AN INTRODUCTION TO HYPERGEOMETRIC EQUATIONS, VIA D-MODULES

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Abstract. A brief introduction to $\mathcal{D}$-modules, followed by an equally brief introduction to GKZ hypergeometric differential equations.

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1. Introduction

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Outline of section.

2.1. Overview of $\mathcal{D}$-modules

In this section we provide an outline of the structures discussed in more detail later on.

2.1.1. Setup. We fix a characteristic zero field K, which the reader can safely assume is $\mathbb{C}$. It will be useful, however, to develop the theory with a general field in mind, since in some applications we will take $K = k(s)$ to be the field of rational functions in one formal variable s, over another characteristic zero field k.

We will denote by $\mathcal{O}_{\mathbb{A}^n} := K[x_1, \ldots, x_n]$ the algebra of polynomial functions on the affine n-space $\mathbb{A}^n$, and by $\mathcal{O}_{T^\vee\mathbb{A}^n} := K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ the algebra of polynomial functions on its cotangent bundle $T^\vee\mathbb{A}^n$, which is a space isomorphic to $\mathbb{A}^{2n}$.

2.1.2. The Weyl algebra. Let $\mathcal{D}_n$ denote the algebra of differential operators on $\mathbb{A}^n$, with polynomial coefficients. Concretely, a differential operators $D \in \mathcal{D}_n$ can be written as a finite sum

$$D = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta$$

where $c_{\alpha,\beta}$ are scalar coefficients and we used multi-index notation:

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \partial^\beta := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$$
$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad \beta = (\beta_1, \ldots, \beta_n)$$
$$\text{and we will write } |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

To ease notation, we will sometimes write ∂_i instead of ∂_{x_i}.

An equivalent description of the Weyl algebra is as the free algebra on the symbols $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ subject to the fundamental commutation relation

$$[\partial_j, x_i] := \partial_j x_i - x_i \partial_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.1.3}$$

2.1.4. $\mathcal{D}$-modules. Our main object of study will be modules over the algebra $\mathcal{D}_n$. Suppose that a function, say $f(x) = e^x$, satisfies a differential equation, such as $\partial_x f = f$, rewritten as $(\partial_x - 1)f = 0$. To such a function we can associate a $\mathcal{D}$-module $\mathcal{M}_f$ defined as the quotient of $\mathcal{D}_n$ by the left ideal generated by the annihilator of f, e.g. $\mathcal{I}_f := \mathcal{D}_n \cdot (\partial_x - 1)$. Notice that while the solution of a differential equation is typically not algebraic, its corresponding $\mathcal{D}$-module has an algebraic description. We will try to

extract useful algebraic information from the module itself and see some applications to the analytic properties of the solutions.

2.1.5. Dequantization. The Weyl algebra $\mathcal{D}_n$ is not commutative, but only very mildly so. To every differential operator $D \in \mathcal{D}_n$ one can associate its principal symbol $\sigma(D)$ consisting of the terms with highest number of derivatives. This is a well-defined function on the cotangent bundle, i.e. an element of $\mathcal{O}_{T^\vee \mathbb{A}^n}$.

More generally, to any $\mathcal{D}_n$-module $\mathcal{M}$ one can associate a module $\operatorname{gr} \mathcal{M}$ over $\mathcal{O}_{T^\vee \mathbb{A}^n}$. The modules that arise this way are not arbitrary – in fact their support is a coisotropic subvariety of $T^\vee \mathbb{A}^n$. Recall that $T^\vee \mathbb{A}^n$ has a natural symplectic structure, and a coisotropic subvariety S of a symplectic variety X is one in which the tangent space at every point is coisotropic, i.e. $T_s S \supseteq T_s S^\perp$ for all $s \in S$, where $\perp$ is taken for the symplectic form on $T_s X$.

2.1.6. Holonomic $\mathcal{D}$-modules. A consequence of the coisotropic property of the support is that its dimension belongs to the set $\{n, n+1, \dots 2n\}$. In symplectic geometry, the most interesting submanifolds are the Lagrangian ones, i.e. the coisotropic manifolds of dimension n, or equivalently satisfying $T_s S^\perp = T_s S$ where again $\perp$ is the symplectic orthogonal in the ambient $T_s X$.

The corresponding $\mathcal{D}$-modules, whose support is concentrated on Lagrangian subvarieties, are called *holonomic* $\mathcal{D}$-modules and are the ones most amenable to be studied by algebraic methods. We will see that PDEs that lead to holonomic $\mathcal{D}$-modules have, on a Zariski-open set, a finite-dimensional space of solutions. So holonomic $\mathcal{D}$-modules are closer to ODEs, where a finite number of scalars determines a solution uniquely, and quite far from equations of interest in analysis such as the Laplace equation for harmonic functions or the wave equation.

2.1.7. Generalized functions. Although far from analysis, the information coming from $\mathcal{D}$-modules can be useful in studying singularities of functions, and generalized functions, aka distributions. Some examples are included in the exercises.

2.1.8. Some examples of $\mathcal{D}$-modules. We list some elementary examples of $\mathcal{D}_1$-modules that the reader can keep in mind while reading subsequent sections.

Suppose that f is some function in a sufficiently rich function space, e.g. smooth, analytic, or perhaps a generalized function. It is rarely an algebraic object, but we can consider the $\mathcal{D}$-module it generates in the ambient function space, and denote it by $\mathcal{M}_f$.

To start, if δ_0 is the Dirac-delta function at zero in $\mathbb{A}^1$, then it is annihilated by $x, x^2, \dots$ i.e. any polynomial vanishing at the origin. So we denote $\mathcal{M}_{\delta_0} := K[\partial] = \mathcal{D}_1 / \mathcal{D}_1 \cdot x$.

If one considers the generalized function δ_0', which returns the first derivative of a function at the origin, then the corresponding module is $\mathcal{M}_{\delta_0'} := \mathcal{D}_1/D_1 \cdot x^2$. See Exercise 2.8.4 for an unexpected isomorphism.

The constant function 1 is annihilated by any derivative, so we have $\mathcal{M}_1 := K[x] = \mathcal{D}_1/\mathcal{D}_1 \cdot \partial$.

In §2.7 we will consider the $\mathcal{D}$-module $\mathcal{M}(p^s)$ where $p(x_1, \ldots, x_n)$ is a polynomial in n variables and s is a formal variable, viewed as an exponent. We will think of it as the $\mathcal{D}_n$-module spanned by $p(x_1, \ldots, x_n)^s$, viewed as a formal expression. Many of the statements below give nontrivial applications in this example.

2.2. Filtered algebras and modules

2.2.1. Setup. Let $\mathcal{D}$ be an associative, but not necessarily commutative algebra with unit. The reader can think of $\mathcal{D}_n$ as defined in §2.1.2, but much of what we say at first will be valid in greater generality.

2.2.2. Definition (Filtration on algebra)**.** A *filtration* on the algebra $\mathcal{D}$ is an increasing collection of additive subgroups

$$\{0\} \subsetneq F_0\mathcal{D} \subseteq F_1\mathcal{D} \ldots \subseteq F_i\mathcal{D} \ldots \subseteq \mathcal{D}$$

satisfying:

Unit: $1 \in F_0\mathcal{D}$.
Multiplicativity: $F_i\mathcal{D} \cdot F_j\mathcal{D} \subseteq F_{i+j}\mathcal{D}$.
Exhaustion: $\cup_{i\geq 0} F_i\mathcal{D} = \mathcal{D}$.

The *associated graded* algebra is defined to be the direct sum of successive quotients

$$\operatorname{gr}^F \mathcal{D} := \bigoplus_{i\geq 0} F_i\mathcal{D}/F_{i-1}\mathcal{D} \tag{2.2.3}$$

which is equipped with a natural multiplication, respecting the grading, by the multiplicativity property of the filtration.

2.2.4. Definition (Almost commutative)**.** A filtered algebra is *almost commutative* if $\operatorname{gr}^F \mathcal{D}$ is commutative, or equivalently if

$$[F_i\mathcal{D}, F_j\mathcal{D}] \subseteq F_{i+j-1}\mathcal{D}$$

where $[A, B] := \{ab - ba \colon a \in A, b \in B\}$.

Assume from now on that $\mathcal{D}$ is almost commutative.

2.2.5. Symbol map. For $D \in \mathcal{D}$, define its *order* to be

$$\operatorname{ord}_F D := \inf\{i \colon D \in F_i\mathcal{D}\}$$

i.e. the smalled term of the filtration in which it occurs. We then define the *symbol* map to be

$$\begin{aligned} \sigma\colon \mathcal{D} &\to \operatorname{gr}_F \mathcal{D} \\ D &\mapsto [D] \in F_i\mathcal{D}/F_{i-1}\mathcal{D} \quad i = \operatorname{ord}_F \mathcal{D} \end{aligned}$$

Note that if $D_1, D_2 \in \mathcal{D}$ and $D_1 \cdot D_2 \neq 0$ then

$$\begin{aligned} \operatorname{ord}(D_1 \cdot D_2) &= \operatorname{ord}(D_1) + \operatorname{ord}(D_2) \\ \sigma(D_1 \cdot D_2) &= \sigma(D_1) \cdot \sigma(D_2) \end{aligned}$$

2.2.6. Example (Geometric and Bernstein filtrations)**.** Let $\mathcal{D} := \mathcal{D}_n$ denote the Weyl algebra. There are two natural filtrations on it, useful for different purposes.

The **geometric filtration** (also called the order filtration), is defined by

$$F_i^G\mathcal{D} := \left\{ \sum c_{\alpha\beta} x^\alpha \partial^\beta \colon |\beta| \leq i \right\}$$

or in other words, $F_i^G\mathcal{D}$ denotes the differential operators that have at most i derivatives. Then

$$\operatorname{gr}^G \mathcal{D} \cong K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

with grading given by the total degree in the $\xi_\bullet$-variables.

The **Bernstein filtration** is defined by

$$F_i^B\mathcal{D} := \left\{ \sum c_{\alpha\beta} x^\alpha \partial^\beta \colon |\alpha| + |\beta| \leq i \right\}$$

or in other words, we assign the x and ∂ variables equal weight. Again

$$\operatorname{gr}^B \mathcal{D} \cong K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$$

but now the grading is given by total degree in all the variables.

Both filtrations make $\mathcal{D}$ into an almost commutative algebra, but for the Bernstein filtration we have the stronger property that

$$[F_i^B\mathcal{D}, F_j^B\mathcal{D}] \subseteq F_{i+j-2}^B\mathcal{D}.$$

Note that the identity $\partial \cdot x - x \cdot \partial = 1$ in the Weyl algebra is one instance of this containment. Another advantage of the Bernstein filtration is that the associated graded pieces are finite-dimensional, which is not the case for the geometric filtration.

2.2.7. Filtered modules. Suppose next that $\mathcal{M}$ is a $\mathcal{D}$-module, and assume that it too has a filtration $F_\bullet\mathcal{M}$. The associated graded module $\operatorname{gr}^F \mathcal{M}$ is defined analogously to Eqn. (2.2.3). We will say that the filtration on $\mathcal{M}$ is *compatible* with the filtration on $\mathcal{D}$, if

$$F_i\mathcal{D} \cdot F_j\mathcal{M} \subseteq F_{i+j}\mathcal{M}$$

and we will only consider compatible filtrations.

2.2.8. Definition (Good filtration)**.** The filtration $F_\bullet\mathcal{M}$ is *good* if $\mathrm{gr}^F\,\mathcal{M}$ is finitely generated over $\mathrm{gr}^F\,\mathcal{D}$.

Two filtrations $F_\bullet\mathcal{M}, F'_\bullet\mathcal{M}$ are called *equivalent* if there exists j_0 such that

$$F_{j-j_0}\mathcal{M} \subseteq F'_j\mathcal{M} \subseteq F_{j+j_0}\mathcal{M} \quad \forall j.$$

2.2.9. Hilbert polynomial of a module. We recall some general facts from commutative algebra. Suppose that $\mathcal{A} = \oplus_{i\geq 0}\mathcal{A}_i$ is a commutative, graded algebra over the field $K = \mathcal{A}_0$, which is generated as an algebra by the finite-dimensional vector space $\mathcal{A}_1$. Suppose now that $\mathcal{M} = \oplus_{i\geq 0}\mathcal{M}_i$ is a finitely generated graded $\mathcal{A}$-module.

Then there exists a polynomial $\chi_{\mathcal{M}}(t) \in \mathbb{Q}[t]$, called the *Hilbert polynomial* of $\mathcal{M}$, such that for $i \gg 0$ we have that

$$\dim \mathcal{M}_i = p_{\mathcal{M}}(i).$$

Note that furthermore $\sum_{l=0}^{i} \dim \mathcal{M}_l$ is also a polynomial in i, of degree one higher than $p_{\mathcal{M}}$. See also Exercise 2.8.2 for some examples.

We now return to the setting of an almost commutative filtered algebra $\mathcal{D}$, equipped with a filtration $F_\bullet\mathcal{D}$ such that $F_i\mathcal{D}$ is finite-dimensional. For instance, the Bernstein filtration on the Weyl algebra satisfies this condition.

2.2.10. Proposition (Polynomial dimension for $\mathcal{D}$-modules)**.** *Let $\mathcal{M}$ be a $\mathcal{D}$-module equipped with a good filtration $F_\bullet\mathcal{M}$. Then there exists a polynomial $p_{\mathcal{M}}(t) \in \mathbb{Q}[t]$ such that for $i \gg 0$ we have that*

$$\dim F_i\mathcal{M} = p_{\mathcal{M}}(i).$$

Proof. Indeed, this follows from the identity

$$\dim F_i\mathcal{M} = \sum_{l=0}^{i} \dim \mathrm{gr}_l^F\,\mathcal{M}$$

and the corresponding statement in the commutative setting. □

2.2.11. Definition (Dimension and Multiplicity)**.** Suppose that $\mathcal{M}$ and $p_{\mathcal{M}}(t)$ are as in Proposition 2.2.10. Write out the leading term of the Hilbert polynomial as

$$p_{\mathcal{M}}(t) = \frac{m}{d!}t^d + O\left(t^{d-1}\right).$$

Then the *algebraic dimension* of $\mathcal{M}$ is defined to be d and denoted $\dim_{alg}\mathcal{M}$, and the *algebraic multiplicity* is defined to be m and denoted $mult_{alg}\mathcal{M}$.

While the algebraic dimension is clearly a nonnegative integer, so is the algebraic multiplicity. This follows from Exercise 2.8.3.

2.2.12. Definition (Characteristic variety and Characteristic cycle)**.** Suppose $\mathcal{D}$ and $\mathcal{M}$ are equipped with compatible, good filtrations, and the associated graded is taken for them.

The *singular support* $SS(\mathcal{M})$ of a $\mathcal{D}$-module $\mathcal{M}$ is the support $\operatorname{supp} \operatorname{gr} \mathcal{M}$ of $\operatorname{gr} \mathcal{M}$ on $\operatorname{Spec}(\operatorname{gr} \mathcal{D})$ in the usual sense of commutative algebra, i.e. the subset cut out by the radical of the annihilator: $\sqrt{\operatorname{Ann} \operatorname{gr} \mathcal{M}} \subset \operatorname{gr} \mathcal{D}$.

If $S \in SS(\mathcal{M})$ is an irreducible component of the singular support, denote by $\operatorname{mult}_S(\mathcal{M})$ the multiplicity of $\operatorname{gr} \mathcal{M}$ along S, i.e. the dimension of the vector space $\operatorname{gr} \mathcal{M} \otimes_{\operatorname{gr} \mathcal{D}} K(S)$ where $K(S)$ is the function field of S.

The *characteristic cycle* of $\mathcal{M}$ is the formal algebraic cycle

$$\widetilde{SS}(\mathcal{M}) := \sum_{S \in SS(\mathcal{M})} \operatorname{mult}_S \mathcal{M}[S].$$

2.2.13. Proposition (Properties of the characteristic variety)**.**

(i) *The characteristic variety and characteristic cycle do not depend on the choice of good filtration on $\mathcal{M}$.*

(ii) *The characteristic cycle is additive in short exact sequences:*

$$\text{if } 0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0 \text{ is exact}$$
$$\text{then } \widetilde{SS}(\mathcal{M}) = \widetilde{SS}(\mathcal{M}') + \widetilde{SS}(\mathcal{M}'').$$

We will see in §2.4 below that in the case of the Weyl algebra, the characteristic variety of a $\mathcal{D}$-module is further linked to the symplectic geometry of cotangent bundles.

2.3. Bernstein inequality

2.3.1. Setup. We now restrict our attention to the noncommutative Weyl algebra $\mathcal{D}_n$, equipped with the Bernstein filtration.

The next result can be seen as an algebraic manifestation of the uncertainty principle.

2.3.2. Theorem (Bernstein inequality)**.** *Suppose that $\mathcal{M}$ is a $\mathcal{D}_n$-module, finitely generated and equipped with a good filtration, and assume $\mathcal{M} \neq 0$. Then $\dim_{alg} \mathcal{M} \geq n$.*

This result is in stark contrast to modules over polynomial algebras, where any dimension between zero and that of the algebra itself can be realized by some module.

Proof. By shifting the indexing of the filtration on $\mathcal{M}$, we can assume that $F_0\mathcal{M} \neq \{0\}$. We will show that the natural action of differential operators on the module is faithful, i.e. we claim that the natural map

$$F_j^B \mathcal{D}_n \to \operatorname{Hom}_K(F_j\mathcal{M}, F_{2j}\mathcal{M})$$

is injective. But we know that $\dim F_l\mathcal{M} = \frac{m}{d!} l^d + O(l^{d-1})$ where d is the algebraic dimension of $\mathcal{M}$, so the dimension of the Hom-space above is $\lesssim j^{2d}$. But $\dim F_j^B \mathcal{D}_n \gtrsim j^{2n}$, so we find that $d \geq n$.

To prove the claim, we proceed by induction on j. For $j = 0$ the claim follows since $F_0^B \mathcal{D}_n$ consists of scalars, which act faithfully on $F_0\mathcal{M}$, which is nonzero by assumption. Next, suppose by contradiction that a differential

operator $D \in F_j^B \mathcal{D}_n$ has the property that $Dm = 0$ for any $m \in F_j\mathcal{M}$. By Exercise 2.8.1, there exists a differential operator $a \in F_1^B \mathcal{D}_n$ such that $[a, D] \neq 0$ and is an element of $F_{j-1}^B \mathcal{D}$. By the inductive assumption, there exists $m' \in F_{j-1}\mathcal{M}$ such that $[a, D]m' \neq 0$. Writing it out explicitly we find $a(Dm') - D(am') \neq 0$, but $am' \in F_j\mathcal{M}$, and $m' \in F_{j-1}\mathcal{M} \subseteq F_j\mathcal{M}$, so D annihilates both by assumption. This yields the desired contradiction. □

It follows from the Bernstein inequality that the algebraic dimension of a $\mathcal{D}_n$-module is at least n. The class of modules for which this lower bound is achieved is particularly important:

2.3.3. Definition (Holonomic $\mathcal{D}$-module)**.** A $\mathcal{D}_n$-module $\mathcal{M}$ is called *holonomic* if it is finitely generated and $\dim_{alg} \mathcal{M} = n$, or if $\mathcal{M} = 0$.

Holonomic $\mathcal{D}_n$-modules are well-behaved and form an abelian category, see Exercise 2.8.6.

The next result gives a practical way do decide if a module is holonomic.

2.3.4. Lemma (Growth bound)**.** *Suppose that $\mathcal{M}$ is a $\mathcal{D}_n$-module, not necessarily finitely generated. Assume that it is equipped with a filtration $F_\bullet\mathcal{M}$, compatible with the Bernstein filtration on $\mathcal{D}_n$, and satisfying*

$$\dim_K F_j\mathcal{M} \leq \frac{c}{n!} j^n + O\left(j^{n-1}\right).$$

Then $\mathcal{M}$ is holonomic, and in particular finitely generated over $\mathcal{D}_n$.

Proof. We only sketch the argument. It is clear that any finitely generated submodule $\mathcal{N} \subset \mathcal{M}$ is holonomic, since finite generation allows us to apply the results on Hilbert polynomials and find that $\dim F_j\mathcal{N} = \frac{m_{\mathcal{N}}}{d!} j^d + O(j^{d-1})$. Combined with the Bernstein inequality, this implies $d = n$. □

2.4. Geometric methods

A good introduction to symplectic geometry for the purposes of algebraic geometry is in the first chapter of [CG10].

2.4.1. Recollections from symplectic geometry. A symplectic form ω on a $2n$-dimensional manifold M is a closed nondegenerate 2-form, i.e. $d\omega = 0$ and ω^n is nowhere vanishing on M. The definition makes sense in the smooth, analytic, complex analytic, or algebraic categories, and many of the constructions below can be done in all of these situations.

A symplectic form ω on M induces a symplectic pairing on each tangent space. Recall also that in a vector space with a symplectic pairing, a subspace is called *isotropic* if the symplectic pairing is trivial when restricted to it, *coisotropic* if its symplectic orthogonal is isotropic. Finally a subspace is called *Lagrangian* if it is both isotropic and coisotropic. Note that Lagrangian spaces have precisely half the ambient dimension, isotropic ones have at most half the ambient dimension, and coisotropic ones have at least half the ambient dimension.

2.4.2. Cotangent bundles. Suppose that X is an n-dimensional manifold. Its cotangent bundle T^*X carries a "tautological" 1-form typically denoted λ, defined as follows. At a point $(x, \xi) \in T^*X$, for a vector in its the tangent space $v \in T_{(x,\xi)}T^*X$, project v to the tangent space of X to obtain $[v] \in T_xX$, and apply ξ to the result, or in other words $\lambda_{(x,\xi)}(v) := \xi([v])$. The tautological symplectic form on T^*X is defined to be $\omega := d\lambda$, which is visibly closed.

The group of nonzero scalars, denoted $\mathbf{G}_m$, which can be $\mathbb{R}^\times, \mathbb{C}^\times$, or their analogues over other fields, acts on the fibers of $T^*X \to X$. It induces the *Euler vector field* Eu on T^*X, which is tangent to the fibers of the projection.

In local coordinates $(x_1, \ldots, x_n)$ on X with dual coordinates $\xi_1, \ldots, \xi_n$ on the fibers of T^*X, the 1-form λ is $\sum \xi_i dx_i$, the symplectic 2-form is $\omega = \sum d\xi_i \wedge dx_i$ and the Euler vector field is $\mathrm{Eu} := \sum \xi_i \partial_{\xi_i}$. In particular we observe the following basic differential-geometric formulas, which can be deduced from the definitions, or from the explicit local descriptions:

$$\begin{aligned} \iota_{\mathrm{Eu}}\lambda = 0 \quad \iota_{\mathrm{Eu}}\omega = \lambda \\ \mathcal{L}_{\mathrm{Eu}}\lambda = \lambda \quad \mathcal{L}_{\mathrm{Eu}}\omega = \omega \end{aligned} \tag{2.4.3}$$

2.4.4. Conormal bundles. Suppose now that $S \subset X$ is a k-dimensional submanifold. Define its *conormal bundle* $N^*_X S \subset T^*X$ to be the set of points of the form $(s, \xi) \in T^*X$ with $s \in S$ and $\xi \in T^*_s X$ such that ξ the tangent space of S at s is in the kernel of ξ, i.e. $\xi|_{T_sS} \equiv 0$.

A dimension count gives that the fiber of $N^*_X S$ above $s \in S$ has dimension $n - k$, so the total dimension of $N^*_X S$ is n, i.e. half the ambient dimension. We will see in Theorem 2.4.6 below that $N^*_X S$ is a Lagrangian submanifold of T^*X. It has the additional property that it is invariant under scaling, and more generally a submanifold $L \subset T^*X$ is called *conical* if it is invariant under the scaling action of $\mathbf{G}_m$.

For example, the conormal bundle of a point $x \in X$ is the fiber $T^*_x X$, and the conormal bundle of X is the zero section in T^*X. More generally, if in local coordinates $(x_1, \ldots, x_n)$ the submanifold S is given by $x_{k+1} = \ldots = x_n = 0$, then

$$N^*_X S \text{ is given by the equations } x_{k+1} = \ldots = x_n = 0 = \xi_1 = \ldots = \xi_k.$$

2.4.5. Allowing singularities. When working with algebraic varieties, it is useful to allow manifolds to have singularities. We will not discuss singular symplectic manifolds, but we will say that a subvariety L of a smooth symplectic manifold (M, ω) is *Lagrangian* if the Zariski-open smooth locus $L^\circ \subset L$ is Lagrangian. So the closure of a smooth Lagrangian manifold is Lagrangian.

When working with conormal bundles, one should be careful. Suppose $S \subset X$ is a not necessarily smooth subvariety of the smooth variety X. Let $S^\circ \subset S$ denote the smooth locus of S. Then $N^*_X S$ is defined as the closure of $N^*_X(S^\circ)$. For example, if S is an irreducible curve in $X = \mathbb{A}^2$, with a singular point at $s_0 \in S$, then $N^*_X S$ is an irreducible 2-dimensional manifold

in $T^*X = \mathbb{A}^4$, while the conormal bundle $N^*_X s_0$ is the fiber $T^*_{s_0}X$, and is not part of $N^*_X S$.

It turns out that all conical Lagrangians can be obtained this way, as the next result of Kashiwara shows, see also [CG10, Lemma 1.3.27].

2.4.6. Theorem (Conical Lagrangians)**.**

(i) *The conormal bundle of a subvariety $S \subset X$ is a conical Lagrangian of $T^\vee X$.*

(ii) *Suppose $L \subset T^\vee X$ is an irreducible, conical Lagrangian and let $S^\circ \subset X$ denote the smooth locus of the projection of L to X. Then $L = \overline{N^*_X(S^\circ)}$.*

2.5. Riemann–Hilbert correspondence and Perverse Sheaves

2.5.1. Setup. The de Rham functor

$$DR(\mathcal{M}) := \mathcal{M} \otimes (\mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \cdots \Omega^n_X)$$

gives a map

$$\mathsf{D}^b(RHol_X) \to \mathsf{D}^b(Constr)$$

to (derived) category of complexes of sheaves with constructible cohomology sheaves. A triangulated category can have different abelian cores, and this functor yields a new core of the category on the right-hand side. It is natural to ask if there is a description of the complexes of sheaves on the right that yield this core. The answer is given by perverse sheaves.

2.5.2. Internal functors on sheaves. Let X be a topological space, equipped with a stratification by manifolds. Let Sh_X denote the category of constructible sheaves on X, with coefficients in a field $\mathbb{K}$ of characteristic zero. Let also $D^b(\mathsf{Sh}_X)$ denote the associated bounded derived category.

On it we have the functors $\otimes$, sheaf-$\mathcal{H}om$, which are defined at the level of Sh_X already, and more interestingly the duality functor $\mathbb{D}_X$ which is in full generality only defined on the derived category (when X is singular).

2.5.3. External functors on sheaves. Let X, Y be reasonable topological spaces, e.g. algebraic varieties over $\mathbb{C}$ or more broadly, definable in an o-minimal structure extending $\mathbb{R}_{alg}$. Let also $f\colon X \to Y$ be a reasonable map (of the same flavor as X, Y).

To a sheaf $\mathcal{F}$ on X we can apply the two basic functors:

pushforward: defined by

$$(f_*\mathcal{F})(U) := \mathcal{F}\left(f^{-1}(U)\right) \text{ for an open set } U \subset Y.$$

proper pushforward: defined by

$$(f_!\mathcal{F})(U) := \{s \in \mathcal{F}\left(f^{-1}(U)\right) : f|_{\operatorname{supp} s}\colon f^{-1}(U) \to U \text{ is proper}\}$$

where $\operatorname{supp} s$ is the support of s, i.e. $x \in X$ such that s restricts to a nonzero section of the stalk $\mathcal{F}_x$.

We have a natural inclusion $f_!\mathcal{F} \hookrightarrow f_*\mathcal{F}$ and when f is a proper map, this is an isomorphism. More interestingly, if $f\colon S \hookrightarrow X$ is a locally closed embedding, then $f_*\underline{\mathbb{C}}_S / f_!\underline{\mathbb{C}}_S \cong \underline{\mathbb{C}}_{\overline{S}\setminus S}$.

To a sheaf $\mathcal{G}$ on Y we can apply the other two basic functors:

pullback: defined as the sheafification of the injective limit presheaf

$$(f^*\mathcal{G})(V) := \varinjlim_{f(V)\subset U} \mathcal{G}(U)$$

pullback with support: If f is a locally closed inclusion, defined as the sheafification of the injective limit presheaf

$$\left(f^!\mathcal{G}\right)(V) := \varinjlim_{f(V)\subset U} \{s \in \mathcal{G}(U)\colon \operatorname{supp} s \subset f(V)\}$$

Pullback is very well-behaved, in fact it is exact, so there's no need to pass to the derived category. To define $f^!$ for a general morphism, it is necessary to pass to the derived category.

2.5.4. Adjunctions. We have that f^* is the right adjoint of f_*

$$\operatorname{Hom}_Y(f_*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_X(\mathcal{F}, f^*\mathcal{G})$$

and similarly for $f^!, f_!$ when they are defined.

2.5.5. Duality.

2.6. Six functor formalism

2.6.1. Setup. Internal functors are the tensor product $\otimes$ and duality $\mathbb{D}$; then $\operatorname{Hom}(A, B) := \mathbb{D}(A) \otimes B$?

External functors, associated to a morphism $f\colon X \to Y$, are:?

2.7. Applications

We keep the notation as in the preceding sections and will develop some applications of holonomic $\mathcal{D}_n$-modules.

2.7.1. Module associated to a hypersurface. We fix now a characteristic zero field k and set $K = k(s)$ to be the field of rational functions in a formal variable s. The algebra $\mathcal{D}_n$ will refer to $k(s)\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$. Fix a nonzero polynomial $p \in k[x_1, \dots, x_n]$; incidentally, note that what we say below works even if $p \in k(s)[x_1, \dots, x_n]$. Define now the $\mathcal{D}_n$-module

$$\mathcal{M}(p^s) := K\left[x_1, \dots, x_n, p^{-1}\right] \cdot p^s$$

in other words, set-theoretically $\mathcal{M}(p^s)$ consists of $K[x_1, \dots, x_n, p^{-1}]$, but with the action of derivatives twisted by the formal expression p^s:

$$\partial_i(f \cdot p^s) := \partial_i f \cdot p^s + s \cdot f \cdot \partial_i p \cdot p^{-1} \cdot p^s$$

obtained by applying the Leibniz rule and the formal differentiation $\partial_i p^s = s \cdot \partial_i p \cdot p^{s-1}$.

2.7.2. Proposition (Finite generation and holonomicity)**.**

(i) *The module $\mathcal{M}(p^s)$ is holonomic.*
(ii) *For any $j \in \mathbb{Z}$, the module $\mathcal{M}(p^s)$ is generated over $\mathcal{D}_n$ by the element $p^j \cdot p^s$.*

Proof. To show that $\mathcal{M}(p^s)$ is holonomic, we will apply the criterion provided by Lemma 2.3.4 and describe the appropriate filtration. Namely, set

$$F_j\mathcal{M}(p^s) := \left\{q \cdot p^{-j} \cdot p^s \colon \deg q \leq j \cdot (\deg p + 1)\right\}$$

The inequality on degrees can be rewritten as $\deg q - j \deg p \leq j$, meaning that it is essentially a filtration by degree, assigning negative degrees to p^{-1}. It is then clear that for monomials we have $x_{i_1} F_j\mathcal{M}(p^s) \subseteq F_{j+1}\mathcal{M}(p^s)$ and for derivatives we compute

$$\partial_{i_1}\left(qp^{s-j}\right) = (\partial_{i_1} q) \cdot p^{s-j} + q \cdot (s-j) \cdot (\partial_{i_1} p) \cdot p^{s-(j+1)}$$

which is also in $F_{j+1}\mathcal{M}(p^s)$. The dimension of $F_j\mathcal{M}(p^s)$ over K is, up to some multiplicative constants $\lesssim (j \deg p)^n \lesssim j^n$, so the bound necessary to apply Lemma 2.3.4 follows.

We will next check that $\mathcal{M}(p^s)$ is generated by some p^{s-j_0}. This implies that for any $k \geq 0$ there exists $D_k \in \mathcal{D}_n$ such that we have the identity $D_k \cdot p^{s-j_0} = p^{s-j_0-k}$. But the field $K = k(s)$ has automorphisms σ_j taking s to $s+j$, so applying this automorphism to the stated identities yields analogous ones for any $j \in \mathbb{Z}$, which in turn implies that any p^{s-j} generates $\mathcal{M}(p^s)$.

It remains therefore to show that *some* p^{s-j_0} generates the module. But set $\mathcal{M}_j \subset \mathcal{M}(p^s)$ to be the module generated by p^{s-j}. We clearly have $\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$, and by the Artinian property of holonomic $\mathcal{D}_n$-modules (see Exercise 2.8.6) it follows that this sequence must eventually stabilize, and it must equal all of $\mathcal{M}$ since $\cup_{j\geq 0}\mathcal{M}_j = \mathcal{M}$. □

2.7.3. Corollary (Existence of Bernstein–Sato polynomial)**.** *Suppose that $p \in k[x_1, \ldots, x_n]$ is a nonzero polynomial. Then there exists a differential operator $D \in k[s]\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ and a polynomial $b(s) \in k[s]$ such that*

$$D \cdot p^{s+1} = b(s) \cdot p^s.$$

Proof. Since p^{s+1} generates $\mathcal{M}(p^s)$ by Proposition 2.7.2, it follows that there exists a differential operator $\widetilde{D} \in \mathcal{D}_n = k(s)\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ such that

$$\widetilde{D} \cdot p^{s+1} = p^s$$

Now clearing the denominators in $\widetilde{D}$ yields the differential operator D and polynomial $b(s)$. □

The monic polynomial $b(s)$ of smallest degree satisfying Corollary 2.7.3 is called the *Bernstein–Sato* polynomial of $p(x_1, \ldots, x_n)$.

2.8. Exercises

2.8.1. Exercise (Geometric and Bernstein filtration)**.** Let $F^G_\bullet$ and $F^B_\bullet$ be the geometric and Bernstein filtrations of the Weyl algebra, as introduced in Example 2.2.6. Verify the assertions made there, namely:

$$\left[F^G_i, F^G_j\right] \subseteq F^G_{i+j-1} \quad \text{and} \quad \left[F^B_i, F^B_j\right] \subseteq F^B_{i+j-2}$$

Show also that for any nonzero $D \in F^B_j\mathcal{D}_n$ there exists an element $a \in F^B_1\mathcal{D}_n$ (in fact, a can be taken to be either a monomial, or a derivation in one of the variables) such that the commutator $[a, D] \in F^B_{j-1}\mathcal{D}_n$ is nonzero.

2.8.2. Exercise (Hilbert polynomials)**.** Consider the commutative algebra $\mathcal{A} = K[x_1, \ldots, x_n]$ with graded piece $\mathcal{A}_i$ consisting of polynomials of total degree i. Consider the module $\mathcal{M} = \mathcal{A}/(x_{d+1}, \ldots, x_n)$ obtained by quotienting by the ideal generated by some of the variables, with induced grading.

(i) Compute $\dim \mathcal{M}_i$ and the associated Hilbert polynomial.
(ii) What is the leading term of the Hilbert polynomial?

2.8.3. Exercise (Integer-valued polynomials)**.** Suppose that $p(t) \in \mathbb{C}[t]$ is a polynomial with complex coefficients, with the property that $p(n) \in \mathbb{Z}$ for every integer $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Prove that $p(t)$ is a $\mathbb{Z}$-linear combination of the polynomials

$$p_d(t) := \binom{t}{d} \quad \text{where} \quad \binom{t}{d} := \frac{t(t-1)\cdots(t-(d-1))}{d!}$$

Note, in particular, that $p_d(t)$ is a degree d polynomial which has rational coefficients but integer values.

Hint: Use induction on the degree and the "discrete derivative" $(\delta p)(t) := p(t+1) - p(t)$.

2.8.4. Exercise (Explicit calculations with D-modules)**.** Show that $\mathcal{M}_{\delta_0'} \cong \mathcal{M}_{\delta_0}^{\oplus 2}$ as $\mathcal{D}_1$-modules, see §2.1.8 for notation.

2.8.5. Exercise (An exact sequence of $\mathcal{D}$-modules)**.** Let $\mathcal{D} := \mathcal{D}_1 = K\langle x, \partial\rangle$ be the Weyl algebra of affine space, equipped with its Bernstein filtration.

Consider the $\mathcal{D}$-module $\mathcal{M}_{1/x} := K[x, x^{-1}] = \mathcal{D}_1/\mathcal{D}_1 \cdot (\partial x)$ which is generated over $\mathcal{D}_n$ by the element $1/x$.

(i) Check that the notation is consistent, i.e. establish an isomorphism of $\mathcal{D}_n$-modules between $K[x, x^{-1}]$ and $\mathcal{D}_1/\mathcal{D}_1 \cdot (\partial x)$, where ∂x is the differential operator annihilating $1/x$.
(ii) Equip $\mathcal{M}_{1/x}$ with a good filtration $F_i\mathcal{M}$ and compute the dimensions $\dim F_i\mathcal{M}$.
(iii) Show that there is an exact sequence

$$0 \to \mathcal{M}_1 \hookrightarrow \mathcal{M}_{1/x} \twoheadrightarrow \mathcal{M}_{\delta_0} \to 0$$

and compute the characteristic cycles of each of these modules.

Recall that $\mathcal{M}_{\delta_0} := K[\partial]$, see §2.1.8.

2.8.6. Exercise (Holonomic $\mathcal{D}$-modules)**.**

(i) Suppose that $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are $\mathcal{D}_n$-modules and we have an exact sequence
$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0.$$
Show that $\mathcal{M}$ is holonomic if and only if $\mathcal{M}'$ and $\mathcal{M}''$ are holonomic.

(ii) Show that any holonomic $\mathcal{D}_n$-module $\mathcal{M}$ has finite length, i.e. it admits a finite nested collection of submodules $\mathcal{M}_i \subsetneq \mathcal{M}_{i+1}$ such that each subquotient $\mathcal{M}_i/\mathcal{M}_{i+1}$ is simple, i.e. without any proper $\mathcal{D}_n$-submodule.

(iii) Show that holonomic $\mathcal{D}_n$-modules are both Artinian and Noetherian, i.e. any increasing or decreasing chain of submodules eventually stabilizes.

3. Hypergeometric equations

Outline of section.

3.1. Basic constructions

For this section, we work over some background field K and express most of our constructions geometrically.

3.1.1. Setup. The $\mathcal{D}$-modules of interest will live on the variety $\mathbf{V}$, an n-dimensional vector space. Let $\mathbf{T}$ be a k-dimensional torus, i.e. $\mathbf{T} \cong \mathbf{G}_m^k$ where $\mathbf{G}_m := \operatorname{Spec} K[z, z^{-1}]$, acting on $\mathbf{V}$.

To describe the action, denote by M the character lattice of $\mathbf{T}$. Then we have a decomposition

$$\mathbf{V} = \bigoplus_{\mathbf{a} \in A} \mathbf{V}_{\mathbf{a}} \tag{3.1.2}$$

where $A \subset M$ is a finite set of characters.

3.1.3. Standing assumptions. We assume for convenience that each $\mathbf{V}_a$ is 1-dimensional, this is not essential but simplifies the discussion. The next assumptions are more important:

(i) The set of weights $A \subset M$ spans M as a $\mathbb{Z}$-module.

(ii) There exists a $\mathbb{Z}$-linear map $h\colon M \to \mathbb{Z}$ such that $h(A) = 1$.

This assumption is equivalent to the existence of a map $\mathbf{G}_m \xrightarrow{\chi_h} \mathbf{T}$ such that the induced action of $\mathbf{G}_m$ on $\mathbf{V}$ is by scalings. Indeed $\operatorname{Hom}(M, \mathbb{Z})$ is the cocharacter lattice of $\mathbf{T}$, h is an element in it, and viewing $\mathbb{Z}$ as the character lattice of $\mathbf{G}_m$ yields directly the equivalence.

(iii) The intersection of $\mathbb{R}_{\geq 0} \cdot A$ with M is equal to $\mathbb{Z}_{\geq 0} \cdot A$, in other words all the integer points in the cone spanned by A can be obtained as positive combinations of elements of A.

This last assumption is needed to ensure good commutative-algebraic properties of some objects defined later on, see [Beu11, Thm. 6.2] and .

3.1.4. The kernel lattice. Let $\mathbb{Z}^A$ denote a free $\mathbb{Z}$-module generated by elements e_a, one for each $a \in A$. Then we have a fundamental short exact sequence

$$0 \to L \hookrightarrow \mathbb{Z}^A \xrightarrow{\pi_A} M \to 0 \tag{3.1.5}$$

where L is defined to be the kernel π_A that sends e_a to $a \in A \subset M$.

We assume that $\operatorname{rk}_{\mathbb{Z}} L = d$ so we have that $n = d + k$ where $n = \#A$ and $d = \operatorname{rk}_{\mathbb{Z}} M$.

3.1.6. Explicit coordinates. It will be convenient to use explicit coordinates, although it is possible to formulate most constructions below more invariantly.

Enumerate the set $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and let $(v_1, \dots v_n)$ be coordinates on $\mathbf{V}$, with v_i corresponding to $\mathbf{V}_{a_i}$ in Eqn. (3.1.2). Let also $(t_1, \dots, t_k)$ denote the coordinates on $\mathbf{T}$, corresponding to characters $\chi_1, \dots, \chi_k \in M$ that give a basis.

The set of weights $A \subset M$ now leads to a $k \times n$ matrix, with

$$\begin{aligned}
\mathbf{a}_1 &= a_1^1\chi_1 + \cdots + a_1^k\chi_k \\
&\vdots \\
\mathbf{a}_j &= a_j^1\chi_1 + \cdots + a_j^k\chi_k \\
&\vdots \\
\mathbf{a}_n &= a_n^1\chi_1 + \cdots + a_n^k\chi_k
\end{aligned}$$

Note that $\{a_j^i\}$ represents the matrix of the linear map $\pi_A \colon \mathbb{Z}^A \to M$ from Eqn. (3.1.5), and each row in the expression above determines a column of the matrix.

This matrix also allows us to make explicit the torus action on the vector space:

$$(t_1, \dots, t_k) \cdot (v_1, \cdots, v_n) = \left(\left[\prod_{i=1}^{k} t_i^{a_i^1} \right] \cdot v_1, \dots, \left[\prod_{i=1}^{k} t_i^{a_i^n} \right] \cdot v_n, \right)$$

where t_i are nonzero scalars, so raising them to integer powers is well-defined. One could abbreviate it slightly by setting $\mathbf{t} := (t_1, \dots, t_k)$ and $\mathbf{t}^{\mathbf{a}_j} := t_1^{a_1^j} \cdots t_k^{a_k^j}$, a notation we will use later on.

3.1.7. Scaling vector fields. The torus $\mathbf{T}$ is equipped with the $\mathbf{T}$-invariant vector fields $t_i\partial_{t_i}$ with $i = 1, \dots, k$. The scaling action of $\mathbf{T}$ on $\mathbf{V}$, described by the set A, determines then a collection of k vector fields on $\mathbf{V}$:

$$Z_i := a_i^1 \left(v_1\partial_{v_1}\right) + \cdots a_i^j \left(v_j\partial_{v_j}\right) + \cdots + a_i^n \left(v_n\partial_{v_n}\right)$$

Note that the vector field $v_j\partial_{v_j}$ yields the scaling action on $\mathbf{V}_{\mathbf{a}_j}$ and the integer a_i^j determines how "fast" the scaling is.

It is a bit more convenient to index the vector fields by elements $m \in M^\vee := \mathrm{Hom}(M, \mathbb{Z})$, which is the cocharacter lattice of the torus $\mathbf{T}$, and we will write Z_m for the corresponding vector field.

3.1.8. The "box" equations. We return now to the fundamental exact sequence Eqn. (3.1.5) and consider the lattice $L \subset \mathbb{Z}^A$. A vector $\mathbf{l} \in L$ has coordinates $\mathbf{l} = (l_1, \ldots, l_n)$ and satisfies

$$\sum_{i=1}^{n} l_i \mathbf{a}_i = 0 \text{ as an element of } M.$$

Note that because $h(\mathbf{a}_i) = 1, \forall i$ it follows that $\sum_{i=1}^n l_i = 0$. Define for every $\mathbf{l} \in L$ the "box operator"

$$\Box_{\mathbf{l}} := \prod_{l_i > 0} \partial_{v_i}^{l_i} - \prod_{l_i < 0} \partial_{v_i}^{|l_i|}$$

Note that the operators $\Box_{\mathbf{l}}$ are homogeneous and satisfy $[\Box_{\mathbf{l}}, \Box_{\mathbf{l}'}] = 0$ (since only derivatives appear in their definition).

3.1.9. The GKZ hypergeometric equations. With all these preliminaries, we are now ready to state the GKZ, also known as A-hypergeometric, system of equations for a function Φ on $\mathbf{V}$. The system depends on a parameter $\mu \in M_{\mathbb{C}}$ and reads:

$$\begin{cases} Z_m \Phi = \mu(m)\Phi & \forall m \in M^\vee \\ \Box_{\mathbf{l}}\, \Phi = 0 & \forall l \in L \end{cases} \tag{3.1.10}$$

Note that for the first set of equations, it suffices to use a basis of k vector fields, for instance the Z_i defined in §3.1.7. For the second set of equations, note that while in general $\Box_{\mathbf{l}} \cdot \Box_{\mathbf{l}'} \neq \Box_{\mathbf{l}+\mathbf{l}'}$, it is true that only finitely many of them suffice to imply all the other ones by the Hilbert basis theorem.

Appendix A. Toric varieties

Outline of section. *Motto: Toric varieties are to general algebraic varieties like abelian groups are to general groups.*

For more details on the notions below, and much more, see Ewald's textbook [Ewa96].

A.1. Some convex geometry

A.1.1. Setup. Let $L_{\mathbb{R}}$ be a finite-dimensional real vector space and $L_{\mathbb{R}}^\vee$ its dual. For many constructions, it will be useful to keep in mind that they can be performed in both spaces, and that we have a natural isomorphism $L_{\mathbb{R}}^{\vee\vee} \cong L_{\mathbb{R}}$. We now describe some constructions based on notions of convexity.

A.1.2. Polytopes and Cones. Define a *polytope* $\mathcal{P}$ to be the convex hull of finitely many points in L. Exhibiting a polytope in this manner is to give a "vertex representation". More generally, one can define a polytope as the intersection of finitely many affine half-spaces $H_{\xi,a} = \{l \in L \colon \xi(l) \geq a\}$ for given $\xi \in L^\vee_\mathbb{R}$ and $a \in \mathbb{R}$. Note that such more general polytopes need not be compact.

A *cone* in $L_\mathbb{R}$ is the intersection of finitely many linear half-spaces, of the form $H_{\xi,0} = \{l \in L \colon \xi(l) \geq 0\}$ for $\xi \in L^\vee_\mathbb{R}$. Note that cones are invariant under scaling by $\mathbb{R}^\times_{\geq 0}$. We will refer to polytopes and cones as convex bodies.

Note also that what our definition of a "cone" is called in the literature a "polyhedral cone", and general cones are convex subsets of a real vector space invariant by positive scalings.

A.1.3. Stratification by faces. Every polytope has a stratification into faces, which are polytopes themselves. Specifically, if a polytope $\mathcal{P}$ is given as the intersection of half-spaces $\xi_i(l) \geq a_i$, then faces are given as intersections of $\mathcal{P}$ with hyperplanes $\xi_i(l) = a_i$. We can define a partial order on the faces of the polytope, with smallest elements given by vertices.

The *relative interior* $\mathcal{F}^\circ$ of a face $\mathcal{F} \subset \mathcal{P}$ is defined to be the set of points of $\mathcal{P}'$ that does not belong to a smaller face of $\mathcal{P}$. This definition applies in particular to $\mathcal{P}$ itself and determines its relative interior $\mathcal{P}^\circ$. Every point $p \in \mathcal{P}$ belongs to the relative interior of a unique face.

A.1.4. Dual convex body. Given a convex body $K \subset L_\mathbb{R}$, we will define its dual body $K^\vee \subset L^\vee_\mathbb{R}$ as follows. For a cone $\mathcal{C} \subset L_\mathbb{R}$, its dual is defined by:

$$\mathcal{C}^\vee := \{\xi \in L^\vee_\mathbb{R} \colon \xi|_\mathcal{C} \geq 0\}$$

while for a polytope $\mathcal{P} \subset L_\mathbb{R}$ we define the dual by

$$\mathcal{P}^\vee := \{\xi \in L^\vee_\mathbb{R} \colon \xi|_\mathcal{P} \leq 1\}.$$

Note that the double dual of a cone is the original cone.

A.1.5. Fans from polytopes. Given a polytope $\mathcal{P}$ and a point $p \in \mathcal{P}$, we can associate to this data the cone

$$\mathcal{C}^\vee_{\mathcal{P},p} = \{\xi \in L^\vee_\mathbb{R} \colon \xi(p') \geq \xi(p), \forall p' \in \mathcal{P}\}.$$

One can check that this cone does not change as p varies in the relative interior of the same face $\mathcal{P}'$ of $\mathcal{P}$, so we can associate such a cone to the face and will denote it by $\mathcal{C}^\vee_{\mathcal{P},\mathcal{P}'}$.

We can also associate a cone in $L_\mathbb{R}$, thinking of an observer located at p and how how $\mathcal{P}$ looks like to that observer:

$$\mathcal{C}_{\mathcal{P},p} := \{l \in L_\mathbb{R} \colon \exists \varepsilon > 0 \text{ s.t. } (p + \varepsilon \cdot l) \in \mathcal{P}\}.$$

Then the cones $\mathcal{C}_{\mathcal{P},p}$ and $\mathcal{C}^\vee_{\mathcal{P},p}$ are naturally in duality.

The collection of all cones $\mathcal{C}^\vee_{\mathcal{P},\mathcal{P}'}$, as $\mathcal{P}'$ ranges over all the faces of a polytope $\mathcal{P}$, are an example of the notion of a fan:

A.1.6. Definition (Fan of cones)**.** A *fan* in $L_{\mathbb{R}}^{\vee}$ is a collection of cones $\mathcal{C}_{\tau}^{\vee} \subset L_{\mathbb{R}^{\vee}}$, such that any face of a cone $\mathcal{C}_{\tau}$ is also a cone $\mathcal{C}_{\tau'}$ in the fan.

A.2. Fans and integral polytopes

A.2.1. Setup.

A.3. Exercises

A.3.1. Exercise (Manipulations in convexity)**.**

(i) Suppose $\mathcal{P} \subset L_{\mathbb{R}}$ is a polytope and $l_0 \in L_{\mathbb{R}}$ is a vector. Relate the dual $\mathcal{P}^{\vee}$ to the dual of the translate, i.e. $(\mathcal{P} + l_0)^{\vee}$.

(ii) The same question, but for a linear transformation $A \cdot \mathcal{P}$ of the original polytope.

Appendix B. The Γ-function

Outline of section. To study a function by the methods of complex analysis, two aspects are important: zeros or poles on the one hand, and boundedness or growth on the other. Most of the basic results on the Γ-function can be viewed with this lens and its essential features are the location of the poles at the negative integers and nonvanishing elsewhere on the one hand, and boundedness on vertical strips (in the right halfplane) and the functional equation $\Gamma(s+1) = s\Gamma(s)$ on the other.

B.1. Algebraic Properties

An elegant derivation of most algebraic properties of the Γ-function is in the article of Remmert [Rem96]. We describe below the key features and sketch the arguments for their justification, leaving the full derivation to the exercises.

B.1.1. Integral representation. One starting point is to define the Γ-function as a Mellin transform of the function e^{-x}:

$$\Gamma(s) := \int_0^{\infty} x^s e^{-x} \frac{dx}{x} \quad \text{for } \operatorname{Re} s > 0.$$

This integral converges absolutely in the right half-plane in $\mathbb{C}$, and a standard manipulation with integration by parts in the integrand yields the *recursion formula*:

$$\Gamma(s+1) = s \cdot \Gamma(s) \quad \text{for } \operatorname{Re} s > 0. \tag{B.1.2}$$

This immediately connects the Γ-function to the factorial:

$$\Gamma(n+1) = n! \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \tag{B.1.3}$$

by evaluating the corresponding integral to find $\Gamma(1) = 1$ and then applying the recursion formula.

B.1.4. Analytic continuation of Γ. The recursion formula of Eqn. (B.1.2) allows us to define $\Gamma(s)$ as a meromorphic function on all of $\mathbb{C}$, by observing that

$$\Gamma(s) := \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{s(s+1)} \cdots = \frac{\Gamma(s+n-1)}{s \cdots (s+n)}$$

By taking n large enough, we can ensure that $\operatorname{Re} s + n - 1 > 0$ and so define $\Gamma(s)$ for arbitrary $s \in \mathbb{C}$.

Observe that while $\Gamma(s)$ is holomorphic in $\operatorname{Re} s > 0$, it acquires poles at $s = 0, -1, \ldots, -n, \ldots$ with

$$\operatorname{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$$

It turns out that $\Gamma(s)$ doesn't vanish anywhere and so $\frac{1}{\Gamma(s)}$ is a holomorphic function, with zeros precisely at the nonpositive integers. See Exercise B.3.3 for a derivation of these properties, as well as an infinite product expansion for $1/\Gamma$.

The location of zeros of $1/\Gamma$ indicates that there should be a relation with the trigonometric functions, which we know to vanish at all the integers. We next develop this property.

B.1.5. Reflection formula. The function $\frac{1}{\Gamma(s)\Gamma(1-s)}$ vanishes precisely at the integers, as does $\sin(\pi \cdot s)/\pi$. Furthermore, standard growth bounds on the Γ-function allow us to deduce that the functions must agree up to a multiplicative scaling, by Hadamard's factorization theorem (holomorphic functions with the same zeros, and bounds on growth, should agree). But a Taylor series comparison at $s = 0$ shows that the multiplicative factor is 1, so

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{1}{\pi}\sin(\pi s) \text{ as holomorphic functions.} \tag{B.1.6}$$

B.1.7. Duplication, and multiplication formulas. By considering the structure of zeros, we expect that $\Gamma(s)\Gamma(s+\frac{1}{2})$ and $\Gamma(2s)$ should be closely related, both having poles precisely at the negative half-integers. Manipulations with the Gauss product formula, see Exercise B.3.3, yield:

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = 2^{1-2s}\pi^{1/2}\Gamma(2s)$$

Analogous "multiplication formulas" exist for arbitrary $k \in \mathbb{N}$ instead of 2.

B.2. Analytic properties

We next establish some estimates for the behavior of the Γ-function.

B.2.1. Proposition (Comparison to factorial)**.** *For $s \in \mathbb{C}$, there exists a constant $C > 0$ such that for any $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\frac{1}{\Gamma(s+n)} \leq C \frac{|n^{-(s-1)}|}{n!}$$
$$\frac{1}{\Gamma(s-n)} \leq C \frac{|n^{s}|}{n!}$$

Proof. See [Beu11, Lemma 16.3], but stated in a mangled way. □

B.2.2. Proposition (Balanced multifactorials)**.** *Suppose that $l = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ and assume that $l_1 + \ldots + l_N = 0$. Then we have*

$$\frac{\prod_{l_i<0} |l_i|!}{\prod_{l_i>0} l_i!} \leq N^{\frac{1}{2}\|l\|_1}$$

where $\|l\|_1 := |l_1| + \ldots + |l_N|$.

B.3. Exercises

B.3.1. Exercise (Decay properties of Γ)**.**

(i) Prove from the definition as a Mellin transform §B.1.1 that $|\Gamma(x + \sqrt{-1}y)| \leq |\Gamma(x)|$ for $x, y \in \mathbb{R}$ and $x > 0$. Deduce that $\Gamma(s)$ is bounded for $\operatorname{Re} s \in [1, 2]$.

(ii) Deduce from the reflection formula Eqn. (B.1.6) that

$$|\Gamma(x + \sqrt{-1}y)| \lesssim_x e^{-\pi|y|} \text{ for } x \in (0, 1)$$

and deduce analogous bounds in arbitrary strips.

B.3.2. Exercise (Wielandt's theorem)**.** Suppose that $F(s)$ is a holomorphic function in the right halfplane $\{\operatorname{Re} s > 0\}$ and satisfies the following properties:

Recursion formula: $F(s+1) = s \cdot F(s)$
Boundedness in vertical strip: $F(s)$ is bounded in the strip $\operatorname{Re} s \in [1, 2)$
Normalization: $F(1) = 1$

We will prove that $F(s) = \Gamma(s)$ in the following steps:

(i) Set $f(s) := F(s) - \Gamma(s)$ and show that it satisfies $f(s+1) = s \cdot f(s)$. Use this to show that it extends to a *holomorphic* function on all of $\mathbb{C}$, still denoted $f(s)$.

(ii) Show that $f(s)$ is bounded in $\operatorname{Re} s \in [0, 1)$.

(iii) Set now $g(s) = f(s)f(1-s)$. Use the recursion formula for $f(s)$ to establish that $g(s+1) = -g(s)$, and hence that $g(s)$ is bounded on all of $\mathbb{C}$. Use this to conclude that $g(s)$ is the zero function.

Many developments of the properties of the Γ-function establish first the reflection formula Eqn. (B.1.6) and use it to deduce that the Γ-function doesn't vanish. We will follow an alternative route to this nonvanishing.

B.3.3. Exercise (Gauss formula and nonvanishing)**.** Define Euler's constant as:

$$\gamma := \lim_{n\to+\infty}\left[\left(\sum_{k=1}^{n}\frac{1}{k}\right)-\log n\right]$$

and consider the function

$$\Delta(s) := ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{s}{k}\right)e^{-\frac{s}{k}}.$$

(i) Show that the infinite product defining $\Delta(s)$ converges uniformly on every compact subset of $\mathbb{C}$.

(ii) Show that we have the alternative formula

$$\Delta(s) = \lim_{n\to+\infty}\frac{s(s+1)\cdots(s+n)}{n!\cdot n^s}$$

(iii) Use the preceding formula to establish that $\frac{1}{\Delta(s)}$ is bounded in the strip $\operatorname{Re} s \in [1,2]$ and that $(s+1)\Delta(s) = \Delta(s)$.

(iv) Use Wielandt's theorem (Exercise B.3.2) to establish that

$$\Gamma(s) = \frac{1}{\Delta(s)} \text{ and that } \Gamma(s) \text{ has no zeros in } \mathbb{C}.$$

B.3.4. Exercise (Proof of the reflection formula)**.** To prove the reflection formula Eqn. (B.1.6) and also the nonvanishing, we will use that $|\Gamma(s)|$ is uniformly bounded in the strip $[1,2]$, see Exercise B.3.1.

(i) Check that $f(s) := \Gamma(s)\Gamma(1-s)\sin(\pi s)$ vanishes nowhere and satisfies $f(s+1) = f(s)$.

(ii) Prove that $|f(s)| \lesssim e^{\pi|\operatorname{Im} s|}$ for $\operatorname{Re} s \in [0,1]$.

(iii) Using part (i) show that there exists a holomorphic $\tilde{f}$ defined on $\mathbb{C}^\times$ such that

$$f(s) = \tilde{f}\left(e^{2\pi\sqrt{-1}s}\right) \text{ and } |\tilde{f}(q)| \lesssim \max\left(|q|, \frac{1}{|q|}\right)^{\frac{1}{2}}.$$

(iv) Conclude that $\tilde{f}$ is a constant function and compute the constant.

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