. This also indicates one of the difficulties of classification: any argument has to be able to distinguish the many different imprimitive orbit closures in a stratum. Even the simplest imprimitive ones, the torus covers (see §4.5.4 below) can pose substantial classification challenges.

4.5.2. Remark (On primitivity)**.** It is a theorem of Möller [Möl06a, Thm. 2.6] that for every translation surface (X, ω) there exists a translation cover $(X, \omega) \xrightarrow{\pi} (X', \omega')$ (so $\pi^*\omega' = \omega$) such that the genus of X' is minimal, and furthermore if this genus is not 1, then the cover is unique and will be called canonical. Under the same assumption, the canonical cover of translation surfaces is $\mathbf{GL}_2(\mathbb{R})$-equivariant and induces a finite map of orbit closures.

4.5.3. Teichmüller curves: generalities. The first examples of nontrivial orbit closures were discovered by Veech [Vee89] and described below in §4.5.8. The corresponding *class* of orbit closures are called Teichmüller curves, while the underlying translation surfaces are called Veech surfaces.

Teichmüller curves have cylinder rank 1, torsion corank 0, but can have a field of affine definition k larger than $\mathbb{Q}$. The degree of k is an essential invariant of the Teichmüller curve: if it is equal to the genus, the curve is called *algebraically primitive*.

4.5.4. Square-tiled surfaces. Consider a translation surface (X, ω) such that its period point has rational coordinates, i.e. $[\omega] \in \mathrm{H}^1(X, Z; \mathbb{Q}[\sqrt{-1}]) \subset \mathrm{H}^1(X, Z; \mathbb{C})$. By appropriately choosing $N \in \mathbb{N}$, we can and will assume that the period coordinates are in $\frac{1}{N}\mathbb{Z}[\sqrt{-1}]$. Choosing one reference point

$p_0 \in Z$, we therefore have a well-defined map

$$X \to \mathbb{C}\Big/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$$
$$p \mapsto N \cdot \int_{p_0}^{p} \omega$$

which is a covering of the square torus with degree $N^2[\omega] \cap [\overline{\omega}]$. The points in Z map to torsion points on this torus.

The orbit $\mathbf{GL}_2^+(\mathbb{R}) \cdot (X, \omega)$ is a Teichmüller curve, with field of affine definition $\mathbb{Q}$. Indeed, the stabilizer of (X, ω) is a finite index subgroup of $\mathbf{SL}_2(\mathbb{Z})$, in fact any finite index subgroup of the congruence subgroup $\Gamma(2)$, containing ± 1, arises in this way by a theorem of Ellenberg–McReynolds [EM12, Thm. 1.2].

Appropriate sequences of such Teichmüller curves equidistribute in any stratum, by Theorem 4.1.8. An illustration is provided in Figure 4.5.5.

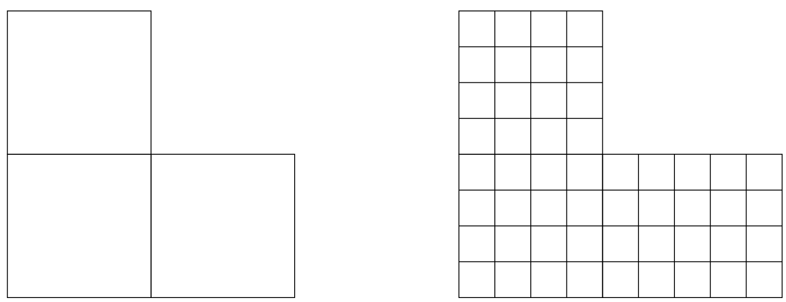

FIGURE 4.5.5. Left: A square-tiled surface with 3 squares. Right: A square-tiled surface with many squares. The two surfaces are nearby in moduli space, but have substantially different orbit closures.

Let us finally note that square-tiled surfaces are typical in the sense of §5.1 and also imprimitive, since they are obtained from a covering construction on an orbit closure in smaller genus, namely the stratum of genus 0 surfaces with 1 marked point $\Omega\mathcal{M}_1(0)$.

4.5.6. Torus covers. A bit more generally, suppose that the period point $[\omega]$ when projected to absolute cohomology $\mathrm{H}^1(X;\mathbb{C})$ has coordinates in $\mathbb{Q}(\sqrt{-1})$. The same construction as before gives a map of the translation surface (X, ω) to a torus, but now *some* of the points in Z might not map to torsion points on the torus, call those points $Z' \subset Z$. The orbit closure of (X, ω) will consist of translation surfaces covering a torus, and such that the ramifications under Z' move freely on the torus. This orbit closure, denoted $\mathcal{H}_{|Z|,|Z'|}(\kappa')$ has cylinder rank 1, torsion corank $|Z'|$, and torsion rank $|Z \setminus Z'|$, and κ' encodes the ramification profile.

4.5.7. Hurwitz spaces and isoperiodic foliation. Let us note that the orbit closure described above has a natural map to the moduli space of tori $\Omega\mathcal{M}_1(0)$. The fibers of this map are no longer orbit closures, and in fact locally their tangent space is a subbundle of $\mathrm{W}_0\mathrm{H}^1_{rel} \cong \tilde{\mathrm{H}}^0(Z)$. This subbundle induces on a stratum a foliation, called the isoperiodic, or relative, or just rel, foliation. So fibers of $\mathcal{H}_{|Z|,|Z'|(\kappa')} \to \Omega\mathcal{M}_1(0)$ give examples of algebraic subvarieties of the stratum, which are furthermore affine-linear in period coordinates (extending Definition 4.1.2 to allow for affine equations). Note that these fibers are *not* $\mathbf{GL}_2(\mathbb{R})$-invariant.

4.5.8. The orbit closure of the regular n-gon. A rich class of examples of translation surfaces whose orbit closures are Teichmüller curves was discovered by Veech [Vee89]. To illustrate it, we continue with the example of the regular n-gon from Example 2.1.6 and Example 2.2.3, with $n = 2g+1$ odd. We will see that the action of $\mathbf{GL}_2(\mathbb{R})$ on X_P has a large stabilizer, namely the preimage inside $\mathbf{SL}_2(\mathbb{R})$ of the lattice $\Delta(2,n,\infty) \subset \mathbf{PSL}_2(\mathbb{R})$ generated by reflections in the sides of an ideal hyperbolic triangle with angles $\pi/2, \pi/n, 0$ (where an angle of 0 means the point is on the ideal boundary). We can again be quite explicit and describe both the lattice, and the corresponding family of algebraic curves in the moduli space.

It is clear that the rotation by an angle of $2\pi/n$, denoted $R \in \mathbf{GL}_2(\mathbb{R})$, fixes the translation surface X_P. Indeed, in the polygonal representation, we can chop the reflected copy of the regular n-gon into n triangles and attach them to the sides of the original copy (to make it look like a cartoon of the sun). Then the asserted rotation visibly preserves the polygon and the gluing.

The second stabilizer that we can exhibit is a unipotent one. Note that X_P decomposes into $g = \frac{n-1}{2}$ horizontal cylinders, each cylinder glued out of two trapezoids. The heights h_k and widths w_k of these cylinders are readily computed, for instance by placing the origin at the center of the regular n-gon and using the n-th root of unity $\zeta := e^{2\pi\sqrt{-1}/n}$:

$$\begin{aligned} h_k &= \left|\mathrm{Re}\left(\zeta^k - \zeta^{k+1}\right)\right| = \tfrac{1}{2}\left|\zeta^k + \zeta^{-k} - \zeta^{k+1} - \zeta^{-k-1}\right| \\ &= \tfrac{1}{2}\left|\zeta^{-1/2} - \zeta^{1/2}\right|\left|\zeta^{k+1/2} - \zeta^{-k-1/2}\right| \end{aligned}$$

and for the widths:

$$\begin{aligned} w_k &= 2\left|\mathrm{Im}\left(\zeta^k + \zeta^{k+1}\right)\right| = \left|\zeta^k - \zeta^{-k} + \zeta^{k+1} - \zeta^{-k-1}\right| \\ &= \left|\zeta^{-1/2} + \zeta^{1/2}\right|\left|\zeta^{k+1/2} - \zeta^{-k-1/2}\right| \end{aligned}$$

We thus see that the ratio is independent of the cylinder:

$$\frac{w_k}{h_k} = \frac{2\cos(\pi/n)}{\sin(\pi/n)} = 2\cot(\pi/n)$$

and so the transformation $T := \begin{bmatrix} 1 & 2\cot(\pi/n) \\ 0 & 1 \end{bmatrix}$ also preserves the surface. This is a cylinder twist, see §6.1.6. Note that the group generated by $R, T \subset \mathbf{PSL}_2(\mathbb{R})$ has an "accidental" relation $(RT)^2 = 1 \in \mathbf{PSL}_2(\mathbb{R})$, hence leads to the asserted triangle group $\Delta(2, 5, \infty)$.

It is also possible to explicitly describe the family of algebraic curves that constitute the orbit closure of the regular n-gon, see [Loc05, Prop. 5.8]. First, consider the family of hyperelliptic curves

$$y^2 = \prod_{k=1}^{n} \left(x - \left(\zeta^k + t\zeta^{-k} \right) \right) \qquad \text{over } \mathbb{P}^1.$$

Equip each member of the family with the 1-form $\frac{dx}{y}$, to obtain a family of translation surfaces. The dihedral group action on $\mathbb{P}^1$ via $t \mapsto 1/t$ and $t \mapsto \zeta t$ lifts to the family, and the quotient is precisely the Teichmüller curve in $\mathcal{M}_g$ giving the orbit closure of the regular n-gon.

4.5.9. Bouw–Möller examples. Generalizing Veech's construction, Bouw and Möller [BM10] discovered a two-parameter family of Teichmüller curves $\mathcal{BM}_{m,n}$, with Veech groups $\Gamma_{m,n}$ commensurable to triangle reflection groups $\Delta(m, n, \infty)$. Interestingly, the inspiration for their construction and proof had its origin in the geometry of variations of Hodge structure. Polygonal models generating some of the Teichmüller curves were found in [BM10] (for $m = 4, 5$) and in general by Hooper [Hoo13].

4.5.10. Intersecting Hilbert modular surfaces. A general construction of McMullen, pioneered in [McM03], shows that an orbit closure $\mathcal{M}$ with cylinder rank 2, arbitrary torsion corank, and with field of linear definition $\mathbb{Q}$, contains an infinite family of suborbit closures $\mathcal{M}_D \subset \mathcal{M}$ of cylinder rank 1 and torsion corank same as that of $\mathcal{M}$, and with field of linear definition $\mathbb{Q}(\sqrt{D})$. Namely, $\mathcal{M}_D$ is obtained by intersecting $\mathcal{M}$ with the eigenform locus over a Hilbert modular surface, under an appropriate period map to an automorphic vector bundle over a moduli space of abelian surfaces. Examples of $\mathcal{M}$ are furnished by the strata in genus 2, as well as the Prym examples discussed in §4.5.11 below.

We will return to this class of examples when discussing the typical-atypical dichotomy in §5.1. Incidentally, note that that various topological invariants of $\mathcal{M}_D$ can be computed since it is obtained as an intersection of varieties, with intersection of the expected dimension (as opposed to larger).

4.5.11. Prym loci. Moving on to higher rank orbit closures, McMullen identified in [McM06a], in genus $g = 2, 3, 4$ Prym loci $\mathcal{P}_g \subseteq \Omega\mathcal{M}_g(2g-2)$ that are of cylinder rank 2 and with linear field of definition $\mathbb{Q}$. By construction, every $(X, \omega) \in \mathcal{P}_g$ is the canonical double cover of a quadratic differential (X_0, q_ω) with genus$(X_0) = g - 2$. From the Riemann–Hurwitz formula, we find that there are $2(5 - g)$ ramification points on X_0, and $\mathcal{P}_g$ covers a stratum of quadratic differentials

$$\mathcal{P}_g \to \mathcal{QM}_{g-2}\left(-1^{9-2g}, 2g-3\right).$$

Note that the case $g = 2$ yields an entire stratum, which simply means that the stratum $\Omega\mathcal{M}_2(2)$ consists of hyperelliptic translation surfaces that cover a quadratic differential with some poles on $\mathbb{P}^1$. Applying the construction described in §4.5.10 leads to infinite families of "Weierstrass" Teichmüller curves $\mathcal{W}_{g,D} \subset \Omega\mathcal{M}_g(2g-2)$ for $g = 2, 3, 4$.

4.5.12. Gothic and related loci. Further examples of primitive orbit closures $\mathcal{G}_a \subset \Omega\mathcal{M}_g(\kappa)$, for quadruples of integers $a = (a_1, a_2, a_3, a_4)$ parametrizing rational-angled quadrilaterals, or covers of $\mathbb{P}^1$, were recently discovered in [EMMW20], with genus going up to 6. These are atypical in the sense of Definition 5.1.3 and some of them provide further examples of infinite families of Teichmüller curves.

4.5.13. Twisted torsion. Kumar and Mukamel found in [KM16, Thm. 3] an example of an orbit closure $\mathcal{KM}$, with field of linear definition $\mathbb{Q}(\sqrt{5})$, where the "twisted torsion" predicted by Theorem 4.4.7 is realized with coefficients in the field, in other words the weight 0 bundle $W_0T\mathcal{KM}$ is defined over $\mathbb{Q}(\sqrt{5})$ and not $\mathbb{Q}$. This is an atypical example in the sense of Definition 5.1.3.

The orbit closure $\mathcal{KM}$ is in $\Omega\mathcal{M}_2(1,1,0)$, in other words the stratum of genus 2 surfaces with a differential with two zeros and one marked point. It has cylinder rank 1 and torsion corank 1, so it has dimension 3, but it also has torsion rank 1, i.e. there is one nontrivial torsion condition. Concretely, $\mathcal{KM}$ parametrizes pairs (X, ω, z_1, z_2, p) with the following properties:

- The zeros of ω are $z_1, z_2 \in X$.
- The Jacobian $\mathrm{Jac}(X)$ has real multiplication by $\mathbb{Z}[\gamma]$ with $\gamma^2 - \gamma - 1 = 0$, with endomorphism $T_\gamma\colon \mathrm{Jac}(X) \to \mathrm{Jac}(X)$.
- The 1-form ω is an eigenform for real multiplication: $T_\gamma^*\omega = \frac{1+\sqrt{5}}{2}\omega$.
- We have the relation between degree 0 divisors on the Jacobian

$$(p - z_1) = T_\gamma \cdot (z_2 - z_1).$$

Note that under the above conditions, we automatically have also the relation

$$(p - z_2) = T_{\gamma'} \cdot (z_1 - z_2) \text{ where } \gamma' = 1 - \gamma.$$

It is also possible to give a pleasant flat-geometric description of this relation, see Figure 4.5.14 and [KM16, §7]. The translation surfaces in this orbit closure are obtained by gluing along a slit a torus with a golden-ratio rescaling of it, such that the relation above between the points z_1, z_2, p holds on the surface.

4.5.15. m-differentials. There is another class of interesting linear submanifolds in the sense of Definition 4.1.2, whose field of linear definition is not real, and hence which do not arise as orbit closures. We refer to [BCG⁺19a] for more details.

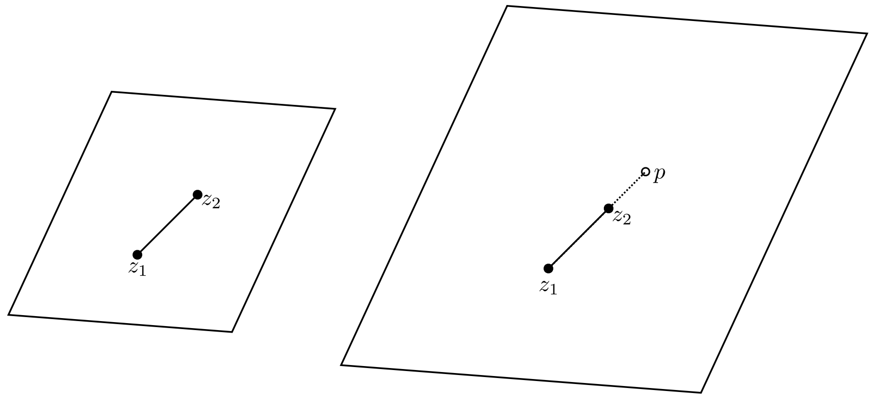

FIGURE 4.5.14. Two proportional slit tori, with a marked point.

Fix $m \in \mathbb{N}$ and consider strata of m-differentials, namely the bundle over $\mathcal{M}_g$ of sections of the m-th tensor power of the canonical bundle $\omega_X^{\otimes m}$. This bundle has degree $m(2g-2)$, and for any $\mu = (m_1, \ldots, m_n)$ with $\sum m_i = m(2g-2)$ we can consider a stratum $\Omega^m\mathcal{M}_g(\mu)$ of such differentials. Locally on a Riemann surface X with m-differential ρ, we can write it as $\rho(z) = f(z)dz^{\otimes m}$, and there is a canonical m-fold cover

$$X_\rho \xrightarrow{\pi} X \text{ such that } \pi^*\rho = \omega^{\otimes m}$$

for a holomorphic 1-form ω on X_ρ. It follows that we have an embedding

$$\Omega^m\mathcal{M}_g(\mu) \hookrightarrow \Omega\mathcal{M}_{g_\mu}(\kappa_\mu)$$

for an appropriate genus g_μ and configuration of zeros κ_μ. One can then verify that the image is a linear submanifold in period coordinates, with field of linear definition the cyclotomic field $\mathbb{Q}\left(\sqrt[m]{1}\right)$. Therefore, as soon as $m \geq 3$, these are not $\mathbf{GL}_2(\mathbb{R})$-invariant subvarieties.

5. Finiteness results

Outline of section. In this section we establish sharp finiteness results for orbit closures. The key dichotomy, in analogy with problems of unlikely intersections, is between typical and atypical orbit closures. We will see that typical ones are dense and equidistribute towards an ambient orbit closure, while the atypical ones are always finite in number.

The finiteness and abundance statements, as well as definitions of typical and atypical, are contained in §5.1. To establish finiteness, and to reformulate (a)typicality, a crucial tool is the algebraic hull of a cocycle, introduced in §5.2. Rigidity properties of algebraic hulls are established in §5.3, using the Hodge-theoretic rigidity properties from §4.3. The finiteness results are then proved in §5.4.

5.1. Typical vs. Atypical

5.1.1. Setup. We continue in this section with the notation of Section 4. Suppose $\mathcal{M} \subset \Omega\mathcal{M}_g(\kappa)$ is an orbit closure in a stratum of genus g Riemann surfaces with $|\kappa|$ marked points, and set $n = |\kappa| - 1$ for convenience of notation. We will define an invariant $\delta(\mathcal{M})$ valued in nonnegative integers that measures how atypical the orbit closure is.

We will denote by $r := \operatorname{rk} \mathcal{M}$ the cylinder rank, and by t the torsion corank of $\mathcal{M}$ (see §4.1.5). Let also d denote the degree of $\mathcal{M}$, i.e. the degree of the number field defining the linear equations of $\mathcal{M}$, see Theorem 4.4.2. The next inequalities follow from the definitions, and the fact that the numbers in question are ranks of local systems with corresponding inclusions:

$$1 \leq r \cdot d \leq g \quad 0 \leq t \leq n.$$

5.1.2. Equations cutting out $\mathcal{M}$ in a stratum. Denote by $c_1(\mathcal{M})$ the codimension of the appropriate moduli space of mixed Hodge structures admitting real multiplication, as well as torsion and eigenform properties as those of $\mathcal{M}$, inside the moduli space of all such mixed Hodge structures with a marked 1-form. This sub-moduli space is described by Theorem 4.4.2 and Theorem 4.4.7, and depends only on fixed topological invariants of $\mathcal{M}$. Denote also by $c_2(\mathcal{M})$ the codimension of $\mathcal{M}$ inside its ambient stratum $\Omega\mathcal{M}_g(\kappa)$. We then set $\delta(\mathcal{M}) := c_1(\mathcal{M}) - c_2(\mathcal{M})$, and an elementary analysis shows that $\delta(\mathcal{M}) \geq 0$. This invariant can be computed by an elementary if tedious calculation, and we do so below in a simplified situation, see Eqn. (5.1.14) for the final answer.

5.1.3. Definition (Typical and Atypical Orbit closure)**.** Call $\mathcal{M}$ *typical inside* $\Omega\mathcal{M}_g(\kappa)$ if $\delta(\mathcal{M}) = 0$, and atypical otherwise.

If $\mathcal{N}$ is another orbit closure and $\mathcal{M} \subset \mathcal{N}$, call $\mathcal{M}$ *typical inside* $\mathcal{N}$ if $\delta_{\mathcal{N}}(\mathcal{M}) = 0$, where $\delta_{\mathcal{N}}$ refers to the invariant calculated in Eqn. (5.1.14) by plugging in the corresponding ranks of $\mathcal{N}$.

We can now state the main result of this section, obtained in [EFW18]:

5.1.4. Theorem (Finiteness of Atypical)**.** *In any stratum $\Omega\mathcal{M}_g(\kappa)$, there are only finitely many maximal atypical orbit closures.*

More generally, in any orbit closure $\mathcal{N}$ there are only finitely many atypical relative to $\mathcal{N}$ orbit closures.

In the statement, "maximal" means with respect to inclusion. The method of proof, which will be described in the following sections, is dynamical and goes by contradiction. It is therefore natural to ask:

5.1.5. Question (Effective finiteness)**.** Can one make the bound on the number of atypical orbit closures effective?

More generally, can one effectively bound their numerical invariants (rank, torsion corank) and arithmetic invariants (discriminant of order of number field, index of lattice)?

See §6.2.2 for an approach using finiteness results on mixed Shimura varieties which could potentially make the results effective.

The Finiteness Theorem 5.1.4 is complemented by an elementary calculation, which we will do at the end of this section:

5.1.6. Proposition (Typical in a stratum)**.** *Suppose $\mathcal{M} \subsetneq \Omega\mathcal{M}_g(\kappa)$ is typical in a stratum of genus g surfaces with n zeros. Then we have the following possibilities for the triple (r, d, t) of rank, degree, and torsion corank of $\mathcal{M}$:*

- **Torus cover:** $(r, d) = (1, 1)$*, and t is arbitrary; $\mathcal{M}$ parametrizes ramified covers of elliptic curves; some ramification points move freely on the elliptic curve, others are fixed over torsion points.*
- **Hilbert modular surface:** $(r, d) = (1, 2)$ *with $g = 2$, and $t = n = 1$; $\mathcal{M}$ parametrizes genus 2 surfaces with 1-form ω an eigenform of real multiplication on the Jacobian.*
- **Weierstrass curve:** $(r, d) = (1, 2)$ *with $g = 2$, and $t = n = 0$; $\mathcal{M}$ parametrizes genus 2 surfaces whose 1-form ω with a double zero is also an eigenform of real multiplication on the Jacobian.*

The first possibility occurs in any stratum $\Omega\mathcal{M}_g(\kappa)$, the second in the stratum $\Omega\mathcal{M}_2(1, 1)$, and the third in $\Omega\mathcal{M}_2(2)$.

The Weierstrass curve examples were first discovered by Calta [Cal04] and McMullen [McM03], and the Hilbert modular surfaces, as well as further "relatively typical" examples (see §5.1.16) of Weierstrass curves were developed by McMullen [McM06a].

As a complement, we can also establish that typical orbit closures arise in large numbers. For the statement, we will group together the last two cases of Proposition 5.1.6 and call them "real quadratics":

5.1.7. Theorem (Abundance of typical)**.** *Suppose $\mathcal{M}$ is an orbit closure, with field of affine definition $\mathbb{Q}$, of cylinder rank g and torsion corank n. Then in the following cases, and only in them, we have typical suborbit closures:*

- **Torus covers:** *Orbit closures with field of affine definition $\mathbb{Q}$, of cylinder rank 1, and arbitrary torsion corank $t \in [0, n]$ are dense in $\mathcal{M}$.*
- **Real quadratics:** *If $g = 2$ and n is arbitrary, then orbit closures defined over a real quadratic field k, of cylinder rank 1 and torsion corank n, are dense in $\mathcal{M}$ as k ranges over all real quadratic fields.*

In the last case already for a fixed real quadratic field k the suborbit closures will be dense and involve the infinitely many orders in k.

We postpone the proof of this result to §5.4.8, when we will have a more direct and less cumbersome way to describe atypical suborbit closures provided by Theorem 5.4.5, in terms of algebraic hulls.

5.1.8. The target moduli space. We now explain the formal calculation of the invariant $\delta(\mathcal{M})$. For simplicity of exposition, we assume that the complement to the factor of the Jacobian with real multiplication is "Hodge-generic" in the corresponding moduli space, or equivalently that the monodromy on that piece is the full symplectic group.

Denote by $\mathcal{A}_g$ the moduli space of g-dimensional of (principally polarized) abelian varieties, and let let $\mathcal{A}_{g,n} \to \mathcal{A}_g$ denote the fibration whose fiber over $A \in \mathcal{A}_g$ is $\operatorname{Sym}^{[n]} A$, the unordered n-tuples of points in A. Then $\mathcal{A}_{g,n}$ parametrizes mixed Hodge structures as described in Proposition 4.2.14. Let also $\Omega\mathcal{A}_{g,n} \to \mathcal{A}_{g,n}$ denote the rank g vector bundle of differentials on the corresponding abelian varieties (the vector bundle $\Omega\mathcal{A}_{g,n}$ is pulled back from $\mathcal{A}_g$). We have the following elementary dimension and codimension calculations, where codimension refers to the dimension of fibers of the corresponding morphism, or equivalently actual codimension, by embedding the smaller space into the larger by a tautological zero section:

$$\dim \mathcal{A}_g = \frac{g(g+1)}{2} \quad \operatorname{codim}_{\mathcal{A}_{g,n}} \mathcal{A}_g = g \cdot n \quad \operatorname{codim}_{\Omega\mathcal{A}_{g,n}} \mathcal{A}_{g,n} = g \tag{5.1.9}$$

so in particular, the dimension of $\Omega\mathcal{A}_{g,n}$ is the sum of the three displayed numbers.

We have a natural tautological map

$$\Omega\mathcal{M}(\kappa) \to \Omega\mathcal{A}_{g,n}$$

mapping (X, ω) to the associated mixed Hodge structure and differential ω on the Jacobian, and our next goal is to calculate the dimension of the locus in which, in view of the Algebraicity Theorem 4.4.10, the image of the orbit closure $\mathcal{M}$ should lie.

5.1.10. Expected sub-moduli space. Let $\mathcal{E}\Omega\mathcal{A}_{r,t,d} \subset \Omega\mathcal{A}_{g,n}$ be defined as the (algebraic) locus of points that have "the same structures as on $\mathcal{M}$", which we now make more precise. Let $\mathcal{E}\mathcal{A}_{r,d} \subset \mathcal{A}_g$ be the locus of abelian varieties that have real multiplication on a factor, with the same data (order in number field, lattice structure) as points in $\mathcal{M}$. Then

$$\dim \mathcal{E}\mathcal{A}_{r,d} = d \cdot \frac{r(r+1)}{2} + \frac{(g-dr)(g-dr+1)}{2}$$

where the first term accounts for d pieces of the Hodge structure with real multiplication, and the second term accounts for the freedom to choose the remaining factor of the abelian variety.

Next, let $\mathcal{E}\mathcal{A}_{r,d,t} \subset \mathcal{A}_{g,n}$ be the bundle over $\mathcal{E}\mathcal{A}_{r,d}$ consisting of n-tuples of points that satisfy the same torsion conditions as those on $\mathcal{M}$. The dimension of the fibers $\mathcal{E}\mathcal{A}_{r,d,t} \to \mathcal{E}\mathcal{A}_{r,d}$ is

$$t \cdot g + (n-t) \cdot (g - rd)$$

where the first factor term accounts for the t points that move freely on the entire abelian variety, and the second factor accounts for the remaining points that in an (rd)-dimensional factor must be locked in a torsion position, but otherwise move freely in the remaining factor.

Finally, we set $\mathcal{E}\Omega\mathcal{A}_{r,d,t} \subset \Omega\mathcal{A}_{g,n}$ to be the bundle of 1-forms over $\mathcal{E}\mathcal{A}_{r,d,t}$ that live in a distinguished subbundle of the factor admitting real multiplication, so the dimension of the fibers of $\mathcal{E}\Omega\mathcal{A}_{r,d,t} \to \mathcal{E}\mathcal{A}_{r,d,t}$ is r.

An elementary (and only mildly tedious) calculation with the above numbers then gives the codimension:

$$\begin{aligned} \operatorname{codim}_{\Omega\mathcal{A}_{g,n}} \mathcal{E}\Omega\mathcal{A}_{r,d,t} = {} & rd(g-r) - r^2 \cdot \frac{d(d-1)}{2} \\ & + (n-t)rd + (g-r). \end{aligned} \tag{5.1.11}$$

The diagram of spaces and maps between them is illustrated in Eqn. (5.1.12).

$$\begin{tikzcd} & & & \mathcal{M} \arrow[dl, "\text{Torelli}"'] \arrow[dd, hook] \\ \mathcal{E}\mathcal{A}_{r,d} \arrow[dd, hook] & \mathcal{E}\mathcal{A}_{r,d,t} \arrow[l, two heads] \arrow[dd, hook] & \mathcal{E}\Omega\mathcal{A}_{r,d,t} \arrow[l, two heads] \arrow[dd, hook] & \\ & & & \Omega\mathcal{M}_g(\kappa) \arrow[dl, "\text{Torelli}"] \\ \mathcal{A}_g & \mathcal{A}_{g,n} \arrow[l, two heads] & \Omega\mathcal{A}_{g,n} \arrow[l, two heads] & \end{tikzcd} \tag{5.1.12}$$

5.1.13. Degree of atypicality. The codimension of $\mathcal{M}$ inside $\Omega\mathcal{M}_g(\kappa)$ is computed, in a more elementary fashion, to be $2(g-r)+(n-t)$, and we define the *degree of atypicality of* $\mathcal{M}$ *inside* $\Omega\mathcal{M}_g(\kappa)$ to be the difference in codimension:

$$\begin{aligned} \delta(\mathcal{M}) := & \operatorname{codim}_{\Omega\mathcal{A}_{g,n}} \mathcal{E}\Omega\mathcal{A}_{r,d,t} - \operatorname{codim}_{\Omega\mathcal{M}_g(\kappa)} \mathcal{M} \\ = & \tfrac{rd}{2}\Big[(g-rd)+(g-r)\Big] - (g-r) + \\ & + (rd-1)(n-t) \end{aligned} \tag{5.1.14}$$

The final expression has the terms arranged so as to emphasize the contribution coming from the weight 1 and the weight 0 parts of the cohomology. The terms in each of the last two rows are nonnegative, by the elementary inequalities in §5.1.1.

When $\mathcal{N}$ is an orbit closure with linear field of definition larger than $\mathbb{Q}$, the preceding calculations need to be modified in order to define typicality relative to $\mathcal{N}$. We will not expand on this, since a simpler description of relative atypicality can be given in terms of algebraic hulls. The statement of Theorem 5.4.5 can be taken as the definition of atypical, in terms of the algebraic hull.

Let us note that covering constructions would lead to *atypical* invariant subvarieties in the corresponding stratum. This is compatible with the Finiteness Theorem 5.1.4 below, since one can perform only finitely many covering constructions to land in a fixed stratum.

It is now an elementary calculation to determine the typical subvarieties of a stratum:

5.1.15. Proof of Proposition 5.1.6. From the expression for $\delta(\mathcal{M})$ in Eqn. (5.1.14), it is clear that setting it equal to zero implies $rd \in \{1,2\}$. So we have the possibilities for (r,d) as $(1,1),(1,2)$ and $(2,1)$. The case $r=d=1$ puts no restriction on the torsion and corresponds to torus covers. When $rd=2$, we must also have $g-rd=0$ so $g=2$. Then we must also have $t=n$. The case $r=2, d=1$ and $t=n$ implies that $\mathcal{M}$ is the corresponding stratum $\Omega\mathcal{M}_2(\kappa)$ with $\kappa=(1,1)$ or (2), and the cases $r=1, d=2$ lead to the final two possibilities on the list. $\square$

5.1.16. Relative typicality. As suggested by Definition 5.1.3, once an orbit closure $\mathcal{N}$ has been found in some stratum, it is meaningful to ask for typical subvarieties inside of it. For instance, it is proved in [EFW18, Thm. 1.7] that if $\mathcal{N}$ has rank 2 and defined over $\mathbb{Q}$ (with torsion corank n), then we automatically have a large supply of invariant subvarieties with $(r,d)=(1,2)$ and $t=n$ analogously to the last two cases of Proposition 5.1.6. Such examples were first found by McMullen and there are also more recent ones, see §4.5.12.

The formulation of the Finiteness Theorem 5.1.4 using the typical-atypical dichotomy is inspired by the work of Baldi, Klingler, and Ullmo [BKU21], who prove a similar kind of result for the Hodge locus of a variation of Hodge structure. The "equidistribution of algebraic hulls" method that is behind the proof, explained in Theorem 5.4.2 below, was also used by Bader, Fisher, Miller, and Stover to prove that (real or complex) hyperbolic manifolds are arithmetic if they contain infinitely many totally geodesic submanifolds [BFMS21, Thm. 1.1].

5.2. Algebraic Hulls

This section contains some general constructions that play a key role in the statements, and proofs, of the general finiteness results in §5.4. Many of the proofs are based on the tension between ergodic actions of general groups with complicated orbit structure, and actions of algebraic groups on algebraic varieties, which have much simpler orbit structure.

5.2.1. Setup. In what follows, we will abbreviate "affine algebraic group defined over $\mathbb{R}$" to simply "algebraic group". Much of what we say works in greater generality, for instance by replacing $\mathbb{R}$ with other local fields. Much of what we need, and more, is contained in [Zim84, §3].

Two results are fundamental for our purposes. The first one allows us to translate many questions about affine algebraic groups to their linear representations.

5.2.2. Theorem (Chevalley Stabilizer Theorem)**.** *Let $\mathbf{G}$ be an affine algebraic group over a field k, and let $\mathbf{H} \subset \mathbf{G}$ be a k-subgroup. Then there exists a linear representation $\rho\colon \mathbf{G} \to \mathbf{GL}(\mathbf{V})$ over k and a vector $v \in \mathbf{V}(k)$ such that $\mathbf{H}$ is the stabilizer of v in $\mathbf{G}$.*

The second result implies a useful property of quotients by algebraic group actions. Namely, it is immediate to check that if a group G acts on a Hausdorff topological space X with locally closed orbits, then the quotient X/G with its induced topology satisfies the T_0-separation axiom: for any two points, there exists an open set that contains one, but not the other. Such actions are called *tame* in [Zim84].

5.2.3. Theorem (Locally closed orbits)**.** *Let* $\mathbf{G}$ *be an affine algebraic group acting algebraically on a variety* $\mathbf{X}$*, all defined over* k*.*

- **Folklore:** *For every* $x \in \mathbf{X}(k)$*, the orbit* $\mathbf{G} \cdot x$ *is a locally closed subset of* $\mathbf{X}$ *in the Zariski topology.*
- **Borel–Serre:** *If* k *is a local field (with its "analytic" topology) and* $x \in \mathbf{X}(k)$ *is a point, the orbit* $\mathbf{G}(k) \cdot x$ *is locally closed in* $\mathbf{X}(k)$ *with its analytic topology.*
- **Margulis/Zimmer:** *If* k *is a local field and* μ *is a Radon probability measure on* $\mathbf{X}(k)$*, then its orbit under* $\mathbf{G}(k)$ *in the space of finite Radon measures* $M(\mathbf{X}(k))$ *is locally closed in the weak-* topology.*

For the last assertion, see [Zim84, Thm. 3.2.6]; note that the case of general $\mathbf{X}$ can be reduced to the one of proper projective $\mathbf{X}$ by reducing to affine charts, then taking projective closures, and then removing the piece of the measure supported on $\overline{\mathbf{X}} \setminus \mathbf{X}$. We let $C_0(\mathbf{X}(k))$ denote the space of continuous functions vanishing at ∞ on $\mathbf{X}(k)$ (the last condition is only relevant if $\mathbf{X}$ is not proper) equipped with the sup norm, and $M(\mathbf{X}(k))$ is the space of finite mass Radon measures on $\mathbf{X}(k)$, which is the Banach space dual of $C_0(\mathbf{X}(k))$ and has, in particular, a weak-* topology.

5.2.4. Group actions. Suppose now that a group G acts on a space X. The group G could be discrete, or a Lie group, while the space X could be a manifold, a topological or measure space. We fix a regularity $\alpha \in \{0^-, 0, \infty, \omega, \ldots\}$ i.e. measurable, continuous, smooth, real-analytic, etc. and assume that all objects below have at least this regularity. In the case of measurable actions, a quasi-invariant measure μ is also assumed.

We also assume an appropriate form of *ergodicity*: a scalar-valued C^α function on X invariant under G must be constant. If the action is continuous, the existence of one dense orbit suffices, and in the case of measurable actions ergodicity is in the usual sense of measure theory.

5.2.5. Definition (Cocycles)**.** A *cocycle* for the action of G on X is a C^α vector bundle $E \to X$ with a C^α-lift of the action of G from X to E, by fiberwise linear transformations. Maps of cocycles, and subcocycles, are defined in the natural way.

For ease of notation, we will typically omit mentioning the group G when referring to a cocycle over X. The fiber of E over a point $x \in X$ will be denoted $E(x)$.

5.2.6. Linear algebra constructions on cocycles. Starting from one G-cocycle $E \to X$, we can apply successively the standard operations of linear algebra: duality, $\otimes$, taking quotients or subcocycles. Let T_E denote the category of all G-cocycles over X that can be obtained in this way, with morphisms given by maps of G-cocycles over X. Then T_E carries the natural structure of a "rigid tensor category" in the sense of [DM82, Def. 1.7] and furthermore by ergodicity, it satisfies the assumption $\mathbf{End}(\mathbf{1}) = \mathbb{R}$ of [DM82, Thm. 2.11], where $\mathbf{1}$ denotes the trivial 1-dimensional cocycle[3]. Therefore there is a natural $\mathbb{R}$-algebraic group[4] associated to a "fiber functor" on T_E. Such fiber functors exist, and it will be in fact convenient to define the group directly, as suggested by Chevalley's Theorem 5.2.2. Note that in particular, we find that T_E is a neutral Tannakian category, in the sense of [DM82, Def. 2.19]. It is also useful to keep in mind that although categories of vector bundles are typically not abelian, under the ergodicity assumption the rank of morphisms (and hence of kernels and cokernels) is constant on a "large" invariant set (full measure, or dense open, according to the regularity).

Recall that we can think of the cocycles in T_E as obtained by taking a linear-algebraic construction LE on E and considering G-invariant subcocycles $L' \subset LE$. For every fiber $E(x)$, the group $\mathbf{GL}(E(x))$ naturally acts on the fiber $LE(x)$.

5.2.7. Definition (Algebraic Hull)**.** The *algebraic hull* of the cocycle $E \to X$ is the family of subgroups $H(x) \subset \mathbf{GL}(E(x))$ defined as the intersection of stabilizers of $L'(x) \subset LE(x)$, ranging over all linear-algebraic constructions LE on E and G-invariant subcocycles $L' \subset LE$.

Let us stress again that all cocycles are assumed to have the fixed regularity α. For example, in the measurable setting, the groups $H(x)$ are defined on a set of full measure only, in a continuous setting on a dense open, and in both settings the set is dynamics-invariant. Note also that by the Noetherian property of algebraic groups, finitely many linear-algebraic constructions and subcocycles suffice to define the algebraic hull.

5.2.8. Proposition (Elementary properties of algebraic hulls)**.** *Suppose G acts on X and $E \to X$ is a G-cocycle, all with regularity α. Let $H(x) \subset \mathbf{GL}(E(x))$ denote the α-algebraic hull.*

(i) *If α' is a lower regularity than α and $H'(x)$ is the corresponding α'-algebraic hull, then $H'(x) \subset H(x)$ for a.e. $x \in X$.*
(ii) *If α is continuous or better, and $Y \subset X$ is a closed G-invariant subset with a dense orbit and admitting a notion of α-regularity, then the α-algebraic hull of $E|_Y$ is contained in $H(y)$ for every $y \in Y$.*

[3] Incidentally, this essential assumption was missing from an earlier treatment [SR72] of Tannakian categories.

[4] In general, only a pro-algebraic group, but our Tannakian category has a tensor generator, namely E itself, so automorphisms of fiber functors are affine algebraic groups.

We will need part (ii) in the situation when α is given by a collection of locally polynomial functions on the spaces in question.

Proof. Part (i) is immediate since if $L \subset T$ are a cocycle and subcocycle of regularity α, then they are also of regularity α', so the conditions cutting out the α-algebraic hull also apply to the α'-algebraic hull.

Part (ii) also follows immediately for the same reasons, since continuous cocycles and subcocycles can be restricted to a closed subset. ☐

5.3. Rigidity of Algebraic Hulls over Orbit Closures

5.3.1. Setup. We now specialize to the case of the action of $\mathbf{GL}_2(\mathbb{R})$ on an orbit closure $\mathcal{M}$. When convenient or necessary, we will restrict to the action of $\mathbf{SL}_2(\mathbb{R})$ on the unit area subset $\mathcal{M}^1 \subset \mathcal{M}$, so that the group action preserves a finite measure. For questions related to the algebraic hull, the two situations are equivalent.

We start with the statement of the main result of this section, and explain the terminology in the coming paragraphs. The theorem below strongly constrains the behavior of algebraic hulls in this setting and is established in [Fil16a, Thms. 1.3-1.5] in the pure case, and [EFW18, App. A] in the mixed case:

5.3.2. Theorem (Rigidity of Algebraic Hulls)**.** *Let $\mathcal{M}$ be an orbit closure of the $\mathbf{GL}_2(\mathbb{R})$-action.*

(i) *The measurable and polynomial in period coordinates algebraic hulls coincide. Equivalently, any measurable $\mathbf{GL}_2(\mathbb{R})$-invariant subbundle of H^1_{rel} (or its tensor powers) over $\mathcal{M}$ can be described in local period coordinates with polynomial functions.*
(ii) *The algebraic hull of H^1 or H^1_{rel} is compatible with the Hodge structure, i.e. any $\mathbf{GL}_2(\mathbb{R})$-invariant subbundle of H^1 or H^1_{rel} (or their tensor powers) carries pointwise an induced (mixed) Hodge structure, compatible with that on the ambient bundle.*

5.3.3. Remark (On Hodge structures and Algebraic Hulls)**.**

(i) Part (i) implies that the algebraic hulls in the highest and lowest possible regularities agree, so after its proof we will speak of "the" algebraic hull, without specifying the regularity. However, for proofs it is necessary to establish (ii) first, for *measurable* invariant subbundles.
(ii) Part (ii) in the case of the cocycle H^1 can be equivalently reformulated to say that the Deligne torus giving at each $x \in \mathcal{M}$ the Hodge structure is contained in the algebraic hull.

5.3.4. Allowed polynomials and the area function. We now describe the "polynomial functions" that appear in the statement of Theorem 5.3.2. Fix a basis of relative integral cohomology $a_i, b_i, i = 1 \ldots g$ and $c_j, j = 1 \ldots n$, with corresponding complex coordinates $z_k, k = 1 \ldots 2g + n$ and

$z_k = x_k + \sqrt{-1} y_k$. Assume the standard normalization for the symplectic pairing $a_i \cup b_j = \delta_{ij}$ and let the period coordinate of the holomorphic 1-form be

$$\omega = \sum_{i=1}^{g} z_i a_i + z_{g+i} b_i + \sum_{j=1}^{n} z_{2g+j} c_j.$$

Then the area function $A(\mathbf{x}, \mathbf{y})$ is defined to be:

$$A(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{g} x_i y_{g+i} - y_i x_{g+i} \tag{5.3.5}$$

which is (up to a real scalar) equal to $\sqrt{-1}[\omega] \cup [\overline{\omega}]$.

From now on, "polynomial functions in period coordinates" will refer to rational functions of the form $P(\mathbf{x}, \mathbf{y})/A(\mathbf{x}, \mathbf{y})^k$ where $P(\mathbf{x}, \mathbf{y})$ is a homogeneous polynomial in the variables x_i, y_i and $k \in \mathbb{Z}_{\geq 0}$. Such functions have a natural degree of homogeneity, and to be invariant by scalings (which is a subgroup of $\mathbf{GL}_2(\mathbb{R})$), the degree of homogeneity should be 0.

5.3.6. Invariant subbundles. The assertion that a bundle $\mathrm{S} \subset \mathrm{H}$ is "polynomial in period coordinates" will mean that in a local flat trivialization of H (or H^1_{rel}), the Plücker coordinates of $\Lambda^k \mathrm{S}$ (with $k = \operatorname{rk} \mathrm{S}$) can be chosen to be polynomial in the sense above.

The simplest example of an $\mathbf{SL}_2(\mathbb{R})$-invariant subbundle over $\mathcal{M}$ is one that is also flat, i.e. invariant under parallel transport on all of $\mathcal{M}$ (rather, one should work on the abstract finite cover $\mathcal{M}^a$, see Definition 4.1.2, but we will continue to omit this from the discussion). Useful examples are the tangent bundle $T\mathcal{M} \subset \mathrm{H}^1_{rel}$ and its projection $p(T\mathcal{M}) \subset \mathrm{H}^1$. Here is a more interesting, and equally important, one:

5.3.7. Example (Tautological Plane)**.** At each $p = (X, \omega) \in \mathcal{M}$, we have the tautological 2-dimensional subspace $\mathrm{T}(p) := \mathit{span}\,(\operatorname{Re}\omega, \operatorname{Im}\omega) \subset \mathrm{H}^1(p)$ and analogously for $\mathrm{T}_{rel} \subset \mathrm{H}^1_{rel}$. This is visibly a $\mathbf{GL}_2(\mathbb{R})$-invariant subbundle, but it is *not* flat. Let us see how it can be described by polynomial functions (for ease of notation, we restrict to H^1).

In local period coordinates, we have the tautological section $\omega(p) \in \mathrm{H}^1_{\mathbb{C}}$ given by

$$\omega(p) = \sum_{i=1}^{g} z_i a_i + z_{i+g} b_i = \mathbf{x} + \sqrt{-1}\mathbf{y}.$$

Then its Plücker coordinate of the tautological 2-plane can be taken to be $\sqrt{-1}\omega \wedge \overline{\omega}/A(\mathbf{x}, \mathbf{y})$, which is visibly a polynomial function as defined earlier.

Equally natural is to take the operator π_{T} of projecting H^1 onto T along the symplectically (and Hodge) orthogonal decomposition $\mathrm{H}^1 = \mathrm{T} \oplus \mathrm{T}^{\perp}$. Then on a vector $\mathbf{v} \in \mathrm{H}^1$, as a function of $(\mathbf{x}, \mathbf{y})$ in period coordinates, the projector is

$$\pi_{\mathrm{T}}(\mathbf{x}, \mathbf{y})(\mathbf{v}) = \frac{A(\mathbf{v}, \mathbf{y})}{A(\mathbf{x}, \mathbf{y})} \sum_{i=1}^{g} (x_i a_i + x_{i+g} b_i) + \frac{A(\mathbf{x}, \mathbf{v})}{A(\mathbf{x}, \mathbf{y})} \sum_{i=1}^{g} (y_i a_i + y_{i+g} b_i)$$

where we recall that $z_i = x_i + \sqrt{-1}y_i$. Indeed, note that $\pi_{\mathrm{T}}(\mathbf{x},\mathbf{y})(\mathbf{x}) = \mathbf{x}$ and similarly for $\mathbf{y}$, and it annihilates any vector symplectically orthogonal to either of $\mathbf{x},\mathbf{y}$.

Note that the Plücker coordinate is only defined up to an arbitrary scaling, while the projector is naturally normalized.

5.3.8. Proof of Hodge compatibility in Theorem 5.3.2. The assertion that invariant subbundles have compatible Hodge structures is, in the case of H^1, a restatement of Theorem 4.3.4. Note that this last result applies to tensor constructions on H^1, whereas Theorem 4.3.4 only addresses H^1_{rel}. In general the algebraic hull of H^1_{rel} is a subgroup of the semidirect product $\mathbf{Sp}(\mathrm{H}^1) \ltimes \mathrm{Hom}(\mathrm{H}^1, \mathrm{W}_0)$, and as such it has a naturally defined unipotent radical, which is a subgroup of $\mathrm{Hom}(\mathrm{H}^1, \mathrm{W}_0)$.

It is immediate to check that the only difference between the algebraic hulls on H^1 and H^1_{rel} is in the unipotent radical. Note that $\mathrm{Hom}(\mathrm{H}^1, \mathrm{W}_0)$ has the structure of an abelian unipotent group (as automorphisms of H^1_{rel}) and so any $\mathbf{GL}_2(\mathbb{R})$-invariant subspace could be the unipotent radical of the algebraic hull. This is controlled by the $\mathbf{GL}_2(\mathbb{R})$-invariant decomposition of $\mathrm{H}^1 = \oplus \mathrm{H}^1_i$. The property that the unipotent radical intersects the piece $\mathrm{Hom}(\mathrm{H}^1_i, \mathrm{W}_0)$ in $\mathrm{Hom}(\mathrm{H}^1_i, \mathrm{W}_{0,i})$ is equivalent to the existence of a $\mathbf{GL}_2(\mathbb{R})$-invariant subbundle isomorphic to H^1_i inside the bundle $p^{-1}(\mathrm{H}^1_i)/\mathrm{W}_{0,i}$. By Theorem 4.3.7 this subbundle must be compatible with the Hodge structure, as asserted. We refer to [EFW18, Lemma A.2] for more details.

5.3.9. Measurable implies polynomial for invariant bundles. We now sketch the steps in the proof of the rigidity part in Theorem 5.3.2. The proof uses a number of dynamical ideas. We outline the steps and then explain what each means, and how it is accomplished; for more details see [Fil16a, §7].

Suppose then that $\mathrm{E} \subset \mathrm{H}$ is a $\mathbf{GL}_2(\mathbb{R})$-invariant subcocycle over an orbit closure $\mathcal{M}$ (where H could be H^1 or some tensor power construction on it). We know that its Hodge orthogonal $\mathrm{F} := \mathrm{E}^{\perp}$ is also $\mathbf{GL}_2(\mathbb{R})$-invariant, and will use in the proof that the Hodge metric is real-analytic in period coordinates.

- **Step 1:** On a.e. local stable leaf, show that E varies real-analytically; same for the unstable.
- **Step 2:** On a.e. local stable leaf, show that E varies polynomially; same for the unstable.
- **Step 3:** Conclude that E varies polynomially on $\mathcal{M}$.

The diagonal subgroup $g_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ acts in period coordinates by expanding the $\mathbf{x}$-coordinate and contracting the $\mathbf{y}$-coordinate. A *local stable leaf* is, in period coordinates, an open subset of the $\mathbf{x} = \mathit{const}$ piece, so under g_t two points in the same stable leaf approach each other exponentially fast.

5.3.10. Proof of Step 1. Real-analyticity of the bundles is proved with the help of the Lyapunov filtration $\mathrm{H}^{\leq\lambda_i}$, which consists of vectors that grow at rate at most $e^{t\lambda_i}$ under g_t (for more on Lyapunov exponents see [Fil19]). It is proved by induction on the index of the Lyapunov exponent, starting from the smallest, that $\mathrm{E}^{\leq\lambda_i}$ varies real-analytically on the stable leaf; when reaching the last piece E one concludes step 1. This follows from two ingredients: the Lyapunov filtration on $\mathrm{H}^{\leq\lambda_i}$ is real-analytic, in fact *flat* along the stable leaves, and the decomposition $\mathrm{H}^{\leq\lambda_i} = \mathrm{E}^{\leq\lambda_i} \oplus \mathrm{F}^{\leq\lambda_i}$ which is orthogonal for the real-analytic Hodge metric. Now each new exponent falls either into E or into F, and can be described as the orthogonal complement for a real-analytic metric of a bundle known to be real-analytic, inside a flat bundle. A special situation arises when λ_i occurs in both E and F, and this is treated using the Ledrappier invariance principle [Led86].

5.3.11. Proof of Step 2. To go from real-analyticity to polynomiality, one makes use of the g_t-action again. This time, Taylor-expanding the real-analytic functions in question, combined with the exponential contraction of the flow, shows that only terms up to a certain order (bounded in terms of the Lyapunov exponents of H) can occur.

5.3.12. Proof of Step 3. The final step combines the polynomiality property on individual stable and unstable manifolds. It is based on an elementary lemma proving that if a measurable function $f(\mathbf{x}, \mathbf{y})$ is polynomial for a.e. $\mathbf{x}$ and a.e. $\mathbf{y}$, then it is in fact a polynomial.

5.3.13. Computation of the algebraic hull. The rigidity properties established in Theorem 5.3.2 can be used to determine the algebraic hull over the orbit closure $\mathcal{M}$. To describe it, let $\mathbf{G}^{mon}$ be the $\mathbb{R}$-Zariski closure of the monodromy group on H^1, and similarly $\mathbf{G}^{mon}_{rel}$ for H^1_{rel}. Note that just like the algebraic hull, this can be viewed as a family of groups, one for each point of $\mathcal{M}$ (viewing that point as the basepoint for the fundamental group).

Let now $\mathbf{G}^{\mathrm{T}}$ and $\mathbf{G}^{\mathrm{T}}_{rel}$ be the subgroups of $\mathbf{G}^{mon}_{\bullet}$ stabilizing the tautological plane T (see Example 5.3.7), viewed either in H^1 or H^1_{rel}. Then we have (see [EFW18, Thm. 1.1]):

5.3.14. Theorem (Algebraic Hull and Monodromy)**.** *The algebraic hull for the $\mathbf{GL}_2(\mathbb{R})$-action on H^1 is $\mathbf{G}^{\mathrm{T}}$, and similarly $\mathbf{G}^{\mathrm{T}}_{rel}$ on H^1_{rel}.*

This result is complemented by the next one, which is established beforehand and in fact necessary for the proof (see [Fil17, Cor. 1.7] for the pure case and [EFW18, Prop. 4.7] for the mixed case):

5.3.15. Theorem (Monodromy on tangent space)**.** *Over the orbit closure $\mathcal{M}$, the monodromy on $\mathrm{H}^1(T\mathcal{M})$ is $\mathbb{R}$-Zariski dense in the symplectic group $\mathbf{Sp}(\mathrm{H}^1(T\mathcal{M}))$. On $T\mathcal{M}$, the monodromy is $\mathbb{R}$-Zariski dense in $\mathbf{Sp}(\mathrm{H}^1(T\mathcal{M})) \ltimes \mathrm{Hom}(\mathrm{H}^1(\mathcal{M}), \mathrm{W}_0(T\mathcal{M}))$.*

The bundles $\mathrm{H}^1(T\mathcal{M})$ and $\mathrm{W}_0(T\mathcal{M})$ are defined in terms of the tangent space $T\mathcal{M}$ by the short exact sequence in Eqn. (4.1.4).

5.3.16. Proof sketch of Theorem 5.3.14. It is clear that the algebraic hull is contained in the monodromy group $\mathbf{G}^{mon}$, and since the tautological plane is also $\mathbf{GL}_2(\mathbb{R})$-invariant, the algebraic hull is contained in $\mathbf{G}^{\mathrm{T}}_{\bullet}$. It remains to prove the converse.

In local period coordinates on $\mathcal{M}$, the space of $\mathbf{GL}_2(\mathbb{R})$-orbits is an open subset of $\mathrm{Gr}^{\circ}(2; T\mathcal{M}_{\mathbb{R}})$, the Grassmannian of real 2-dimensional planes in $T\mathcal{M}_{\mathbb{R}}$, with $\circ$ denoting that we require the projection to $\mathrm{H}^1(T\mathcal{M})$ to be nondegenerate for the symplectic pairing. This Grassmannian is equivalently a homogeneous space for the algebraic group $\mathbf{H} := \mathbf{Sp}(\mathrm{H}^1(T\mathcal{M})) \ltimes \mathrm{Hom}(\mathrm{W}_0(T\mathcal{M}), \mathrm{H}^1(T\mathcal{M}))$, with stabilizer of a point the subgroup $\mathbf{H}^{\mathrm{T}}$ stabilizing a (tautological) 2-plane.

By Theorem 5.3.2 we know that $\mathbf{GL}_2(\mathbb{R})$-invariant bundles are described, in local period coordinates, by polynomial functions. Combined with the above local description of the space of $\mathbf{GL}_2(\mathbb{R})$-orbits, we obtain on open patches in $\mathbf{H}/\mathbf{H}^{\mathrm{T}}$, algebraic bundles. Analytically continuing the construction along loops in $\mathcal{M}$, we obtain bundles that are *equivariant* for the monodromy Γ, and since the bundles are algebraic, they must be equivariant for the $\mathbb{R}$-Zariski closure $\overline{\Gamma}^{Zar} =: \mathbf{G}^{mon}$.

Now by Theorem 5.3.15, the group $\mathbf{G}^{mon}$ surjects onto $\mathbf{H}$. From the characterization of algebraic $\mathbf{H}$-equivariant bundles on $\mathbf{H}/\mathbf{H}^{\mathrm{T}}$ as $\mathbf{H}^{\mathrm{T}}$-representations, Theorem 5.3.14 follows. □

We can now combine Theorem 5.3.14 with Theorem 5.3.15 to evaluate the algebraic hull on $T\mathcal{M}$:

5.3.17. Corollary (Algebraic Hull on tangent space)**.** *Over an orbit closure $\mathcal{M}$, the algebraic hull on the weight* 1 *part of its tangent space* $\mathrm{H}^1(T\mathcal{M})$ *is the stabilizer of the tautological plane inside* $\mathbf{Sp}(\mathrm{H}^1(T\mathcal{M}))$*. On the full tangent space, it is the stabilizer of the tautological plane in inside* $\mathbf{Sp}(\mathrm{H}^1(T\mathcal{M})) \ltimes \mathrm{Hom}(\mathrm{H}^1(T\mathcal{M}), \mathrm{W}_0(T\mathcal{M}))$*.*

Concretely, if the orbit closure has cylinder rank r, degree d, and torsion corank t, then on the weight 1 part the algebraic hull is $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2(rd-1)}(\mathbb{R})$, while on the absolute part it has additionally a unipotent radical isomorphic to $\mathbb{R}^{t\times 2rd}$. Note also that this theorem applies, in particular, to the strata $\Omega\mathcal{M}_g(\kappa)$ and will be useful for the finiteness results of §5.4.

We have not discussed Lyapunov exponents in this survey, but let us note that once the algebraic hull has been identified, and established to be rigid, it is natural to ask:

5.3.18. Question (Simple Lyapunov spectrum)**.** Is the Lyapunov spectrum as simple as possible, given the algebraic hull? Concretely, each semisimple factor of the algebraic hull has a split Cartan subalgebra, and the Lyapunov vector is an element of its dual. Then the Lyapunov vector should not lie on any of the walls of the Weyl chamber.

The characterization of zero Lyapunov exponents is obtained in [Fil17]. For this more general question, it is likely that the methods introduced by Bader–Furman [BF14, §5] will be useful.

5.4. Finiteness

5.4.1. Setup. We are now ready to prove the finiteness of atypical orbit closures stated in Theorem 5.1.4. With the tools and notions we have developed, it is in fact easier to prove a slightly stronger result. Recall that by Theorem 4.1.8, whenever $\mathcal{M}_i$ and μ_i is a sequence of orbit closures with their natural probability measures, there exists another orbit closure $\mathcal{M}$ and with measure μ, and a subsequence of the original still denoted $\mathcal{M}_i, \mu_i$, such that $\mathcal{M}_i \subset \mathcal{M}$ and $\mu_i \rightharpoonup^* \mu$.

5.4.2. Theorem (Equidistribution of Algebraic Hull)**.** *Let $\mathcal{M}, \mu$ be an orbit closure with corresponding measure μ, and let $\mathcal{M}_i \subset \mathcal{M}$ be a sequence of orbit closures with corresponding measures satisfying $\mu_i \rightharpoonup^* \mu$. Let $\mathbf{A}$ be the algebraic hull of $\mathcal{M}$ and $\mathbf{A}_i$ of $\mathcal{M}_i$ on H^1_{rel}.*
Then $\mathbf{A}_i \subset \mathbf{A}$ and there exists $i_0 \geq 0$ such that for $i \geq i_0$, we have that $\mathbf{A}_i$ and $\mathbf{A}$ differ at most by a compact factor and finite index.

5.4.3. Remark (On compact factors and finite index)**.**

(i) On the case of H^1, the algebraic hulls are semisimple by Theorem 4.3.4 and Theorem 5.3.14. The assertion that $\mathbf{A}_i$ and $\mathbf{A}$ agree up to finite index and compact factors can then be stated at the level of Lie algebras: they have the same noncompact factors in the Lie algebra decomposition. Note in particular that both might have compact factors, but the inclusion might be strict.

(ii) We will restrict the discussion below to the case of H^1, the case of H^1_{rel} being analogous, and the algebraic hulls acquiring a unipotent radical. The assertion of the theorem then includes the property that for sufficiently large i, the groups $\mathbf{A}_i$ and $\mathbf{A}$ will have the same unipotent radical.

(iii) In Theorem 5.4.2 we use the words "the same" to assert a pointwise statement, for every point in the orbit closure, for which the algebraic hull is well-defined, and the groups $\mathbf{A}_i(x)$ and $\mathbf{A}(x)$ are viewed inside $\mathbf{GL}(\mathrm{H}^1(x))$.

5.4.4. Proof outline of Theorem 5.4.2. The containment $\mathbf{A}_i \subset \mathbf{A}$ follows from $\mathcal{M}_i \subset \mathcal{M}$ and Theorem 5.3.2 (combined with Proposition 5.2.8), since the measurable and continuous algebraic hulls coincide.

For any semisimple $\mathbb{R}$-algebraic group $\mathbf{G}$, we set $\mathbf{G}^{\Phi} \subset \mathbf{G}$ to be the smallest normal algebraic subgroup such that $\mathbf{G}/\mathbf{G}^{\Phi}$ is compact. Equivalently, it is the Zariski closure of the exponential of the noncompact part of the Lie algebra of $\mathbf{G}$. The property of $\mathbf{G}^{\Phi}$ that we will use is that if $\mathbf{G}(\mathbb{R})$ preserves a measure on some projective space $\mathbf{P}(V_{\mathbb{R}})$ for a $\mathbf{G}$-irreducible representation $V_{\mathbb{R}}$, then $\mathbf{G}^{\Phi}$ acts trivially on $V_{\mathbb{R}}$ (see [Zim84, Cor. 3.2]).

For the two semisimple groups $\mathbf{A}_i \subseteq \mathbf{A}$ to be "the same up to compact factors", it suffices to show that $\mathbf{A}_i^{\Phi} = \mathbf{A}^{\Phi}$. To do so, we will construct a $\mathbf{A}$-representation V such that, on the one hand $\mathbf{A}_i$ can be defined as the stabilizer of a line $L_i \subset V$, and on the other hand for any $V' \subset V$ which is $\mathbf{A}$-irreducible, $\mathbf{A}$ preserves a measure on $\mathbb{P}(V')$. This last property implies $\mathbf{A}^{\Phi}$ acts trivially on V and yields the conclusion.

We already know that $\mathbf{A}_i^{\Phi} \subset \mathbf{A}^{\Phi}$, and suppose by contradiction that along some subsequence the containment is strict. An argument with algebraic groups (see [EFW18, §5.1,pg. 297]) shows that there exists one representation V and cocycle V (independent of i) such that there exists a family of lines $L_i \subset V$ and subcocycles $\mathrm{L}_i \subset \mathrm{V}$, such that $\mathbf{A}_i$ is defined as the stabilizer of L_i. For any $\mathbf{A}$-invariant decomposition $\mathrm{V} = \mathrm{V}' \oplus \mathrm{V}''$, with V' being $\mathbf{A}$-irreducible, we can assume that L_i is not contained in V'', otherwise we can pass to a subsequence and work with V'' in what follows, and still obtain a contradiction.

Denote by $\mathrm{L}_i' \subset \mathrm{V}'$ the projection of L_i to V'. The measure μ_i on $\mathcal{M}_i$ admits a unique lift $\tilde{\mu}_i$ on $\mathbb{P}(\mathrm{V}')$ with the property that it projects to μ_i, and the fiberwise measures are δ-masses on $\mathbb{P}(\mathrm{L}_i')$. Note that the measures $\tilde{\mu}_i$ are $\mathbf{SL}_2(\mathbb{R})$-invariant for the lifted action to $\mathbb{P}(\mathrm{V}')$, since the bundles L_i are.

Since the fibers of $\mathbb{P}(\mathrm{V}') \to \mathcal{M}$ are compact, we can take a weak-* limit of the $\tilde{\mu}_i$, denoted $\tilde{\mu}$, and since $\mu_i \rightharpoonup^* \mu$ we also have that $\tilde{\mu}$ projects to μ. Invariance under $\mathbf{SL}_2(\mathbb{R})$ is preserved.

We now claim that $\mathbf{A}$ preserves the disintegrations $\tilde{\mu}(x)$ on the fibers (with $x \in \mathcal{M}$), and this will conclude the proof. Let $\mathbf{S}(x) \subset \mathbf{PGL}(\mathrm{V}'(x))$ be the group stabilizing the measure $\tilde{\mu}(x)$; these form a *measurable* family of subgroups, and are algebraic by [Zim84, Thm. 3.2.4]. The family of subgroups is also $\mathbf{SL}_2(\mathbb{R})$-equivariant, and in fact μ-a.e. x, y the groups $\mathbf{S}(x), \mathbf{S}(y)$ can be conjugated by an isomorphism of the fibers. Indeed, we have a measurable map $\mathcal{M} \to M^1(\mathbb{P}(V'))/\mathbf{PGL}(V')$ from $\mathcal{M}$ to the space of probability measures on the projective space $\mathbb{P}(V')$, modulo linear transformations. The space of such orbits is T_0 by Theorem 5.2.3, so μ-a.e. the image lies in one orbit. It follows again by Chevalley's theorem that there is some representation W of $\mathbf{PGL}(V')$, and associated cocycle W, with 1-dimensional line bundle $\mathrm{K} \subset \mathrm{W}$, such that $\mathbf{S}(x)$ is the stabilizer of $\mathrm{K}(x)$. The family of lines K is $\mathbf{SL}_2(\mathbb{R})$-invariant, hence stabilized by the algebraic hull $\mathbf{A}$ by its definition, so we conclude that $\mathbf{A} \subset \mathbf{S}$. □

We can finally conclude the asserted finiteness results, in a slightly stronger form. Recall that the algebraic hull of an orbit closure is a pointwise defined family of algebraic groups, all conjugate to each other.

5.4.5. Theorem (Finiteness)**.**

(i) *An orbit closure $\mathcal{M}$ is typical inside a stratum if and only if its algebraic hull coincides with that of the stratum, and atypical otherwise.*
(ii) *Every stratum has finitely many atypical orbit closures which contain all other atypical orbit closures.*

(iii) *Analogously, an orbit closure $\mathcal{M}$ contained in another one $\mathcal{N}$ is typical inside it if the algebraic hull of $\mathcal{M}$ coincides with that of $\mathcal{N}$ up to compact factors, and atypical otherwise.*
(iv) *Every orbit closure $\mathcal{N}$ contains finitely many atypical orbit closures which contain all other atypical orbit closures (all relative to $\mathcal{N}$).*

Proof. Given an orbit closure $\mathcal{N}$, let us see first why orbit closures inside it with strictly smaller (up to compact factors) algebraic hull must be contained in a finite set of smaller orbit closures. Indeed, if there was such an infinite such sequence $\mathcal{M}_i \subset \mathcal{N}$, up to passing to a subsequence there is $\mathcal{N}'$ containing all of them, with $\mathcal{M}_i$ equidistributed in $\mathcal{N}'$ (by Theorem 4.1.8). Because of equidistribution we must have $\mathcal{N}' \subseteq \mathcal{N}$, and by the assumption on the algebraic hulls and Theorem 5.4.2 we must have $\mathcal{N}' \subsetneq \mathcal{N}$.

It remains to establish (iii) (which clearly implies (i)), i.e. compare the definition of atypical in terms of codimensions and in terms of algebraic hulls. We will first treat the splitting of H^1 and eigenform conditions together, and then the torsion part. Note that the contribution to the codimension "defect" from the absolute cohomology and the weight 0 part are both nonnegative, so for a typical suborbit closure, it will suffice to equate both contributions to 0. All dimensions computed below are over $\mathbb{C}$.

For any orbit closure $\mathcal{N}$ let $\mathcal{A}_{\mathcal{N}} \subset \mathcal{A}_g$ denote the subvariety parametrizing abelian varieties with the same splitting as on $\mathcal{N}$. This is determined by the Zariski closure of monodromy over $\mathcal{N}$, denoted $\mathbf{M}_{\mathcal{N}}$, and a lattice inside its real points. Let also $\Omega\mathcal{A}_{\mathcal{N}} \to \mathcal{A}_{\mathcal{N}}$ denote the bundle of eigenforms of the same type as on $\mathcal{N}$, it has fiber dimension equal to the rank $r_{\mathcal{N}}$ of $\mathcal{N}$. Let also $\Omega\mathcal{A}^u_{\mathcal{N}} \to \Omega\mathcal{A}_{\mathcal{N}}$ denote the bundle parametrizing the mixed Hodge structures of the same type as on $\mathcal{N}$. Again, this is determined by the monodromy over $\mathcal{N}$, and if we denote by $\mathbf{M}^u_{\mathcal{N}} \subset \mathbf{M}_{\mathcal{N}}$ the unipotent radical, then the dimension of the fibers is $\frac{1}{2}\dim \mathbf{M}^u_{\mathcal{N}}$. Indeed, the unipotent radical is a direct sum of pieces of the form $\operatorname{Hom}\left(\mathrm{H}^1_\alpha, \mathrm{W}_{0,\alpha'}\right)$, where H^1_α is some (symplectic) factor of H^1 and $\mathrm{W}_{0,\alpha'}$ is some factor of W_0. But the corresponding parameter space consists of points on the abelian variety (by Proposition 4.2.14) corresponding to H^1_α, which has complex dimension $\frac{1}{2}\dim \mathrm{H}^1_\alpha$.

Now the dimension of $\mathcal{N}$ is $2r_{\mathcal{N}} + t_{\mathcal{N}}$, where $t_{\mathcal{N}}$ is the dimension of $\mathrm{W}_0(T\mathcal{N})$. It follows that the (formally computed) codimension of $\mathcal{N}$ in $\Omega\mathcal{A}^u_{\mathcal{N}}$ is

$$\dim \mathcal{A}_{\mathcal{N}} + r_{\mathcal{N}} + \tfrac{1}{2}\dim \mathbf{M}^u_{\mathcal{N}} - (2r_{\mathcal{N}} + t_{\mathcal{N}}) = (\dim \mathcal{A}_{\mathcal{N}} - r_{\mathcal{N}}) + \left(\tfrac{1}{2}\dim \mathbf{M}^u_{\mathcal{N}} - t_{\mathcal{N}}\right).$$

To connect the calculation to the algebraic hull $\mathbf{A}_{\mathcal{N}}$, we recall that we have $\mathbf{A}_{\mathcal{N}} \subset \mathbf{M}_{\mathcal{N}}$. We can define the associated symmetric space for the algebraic hull $\widetilde{\mathcal{AH}}_{\mathcal{N}}$ which is the quotient of $\mathbf{A}_{\mathcal{N}}(\mathbb{R})$ modulo its maximal compact. We now note that $\dim \mathcal{A}_{\mathcal{N}} = \dim \widetilde{\mathcal{AH}}_{\mathcal{N}} + (r_{\mathcal{N}} - 1)$, since the only difference in the symmetric spaces (once we pass to the universal cover of $\mathcal{A}_{\mathcal{N}}$) is that a Siegel space factor of rank $r_{\mathcal{N}}$ splits as a product of a rank 1 and a rank

$(r_{\mathcal{N}}-1)$ Siegel spaces, and the dimension of a Siegel space of rank r is $\frac{1}{2}r(r+1)$.

Analogously, let $\mathbf{A}^u_{\mathcal{N}} \subset \mathbf{M}^u_{\mathcal{N}}$ denote the unipotent radical of the algebraic hull. Then $\dim \mathbf{A}^u_{\mathcal{N}} = \dim \mathbf{M}^u_{\mathcal{N}} - 2t_{\mathcal{N}}$, corresponding to the requirement that $\mathbf{A}^u_{\mathcal{N}}$ must act trivially on the rank 2 tautological bundle in the tangent space directions, which have dimension $t_{\mathcal{N}}$.

Putting the last two calculations together, it follows that we can rewrite the above formal codimension as

$$\left(\dim \widetilde{\mathcal{AH}}_{\mathcal{N}} - 1\right) + \tfrac{1}{2}\dim \mathbf{A}^u_{\mathcal{N}}.$$

If $\mathcal{M} \subset \mathcal{N}$ is a suborbit closure, we have by Theorem 5.3.2 that $\mathbf{A}_{\mathcal{M}} \subset \mathbf{A}_{\mathcal{N}}$ and similarly for unipotent radicals $\mathbf{A}^u_{\mathcal{M}} \subset \mathbf{A}^u_{\mathcal{N}}$. But $\mathcal{M}$ is typical inside $\mathcal{N}$ if and only if its formal codimension equals that of $\mathcal{N}$, which by the above calculation implies $\mathbf{A}^u_{\mathcal{M}} = \mathbf{A}^u_{\mathcal{N}}$, and $\dim \widetilde{\mathcal{AH}}_{\mathcal{M}} = \dim \widetilde{\mathcal{AH}}_{\mathcal{N}}$. However, the semisimple parts are nested, and to have equality of dimension in the corresponding symmetric spaces, the groups must agree up to compact factors. $\square$

5.4.6. Monodromy and Lyapunov spectrum of square-tiled surfaces. Although the main application of the Algebraic Hull Equidistribution Theorem 5.4.2 is to finiteness questions, it turns out that it gives interesting information about the monodromy and Lyapunov spectrum of orbit closures. A natural class of examples to which the result applies is square-tiled surfaces in a stratum which generate a sequence of Teichmüller curves $\mathcal{T}_i \subset \mathcal{M}_g(\kappa)$ that also equidistribute inside the stratum. On the one hand, the algebraic hull of a stratum on H^1 is $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2g-2}(\mathbb{R})$, by Corollary 5.3.17. For sufficiently large i, the algebraic hull of $\mathcal{T}_i$ is the same (since there are no compact factors). This implies, by the relation between algebraic hulls and monodromy from Theorem 5.3.14, that the monodromy of H^1 over $\mathcal{T}_i$ has Zariski closure equal to $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2g-2}(\mathbb{R})$. We conclude:

5.4.7. Theorem (Monodromy of square-tiled surfaces)**.** *For all square-tiled surfaces in a stratum $\Omega\mathcal{M}_g(\kappa)$ outside of a finite set of proper atypical suborbit closures, the Zariski closure of monodromy over the corresponding Teichmüller curve is $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2g-2}(\mathbb{R})$.*

In particular, the Lyapunov spectrum of the geodesic flow is simple.

Proof. The first assertion was explained above, it remains to justify the claim on Lyapunov exponents. By the Eskin–Matheus coding-free simplicity criterion [EM15, Thm. 1.1], it follows that on the factor that's not the tangent space of the Teichmüller curve, the spectrum is simple. Since on the $\mathbf{SL}_2$-factor the exponents are $1, -1$, by Forni's spectral gap property $1 > \lambda_2$, see [For02, Thm. 0.1], the claim follows. $\square$

This gives an extension to all strata of some results of Matheus, Möller, and Yoccoz [MMY15, §1.2] on the simplicity of the Lyapunov spectrum. Let us also note that the finitely many exceptions are necessary, as illustrated by the examples in [FFM18, Thm. 1.1].

Note also that an alternative route to the above result is via [BEW20, Thm. 2.8] of Bonatti–Eskin–Wilkinson, which shows that in the setting of the $\mathbf{SL}_2(\mathbb{R})$-action on strata, the Lyapunov spectrum is "continuous": if a sequence of ergodic $\mathbf{SL}_2(\mathbb{R})$-invariant measures μ_i equidistributes to another such measure μ, then the Lyapunov exponents of μ_i converge to those of μ. Combined with the simplicity of the Lyapunov spectrum on strata established by Avila–Viana [AV07, Thm. 1.1], the result follows.

5.4.8. Proof of Abundance of Typical, Theorem 5.1.7. We now tie on loose end and characterize the orbit closures that can possibly admit an infinite family of typical suborbit closures, and show that they in fact do contain them.

To characterize orbit closures that could admit an infinite family of typical suborbit closures, we will use Theorem 5.4.5. Suppose $\mathcal{M}$ is such an orbit closure, with suborbit closures $\mathcal{M}_i$ equidistributing to it; we can assume by Theorem 5.4.2 that they have the same algebraic hulls, up to compact factors. Let $\mathrm{H}^1_{\mathcal{M}}$ be the projection to absolute cohomology of the tangent bundle of $\mathcal{M}$, say of rank $2g$. Using Corollary 5.3.17, the algebraic hull of $\mathcal{M}$ on it is $\mathbf{SL}_2(\mathbb{R}) \times \mathbf{Sp}_{2g-2}(\mathbb{R})$, and using it again we find that either $\mathcal{M}_i$ has cylinder rank 1, or $\mathrm{H}^1_{\mathcal{M}_i} = \mathrm{H}^1_{\mathcal{M}}$. In the second case, if we assume that $g > 1$, then we find that the unipotent part of the algebraic hull of $\mathcal{M}$ strictly contains that for $\mathcal{M}_i$, unless they also have the same torsion corank, which means they are equal.

In the first case, if $\mathcal{M}_i$ is has linear field of definition $\mathbb{Q}$ as well, then it parametrize torus covers and this gives the first possibility in Theorem 5.1.7. Otherwise, the field of linear definition must necessarily be quadratic, so that $2g - 2 = 2$ and so $g = 2$, leading to the second case. Note that in this last case, the agreement of unipotent parts of the algebraic hull forces the torsion coranks of $\mathcal{M}$ and $\mathcal{M}_i$ to agree.

It remains to prove abundance, i.e. density of typical orbit closures in the above two cases. For torus covers, this is clear since points in $\mathcal{M}$ with necessary rationality conditions are dense. For the degree 2 case, see [EFW18, §6.2].

6. Classification of orbit closures

Outline of section. The question of classifying orbit closures can be understood in many ways and in this section we describe some of them, as well as progress so far. Methods based on the cylinder deformation theory of Wright are discussed in §6.1. The algebro-geometric and arithmetic points of view are taken up in §6.2, where we discuss the connection to problems of unlikely intersections, as well as an algebro-geometric point of view on the cylinder deformation theorem. We end with a discussion of algorithmic questions, both practical and theoretical, in §6.3.

6.1. Topological methods

We outline some recent progress on the classification of orbit closures, using as an essential tool the Cylinder Deformation Theorem [Wri15a, Thm. 1.1] of Wright. After describing this result and some of its consequences in the initial paragraphs, we include a few (of very many) classification theorems obtained using these methods.

6.1.1. The basic exact sequences, in homology. We will continue to denote by $Z \subset X$ the zeros of ω. The geometric version of the basic exact sequence from Eqn. (6.1.2) and its dual will provide a natural setting for the arguments below:

$$\begin{array}{c} 0 \longrightarrow \tilde{\mathrm{H}}_0(Z) \longrightarrow \mathrm{H}_1(X \setminus Z) \longrightarrow \mathrm{H}_1(X) \longrightarrow 0 \\ \\ 0 \longleftarrow \mathrm{H}_0^{red}(Z) \longleftarrow \mathrm{H}_1(X, Z) \longleftarrow \mathrm{H}_1(X) \longleftarrow 0 \end{array} \tag{6.1.2}$$

The second row is naturally identified, via the pairing between homology and cohomology, with the *dual* of the tangent space to the stratum $\Omega\mathcal{M}_g(\kappa)$. So it encodes the *cotangent bundle*. The Poincaré duality pairing between the first row and the second then identifies naturally the first row with the *tangent bundle* of the stratum.

6.1.3. Relative deformations. Note that while in the second row, the group $\mathrm{H}_0^{red}(Z)$ denotes formal linear combinations of points in Z with coefficients adding up to zero (the "reduced" cohomology), in the first row $\tilde{\mathrm{H}}_0(Z)$ denotes the group of all formal linear combinations of points in Z, modulo the element which takes each point in Z with coefficient 1.

The deformations of (X, ω) corresponding to a point $z \in Z$, and its class $[z] \in \tilde{\mathrm{H}}_0(Z)$, are called *relative* deformations. Specifically, $[z]$ denotes the homology class in $\mathrm{H}_1(X \setminus Z)$ going once clockwise around z and moving in $\Omega\mathcal{M}_g(\kappa)$ by $t[z]$ amounts to adding to each relative period $[\gamma]$ the quantity $t([z] \cap [\gamma])$.

6.1.4. Definition (Horizontal cylinder)**.** A *horizontal cylinder* of a translation surface (X, ω) is a connected open subset $C \subset X$, saturated by closed leaves of the foliation induced by $\ker \operatorname{Im} \omega$, and maximal with this property. Any closed leaf will be called a *core curve* of C.

Note that any core curve has an appropriate neighborhood foliated by core curves, and we take the maximal connected neighborhood with this property. The boundary $\partial C = \overline{C} \setminus C$ consists of finitely many zeros of ω connected by horizontal saddle connections. Note that the boundary might be connected as a subset of X but it has two natural maps from a core curve, obtained by "pushing" up or down the core curve. A core curve also has a natural orientation, specified by requiring ω to have positive integral over positively oriented subsets of the core curve.

6.1.5. Geometry of a cylinder. The homology class $[\gamma_C]$ of any core curve γ_C is independent of choices. We will denote by $[\gamma_C]$ its homology class in $\mathrm{H}_1(X \setminus Z)$, so it will give *tangent vectors*, and by $[\gamma_C^*]$ its homology class in $\mathrm{H}_1(X, Z)$, so it will give *cotangent vectors*.

The *circumference*, or *width*, of a cylinder C is defined to be $\int_{[\gamma_C^*]} \omega$, i.e. the length of any of the core curves, and will be denoted w_C. To define the *height* of C, denoted h_C, let $\alpha \subset \overline{C}$ be any oriented path connecting two zeros on the boundary, and such that $[\alpha] \cap [\gamma_C^*] = 1$. Then we set $h_C := \operatorname{Im} \int_\alpha \omega$, and with our orientation convention on γ_C this gives $h_C > 0$. Note that the imaginary part of the integral is independent of the homotopy class of α, since changing the homotopy class amounts to adding the width (a real number) to the integral.

6.1.6. Cylinder deformations. To every horizontal cylinder $C \subset (X, \omega)$ we associate the tangent vector $\mathbf{v}_C \in T_{(X,\omega)} \Omega\mathcal{M}_g(\kappa)$ equal to $h_C[\gamma_C^*] \in \mathrm{H}_1(X \setminus Z; \mathbb{R})$. The deformation of (X, ω) along $\mathbf{v}_C$ will be denoted $u_t^C(X, \omega)$, and corresponds to cutting out C, applying the horocycle flow u_t only to C, and then gluing it back in. By a cutting and pasting, it can be directly verified that this transformation is periodic, with period $\frac{w_C}{h_C}$. The frequency, i.e. the inverse period, is called the *modulus* of the cylinder and denoted $m_C := \frac{h_C}{w_C}$. A cylinder deformation is illustrated in Figure 6.1.7.

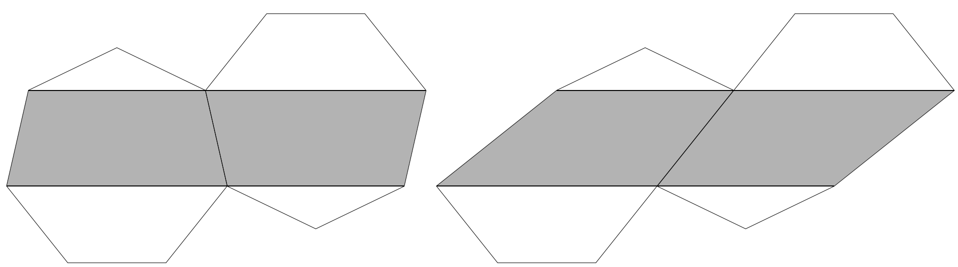

FIGURE 6.1.7. Left: A cylinder on the double heptagon surface. Right: A cylinder deformation applied only to the cylinder.

6.1.8. Definition (Horizontally periodic surface)**.** We will say that (X, ω) is *horizontally periodic* if it is covered by (the closures of) horizontal cylinders.

A fundamental result of Smillie–Weiss [SW04, Thm. 5] describes the minimal sets for the horocycle flow in terms of horizontally periodic surfaces. A key ingredient in the proof is a recurrence result of for the horocycle flow obtained previously by Minsky–Weiss [MW02, §1]. We now proceed to describe these orbit closures.

6.1.9. Tori of horizontally periodic surfaces. Suppose that (X,ω) is covered by cylinders $C_1,\ldots,C_k$ and is therefore horizontally periodic. Each of them determines a vector $\mathbf{v}_{C_i}$ in the tangent space at (X,ω), and their sum is the vector giving the horocycle flow. Let $L\subset \mathrm{H}_1(X\setminus Z)$ denote the span of these vectors. Note that the vectors are linearly independent, since the core curves are, since for each core curve there is a saddle connection intersecting only it. Note also that L is isotropic for the intersection pairing when projected to absolute homology.

We can apply independently any of the flows $u_t^{C_i}$ to (X,ω), each with its own period, and hence obtain a k-dimensional torus in $\Omega\mathcal{M}_g(\kappa)$. The torus is naturally isomorphic to $\prod\left(\mathbb{R}\mathbf{v}_{C_i}/\mathbb{Z}\frac{\mathbf{v}_{C_i}}{m_{C_i}}\right)$, and we shall call it the "full torus" supporting (X,ω). Now the orbit $u_t(X,\omega)$ will stay in this torus, and its orbit closure will equal the subtorus cut out by the following equations:

$$\text{If } \sum_{i=1}^{k} a_i m_{C_i}=0 \text{ with } a_i\in\mathbb{Z} \text{ then require } \sum_{i=1}^{k} a_i\frac{\mathbf{v}_{C_i}}{m_i}=0.$$

Note that we view $\frac{\mathbf{v}_{C_i}}{m_i}=w_{C_i}[\gamma_{C_i}]$ as the generators of the lattice under which the flows $u_t^{C_i}$ are periodic. Note also that the orbit closure of the flow in the direction of the vector $(1,\ldots,1)$ on $\prod(\mathbb{R}/\mathbb{Z}\frac{1}{m_i})$ is equivalent, under a diagonal linear transformation, to the orbit closure of $(m_1,\ldots,m_n)$ on $\prod(\mathbb{R}/\mathbb{Z})$.

For future reference, observe that if $m_1,\ldots,m_l$ are $\mathbb{Q}$-linearly independent from $m_{l+1},\ldots,m_k$ (meaning that their $\mathbb{Q}$-linear spans intersect only at $\{0\}$), then the orbit closure of $(m_1,\ldots,m_k)$ on $(\mathbb{R}/\mathbb{Z})^k$ is equal to the product of orbit closures of $(m_1,\ldots,m_l,0,\ldots,0)$ and $(0,\ldots,0,m_{l+1},\ldots,m_k)$.

6.1.10. $\mathcal{M}$-parallelism. Suppose now that the horizontally periodic surface (X,ω) is contained in an orbit closure $\mathcal{M}$. We have already described the horocycle orbit closure in §6.1.9, but in can be the case that $\mathcal{M}$ intersects the full torus in a larger set.

6.1.11. Definition ($\mathcal{M}$-parallel cylinders and saddle connections)**.** Suppose that $[\alpha^*],[\beta^*]\in\mathrm{H}_1(X,Z;\mathbb{Z})$ denote the classes of a core curve of a horizontal cylinder, or of a horizontal saddle connection, on (X,ω). Call the cylinders or saddle connections α and β *$\mathcal{M}$-parallel* if their images in $T^*\mathcal{M}_\mathbb{R}$ are proportional by a (possibly real) scalar.

Note that we have a natural map $\mathrm{H}_1(X,Z;\mathbb{R})\to T^*\mathcal{M}_\mathbb{R}$, i.e. every relative homology class gives a cotangent vector. In fact, because of the local triviality of the bundles involved, any relative homology class gives a well-defined function on a stratum, and by restriction on $\mathcal{M}$, whose differential is precisely the cotangent vector just described. The property of $\mathcal{M}$-parallelism is then equivalent to (local on $\mathcal{M}$) proportionality of the two functions.

6.1.12. Remark (On $\mathcal{M}$-parellism)**.** We collect some elementary remarks:

(i) Being $\mathcal{M}$-parallel is an equivalence relation, and the definition includes the possibility that a cylinder could be $\mathcal{M}$-parallel to a saddle connection.
(ii) The property of $\mathcal{M}$-parallelism depends on the cohomology class in $\mathrm{H}_1(X, Z)$ of a core curve of a cylinder, while the conclusion of Theorem 6.1.13 below refers to the class in $\mathrm{H}_1(X \setminus Z)$.
(iii) The smaller an orbit closure $\mathcal{M}$, the more likely are two horizontal cylinders to be $\mathcal{M}$-parallel. For instance, if $\mathcal{M}$ has cylinder rank 1, then all horizontal cylinders on (X, ω) must be $\mathcal{M}$-parallel since their span is an isotropic subspace. It will follow from Theorem 6.1.13 that in fact if there is at least one horizontal cylinder, then (X, ω) must be horizontally periodic.

We can now state Wright's result, [Wri15a, Thm. 1.1, Thm. 5.1]:

6.1.13. Theorem (Cylinder deformation theorem)**.** *Denote by $C_1, \ldots, C_k$ the horizontal cylinders on $(X, \omega) \in \mathcal{M}$, and let $\mathcal{C}_1, \ldots, \mathcal{C}_l$ denote their partition into equivalence classes of $\mathcal{M}$-parallel cylinders. Define the vector $\mathbf{v}_{\mathcal{C}_i} := \sum_{C \in \mathcal{C}_i} \mathbf{v}_{C_i}$ in the tangent space at (X, ω) to the stratum.*

(i) *Suppose that (X, ω) is horizontally periodic. Then $T_{(X,\omega)}\mathcal{M}$ contains all the vectors $\mathbf{v}_{\mathcal{C}_i}$.*
(ii) *The same holds more generally, even if (X, ω) is not horizontally periodic.*

Note that while the second assertion is visibly stronger than the first, the proof requires the weaker statement first. We also have an immediate consequence, which does not assume apriori knowledge of the ambient orbit closure:

6.1.14. Corollary (Getting tangent vectors)**.** *Let $C_1, \ldots, C_k$ be all the horizontal cylinders of (X, ω). Then*

$$\mathbf{v}_{C_1} + \cdots + \mathbf{v}_{C_k}$$

is in the tangent space to any *orbit closure $\mathcal{M}$ containing (X, ω).*

If (X, ω) is horizontally periodic then this is simply the tangent vector to the horocycle orbit, but otherwise one obtains a vector which is not tangent to the local $\mathbf{GL}_2(\mathbb{R})$-orbit of (X, ω).

Before sketching the proof of Theorem 6.1.13, we need some further preliminaries.

6.1.15. Real deformations. Recall that the local period coordinates are valued in the complex vector space $\mathrm{H}_1(X \setminus Z; \mathbb{C})$ which has a decomposition into real and imaginary parts. The orbit closure $\mathcal{M}$ is determined by a real subspace $T\mathcal{M}_{\mathbb{R}} \subset \mathrm{H}_1(X \setminus Z; \mathbb{R})$, and the orbit closure itself is locally described by the complexification. The local product structure into real and imaginary coordinates coincides with the unstable/stable foliation for the Teichmüller geodesic flow.

We define a *real deformation* of (X, ω) to be one which stays entirely in the real direction, in other words only the real parts of periods are changed, and not the imaginary parts. For example, the matrices

$$\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \text{ as well as the vectors } \mathbf{v}_{C_i}$$

give real deformations.

6.1.16. Real deformations and horizontal cylinders. Observe that a sufficiently small real deformation does not destroy a horizontal cylinder C, since its core curve γ_C continues to have a real period and furthermore the height remains unchanged. A horizontally periodic surface continues to be horizontally periodic under sufficiently small real deformations.

The modulus $m_C = \frac{h_C}{w_C}$ can change only through the quantity $w_C = \int_{[\gamma_C]} \omega$. Therefore, if C_1, C_2 are $\mathcal{M}$-parallel, then the ratio m_{C_1}/m_{C_2} stays unchanged under real deformations staying in $\mathcal{M}$, essentially by the definition of $\mathcal{M}$-parallelism. Conversely, if C_1, C_2 are not $\mathcal{M}$-parallel, then by an arbitrarily small real deformation one can make their moduli $\mathbb{Q}$-linearly independent. A bit more generally, one can accomplish the same for any collection of cylinders which are not pairwise $\mathcal{M}$-parallel, see [Wri15a, Lemma 4.9]

Proof of Theorem 6.1.13. Let us sketch the basic ideas in the proof. Start with a horizontally periodic surface (X, ω), with cylinders $C_1, \ldots, C_k$, and suppose $\mathcal{C} = \{C_1, \ldots, C_l\}$ is an equivalence class of $\mathcal{M}$-parallel ones. By the discussion on §6.1.16, a sufficiently small real deformation of (X, ω), staying within $\mathcal{M}$, can arrange the moduli of cylinders in $\mathcal{C}$ to be $\mathbb{Q}$-linearly independent from the rest. By the discussion in §6.1.9, the u_t-orbit closure of (X, ω) in its associated full torus in the stratum has $\mathbf{v}_{\mathcal{C}}$ as tangent vector, and therefore so does $\mathcal{M}$.

For a general surface (X, ω) which has some cylinders $C_1, \ldots, C_k$, but is not horizontally periodic, the theorem of Smillie–Weiss ensures that there exists a horizontally periodic $(X', \omega') \in \mathcal{M}$ and a sequence of times $t_i \to +\infty$ such that $u_{t_i}(X, \omega) \to (X', \omega')$. By taking t_i sufficiently large, we can assume that $u_{t_i}(X, \omega)$ is within a sufficiently small period coordinate chart at (X', ω') such that the intersection of $\mathcal{M}$ with the chart is a linear space passing through (X', ω') and $u_{t_i}(X, \omega)$.

Note that the horizontal cylinders on (X, ω) and $u_{t_i}(X, \omega)$ are naturally in bijection. Denote by $C'_1, \ldots, C'_{k'}$ the horizontal cylinders on (X', ω'). Any local path in $\mathcal{M}$ connecting (X', ω') to $u_{t_i}(X, \omega)$ must necessarily involve some imaginary directions as well, since one surface is horizontally periodic but the other one isn't. We can arrange the path to be first purely imaginary, then purely real, and therefore obtain a natural correspondence between the cylinders on (X, ω) and a proper subset of those at (X', ω'). We claim that this correspondence respects the $\mathcal{M}$-parallelism equivalence relation, and so the assertion about the tangent vector at (X, ω) follows from that at (X', ω').

If two cylinders on (X', ω') are $\mathcal{M}$-parallel, under any small deformation of (X', ω') (either real or complex) they will either stay horizontal together, or cease to be horizontal together. Since the condition defining $\mathcal{M}$-parallelism is locally invariant under flat parallel transport, the needed assertion follows. $\square$

6.1.17. Some further consequences. Suppose that $C_1, \ldots, C_l$ are cylinders on (X, ω), that are furthermore an $\mathcal{M}$-parallel equivalence class $\mathcal{C}$ for an orbit closure $\mathcal{M}$ containing (X, ω). Let h_i, w_i be their respective heights and widths, and denote the moduli by $m_i = h_i/w_i$, and core curves γ_i. We have the following properties:

(i) Any ratio of moduli m_i/m_j is in $\mathbb{Q}$.
(ii) Any ratio of widths w_i/w_j is in the field of linear definition $k_{\mathcal{M}}$ of $\mathcal{M}$.
(iii) The set of ratios $1, w_2/w_1, w_3/w_1, \ldots, w_l/w_1$ generate the field $k_{\mathcal{M}}$.

The first assertion follows from the fact that the orbit closure of $u_t^{\mathcal{C}}$ is 1-dimensional, when considered on a full torus associated to a horizontally periodic horocycle flow limit (X', ω') of (X, ω).

The second assertion follows from the identity $w_i[\![\gamma_j^*]\!] = w_j[\![\gamma_i^*]\!]$ where $[\![\gamma]\!]$ denotes the projection of $[\gamma]$ to $T^*\mathcal{M}_{\mathbb{R}}$. This identity is verified at (X, ω) since we know the projected cohomology classes are proportional, and the constant of proportionality can be evaluated by pairing against the holomorphic 1-form ω. It follows that $w_i[\gamma_j^*] - w_j[\gamma_i^*]$ is a (local) equation for $\mathcal{M}$. Rewriting the equation as $[\gamma_j^*] - \frac{w_j}{w_i}[\gamma_i^*]$ and using that $[\gamma_i^*], [\gamma_j^*]$ can be made part of a $\mathbb{Q}$-basis of homology, the assertion follows.

For the last assertion, we note that the vector

$$\frac{1}{h_1}\mathbf{v}_{\mathcal{C}} = [\gamma_1] + \frac{h_2}{h_1}[\gamma_2] \cdots + \frac{h_l}{h_1}[\gamma_l]$$

is in the tangent space $T\mathcal{M}_{\mathbb{R}}$ by Theorem 6.1.13 and belongs to homology with coefficients in $\mathbb{Q}(w_2/w_1, \ldots, w_l/w_1)$ since $h_i/h_1 = (m_i/m_1) \cdot (w_i/w_1)$ and the ratios of moduli are rational. Now the monodromy of $\pi_1(\mathcal{M})$ acts with integer coefficients and irreducibly on the absolute cohomology part $T\mathcal{M}$ by [Wri14, Thm. 5.1] so the orbit of the above vector spans $T\mathcal{M}$. It follows that $k_{\mathcal{M}} \subset \mathbb{Q}(\{w_i/w_1\}_{i=1}^l)$.

6.1.18. Applications to orbit closure classification. The above results, and their further extensions in [Wri15a], indicate an approach to classifying orbit closures. We indicate some techniques and papers that use them, and the reader can find a wealth of additional references in these works.

Start from a horizontally periodic surface (X, ω). We would like to find its orbit closure and at the start, we only have the $\mathbf{GL}_2(\mathbb{R})$-directions. There are cylinders in other directions on (X, ω), and one can always arrange this other direction to be vertical by applying an element of $\mathbf{GL}_2(\mathbb{R})$. If the surface is not vertically periodic, then Corollary 6.1.14 gives a new tangent vector to the orbit closure. One can then deform the surface in this direction and see if the moduli of cylinders change, or new cylinders in other directions emerge. If on the other hand the surface (X, ω) is completely periodic, i.e. if there is a cylinder in one direction then it is covered by cylinders in that direction, indicate that the orbit closure is of cylinder rank 1. Again one can try to search and play with the cylinders and their moduli, and determine also the torsion corank of the orbit closure. This outline has been carried out effectively in many cases in low genus, see for instance [ANW16] for one of the first implementations of this approach.

One can refine this approach by adding an inductive technique: degenerate the translation surface (X, ω) and reduce to lower genus. In order to do so, it is useful to have a compactification of strata and the orbit closures in them. For the methods based on the cylinder deformation theorem, one such useful compactification was introduced by Mirzakhani and Wright [MW17] and subsequently used to establish the following classification result:

6.1.19. Theorem (Orbit closures of full cylinder rank, [MW18])**.** *If the cylinder rank of an orbit closure $\mathcal{M}$ equals the genus of the ambient connected component of a stratum $\Omega\mathcal{M}_g(\kappa)$, then either $\mathcal{M}$ is that connected component, or it is a sublocus of hyperelliptic translation surfaces.*

One of the latest refinements of these techniques is due to Apisa–Wright [AW21], who show that if the cylinder rank of $\mathcal{M}$ is at least $\frac{g}{2} + 1$, then $\mathcal{M}$ is either a connected component of a stratum, or it is a locus of covers of a stratum of quadratic differentials.

6.2. Algebro-Geometric and Arithmetic Methods

In this section we give an overview of some of the results towards classification of orbit closures that are based on methods from algebraic geometry and arithmetic.

For the first few paragraphs, we will freely use the language of Shimura varieties to give a flavor of the relation between finiteness questions for orbit closures, and finiteness questions considered in that context.

6.2.1. Unlikely intersections. It follows from the results explained in §5.1 that the problem of classifying orbit closures reduces to understanding the atypical ones. This terminology is suggested by a broad circle of arithmetic and geometric problems that go under the umbrella term “unlikely intersections”. An introduction to this class of problems is provided by Zannier’s lectures [Zan12].

In these questions, one has an ambient algebraic variety $\mathcal{A}$, a fixed subvariety $\Omega \subset \mathcal{A}$, and a family of “special” subvarieties $\mathcal{S}_i$ indexed by some arithmetic or combinatorial data, but usually a countable set without moduli.

To simplify the discussion, assume that $\mathcal{A}$ is the smallest special subvariety containing Ω, otherwise replace $\mathcal{A}$ be the "special hull" $\mathcal{H}_\Omega$, the smallest special subvariety containing Ω. Additionally, suppose the inequality $\dim \Omega + \dim \mathcal{S}_i < \dim \mathcal{A}$ for all $\mathcal{S}_i$ considered, and so by dimensional considerations one expects the intersection to be empty. The typical type of conclusion one would like to draw is that if Ω intersects nontrivially infinitely many of the $\mathcal{S}_i$, then this must be accounted for by some special subvariety $\mathcal{C}_\Omega \subset \Omega$. Usually, the pattern of intersections of special subvarieties is "known", i.e. described by some arithmetic or combinatorial data.

6.2.2. Zilber–Pink type problems. One can extend the above discussion to the case when we do expect intersections for dimension reasons, and the exceptional intersections are those that have dimension larger than expected. A detailed exposition can be found in Ullmo's article [HRS+17, Ch. 1].

With notation as above, for a special subvariety $\mathcal{S}_i$, call an irreducible component $\mathcal{V}$ of the intersection $\mathcal{S}_i \cap \Omega$ *atypical* if $\dim \mathcal{V} > \dim \Omega - \operatorname{codim}_{\mathcal{A}} \mathcal{S}_i$. Then a Zilber-type conjecture is that all atypical intersections are contained in a proper subvariety of Ω.

6.2.3. Application to Orbit Closures. As explained in §5.1, from Theorem 4.4.10 it follows that atypical orbit closures can be characterized as atypical intersections of the image of the stratum $\Omega\mathcal{M}_g(\kappa)$ in an automorphic vector bundle over a mixed Shimura variety.

In fact, the case of positive-dimensional unlikely intersections on pure Shimura varieties has been established by Baldi, Klingler, and Ullmo [BKU21, Thm. 2.1] (in fact the authors establish a more general statement). Extending their result to mixed Shimura varieties, and including automorphic vector bundles, would imply the finiteness results in Theorem 5.1.4. It is also possible that these proofs could be made effective and hence answer Question 5.1.5.

6.2.4. Implementations of the unlikely intersections approach. We now proceed to describe works which have effectively used the above "unlikely intersections" approach to obtain classification results. Let us note that these methods have so far been complementary to those described in §6.1, both in technique but also in which flavors of orbit closures are covered. The methods described below are most powerful for analyzing Teichmüller curves, the lowest-dimensional orbit closures, whereas methods based on flat geometry are especially effective for analyzing large-dimensional orbit closures.

6.2.5. The decagon. In the stratum $\Omega\mathcal{M}_2(1,1)$ a Teichmüller curve which is not generated by a square-tiled surface (i.e. which is primitive) is atypical. McMullen proved in [McM06b, Thm. 1.1] that the only such orbit closure is the one generated by the regular decagon. The extra constraint which makes such a Teichmüller curve atypical comes from the torsion condition, see Theorem 4.4.7, which in the case of Teichmüller curves is due to Möller [Möl06a, Thm. 3.3].

To classify the primitive Teichmüller curves in $\Omega\mathcal{M}_2(1,1)$, the first observation is that such a curve is noncompact and necessarily has cusps. Taking the limit in the Deligne–Mumford compactification, one obtains a stable differential (X,ω) with $X \in \mathcal{M}_{0,4}$. The real multiplication property on $\mathrm{Jac}(X) \cong \mathbb{G}_m^2$, the eigenform condition on ω, and the torsion property all have their analogues in this context. They translate to the existence of $\alpha, \beta \in \mathbb{Q}$ such that $\frac{\sin(\pi\alpha)}{\sin(\pi\beta)} \in \mathbb{Q}(\sqrt{D})$ for some $D > 0$. The finitely many possibilities are then classified in [McM06b, Thm. 1.5]. Of these, all but the one corresponding to the regular decagon are excluded by a further test coming from ratios of moduli of cylinders..

An analogous result was recently established by Winsor [Win22b, Thm. 1.1], showing that the regular 14-gon generates the unique algebraically primitive (i.e. with cubic field of linear definition) Teichmüller curve in $\Omega\mathcal{M}_3(2,2)$. It is natural to ask:

6.2.6. Question (Uniqueness for regular n-gons)**.** Do the regular n-gons generate the unique primitive and atypical Teichmüller curves in their respective strata?

More generally, are the Bouw–Möller examples the unique primitive and atypical Teichmüller curves in their respective strata?

6.2.7. Cusps of Teichmüller curves. Pursuing the strategy initiated by McMullen in [McM06b], it is natural to investigate which stable curves occur in the boundary of the locus of curves which admit real multiplication on the Jacobian, perhaps on a factor. This has been studied by Bainbridge–Möeller in [BM12], who give a characterization in genus 3 in Thm. 1.1, and §5 of loc.cit. contains necessary conditions in higher genus. These techniques combined with methods from the theory of unlikely intersections are used in [BHM16, Thm. 1.1] to give an effective finiteness statement for algebraically primitive Teichmüller curves, for most strata in genus 3.

6.2.8. Degree of atypicality. Since atypical orbit closures $\mathcal{M}$ have a degree of atypicality $\delta(\mathcal{M})$ defined in Eqn. (5.1.14), it might be meaningful to organize the classification of atypical subvarieties by their degree of atypicality. Note that the formula for the degree of atypicality frequently simplifies:

$$\delta(\mathcal{M}) = (r-1)\big[(g-r)+(n-t)\big] \quad \text{if } \mathcal{M} \text{ is linearly defined over } \mathbb{Q}.$$

At the opposite extreme, for algebraically primitive Teichmüller curves we have

$$\delta(\mathcal{T}) = (g-1)\left[\frac{g}{2}+n-1\right].$$

6.2.9. Question (Classification of very atypical)**.** Let us say that a primitive orbit closure $\mathcal{M}$ is *very atypical* if $\delta(\mathcal{M})$ is larger than the dimension of the ambient stratum. Can one classify all the very atypical orbit closures?

Note that the Veech–Bouw–Möller family of examples is very atypical.

6.2.10. The cylinder package. We end the discussion of algebro-geometric methods with another look at the Cylinder Deformation Theorem 6.1.13, following Benirschke, Dozier, and Grushevsky [BDG20].

First, let us note that one gets simultaneously several pieces of information on the orbit closure $\mathcal{M}$ containing (X, ω) with horizontal cylinders C_i in an $\mathcal{M}$-equivalence class $\mathcal{C}$, with core curves γ_{C_i}:

(i) A tangent vector $\mathbf{v}_{\mathcal{C}} \in T_{(X,\omega)}\mathcal{M}$ with

$$\mathbf{v}_{\mathcal{C}} = \sum_{C_i \in \mathcal{C}} h_i[\gamma_{C_i}] \in \mathrm{H}_1(X \setminus Z; \mathbb{R}).$$

(ii) Equations for $\mathcal{M}$ of the form

$$w_j[\gamma^*_{C_i}] - w_i[\gamma^*_{C_j}] \in \mathrm{H}_1(X, Z; \mathbb{R}).$$

(iii) Monodromy transformations for loops on $\mathcal{M}$:

$$T_{\mathcal{C}}([\alpha]) = [\alpha] + \sum([\alpha] \cap [\gamma_{C_i}]) \cdot n_i \cdot [\gamma_{C_i}] \quad \text{for } [\alpha] \in \mathrm{H}_1(X, Z; \mathbb{Z})$$

where $n_i \in \mathbb{N}$ are determined from taking the smallest $t_0 > 0$ such that $u^{\mathcal{C}}_{t_0}(X, \omega) = (X, \omega)$ and we thus have the relation between heights and widths of cylinders: $th_i = n_i w_i$. Recall that the ratios of moduli of $\mathcal{M}$-parallel cylinders are in $\mathbb{Q}$, see §6.1.17.

We will refer to the above list of properties as a "cylinder package".

6.2.11. Cylinder package in meromorphic strata. We now consider a stratum of *meromorphic* differentials $\Omega\mathcal{M}_g(\kappa)$, as in §3.3. Assume that $\mathcal{M} \subset \Omega\mathcal{M}_g(\kappa)$ is algebraic, and also $\mathbb{R}$-linear in period coordinates in the sense of Definition 4.1.2; note that [BDG20] work more generally with $\mathbb{C}$-linear manifolds.

Cylinders in this context will refer to cylinders of bounded height, and $\mathcal{M}$-parallelism is defined analogously. Then [BDG20, Thm. 1.6,1.9] imply that the "cylinder package" in the sense of §6.2.10 holds on $\mathcal{M}$ as well.

Let us note, moreover, that [BDG20, Thm. 1.4] gives a more precise structure theorem for the linear equations for $\mathcal{M}$ near a boundary stratum in the compactification $\Xi\overline{\Omega}\mathcal{M}_k(\kappa)$.

6.3. Algorithmic aspects

We end by formulating questions regarding algorithms that can be used to analyze individual translation surfaces, as well as their orbit closures. Some questions are of theoretical nature – one would like to know that algorithms exist and they terminate, and others are practical – one would like to have openly available computer programs that implement these algorithms. Some work towards this has been done by Hooper, Delecroix, Rüth, Lelièvre, Chapoton, Eskin, see [HDR+22], as well as McMullen, Mukamel, and many others. We refer also to [DRW21, App. B] for a discussion of some of the above software, as well as related algorithmic questions.

For the questions formulated below, it is meaningful to separate the discussion for typical and atypical orbit closures. The case of typical orbit closures is likely to be algorithmically more approachable.

6.3.1. Cylinders: complexity, enumeration. Suppose (X, ω) is a translation surface, with period coordinates in a number field $k \subset \mathbb{R}$. Then for any cylinder C on the surface, the height, width, modulus, and slope are elements of the field k, and we will refer to them as the numerical invariants of C.

6.3.2. Question (Complexity and Enumeration)**.** Define a notion of complexity $H(C)$ of the cylinder C. Bound the number-theoretic complexity of the numerical invariants of C, i.e. give bounds on the Weil height in terms of the complexity $H(C)$.

Devise an algorithm that enumerates the cylinders on (X, ω). Establish an upper bound, perhaps polynomial, on the time needed to describe all cylinders of bounded complexity.

In the case of surfaces (X, ω) whose orbit closure is a Teichmüller curve whose linear field of definition is real quadratic, the above questions have been answered by McMullen in [McM22].

Let us note that the case of (X, ω) that has period coordinates in $\mathbb{Q}$, i.e. is a ramified torus cover, can also be handled algorithmically. Therefore, in view of the classification of typical orbit closures in Proposition 5.1.6, it would be particularly interesting to answer:

6.3.3. Question (Cylinders and Hilbert modular surfaces)**.** Classify cylinders on translation surfaces $(X, \omega) \in \Omega\mathcal{M}_2(1,1)$ that generate a Hilbert modular surface.

This would settle Question 6.3.2 for all typical orbit closures.

The Cylinder Deformation Theorem 6.1.13, combined with an efficient enumeration of cylinders on a surface answering Question 6.3.2, would open the way to rigorously analyze orbit closures in a computer-aided way:

6.3.4. Question (Orbit closure enumeration)**.** Given $\kappa = (k_1, \ldots, k_n)$, give a terminating algorithm that enumerates all atypical orbit closures in the stratum $\Omega\mathcal{M}_g(\kappa)$.

By the Finiteness Theorem 5.1.4, there are only finitely many atypical orbit closures. In particular, the atypical locus is a finite union of connected algebraic varieties, so at least theoretically computable. The question should be compared to an analogous Diophantine one: give a terminating algorithm that starting from a genus $g \geq 2$ smooth algebraic curve over a number field enumerates all its rational points, which are known to be finite a priori.

A related question is to compute orbit closures starting from given initial data:

6.3.5. Question (Computation of orbit closure)**.** Given a translation surface (X, ω) with period coordinates in a totally real number field, or more generally an intersection P of a plane defined over a totally real number field with a period coordinate chart, compute the closure of this set under $\mathbf{GL}_2(\mathbb{R})$.

For the previous two questions, some algorithms have already been implemented, see [HDR+22] using earlier code of Alex Eskin. It would be of interest to prove termination, and establish upper bounds for the running times, of these algorithms.

6.3.6. Veech surfaces. Recall that (X, ω) is called a Veech surface if its orbit closure is a Teichmüller curve. We will call a Teichmüller curve $\mathcal{M}$ *absolutely atypical* if there is no orbit closure $\mathcal{M}'$ (perhaps a stratum) that contains $\mathcal{M}$ and such that $\mathcal{M}$ is typical relative to $\mathcal{M}'$. Note that square-tiled surfaces can lead to atypical Teichmüller curves, if the monodromy on the complement to the tangent space is smaller than the full symplectic group. The other currently known examples, which are also primitive, are provided by the Bouw–Möller family, as well as three exceptional cases obtained from the unfolding of the triangles $(3, 4, 5), (2, 3, 4), (3, 5, 7)$ where an (a, b, c) triangle refers to one whose angles are proportional to the listed numbers.

6.3.7. Question (Absolutely atypical Teichmüller curves)**.** Are there any other absolutely atypical primitive Teichmüller curves *in any genus* at all?

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